Theoretical Computer Science 573 (2015) 43–53

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

# The max-distance network creation game on general host graphs ☆

Davide Bilò [a], Luciano Gualà [b], Stefano Leucci [b], Guido Proietti [c,d,*]

[a] Dipartimento di Scienze Umanistiche e Sociali, Università di Sassari, Italy
[b] Dipartimento di Ingegneria dell'Impresa, Università di Roma "Tor Vergata", Italy
[c] Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università degli Studi dell'Aquila, Italy
[d] Istituto di Analisi dei Sistemi ed Informatica, CNR, Rome, Italy

ARTICLE INFO

*Article history:*
Received 28 February 2014
Received in revised form 12 December 2014
Accepted 23 January 2015
Available online 28 January 2015
Communicated by X. Deng

*Keywords:*
Network creation games
Pure Nash equilibrium
Price of Anarchy
Host graph

ABSTRACT

In this paper we study a generalization of the classic *network creation game* in the scenario in which the $n$ players sit on a given arbitrary *host graph*, which constrains the set of edges a player can activate at a cost of $\alpha \geq 0$ each. This finds its motivations in the physical limitations one can have in constructing links in practice, and it has been studied in the past only when the routing cost component of a player is given by the sum of distances to all the other nodes. Here, we focus on another popular routing cost, namely that which takes into account for each player her *maximum* distance to any other player. For this version of the game, we first analyze some of its computational and dynamic aspects, and then we address the problem of understanding the structure of associated pure Nash equilibria. In this respect, we show that the corresponding price of anarchy (PoA) is fairly bad, even for very simple host graph topologies. More precisely, we first exhibit a lower bound of $\Omega(\sqrt{n/(1+\alpha)})$ for any $\alpha = o(n)$. Notice that this implies a counter-intuitive lower bound of $\Omega(\sqrt{n})$ even for $\alpha = 0$, i.e., when edges can be activated for free. Then, we show that when the host graph is restricted to be either $k$-regular (for any constant $k \geq 3$), or a 2-dimensional grid, the PoA is still $\Omega(1 + \min\{\alpha, \frac{n}{\alpha}\})$, which is proven to be tight for $\alpha = \Omega(\sqrt{n})$. On the positive side, if $\alpha \geq n$, we show that the PoA is at most 2. Finally, in the case in which the host graph is very sparse (i.e., $|E(H)| = n - 1 + k$, with $k = O(1)$), we prove that the PoA is $O(1)$, for any $\alpha$.



## 1. Introduction

In a *network creation game* (NCG), we are given $n$ players identified as the nodes of a graph, and each player attempts to connect herself to all the other players. In such a decentralized process, each player aims to selfishly optimize a certain *routing cost* she has to pay for communicating on the network with the other players. Thus, her action consists of choosing a suitable subset of players, which are then made adjacent through the activation of the corresponding set of incident edges.

☆ A preliminary version of this paper was presented at the *8th International Workshop on Internet & Network Economics (WINE'12)*, Liverpool, UK, December 9–12, 2012, and appeared in Vol. 7695 of Lecture Notes in Computer Science, Springer, 393–406. This work was partially supported by the Research Grant PRIN 2010 "ARS TechnoMedia", funded by the Italian Ministry of Education, University, and Research.

* Corresponding author.
*E-mail addresses:* davide.bilo@uniss.it (D. Bilò), guala@mat.uniroma2.it (L. Gualà), stefano.leucci@graduate.univaq.it (S. Leucci), guido.proietti@univaq.it (G. Proietti).

http://dx.doi.org/10.1016/j.tcs.2015.01.044
0304-3975/

Unavoidably, activating a link incurs a charge to the player, and so this *building cost* has to be strategically balanced with the aforementioned routing cost.

Due to their generality, and depending on how all the ingredients are mixed, it is evident that NCGs can model very different practical situations, like social networks, peer-to-peer networks, client–server systems, etc. Actually, the original formulation of the game was devised by economists in [7], with the aim of studying the strategic issues of a mutual-relationships social system in which individuals establish autonomously their costly personal contacts, and with the idea that closer connections are more valuable than farther ones. Accordingly, in the formal model each player has no limitations in choosing a subset of adjacent players, and her cost function is the *sum* of the building cost and the routing cost, where this latter is a function of the *distances* in the underlying graph to all the other players.

With the recent advent of the algorithmic game theory, the interest on NCGs has been reawakened. This is especially due to the fact that NCGs are fit to model the decentralized construction of *communication* networks, in which the constituting components (e.g., routers and links) are activated and maintained by different owners, as in the Internet. According to their performance measurement philosophy, computer scientists put a new special emphasis on the challenge of understanding how the social cost for the (very large) system as a whole is affected by the selfish behavior of the players. This goal has been pursued through the study of precise *quantitative* bounds with respect to concrete efficiency measures of a strategic system (e.g., the speed of convergence towards an equilibrium status, etc.), while in contrast economists traditionally provided only *qualitative* statements about the inefficiency of equilibria (e.g., violations of Pareto efficiency, etc.). In particular, for NCGs this new algorithmic flavor originated from the paper of Fabrikant et al. [6], and was then followed by a sequel of papers, as detailed in the following.

*Previous work.* The canonical form of a NCG in the computer science community is the so-called *sum-distance* version, often abbreviated by SumNCG. It was initially provided in [6], and it can be formalized as follows: We are given a set of $n$ players, say $V$, where the strategy space of player $v \in V$ is the power set $2^{V \setminus \{v\}}$. Given a combination of strategies $\sigma = (\sigma_v)_{v \in V}$, let $G(\sigma)$ denote the underlying *undirected* graph whose node set is $V$, and whose edge set is $E(\sigma) = \{(v, u) : v \in V \wedge u \in \sigma_v\}$.[1] Then, the *cost* incurred by player $v$ under $\sigma$ is

$$C_v(\sigma) = \alpha \cdot |\sigma_v| + \sum_{u \in V} d_{G(\sigma)}(u, v) \tag{1}$$

where $\alpha \geq 0$ is the *activation cost* of a link, and $d_{G(\sigma)}(u, v)$ is the distance between nodes $u$ and $v$ in $G(\sigma)$. Accordingly, the global *social cost* is given by the sum of all players' costs. Thus, the cost function implements the inherently antagonistic goals of a player, which on one hand attempts to buy as little edges as possible, and on the other hand aims to be as close as possible to the other nodes in the resulting network. These two criteria are suitably balanced in (1) by making use of the parameter $\alpha \geq 0$. Consequently, the set of *Nash Equilibria*[2] (NE) of the game depends on $\alpha$. Actually, if we characterize such a set in terms of the *Price of Anarchy* (PoA), i.e., the ratio between the social cost of a costlier NE and the optimal (centralized) social cost, then this has been shown to be constant for all values of $\alpha$ except for $n^{1-\varepsilon} \leq \alpha \leq 65n$, for any $\varepsilon \geq 1/\log n$ (see [13,15]).

A first natural variant of SumNCG was introduced in [4], where the authors redefined the player's cost function as follows

$$C_v(\sigma) = \alpha \cdot |\sigma_v| + \max_{u \in V} d_{G(\sigma)}(u, v), \tag{2}$$

i.e., they replaced the total distance to all the other nodes by the *eccentricity* of $v$ in $G(\sigma)$. This variant, named MaxNCG, captures the maximum routing delay, as opposed to the average routing delay addressed by the sum-distance model. Thus, for instance, it is of interest in Internet routing, where the maximum latency has to be maintained within a small bound. This model received further attention in [15], where the authors improved the PoA of the game on the whole range of values of $\alpha$, obtaining in this case that the PoA is constant for all values of $\alpha$ except for $129 > \alpha = \omega(1/\sqrt{n})$.

Besides these two basic models, many variations on the theme have been defined. In an effort of defining $\alpha$-free models, in [10] the authors proposed a variant in which a player, when forming the network, has a limited budget to establish links to other players, while her cost function simply consists of the total distance to other nodes. Since in [10] links and hence the resulting graph are seen as directed, a natural variant of the model was given in [5], where the undirected case was considered. Afterwards, in [2] the authors proposed a model complementing the one given in [5]. More precisely, they assumed that the cost function of each player now only consists of the number of bought edges (without any budget on them), and a player needs to connect to the network by satisfying the additional constraint of staying within a given either maximum or total distance to the rest of the players. Then, in [1] the authors proposed a further variant, called BasicNCG, in which given some existing network, the only improving transformations allowed are *edge swap*, i.e., a player can only modify a *single* incident edge, by either replacing it with a new incident edge, or by removing it. Recently, this model has been extended to the case in which edges are oriented and players can swap only outleading edges [14]. Notice that,

[1] Notice that, in contrast with standard notation in graph theory, but according to other papers on NCGs, we use $(u, v)$ instead of $\{u, v\}$ to denote an *undirected* edge of $G$, to remind that $u$ is the owner of the edge.

[2] In this paper, we only focus on *pure* strategies Nash equilibria.

differently from the previous models, the problem of computing a best-response strategy of a player in BasicNCG is *not* NP-hard. This inspired in [12] a subsequent model, in which a player can swap, add, or delete a single edge. Here the best response strategy is still computable in polynomial time, while the player has more freedom to act.

Generally speaking, in all the above models the obtained results on the PoA are asymptotically worse than those we get in the two basic models, and we refer the reader to the cited papers for the actual bounds.

*Our results.* In this paper we concentrate on a seemingly underplayed generalization of NCGs, namely that in which for each player the set of possible adjacent nodes is constrained by a given connected, undirected graph $H$, which in the end will host the created network. In communication infrastructures, this restriction finds its practical motivations in the physical limitations one could have in constructing links. This variant was originally studied for SumNCG in [3], where it is shown that the PoA is upper bounded by $O(1+\sqrt{\alpha})$ and $O(1+\min\{\sqrt{n}, n^2/\alpha\})$ for $\alpha < n$ and $\alpha \geq n$, respectively, and lower bounded by $\Omega(1+\min\{\alpha/n, n^2/\alpha\})$. Here, we focus on the max-distance version, that we call MaxNCG$(H,\alpha)$,[3] and we show that also in this case the PoA is fairly bad,[4] since we exhibit a lower bound of $\Omega(\sqrt{\frac{n}{1+\alpha}})$ for $0 \leq \alpha = o(n)$. Quite surprisingly, this implies that the PoA is $\Omega(\sqrt{n})$ also when the players can build edges for free! Moreover, even when the host graph is restricted to some basic layout patterns, the lower bound to the PoA can be shown to be highly sensitive to the value of $\alpha$. This is a quite interesting result, since it conveys the message that the loss of efficiency induced by decentralization can be very large also on simple host graph topologies, depending on $\alpha$. More precisely, we show that the PoA is $\Omega(1+\min\{\alpha, \frac{n}{\alpha k}\})$ for $k$-regular graphs, for $3 \leq k = o(\frac{n}{\alpha})$, and $\Omega(1+\min\{\alpha, \frac{n}{\alpha}\})$ for 2-dimensional grids (which are also planar and bipartite). Since we can prove a general upper bound of $O(1+\frac{n}{\alpha+\rho_H})$, where $\rho_H$ is the *radius* of $H$,[5] it follows that these lower bounds are asymptotically tight, whenever $k$ is constant and $\alpha = \Omega(\sqrt{n})$. On the positive side, if $\alpha \geq n$, we show that the PoA is at most 2 (this is a direct consequence of the fact that in this case any equilibrium is a tree). Moreover, we show that the social cost of every stable tree (if any), is $O(\min\{1+\alpha, \rho_H\})$ times the optimum. This generalizes a result in [15] holding for MaxNCG, which states that the social cost of every acyclic equilibrium is $O(1)$ times the optimum. Finally, in the meaningful practical case in which the host graph is sparse (i.e., $|E(H)| = n-1+h$, with $h = O(n)$), we prove that the PoA is $O(1+h)$, and so for very sparse graphs (i.e., $h = O(1)$) we obtain that the PoA is constant.

Preliminarily to the above study, we also provide some results concerning the computational and dynamic aspects of the game. First, we prove that computing a best response for a player is NP-hard for any $0 < \alpha = o(n)$, thus extending a similar result given in [15] for MaxNCG when $\alpha = 2/n$. Then, we prove that MaxNCG$(H,\alpha)$ is not a potential game, by showing that an improving response dynamic does not guarantee to converge to an equilibrium, even if we assume a minimal *liveness* property that no player is prevented from moving for arbitrarily many steps. This implies that an improving response dynamic may not converge for the MaxNCG game as well (after relaxing such a liveness property). A deeper discussion on dynamics in NCGs can be found in [9,11].

*Paper organization.* The paper is organized as follows: in Section 2 we analyze the computational/convergence aspects of the game, while Sections 3 and 4 discuss the upper and lower bounds to the PoA, respectively. Finally, in Section 5 we conclude the paper by providing some directions for future research.

## 2. Preliminary results

Let us start by recalling that a NE is a strategy profile $\sigma$ in which no player can decrease her cost by changing her strategy, assuming that the strategies of the other players are fixed. Thus, in our case, when a strategy profile $\sigma$ is a NE of MaxNCG$(H,\alpha)$, we will say that the corresponding subgraph $G(\sigma)$ of $H$ is *stable*. Conversely, a subgraph $G$ of $H$ is said to be stable if there exists a NE $\sigma$ of MaxNCG$(H,\alpha)$ such that $G = G(\sigma)$.

First of all, we observe that, as for the sum-distance version of the problem studied in [3], it is open to establish whether MaxNCG$(H,\alpha)$ always admits an equilibrium. This problem is particularly intriguing, since the topology of $H$ could play a discriminating role on that. We conjecture an affirmative answer to this question for any $\alpha > 0$ (for $\alpha = 0$ it is trivially true, as any strategy profile $\sigma$ such that $G(\sigma) = H$ is a NE). However, since MaxNCG$(H,\alpha)$ is not a potential game, as we will prove soon, it follows that this problem is really challenging, and to solve it either more elaborate fixed-point arguments need to be used, or some structural property of the host graph has to be exploited. As a first step towards this direction, we can provide the following:

[3] Notice that differently form the canonical version, for the sake of clarity we make explicit here the dependency of the game from both the host graph and the parameter $\alpha$.

[4] According to the spirit of the game, we concentrate on connected equilibria only. In fact, to avoid pathological disconnected equilibria, we can slightly modify the player's cost function (2) as it was done in [5], in order to incentivize the players to converge to connected equilibria. Alternatively, this can be obtained by assuming that initially the players sit on a connected network (embedded in the host graph), and they move (non-simultaneously) with a myopic best/improving response dynamics.

[5] Recall that the radius of a graph is the minimum eccentricity among all nodes in the graph.

**Proposition 1.** *If the host graph $H$ has radius $\rho_H$ and $\alpha \geq \rho_H$, then* MaxNCG$(H, \alpha)$ *admits an equilibrium.*

**Proof.** Let $T$ be a breadth-first search tree rooted at a center of $H$, and in which each node owns the edges towards its children. Observe that $T$ has radius $\rho_H$. We show that $T$ is an equilibrium. Indeed, notice that each vertex $u$ has no unactivated edges towards the vertices of its subtrees in $T$. This immediately implies that $u$ cannot improve its cost by changing to a strategy having at most the same number of bought edges. Moreover, in any other strategy, $u$ must incur a cost of at least $\alpha \geq \rho_H$ to buy some additional edges, while its eccentricity, which is at most $2\rho_H$, cannot decrease by more than $\rho_H$, since it can never be less than $\rho_H$. $\square$

Besides that, and similarly to other NCGs, we also have the bad news that the problem of computing a best response of a player is NP-hard, as stated in the following theorem.

**Theorem 1.** *For every $0 < \alpha = o(n)$, the problem of computing a best response strategy of a player in* MaxNCG$(H, \alpha)$ *is* NP*-hard.*

**Proof.** The reduction is from the *3-Exact 3-Cover* problem (3X3C for short), which was shown to be NP-complete in [8]. Such a problem, given (i) a set $O$ of $3k$ objects, with $k \in \mathbb{N}$, and (ii) a set $\mathcal{S}$ of subsets of $O$ each having cardinality equal to 3 and such that every $o \in O$ is a member of at most three sets in $\mathcal{S}$, asks for determining whether there exists a subset $\mathcal{S}' \subseteq \mathcal{S}$ of cardinality equal to $k$ that covers $O$, i.e., $\bigcup_{S \in \mathcal{S}'} S = O$. The reduction is the following. Let $\eta > (k+1)\alpha$. For a given instance of the 3X3C problem with $3k$ objects, we build a host graph $H$ having $n = \Theta(\eta k)$ vertices. More precisely, $H$ contains an object vertex $v_o$ for every object $o \in O$, and a set vertex $u_S$ for every $S \in \mathcal{S}$. For every $o \in O$ and for every $S \in \mathcal{S}$, $H$ contains a path of length $\eta$ having $v_o$ and $u_S$ as endpoints iff $o \in S$. All paths are vertex-disjoint except for at most their endpoints. Finally, $H$ contains an additional path $P$ of length $2\eta$ which is vertex-disjoint with respect to all the other paths, and an additional vertex $x$ linked to all the set vertices and to one endpoint of $P$, that we denote by $z$.

Let $\sigma$ be any strategy profile such that $G(\sigma)$ contains all edges of $H$ except those incident to $x$. We claim that any best response strategy of node $x$ has cost equal to $\alpha(k+1) + 2\eta + 1$ iff $\mathcal{S}$ contains a subset of cardinality $k$ that covers $O$.

To prove one direction, observe that if $\mathcal{S}' \subseteq \mathcal{S}$ has size $k$ and covers $O$, then, by buying the edge towards $z$ and all the $k$ edges $(x, u_S)$ for every $S \in \mathcal{S}'$, $x$ would incur a cost equal to $\alpha(k+1) + 2\eta + 1$. Indeed, it is easy to see that each vertex of $G(\sigma)$ which is neither $x$ nor a vertex in $P$ is at distance of at most $\eta$ from some object vertex. Furthermore, each object vertex is at distance of at most $\eta$ from some set vertex $u_S$ such that $S \in \mathcal{S}'$.

To prove the other direction, let $\sigma'$ be the strategy profile obtained from $\sigma$ by changing $x$'s strategy with one of its best response strategies, and assume that $C_x(\sigma') = \alpha(k+1) + 2\eta + 1$. Let $\mathcal{S}'$ be the subset of $\mathcal{S}$ containing set $S$ iff $x$ is buying the edge towards $u_S$ in $\sigma'$. First of all, observe that if $\mathcal{S}'$ does not cover $O$, or $x$ has not bought the edge towards $z$, then the routing cost of $x$ is at least $3\eta + 1 > \alpha(k+1) + 2\eta + 1$. Consequently, $x$ has bought the edge towards $z$, and $\mathcal{S}'$ covers $O$. Therefore $|\mathcal{S}'| \geq k$. As the distance from $x$ to the endpoint of $P$ different from $z$ is $2\eta + 1$, and since $C_x(\sigma') = \alpha(k+1) + 2\eta + 1$, it follows that $|\mathcal{S}'| = k$. $\square$

Notice that the result stated in Theorem 1 holds for any $0 < \alpha = o(n)$, and so this extends the NP-hardness proof given in [15] for MaxNCG when $\alpha = 2/n$, that of course carries on to our game by assuming that the host graph is complete.

We now discuss a negative result about the convergence of the improving response dynamics. To the best of our knowledge, this is the first result showing that an improving response dynamics on a max-version of a NCG might not converge to an equilibrium. A deeper discussion about dynamics in NCGs can be found in [9,11].

**Theorem 2.** *For every value of $\alpha < \frac{n}{2} - 6$,* MaxNCG$(H, \alpha)$ *is not a potential game. Moreover, if $\alpha > 0$, an improving response dynamics may not converge to an equilibrium.*

**Proof.** To prove the first part of the statement, we show the non existence of a potential function by providing a cyclic sequence of strategies where (i) the cost of each moving player does not increase when she changes her strategy, (ii) at least one moving player improves her cost, and (iii) at the end of the cycle, the costs of moving players will coincide with the initial ones.

Let $l > \alpha + 6$ be an integer satisfying $l \equiv 1 \pmod 3$, and consider a host graph $H$ similar to the one shown in Fig. 1. $H$ is composed by a cycle of $l$ nodes labeled from 1 to $l$, by a path of $l-2$ edges having $x$ and $z$ as endpoints, and by all the edges between $x$ and the nodes of the cycle. The strategy profile $\sigma_a$ being played is shown by using a graphical notation explained in the caption.

In such a status, node 1 is paying $\alpha + l - 1$, whilst changing the strategy to $\sigma_b$ by removing the edge $(1, 2)$ yields a cost of $l - 1$, thus saving $\alpha$. Observe that now $C_x(\sigma_b)$ is $\alpha + l$, and so $x$ has interest in swapping the edge $(x, 1)$ with the edge $(x, 4)$, thus obtaining the strategy $\sigma_c$ where it saves 1. In such a configuration $C_1(\sigma_c)$ has increased to $2l - 4$, therefore node 1 can buy back the edge $(1, 2)$, as shown in strategy $\sigma_d$, thus reducing its cost to $\alpha + l + 2$, i.e., saving $l - (\alpha + 6) > 0$.

Notice how configuration $\sigma_d$ is similar to $\sigma_a$, with the only difference being the edge bought by $x$. Since $l \equiv 1 \pmod 3$, by repeating $l$ times these strategy changes, every node in the cycle $\langle 1, \ldots, l \rangle$ will play a move at least once, and the resulting configuration will be identical to $\sigma_a$, hence the players will cycle.

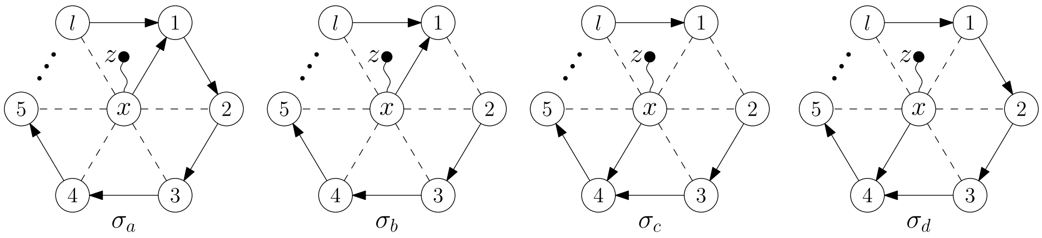


**Fig. 1.** Representation of the strategy changes used in the proof of Theorem 2. On the left side, the initial configuration, where directed edges exit from their respective owner node, dashed edges are non-bought edges of $H$, and the spline denotes a path of length $l-2$ between $z$ and $x$, whose edges are arbitrarily owned.

To prove the latter part of the claim it suffices to note that after each cycle: (i) for $\alpha > 0$ each strategy change is an improving response, and (ii) that the nodes in the path from $x$ to $z$ other than $x$, can never change their strategy. □

Actually, the above proof shows that the improving response dynamics may not converge also when the following *liveness property* has to be guaranteed: each player has a chance to make an improving move every $O(n)$ steps. Indeed, as observed in the proof, the players sitting on the path appended to $x$ do not move just because they cannot! On the other hand, if no liveness condition has to be guaranteed, then we can extend the above proof to the case in which $H$ is a complete graph. Here, it suffices to prevent the players sitting on the path appended to $x$ from moving. This shows that for any $\alpha > 0$, the improving response dynamics may not converge also on complete host graphs, i.e., for the classic MaxNCG.

## 3. Upper bounds

In this section we prove some upper bounds to the PoA for the game. In what follows, for a generic graph $G$, we denote by $\rho_G$ and $\delta_G$ its radius and its diameter, respectively, and by $\varepsilon_G(v)$ the eccentricity of node $v$ in $G$. Moreover, we denote by $SC(\sigma)$ the social cost of a strategy profile $\sigma$ (i.e., the sum of players' individual costs), and by Opt a strategy profile minimizing the social cost.

We start by proving the following result:

**Lemma 1.** *Let $H$ be an arbitrary host graph. Then, for any $\alpha \geq 0$ and for every stable graph $G = G(\sigma)$ of* MaxNCG$(H, \alpha)$*, we have* $SC(\sigma)/SC(\text{Opt}) = O(1 + \frac{\rho_G}{\alpha+\rho_H})$.

**Proof.** Let $u$ be a center of $G$, and let $T$ be a shortest path tree of $G$ rooted at $u$. Clearly, the diameter of $T$ is at most $2\rho_G$. Now, for every node $v$, let us denote by $k_v$ the number of edges of $T$ bought by $v$ in $\sigma$. The key argument is that if a node $v$ bought only the $k_v$ edges of $T$, its eccentricity would be at most $\varepsilon_T(v) \leq 2\rho_G$. Hence, since $\sigma$ is a NE, we have that $C_v(\sigma) \leq \alpha k_v + 2\rho_G$. By summing up the inequalities over all nodes, we obtain

$$SC(\sigma) = \sum_v C_v(\sigma) \leq \alpha \sum_v k_v + 2n\rho_G = \alpha(n-1) + 2n\rho_G.$$

Now, since $SC(\text{Opt}) \geq \alpha(n-1) + n\rho_H$, we have

$$\frac{SC(\sigma)}{SC(\text{Opt})} \leq \frac{\alpha(n-1) + 2n\rho_G}{\alpha(n-1) + n\rho_H} \leq 1 + \frac{2n\rho_G}{\alpha(n-1) + n\rho_H} = O\left(1 + \frac{\rho_G}{\alpha + \rho_H}\right). \qquad \square$$

As an immediate consequence, we obtain the following:

**Theorem 3.** *For every host graph $H$ and for $\alpha = O(n)$, the PoA of* MaxNCG$(H, \alpha)$ *is* $O(\frac{n}{\alpha+\rho_H})$.

Another interesting consequence of Lemma 1 concerns sparse host graphs:

**Theorem 4.** *If the host graph $H$ has $n-1+h$ edges, and $h = O(n)$, then for any $\alpha \geq 0$ the PoA of* MaxNCG$(H, \alpha)$ *is* $O(h+1)$.

**Proof.** Let $G = G(\sigma)$ be stable. Since $G$ must be connected, we have that $|E(H) \setminus E(G)| \leq h$. This is sufficient to provide an upper bound to the diameter of $G$. Indeed, in [16] it is shown that the diameter of a connected graph obtained after deleting $\ell$ edges from a graph of diameter $\delta$, is at most $(\ell+1)\delta$. This implies that in our case $\delta_G \leq (1+h)\delta_H$. Since $\rho_G \leq \delta_G$ and $\delta_H \leq 2\rho_H$, from the previous inequality we have that $\rho_G \leq 2(1+h)\rho_H$, and then the claim follows from Lemma 1. □

Thus, Theorem 4 implies that, for very sparse host graphs $H$, i.e., $|E(H)| = n-1+h$ and $h = O(1)$, we have that the PoA is $O(1)$.

A closer inspection of Lemma 1 shows also that when $\alpha \geq n$, then the PoA is at most 3. Actually, such a bound can be lowered to 2, by adapting a similar result that has already been proved in [4] for the case in which $H$ is a complete graph. In fact, it turns out that the same proof extends to any host graph $H$, and so we can give the following:

**Theorem 5.** *(See [4].) For every host graph $H$, if $\alpha \geq n$ then the PoA of* MaxNCG$(H,\alpha)$ *is at most* 2.

Therefore, from Theorems 3 and 5 we immediately obtain a general (i.e., for any $\alpha$) upper bound of $O(1+\frac{n}{\alpha+\rho_H})$ for MaxNCG$(H,\alpha)$.

We end this section by showing that when $\alpha$ is small or the host graph has small diameter, then every stable tree (if any) is a good equilibrium. This generalizes a result in [15] holding for MaxNCG, which states that the social cost of every acyclic equilibrium is $O(1)$ times the optimum. We start by giving the following:

**Lemma 2.** *Let $H$ be an arbitrary host graph, and let $(u,v) \in E(H)$ be an edge of the host graph. Then, for any $\alpha \geq 0$ and for every stable graph $G$ of* MaxNCG$(H,\alpha)$*, we have $|\varepsilon_G(u) - \varepsilon_G(v)| \leq 1+\alpha$.*

**Proof.** W.l.o.g. assume $\varepsilon_G(u) \geq \varepsilon_G(v)$. If $(u,v) \in E(G)$, then the claim trivially holds. Otherwise, if $u$ buys the edge $(u,v)$, then its eccentricity will decrease at least by $\varepsilon_G(u) - \varepsilon_G(v) - 1$, while its building cost will increase by $\alpha$. Since $G$ is stable, we then have that $\varepsilon_G(u) - \varepsilon_G(v) - 1 \leq \alpha$, and the claim follows. □

From this lemma it immediately follows:

**Corollary 1.** *Let $H$ be an arbitrary host graph, and let $u, v \in V$. Then, for any $\alpha \geq 0$ and for every stable graph $G$ of* MaxNCG$(H,\alpha)$*, we have $|\varepsilon_G(u) - \varepsilon_G(v)| \leq (1+\alpha)\, d_H(u,v)$.*

The above corollary implies the following:

**Lemma 3.** *Let $H$ be an arbitrary host graph and $\alpha \geq 0$, and let $G = G(\sigma)$ be a stable graph of* MaxNCG$(H,\alpha)$*. If there are two nodes $u, v \in V$, and two real-value constants $c > 1, k$ such that $\varepsilon_G(v) \geq c \cdot \varepsilon_G(u) + k$, then $\frac{\delta_G}{\delta_H} \leq 2 \cdot \frac{1+\alpha-\frac{k}{\delta_H}}{c-1}$.*

**Proof.** We have

$$\begin{aligned}\varepsilon_G(v) - \varepsilon_G(u) &\geq c \cdot \varepsilon_G(u) + k - \varepsilon_G(u) \\ &\geq (c-1)\varepsilon_G(u) + k \geq (c-1)\rho_G + k \geq (c-1)\frac{1}{2}\delta_G + k.\end{aligned}$$

Moreover, from Corollary 1, we have

$$\varepsilon_G(v) - \varepsilon_G(u) \leq (1+\alpha) d_H(u,v) \leq (1+\alpha)\delta_H,$$

from which, we obtain

$$(c-1)\frac{1}{2}\delta_G + k \leq (1+\alpha)\delta_H$$

and hence the claim. □

We are now ready to give the result concerning stable trees:

**Theorem 6.** *Let $H$ be an arbitrary host graph and $\alpha \geq 0$, and let $G = G(\sigma)$ be a stable tree of* MaxNCG$(H,\alpha)$*. Then, $\frac{SC(\sigma)}{SC(\text{Opt})} = O(\min\{1+\alpha, \rho_H\})$.*

**Proof.** Let us consider a center $u$ of $G$, and let $v$ be a node in the periphery of $G$, namely $\varepsilon_G(v) = \delta_G$. Since $G$ is a tree, we have $\varepsilon_G(v) = \delta_G \geq 2\rho_G - 1 = 2\varepsilon_G(u) - 1$. Now, using Lemma 3, with $c = 2$ and $k = -1$, we immediately have that $\frac{\delta_G}{\delta_H} \leq 2(1+\alpha+1/\delta_H) \leq 2(2+\alpha)$. Since $\rho_G \leq \delta_G$ and $\delta_H \leq 2\rho_H$, from the previous inequality we have that $\rho_G \leq 4(2+\alpha)\rho_H$, and if we plug this in Lemma 1 we obtain

$$\frac{SC(\sigma)}{SC(\text{Opt})} = O\left(1 + \frac{\alpha\rho_H}{\alpha+\rho_H}\right)$$

from which the claim follows. □

**Table 1**
Obtained lower bounds to the PoA.

| $\alpha$ | $O(\sqrt[3]{n})$ | $O(\sqrt{n})$ | $\Omega(\sqrt{n})$ |
|---|---|---|---|
| PoA | $\Omega(\sqrt{\frac{n}{1+\alpha}})$ | $\Omega(\alpha)$ | $\Omega(1+\frac{n}{\alpha})$ |

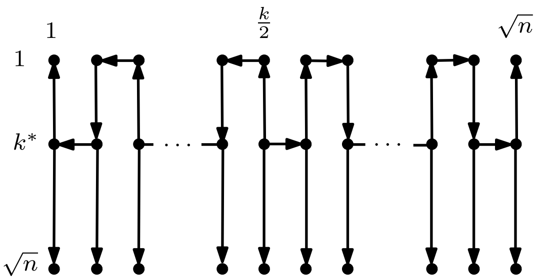


**Fig. 2.** The stable graph $G$ when the host graph $H$ is a square grid of $n$ vertices.

## 4. Lower bounds

In this section we prove some lower bounds to the PoA of the game, as summarized in Table 1.

Before getting to the technical details, let us discuss the significance of the above bounds. First of all, we notice that the lower bound for $\alpha = \Omega(\sqrt{n})$ is tight, due to the general upper bound of $O(1 + \frac{n}{\alpha+\rho_H})$ that is obtained by Theorems 3 and 5. Moreover, observe that we can obtain such a lower bound for two notable types of host graphs, namely for (a representative of) $k$-regular graphs (for any constant $k \geq 3$) and for 2-dimensional grids.[6] We view this as a meaningful result, since it conveys the message that it is not needed to provide artificial host graphs (i.e., with strong complementarities between edges) in order to get bad lower bounds on the PoA. Concerning the case $\alpha \in \Omega(\sqrt[3]{n}) \cap O(\sqrt{n})$, we notice that the lower bound still holds for the classes of $k$-regular graphs (for any constant $k \geq 3$) and 2-dimensional grids, but now it is not tight. Finally, the lower bound of $\Omega(\sqrt{\frac{n}{1+\alpha}})$ for $\alpha = O(\sqrt[3]{n})$ is quite surprising because it implies a very large lower bound of $\Omega(\sqrt{n})$ for tiny values of $\alpha$. Summarizing, we point out that we get a polynomial lower bound for $\alpha = O(n^{1-\varepsilon})$, for any $\varepsilon > 0$, in strong contrast with the almost everywhere constant upper bound to the PoA of MaxNCG.

**Theorem 7.** *If the host graph $H$ is a 2-dimensional square grid, then for any $\alpha \geq 0$ the PoA of* MaxNCG$(H, \alpha)$ *is* $\Omega(1 + \min\{\alpha, \frac{n}{\alpha}\})$.

**Proof.** Let $k = 2p$ where $p$ is an odd number, and let $H$ be a 2-dimensional square grid of $n = k \times k$ vertices. In the rest of the proof, we assume that the vertex in the $i$-th row and $j$-th column of the grid is labeled with $\langle i, j\rangle$, where $1 \leq i, j \leq k$.

For every $1 \leq j \leq k$, let $P_j$ be the path in $H$ which spans all the vertices of the $j$-th column of $H$. Let $k^* = \min\{1 + \lfloor \frac{\alpha}{2} \rfloor, k\}$. Let $F$ be the set of edges linking vertex $\langle 1, j\rangle$ with vertex $\langle 1, j+1\rangle$ iff $j$ is even, and let $F'$ be the set of edges linking vertex $\langle k^*, j\rangle$ with $\langle k^*, j+1\rangle$ iff $j$ is odd.

Let $G$ be the subgraph of $H$ whose edge set is $E(G) = F \cup F' \cup \bigcup_{j=1}^{k} E(P_j)$ (see also Fig. 2). Observe that $G$ is a tree of radius greater than or equal to $\frac{1}{2}kk^* = \Omega(\sqrt{n} \cdot \min\{1+\alpha, \sqrt{n}\})$. Observe also that $\langle k^*, p\rangle$ and $\langle k^*, p+1\rangle$ are the two centers of $G$. Let $\langle k^*, p\rangle$ be the root of $G$, and let $\bar{G}$ be the directed version of $G$ where all the root-to-leaf paths are directed towards the leaves. Finally, let $\sigma$ be the strategy profile induced by $\bar{G}$, i.e., each node $v$ is buying exactly the edges in $\bar{G}$ outgoing from $v$. Clearly, $G(\sigma) = G$.

We prove that $\sigma$ is a NE by showing that every vertex $\langle i, j\rangle$, with $1 \leq i \leq k$ and $1 \leq j \leq p$, is playing a best response strategy. Indeed, if we show that $\langle i, j\rangle$ is playing a best response strategy, then, by symmetry, also $\langle i, k - j + 1\rangle$ is playing a best response strategy.

Let $i$ and $j$ be two fixed integers such that $1 \leq i \leq k$ and $1 \leq j \leq p$, and let $t$ be the number of edges bought by $\langle i, j\rangle$ in $\sigma$. Since $G$ is a tree and since $\langle k^*, p\rangle$ and $\langle k^*, p+1\rangle$ are the two centers of $G$, there exists a vertex $\langle i', j'\rangle$, with $1 \leq i' \leq k$ and $p+1 \leq j' \leq k$, such that the distance in $G$ from $\langle i, j\rangle$ to $\langle i', j'\rangle$ is exactly equal to the eccentricity of $\langle i, j\rangle$ in $G$.[7] Notice also that the (unique) path in $G$ from $\langle i, j\rangle$ to $\langle i', j'\rangle$ traverses the root as well as the vertex $\langle k^*, p+1\rangle$. First of all, observe that if we add to $G$ all the edges adjacent to $\langle i, j\rangle$ in $H$, then the distance from $\langle i, j\rangle$ to $\langle i', j'\rangle$ decreases by at most $\alpha$. Since the cost of activating new links is at least $\alpha$, $\langle i, j\rangle$ cannot improve its cost function by buying more than $t$ edges. Now we prove that $\langle i, j\rangle$ cannot improve its cost function by buying at most $t$ edges. First of all, observe that $t$ is the minimum number of edges $\langle i, j\rangle$ has to buy to guarantee connectivity. Moreover, to guarantee connectivity, $\langle i, j\rangle$ has to buy an edge

[6] Notice that a 2-dimensional grid is also planar and bipartite.
[7] Notice that $\langle i', j'\rangle$ is either the upper or the lower rightmost vertex of the grid.

towards some vertex of every subtree of $G$ rooted at any of its $t$ children. Since the subtree of $G$ rooted at $\langle i, j\rangle$ does not contain $\langle i', j'\rangle$ when $\langle i, j\rangle$ is not the root, $\langle i, j\rangle$ cannot improve its eccentricity, and thus its cost function, by buying an edge towards some vertex of every subtree of $G$ rooted at any of its $t$ children. Furthermore, if $\langle i, j\rangle$ is the root of $G$, then $\langle i, j\rangle$ cannot improve its eccentricity, and thus its cost function, by buying an edge towards some vertex of every subtree of $G$ rooted at any of its $t$ children, as $\langle i, j\rangle$ is already buying the unique edge of $H$ linking it to the subtree of $G$ rooted at $\langle k^*, 1+p\rangle$.

To complete the proof, observe that $SC(\text{OPT})$ is upper bounded by the social cost of $H$, i.e., $SC(\text{OPT}) = O(n(\alpha + \sqrt{n}))$. Since $SC(\sigma) \geq \alpha(n-1) + \frac{1}{2}kk^*n = \Omega(n^{3/2}\min\{1+\alpha, \sqrt{n}\})$, we have that

$$\frac{SC(\sigma)}{SC(\text{OPT})} = \frac{\Omega(n^{3/2}\min\{1+\alpha, \sqrt{n}\})}{O(n(\alpha+\sqrt{n}))} = \Omega\big(1+\min\{\alpha, n/\alpha\}\big). \qquad \square$$

We now show that a similar lower bound holds also when the host graph is $k$-regular.

**Theorem 8.** *Let $H$ be an arbitrary host graph and $\alpha \geq 0$, and assume that $H$ is $k$-regular, with $3 \leq k = o(\frac{n}{1+\alpha})$. Then, the PoA of* MaxNCG$(H, \alpha)$ *is $\Omega(1+\min\{\alpha, \frac{n}{\alpha k}\})$.*

**Proof.** First of all, observe that for $\alpha = O(1)$ and $\alpha = \Omega(n)$ the claim trivially holds since the lower bound becomes $\Omega(1)$. Therefore, we consider the case $\alpha = \omega(1)$ and $\alpha = o(n)$. For the sake of readability, we provide the complete proof for even values of $k$, while we only sketch the proof for odd values of $k$, as the construction is very similar.

Let $l$ be the largest integer such that $l \leq \lfloor \alpha + 1 \rfloor$, and let $\eta = gl + 1$, where $g$ is an arbitrary positive integer which will generate the class of instances of $H$ onto which we prove the lower bound. Notice that if the number of players $n$ is sufficiently large, then $l \geq 3$. We will use a host graph $H$ composed by: (i) a path $P$ of $\eta$ nodes, numbered from 0 to $\eta - 1$, (ii) a set of shortcut edges on $P$ (as described in the following), and (iii) a set of gadgets appended to $P$ which are used to increase to $k$ the degree of the path's vertices (as described in the following).

Concerning the shortcut edges, let $u_i$ be the node on $P$ numbered $i \cdot l$, for $i = 0, \ldots, g$. Then, a shortcut edge connects $u_i$ to $u_{i+1}$, for $0 \leq i < g$. Notice that any node on $P \setminus \{u_1, \ldots, u_{g-1}\}$ has now degree 2, while the degree of all the nodes $u_1, \ldots, u_{g-1}$ is equal to 4.

Concerning the gadgets, for each node $u$ on $P$ that has degree $d < k$, we augment $H$ as follows:

– we add a complete, loop-free graph $K_u$ of $k+1$ vertices to $H$;
– we remove $\frac{k-d}{2}$ vertex-disjoint edges from $K_u$; this way, $d$ vertices of $K_u$ have degree $k$, while the remaining $k-d$ vertices have degree $k-1$;
– we connect $u$ to the $k-d$ nodes of $K_u$ with degree $k-1$.

At the end of this process the resulting host graph $H$ is $k$-regular. Consider now a strategy profile $\sigma$ such that:

– all the edges of the path $P$ are bought (arbitrarily) by vertices other than $u_i$, $i = 0, \ldots, g$;
– each vertex of a gadget that has an edge towards a node on $P$ buys it;
– the remaining vertices of the gadgets buy a single edge towards a vertex adjacent to a node of $P$.

An example of the resulting configuration for $k = 4$ along with the edges of the host graph is shown in Fig. 3(a).

We now show that $\sigma$ is a NE. Indeed, every node $u_i$ can only change its strategy by buying either one or two edges, but this can decrease its eccentricity by at most $l-1$, while it will increase its building cost of at least $\alpha \geq l-1$. Moreover, the remaining nodes in $P$ cannot change their strategy, as doing so will cause the disconnection of the graph. Finally, the nodes of the gadgets buy just a single edge, and no other choice can decrease their eccentricity.

Clearly, $SC(\sigma) = \Omega(\alpha n + n\eta)$, as $G(\sigma)$ is a tree with radius $\Theta(\eta)$. Let now $\widehat{G}$ be the graph obtained by adding to $G(\sigma)$ the shortcut edges of $H$. The number of edges of $\widehat{G}$ is $n - 1 + g \leq 2n$, and its diameter is bounded by $2 \cdot \varepsilon_{\widehat{G}}(u_0) \leq 2 \cdot (g + l + 2)$, as $u_0$ can take advantage of the shortcut edges. As a consequence, with a small abuse of notation, we have $SC(\widehat{G}) = O(\alpha n + n(g+l))$.

Using the relations $l = \Theta(\alpha)$, $\eta = \Theta(lg)$, and $n = \Theta(\eta k)$, we have that

$$PoA \geq \frac{SC(\sigma)}{SC(\widehat{G})} = \frac{\Omega(\alpha n + n\eta)}{O(\alpha n + n(g+l))} = \frac{\Omega(\alpha + \eta)}{O(\alpha + g + l)} = \frac{\Omega(\eta)}{O(\alpha + g)} = \Omega\left(\frac{\eta}{\alpha + \frac{\eta}{\alpha}}\right) = \Omega\left(\frac{n}{\alpha k + \frac{n}{\alpha}}\right)$$

from which the claim easily follows.

If $k$ is odd, then a host graph similar to the one shown in Fig. 3(b) (for the case $k = 3$) is considered. Notice that the shortcut edges are now vertex-disjoint, and each node incident to them has degree 3, but for $u_0$ and $u_g$ that have degree 2. We first append a gadget to every node $x$ of $P$ with degree 2, in order to obtain a 3-regular graph. This gadget is obtained from a clique of 4 vertices by subdividing any of its edges, say $(a, b)$, into two edges $(a, u)$, $(u, b)$. The new vertex $u$ is then connected to $x$ using a new edge (bought by $u$). If $k = 3$ we are done. Otherwise, we can append to each node of the graph

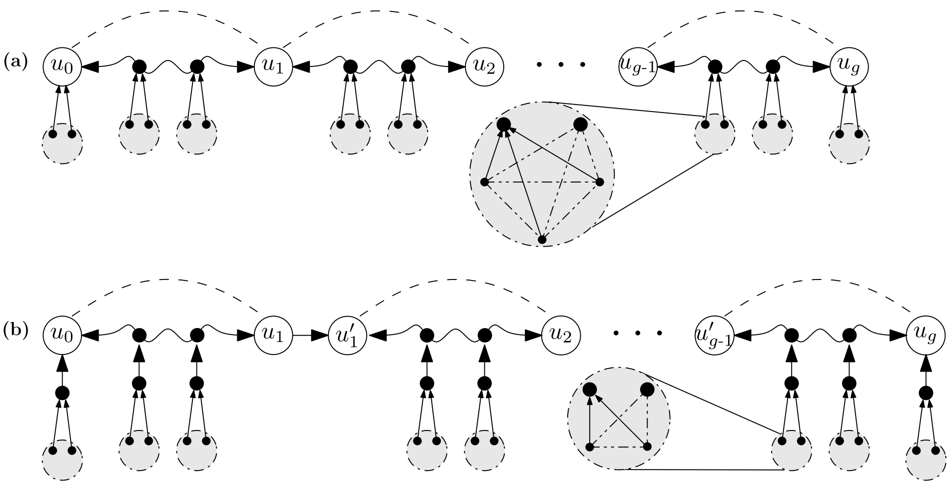


**Fig. 3.** Representation of the host graph and the equilibrium used in the proof of Theorem 8 for (a) $k = 4$, and (b) $k = 3$.

we have just constructed a gadget similar to the one shown for the even case, in order to increase the degree of each node to $k$. □

Notice that we can extend the previous lower bound to outerplanar and series-parallel graphs: it suffices to consider the host graph composed by the path plus the shortcut edges, without any additional gadget. Hence, the following holds:

**Theorem 9.** *For any $\alpha \geq 0$, the PoA of* MaxNCG$(H, \alpha)$ *is $\Omega(1 + \min\{\alpha, \frac{n}{\alpha}\})$, even when the host graph $H$ is an outerplanar or a series-parallel graph.*

We end this section by proving a non-constant lower bound to the PoA when $\alpha = o(n)$. Remarkably, this will imply a high lower bound to the PoA for small (even null) values of $\alpha$. Our lower bounding construction is a non-trivial modification of the 2D-torus-rotated-45° construction used in [1] to prove a lower bound for BasicNCG.

**Theorem 10.** *For $\alpha = o(n)$, the PoA of* MaxNCG$(H, \alpha)$ *is $\Omega(\sqrt{\frac{n}{1+\alpha}})$.*

**Proof.** Let $k \in \mathbb{N}$ and let $\bar{H}$ be an edge-weighted 2D-torus-rotated-45° consisting of $2k^2$ vertices, that we call *junction vertices*. For every pair of integers $0 \leq i, j < 2k$, with $i + j$ even, there is exactly one vertex of $\bar{H}$ labeled with $\langle i, j\rangle$. We treat the two integers of a vertex label as modulo $2k$. Each vertex $\langle i, j\rangle$ has exactly four neighbors in $\bar{H}$: $\langle i \pm 1, j \pm 1\rangle$. All edge weights are equal to $\ell = 2(1 + \lceil\alpha\rceil)$. For every pair of integers $0 \leq i, j < 2k$, let $X_{i,j} = \{\langle i', j'\rangle \mid i' = i \text{ or } j' = j\}$. The properties satisfied by $\bar{H}$ are the following:

(i) $\bar{H}$ is *vertex transitive*, i.e., any vertex can be mapped to any other one by a vertex automorphism, i.e., a relabeling of vertices that preserves edges;
(ii) the distance between two vertices $\langle i, j\rangle$ and $\langle i', j'\rangle$ in $\bar{H}$ is equal to $\ell \cdot \max\{\bar{d}(i, i'), \bar{d}(j, j')\}$, where $\bar{d}(h, h') = \min\{|h - h'|, 2k - |h - h'|\}$;
(iii) the eccentricity of each vertex in $\bar{H}$ is equal to $\ell k$;
(iv) for every $0 \leq i, j < 2k$, the distance from every vertex $v \in X_{i,j}$ to vertex $\langle |i - k|, |j - k|\rangle$ is equal to $\ell k$;
(v) for every edge $e$ of $\bar{H}$, the eccentricity of both endpoints of $e$ in $\bar{H} - e$ is greater than or equal to $\ell(k + 1)$;
(vi) for every edge $e$ of $\bar{H}$ and for every vertex $\langle i, j\rangle$, the distance from $\langle i, j\rangle$ to the closest endpoint of $e$ is less than or equal to $\ell(k - 1)$.

It is easy to see that (i) holds, and it is also easy to see that (iv) holds once that (ii) has been proved. To prove (ii), it is enough to observe that each label can change by $\pm 1$ each time we move from one vertex to any of its neighbors. To prove (iii), we use (i) and the fact that the distance from vertex $\langle i', j'\rangle$ to $\langle k, k\rangle$, which is equal to $\max\{|k - i'|, |k - j'|\}$, is maximized for $i' = j' = 0$. To prove (v), we first use (i) to assume that, w.l.o.g, $e$ is the edge linking $\langle k, k\rangle$ with $\langle k - 1, k - 1\rangle$. Next, we observe that any path in $\bar{H} - e$ going from $\langle k, k\rangle$ to $\langle 1, 1\rangle$ must traverse a neighbor $v$ of $\langle k, k\rangle$ in $\bar{H} - e$, and the distance between $v$ and $\langle 1, 1\rangle$ in $H$ is equal to $\ell k$ because one of the two integers in the label of $v$ is equal to $k + 1$. Finally, to prove (vi), we first use (i) to assume that, w.l.o.g., $i = j = k$, i.e., $\langle i, j\rangle$ is $\langle k, k\rangle$, and the two endpoints of $e$ are, respectively, $\langle i', j'\rangle$ and $\langle i' + 1, j' + 1\rangle$, where $0 \leq i', j' < k$. Then, using (ii), it is easy to see that the distance from $\langle k, k\rangle$ to $\langle i' + 1, j' + 1\rangle$ is less than or equal to $\ell(k - 1)$.

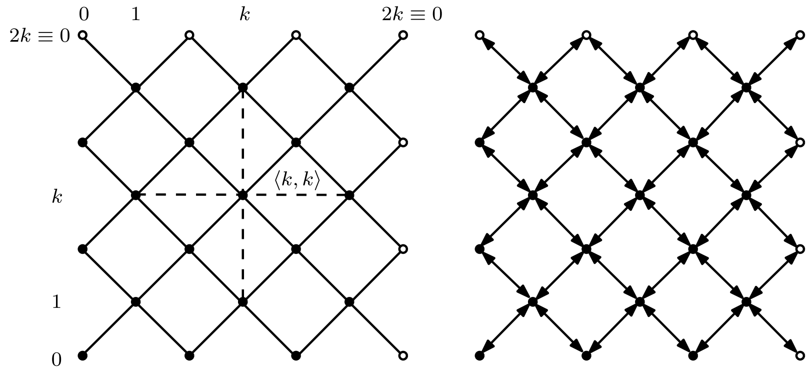


**Fig. 4.** The lower bound construction of Theorem 10. On the left side, the host graph $H$ is depicted. For the sake of readability, only junction vertices are visible and not all the edges are shown. The white vertices of row $2k \equiv 0$ are copies of the vertices of row 0 while the white vertices of column $2k \equiv 0$ are copies of the vertices of column 0. The solid edges are paths of length $\ell$, while the dashed edges are all the other edges adjacent to vertex $\langle k, k\rangle$. On the right side, the stable graph $G$ is depicted.

Let $G$ be an unweighted graph obtained from $\bar{H}$ by replacing each edge of $\bar{H}$ with a path of length $\ell$ via the addition of $\ell - 1$ new vertices per edge of $\bar{H}$. Let $H$ be the host graph obtained from $G$ by adding, for every junction vertex $\langle i, j\rangle$, an edge between $\langle i, j\rangle$ and every vertex in $X_{i,j}$ (see also Fig. 4). Notice that the number of vertices of $H$ is

$$n = 2k^2 + 4k^2(\ell - 1) = \Theta\big((1+\alpha)k^2\big). \tag{3}$$

In what follows, we call *path vertices* the vertices in $H$ which are not in $\bar{H}$. Let $\sigma$ be any strategy profile such that $G(\sigma) = G$, and all edges of $G(\sigma)$ have a unique owner and are bought by players sitting on path vertices. In other words, no edge of $G(\sigma)$ is bought by some player sitting on junction vertices. We now prove that $\sigma$ is a NE.

We start by proving that players sitting on junction vertices are playing a best response strategy. Let $\langle i, j\rangle$ be a junction vertex. Observe that $\langle i, j\rangle$ is not buying any edge, therefore it suffices to show that $\langle i, j\rangle$ cannot improve its cost by buying edges. First of all, observe that by (iii) and (vi), the eccentricity of $\langle i, j\rangle$ in $G$ is equal to $\ell k$. Indeed, if $v$ is a path vertex of a path $P$ which replaced an edge $e$ of $\bar{H}$, then the distance from $\langle i, j\rangle$ to the closest endpoint of $P$ (which corresponds to the closest endpoint of $e$) is less than or equal to $\ell(k-1)$. Therefore, the distance from $\langle i, j\rangle$ to $v$ is less than or equal to $\ell k$. To prove that $\langle i, j\rangle$ is in equilibrium, simply observe that if we add to $G$ all edges of $H$ incident to $\langle i, j\rangle$, i.e., all edges linking $\langle i, j\rangle$ to vertices in $X_{i,j}$, then by (iv) the distance from $\langle i, j\rangle$ to $\langle |i-k|, |j-k|\rangle$ is still $\ell k$.

Now, we prove that players sitting on path vertices are playing a best response strategy. Let $v$ be a path vertex. First of all, by (vi) the eccentricity of $v$ in $G$ is less than or equal to $\ell k + \frac{1}{2}\ell$. Indeed, if $v$ is a vertex of a path $P$ which replaced an edge $e$ of $\bar{H}$, then the distance from any junction vertex to the closest endpoint of $P$ (which corresponds to the closest endpoint of $e$) is less than or equal to $\ell(k-1)$. Therefore, the distance from $v$ to every junction vertex is less than or equal to $\ell k$, and the distance from $v$ to every other path vertex is less than or equal to $\ell k + \frac{1}{2}\ell$. Now, observe that $G$ already contains all edges of $H$ incident to $v$, and, moreover, the degree of $v$ in $G$ is equal to 2. Therefore, $v$ might improve its cost by removing exactly one edge incident to itself, i.e., by buying fewer edges than those it is buying in $\sigma$. However, if $v$ removes any of its incident edges in $G$, thus saving an amount equal to $\alpha$ from its building cost, then by (v) the eccentricity of the unique junction vertex closest to $v$ becomes greater than or equal to $\ell(k+1)$, and thus the eccentricity of $v$ also becomes greater than or equal to $\ell(k+1)$. Since $\ell(k+1) - \alpha \geq \ell(k+1) - \frac{\ell-2}{2} > \ell k + \frac{1}{2}\ell$, it follows that $v$ does not improve its cost by buying fewer edges than those it is buying in $\sigma$.

To complete the proof, we have to show that the PoA is $\Omega(\sqrt{\frac{n}{1+\alpha}})$. First of all, observe that the radius of $H$ is $\Theta(\ell) = \Theta(1+\alpha)$. Let $T$ be a breadth-first-search tree rooted at $\langle k, k\rangle$. Clearly, the radius of $T$ is also $\Theta(1+\alpha)$. Furthermore, the social cost of $\textsc{Opt}$ is upper bounded by the social cost of $T$, i.e., $SC(\textsc{Opt}) \leq \alpha(n-1) + n \cdot O(1+\alpha) = O((1+\alpha)n)$. As $SC(\sigma) \geq \alpha(4\ell k^2) + n\ell k = \Omega(n\ell k) = \Omega((1+\alpha)nk)$, we have that

$$\frac{SC(\sigma)}{SC(\textsc{Opt})} = \frac{\Omega((1+\alpha)nk)}{O((1+\alpha)n)} = \Omega(k) = \Omega\left(\sqrt{\frac{n}{1+\alpha}}\right),$$

where the last equality follows from (3). □

## 5. Conclusions

In this paper, we have studied the MaxNCG in the scenario in which the strategy space of all the players is constrained by a host graph. Our game is interesting for two reasons. First of all, it models the practical situation in which not all the links can be constructed due to physical limitations. Furthermore, the NP-hardness barrier of computing a best response strategy of a player in the MaxNCG, which clearly carries over into our game, can be easily broken if we restrict our game to the class of host graphs of (constant) bounded degree.

In our paper, we proved a strong lower bound of $\Omega(1+\min\{\alpha, n/\alpha\})$ to the PoA for two meaningful types of host graphs, namely $k$-regular graphs and 2-dimensional square grids. Our lower bound asymptotically matches the general upper bound we provided for every $\alpha = \Omega(\sqrt{n})$. Since the PoA in the classical MaxNCG is mostly constant (see [15]), we have that the drawback of having a polynomial-time computability of best response strategies is the existence of stable networks of large social cost.

Finally, we concluded our paper by proving another lower bound of $\Omega(\sqrt{\frac{n}{1+\alpha}})$ to the PoA. Observe that all our lower bounds are never smaller than the lower bound of $\Omega(1+\min\{\alpha/n, n^2/\alpha\})$ known for the corresponding sum version of the game (see [3]). This is mainly due to the fact that reducing the routing cost of a player in our game is costlier, in terms of the building cost incurred by that player, than in the corresponding sum version of the game.

Concerning our future work, we plan to further investigate the challenging problem of establishing whether a NE always exists for MaxNCG$(H, \alpha)$, as well as for SumNCG$(H, \alpha)$. Moreover, we also plan to study other reasonable (social) cost functions, to see how this impacts on the space of NE. In particular, we look with interest at the variants discussed in the introduction in which the $\alpha$ factor is dropped out. Finally, we believe it would be remarkable to provide lower/upper bounds to the PoA depending on meaningful global features of the host graph, like the node degree distribution, or the average node-to-node distance.

## Acknowledgements

We wish to thank the anonymous reviewers for their insightful comments and for their helpful suggestions for improving the paper.

## References

[1] N. Alon, E.D. Demaine, M. Hajiaghayi, T. Leighton, Basic network creation games, SIAM J. Discrete Math. 27 (2) (2013) 656–668.
[2] D. Bilò, L. Gualà, G. Proietti, Bounded-distance network creation games, in: Proceedings of the 8th International Workshop on Internet & Network Economics (WINE'12), in: Lecture Notes in Computer Science, vol. 7695, Springer, 2012, pp. 72–85. An extended version of this paper has been accepted to *ACM Transactions on Economics and Computation*.
[3] E.D. Demaine, M. Hajiaghayi, H. Mahini, M. Zadimoghaddam, The price of anarchy in cooperative network creation games, ACM SIGecom Exch. 8 (2) (2009) 2.
[4] E.D. Demaine, M. Hajiaghayi, H. Mahini, M. Zadimoghaddam, The price of anarchy in network creation games, ACM Trans. Algorithms 8 (2) (2012) 13.
[5] S. Ehsani, M. Fazli, A. Mehrabian, S.S. Sadeghabad, M. Saghafian, S. Shokatfadaee, M. Safari, On a bounded budget network creation game, in: Proceedings of the 23rd ACM Symposium on Parallelism in Algorithms and Architectures, SPAA'11, ACM Press, 2011, pp. 207–214.
[6] A. Fabrikant, A. Luthra, E. Maneva, C.H. Papadimitriou, S. Shenker, On a network creation game, in: Proceedings of the 22nd Symposium on Principles of Distributed Computing, PODC'03, ACM Press, 2003, pp. 347–351.
[7] M.O. Jackson, A. Wolinsky, A strategic model of social and economic networks, J. Econom. Theory 71 (1) (1996) 44–74.
[8] R.M. Karp, Reducibility among combinatorial problems, in: R.E. Miller, J.W. Thatcher (Eds.), Complexity of Computer Computations, Plenum Press, New York, 1972, pp. 85–103.
[9] B. Kawald, P. Lenzner, On dynamics in selfish network creation, in: Proceedings of the 25th ACM Symposium on Parallelism in Algorithms and Architectures, SPAA'13, ACM, 2013, pp. 83–92.
[10] N. Laoutaris, L.J. Poplawski, R. Rajaraman, R. Sundaram, S.-H. Teng, Bounded budget connection (BBC) games or how to make friends and influence people, on a budget, in: Proceedings of the 27th ACM Symposium on Principles of Distributed Computing, PODC'08, ACM Press, 2008, pp. 165–174.
[11] P. Lenzner, On dynamics in basic network creation games, in: Proceedings of the 4th International Symposium on Algorithmic Game Theory, SAGT'11, in: Lecture Notes in Computer Science, vol. 6982, Springer, 2011, pp. 254–265.
[12] P. Lenzner, Greedy selfish network creation, in: Proceedings of the 8th International Workshop on Internet and Network Economics, WINE'12, in: Lecture Notes in Computer Science, vol. 7695, Springer, 2012, pp. 142–155.
[13] A. Mamageishvili, M. Mihalák, D. Müller, Tree Nash equilibria in the network creation game, in: Proceedings of the 10th International Workshop on Algorithms and Models for the Web Graph, WAW'13, in: Lecture Notes in Computer Science, vol. 8305, Springer, 2013, pp. 118–129.
[14] M. Mihalák, J.C. Schlegel, Asymmetric swap-equilibrium: a unifying equilibrium concept for network creation games, in: Proceedings of the 37th International Symposium on Mathematical Foundations of Computer Science, MFCS'12, in: Lecture Notes in Computer Science, vol. 7464, Springer, 2012, pp. 693–704.
[15] M. Mihalák, J.C. Schlegel, The Price of Anarchy in network creation games is (mostly) constant, Theory Comput. Syst. 53 (1) (2013) 53–72.
[16] A.A. Schoone, H.L. Bodlaender, J. van Leeuwen, Improved diameter bounds for altered graphs, in: Proceedings of the 12th Workshop on Graph-Theoretic Concepts in Computer Science, WG'86, in: Lecture Notes in Computer Science, vol. 246, 1986, pp. 227–236.