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Local Boundedness for Vector Valued Minimizers of Anisotropic Functionals

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Abstract. For variational integrals $\mathcal{F}(u) = \int_\Omega f(x, Du)\,dx$ defined on vector valued mappings $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, we establish some structure conditions on f that enable us to prove local boundedness for minimizers $u \in W^{1,1}(\Omega; \mathbb{R}^N)$ of $\mathcal{F}$. These structure conditions are satisfied in three remarkable examples: $f(x, Du) = g(x, |Du|)$, $f(x, Du) = \sum_{j=1}^{n} g_j(x, |u_{x_j}|)$ and $f(x, Du) = a(x, |(u_{x_1}, \ldots, u_{x_{n-1}})|) + b(x, |u_{x_n}|)$, for suitable convex functions $t \to g(x, t)$, $t \to g_j(x, t)$, $t \to a(x, t)$ and $t \to b(x, t)$.

Keywords. Regularity, boundedness, minimizer, variational integral, elliptic system

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1. Introduction

We are concerned with regularity of minimizers of integral functionals

$$\mathcal{F}(u) = \int_\Omega f(x, Du(x))\,dx \tag{1}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n \geq 2$ and Du denotes the gradient of a vector-valued function $u : \Omega \to \mathbb{R}^N$. Moreover $f : \Omega \times \mathbb{R}^{N\times n} \to [0, +\infty)$ is a Caratheodory function, that is, $f(x, z)$ is measurable with respect to x and continuous with respect to z. The study includes also weak solutions of nonlinear elliptic systems

$$\sum_{i=1}^{n} D_{x_i}\left(a_i^\alpha(x, Du(x))\right) = 0\,, \quad \alpha = 1, \ldots, N,$$

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where the vector field $a = (a_i^\alpha) : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ is the gradient with respect to z of the function $f(x,z)$, i.e.,

$$a_i^\alpha(x,z) = \frac{\partial f}{\partial z_i^\alpha}(x,z).$$

We consider minimizers $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ of (1), that is, $u \in W^{1,1}(\Omega;\mathbb{R}^N)$ with finite energy

$$\mathcal{F}(u) < +\infty \tag{2}$$

and

$$\mathcal{F}(u) \le \mathcal{F}(u+\varphi) \tag{3}$$

for every $\varphi \in W_0^{1,1}(\Omega;\mathbb{R}^N)$. In the vectorial case it is usual to look for boundedness of minimizers by assuming some structure condition on f. In fact a counterexample of De Giorgi shows that minimizers and weak solutions of systems do not need to be bounded, [9]. See also Frehse [13], Nečas [30] and Sverak-Yan [32]. However, in the case where $f(x,z) = |z|^p$, $p \ge 2$, Uhlenbeck proved in [34] that minimizers are $C^{1,\alpha}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, a result that was later extended by Tolksdorf [33], Fusco-Hutchinson [14], Giaquinta-Modica [18], Acerbi-Fusco [1], Marcellini [24], Esposito-Leonetti-Mingione [12], Leonetti-Mascolo-Siepe [20], Marcellini-Papi [25]. As a first step towards regularity we want to analize the local boundedness of minimizers u. We assume the p,q-growth condition: There exist constants $c_1, c_3 \in (0,+\infty)$, $c_2, c_4 \in [0,+\infty)$, $p,q \in [1,+\infty)$ with $p \le q$, such that

$$c_1|z|^p - c_2 \le f(x,z) \le c_3|z|^q + c_4 \tag{4}$$

for almost every $x \in \Omega$ and for every $z \in \mathbb{R}^{N\times n}$. Such a growth assumption is not strong enough to ensure boundedness even in the scalar case $N = 1$, when q is large with respect to p (see Giaquinta [17], Marcellini [22,23] and Hong [19]). This leads to require that q is not too far from p. The previous p,q-growth arises in the study of

$$f(x,Du) = g(x,|Du|) \tag{5}$$

and in the anisotropic energy densities:

$$f(x,Du) = \sum_{j=1}^{n} g_j(x,|u_{x_j}|), \tag{6}$$

$$f(x,Du) = a(x,|(u_{x_1},\dots,u_{x_{n-1}})|) + b(x,|u_{x_n}|), \tag{7}$$

for suitable convex functions $t \to g(x,t)$, $t \to g_j(x,t)$, $t \to a(x,t)$ and $t \to b(x,t)$. In the last years the study of regularity under non standard growth condition has increased. In the scalar case the local boundeness has been proved by Moscariello-Nania [28] and Fusco-Sbordone [15, 16], by Mascolo-Papi [26] and Cianchi [5] with some techniques related with the Orlicz spaces, by Lieberman [21] and more recently by Cupini-Marcellini-Mascolo [6]. In the vectorial case, Dall'Aglio-Mascolo in [8] proved the local boundedness of minimizers of (5) when g is a N-function with Δ_2-property. In this paper we give some structure assumptions in order to garantee the boundedness of minimizers. These assumptions allow us to give a *unified* proof (see Theorem 2.1) of local boundeness for (5), (6), and (7), with g, g_i, a, b satisfying the Δ_2-property and growth condition (4), provided p and q are not too far apart. We remark that examples (6) and (7) are interesting even in the isotropic case $p = q$ since they go away from Uhlenbeck-structure (5). For the local boundedness of solutions to quasi-linear systems see Cupini-Marcellini-Mascolo [7]. We remark that boundedness of minimizers is an important tool in order to achieve higher integrability of Du as in D'Ottavio [10], Esposito-Leonetti-Mingione [11], Bildhauer-Fuchs [3, 4]. See also Apushkinskaya-Bildhauer-Fuchs [2]. The plan of the paper is the following: In Section 2 we give precise assumptions and state the main theorem. Section 3 contains preliminary results. In Section 4 we discuss examples (5), (6) and (7). Section 5 is devoted to the proof of the theorem, which is based on suitable Caccioppoli estimates and Moser iteration method, [29]. We thank the referees for useful remarks.

2. Assumptions and result

We consider the functional (1) where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ and Ω is a bounded open set, $n \geq 2$ and $N \geq 1$. Let $f : \Omega \times \mathbb{R}^{N\times n} \to [0, +\infty)$ be such that: for almost every $x \in \Omega$ we have

$$z \to f(x,z) \quad \text{is } C^1(\mathbb{R}^{N\times n}) \tag{8}$$

for every $z \in \mathbb{R}^{N\times n}$, for any $i \in \{1, \dots, n\}$ and $\alpha \in \{1, \dots, N\}$, we have

$$x \to f(x,z) \quad \text{and} \quad x \to \frac{\partial f}{\partial z_i^\alpha} f(x,z) \quad \text{are measurable.} \tag{9}$$

In the sequel we will write "for a.e. x" instead of "for almost every x". Let us assume:

(H1) *Behaviour of* $\frac{\partial f}{\partial z}$: There exist $\nu, L \in (0, +\infty)$, such that for a.e. $x \in \Omega$, for every $z, v, w \in \mathbb{R}^{N\times n}$ and $t \in [-1, 1]$ we have

$$\nu f(x,z) \leq \sum_{i=1}^{n} \sum_{\alpha=1}^{N} \frac{\partial f}{\partial z_i^\alpha}(x,z) z_i^\alpha \tag{10}$$

and

$$\left| \sum_{i=1}^{n} \sum_{\alpha=1}^{N} \frac{\partial f}{\partial z_i^\alpha}(x, v + tw) w_i^\alpha \right| \leq \frac{\nu}{2} f(x,v) + L f(x,w); \tag{11}$$

(H2) *Monotonicity condition*: There exists $H \in [1, +\infty)$ such that for a.e. $x \in \Omega$ and for every $z, w \in \mathbb{R}^{N\times n}$ we have

$$|z_i| \le |w_i| \quad \forall i = 1, \dots, n \quad \Longrightarrow \quad f(x,z) \le Hf(x,w); \tag{12}$$

(H3) *Sign condition*:

$$0 \le \sum_{i=1}^{n} \sum_{\alpha=1}^{N} \frac{\partial f}{\partial z_i^\alpha}(x,z) y^\alpha \sum_{\beta=1}^{N} y^\beta z_i^\beta, \tag{13}$$

for a.e. $x \in \Omega$, for every $z \in \mathbb{R}^{N\times n}$ and $y \in \mathbb{R}^N$;

(H4) *p, q growth*: There exist $c_1, c_3 \in (0, +\infty)$, $c_2, c_4 \in [0, +\infty)$, $p, q \in [1, +\infty)$ with $p \le q$, such that

$$c_1|z|^p - c_2 \le f(x,z) \le c_3|z|^q + c_4, \tag{14}$$

for a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^{N\times n}$.

Let us state our main result:

Theorem 2.1. *Let f satisfy* (H1)–(H4) *and $u \in W^{1,1}(\Omega; \mathbb{R}^N)$ be a minimizer of $\mathcal{F}$. If*

$$p < n \quad \text{and} \quad q < \frac{pn}{n-p} = p^* \tag{15}$$

then $u \in L^\infty_{loc}(\Omega; \mathbb{R}^N)$. Moreover, for every ball $B(x_0, \sigma)$, with $\sigma \le 1$ and $\overline{B(x_0, \sigma)} \subset \Omega$, it results that

$$||u||_{L^\infty(B(x_0,\frac{\sigma}{2}))} \le C \left(\int_{B(x_0,\sigma)} (1 + |u|^{p^*})\, dx \right)^{\frac{p^*-p}{p^*(p^*-q)}} \tag{16}$$

for a suitable constant $C \in (1, +\infty)$ depending only on $\sigma, n, p, q, \nu, L, c_1, c_2, c_3, c_4$.

Remark 2.2. The right hand side in (13), called “indicator function” in the framework of elliptic systems, seems to play an important role in deriving regularity properties (see [27] where the isotropic case $p = q$ has been dealt with).

3. Properties of f and Euler-Lagrange system

We first note that positivity of f and coercivity (10) give

$$f(x,0) = 0 \tag{17}$$

for a.e. $x \in \Omega$. We have the following

Proposition 3.1. *Let $f : \Omega \times \mathbb{R}^{N\times n} \to [0, +\infty)$ satisfy* (8) *and* (11). *Then*

$$|f(x, v + tw) - f(x, v)| \leq \frac{\nu}{2} f(x, v) + Lf(x, w) \tag{18}$$

and

$$f(x, v + tw) \leq \left(\frac{\nu}{2} + 1\right) f(x, v) + Lf(x, w) \tag{19}$$

for a.e. $x \in \Omega$, for every $v, w \in \mathbb{R}^{N\times n}$, for any $t \in [-1, 1]$. Moreover for a.e. $x \in \Omega$, for every $w \in \mathbb{R}^{N\times n}$, for any $t \in \mathbb{R}$ with $|t| \leq k \in \mathbb{N}$ it results that

$$f(x, tw) \leq 2f(x, w) \sum_{i=1}^{k+1} (\tilde{L})^i \tag{20}$$

where

$$\tilde{L} = \max\left\{\frac{\nu}{2} + 1; L\right\}. \tag{21}$$

Proof. Let us evaluate the difference

$$f(x, v+tw) - f(x, v) = \int_0^1 \frac{d}{ds}[f(x, v+stw)]ds = \int_0^1 \sum_{i=1}^{n} \sum_{\alpha=1}^{N} \frac{\partial f}{\partial z_i^\alpha}(x, v+stw)tw_i^\alpha ds$$

then, using (11) we get

$$\begin{aligned} |f(x, v + tw) - f(x, v)| &\leq \int_0^1 \left| \sum_{i=1}^{n} \sum_{\alpha=1}^{N} \frac{\partial f}{\partial z_i^\alpha}(x, v + stw)tw_i^\alpha \right| ds \\ &\leq \int_0^1 \left[\frac{\nu}{2} f(x, v) + Lf(x, w)\right] |t| ds \\ &= \left[\frac{\nu}{2} f(x, v) + Lf(x, w)\right] |t| \\ &\leq \frac{\nu}{2} f(x, v) + Lf(x, w). \end{aligned} \tag{22}$$

Thus (18) holds true and (19) follows at once. Let $\tilde{L}$ be as in (21), then (19) gives

$$f(x, v + tw) \leq \tilde{L}[f(x, v) + f(x, w)] \tag{23}$$

for a.e. $x \in \Omega$, for every $v, w \in \mathbb{R}^{N\times n}$, for any $t \in [-1, 1]$. When $v = 0$, since $f(x, 0) = 0$, we get

$$f(x, tw) \leq \tilde{L} f(x, w), \tag{24}$$

and for $t = -1$ we have

$$f(x, -w) \leq \tilde{L} f(x, w) \tag{25}$$

for a.e. $x \in \Omega$, for every $w \in \mathbb{R}^{N\times n}$. Assume that $s \in (1,2]$, then $0 < s-1 \le 1$ and we can use (23) as follows

$$f(x,sw) = f(x,w+(s-1)w) \le \tilde{L}[f(x,w)+f(x,w)] = 2\tilde{L}f(x,w).$$

Iterating the procedure, for every $k \in \mathbb{N}$, for any $s \in (k,k+1]$, for a.e. $x \in \Omega$ and for every $w \in \mathbb{R}^{N\times n}$ we have

$$f(x,sw) \le 2f(x,w)\sum_{j=1}^{k}(\tilde{L})^j. \tag{26}$$

Now, if $k \in \mathbb{N}$ and $t \in [-(k+1),-k)$, then $-t \in (k,k+1]$ and we can use (25), (26) as follows $f(x,tw) = f(x,-(-t)w) \le \tilde{L}f(x,(-t)w) \le 2\tilde{L}f(x,w)\sum_{j=1}^{k}(\tilde{L})^j$ $= 2f(x,w)\sum_{i=2}^{k+1}(\tilde{L})^i$ so that

$$f(x,tw) \le 2f(x,w)\sum_{i=1}^{k+1}(\tilde{L})^i \tag{27}$$

if $t \in [-(k+1),-k)$. Inequalities (24), (26) and (27) merge into (20). □

Remark 3.2. Left hand side of (14) gives that

$$0 < f(x,z) \quad \text{when} \quad |z|^p > \frac{c_2}{c_1} \tag{28}$$

for a.e. $x \in \Omega$. By means of (28), (17) and (19) with $v = 0$ and $t = 1$, we get $0 < f(x,z) \le \left(\frac{\nu}{2}+1\right)f(x,0) + Lf(x,z) = Lf(x,z)$ so that $1 \le L$. On the other hand (28), (10) and (11) with $v = 0$, $w = z$ and $t = 1$ imply

$$0 < \nu f(x,z) \le \sum_{i=1}^{n}\sum_{\alpha=1}^{N}\frac{\partial f}{\partial z_i^\alpha}(x,z)z_i^\alpha \le \frac{\nu}{2}f(x,0) + Lf(x,z) = Lf(x,z)$$

then

$$\nu \le L. \tag{29}$$

Previous properties of f allow us to show that minimizers of (1) satisfy the Euler system as follows.

Theorem 3.3. *Let $f : \Omega \times \mathbb{R}^{N\times n} \to [0,+\infty)$ satisfy (8), (9) and (11). Let $u \in W^{1,1}(\Omega;\mathbb{R}^N)$ minimize $\mathcal{F}$ so that (2) and (3) hold true. Then u verifies the Euler system*

$$\int_\Omega \sum_{i=1}^{n}\sum_{\alpha=1}^{N}\frac{\partial f}{\partial z_i^\alpha}(x,Du)D_iv^\alpha\,dx = 0 \tag{30}$$

for every $v \in W_0^{1,1}(\Omega;\mathbb{R}^N)$ with finite energy $\mathcal{F}(v) < +\infty$.

Proof. Note that both u and v have finite energy. Then assumptions (8) and (11) give additivity property (19), so that

$$0 \le f(x, Du(x) + tDv(x)) \le \left(\frac{\nu}{2} + 1\right) f(x, Du(x)) + Lf(x, Dv(x))$$

thus $u + tv$ has finite energy for every $t \in [-1, 1]$. Moreover, assumption (11) with $t = 0$ ensures that

$$x \to \sum_{i=1}^{n} \sum_{\alpha=1}^{N} \frac{\partial f}{\partial z_i^\alpha}(x, Du(x)) D_i v^\alpha(x) \in L^1(\Omega).$$

Let us set $\phi(t) = \mathcal{F}(u + tv)$. Then $\phi : [-1, 1] \to \mathbb{R}$ and $\phi(0) = \min_{[-1,1]} \phi$. We claim that

$$\phi'(0) = \int_\Omega \sum_{i=1}^{n} \sum_{\alpha=1}^{N} \frac{\partial f}{\partial z_i^\alpha}(x, Du) D_i v^\alpha \, dx. \tag{31}$$

If so, since ϕ achieves its minumum value at $t = 0$, then $\phi'(0) = 0$ and (30) follows at once. Let us prove claim (31). Observe that

$$\frac{\phi(t) - \phi(0)}{t} = \int_\Omega \frac{f(x, Du + tDv) - f(x, Du)}{t} dx \tag{32}$$

and

$$\lim_{t \to 0} \frac{f(x, Du(x) + tDv(x)) - f(x, Du(x))}{t} = \sum_{i=1}^{n} \sum_{\alpha=1}^{N} \frac{\partial f}{\partial z_i^\alpha}(x, Du(x)) D_i v^\alpha(x).$$

On the other hand assumption (11) gives us (22) and we get

$$\left| \frac{f(x, Du(x) + tDv(x)) - f(x, Du(x))}{t} \right| \le \frac{\nu}{2} f(x, Du(x)) + Lf(x, Dv(x));$$

since $x \to f(x, Du(x)) \in L^1(\Omega)$ and $x \to f(x, Dv(x)) \in L^1(\Omega)$, then we can pass to limit as $t \to 0$ under the integral sign in (32) and (31) is proved. This ends the proof of Theorem 3.3. $\square$

4. Examples

In this section we give some densities f verifing assumptions (H1)–(H3).

4.1. Notations and preliminaries. We recall properties of generalized N-functions of Δ_2-class ([31]). Let $g : \Omega \times [0, +\infty) \to [0, +\infty)$ be a generalized N-function, i.e., for a.e. $x \in \Omega$,

$$t \to g(x,t) \text{ is convex, increasing and } C^1([0,+\infty)), \tag{33}$$

$$\frac{\partial g}{\partial t}(x,0) = 0 = g(x,0) < g(x,t) \quad \text{if } 0 < t. \tag{34}$$

Moreover, for every $t \in [0, +\infty)$,

$$x \to g(x,t) \quad \text{and} \quad x \to \frac{\partial g}{\partial t}(x,t) \quad \text{are measurable.} \tag{35}$$

In addition, we assume Δ_2-property uniformly with respect to x: There exists a constant $k_2 > 0$ such that, for a.e. $x \in \Omega$,

$$g(x, 2t) \le k_2 g(x,t) \quad \forall t \ge 0. \tag{36}$$

Now we recall known properties of function $g : \Omega \times [0, +\infty) \to [0, +\infty)$ satisfying (33), (34) and (36), see [31]. Fix $x \in \Omega$. For every s and t in $[0, +\infty)$ convexity gives

$$g(x,s) \ge g(x,t) + \frac{\partial g}{\partial t}(x,t)(s-t). \tag{37}$$

We use $s = 0$ in (37). Since $g(x,0) = 0$, it results that

$$g(x,t) \le \frac{\partial g}{\partial t}(x,t)t \quad \forall t \ge 0. \tag{38}$$

We use (37) with $s = 2t$ and Δ_2-property. We get $g(x,t) + \frac{\partial g}{\partial t}(x,t)(t) \le g(x,2t) \le k_2 g(x,t)$ then

$$\frac{\partial g}{\partial t}(x,t)t \le (k_2 - 1)g(x,t) \quad \forall t \ge 0. \tag{39}$$

Inequalities (38), (39) and (34) show that $1 \le k_2 - 1$, then $2 \le k_2$. A careful inspection shows that $2 = k_2$ cannot happen under our assumptions, then $2 < k_2$. By iterating inequality (36) we get, for every $m \in \mathbb{N}$,

$$g(x, 2^m t) \le k_2^m g(x,t) \quad \forall t \ge 0.$$

Therefore

$$g(x, \lambda t) \le k_2 \lambda^{\frac{\ln(k_2)}{\ln(2)}} g(x,t) \quad \forall \lambda \ge 1, \ \forall t \ge 0$$

and for every $r, t \in [0, +\infty)$

$$g(x, rt) \le k_2 \max\left\{1, r^{\frac{\ln(k_2)}{\ln(2)}}\right\} g(x, t).$$

Convexity (33) and Δ_2-property (36) imply that, for every $t_1, t_2 \in [0, +\infty)$

$$g(x, t_1+t_2) = g\left(x, 2\left(\frac{1}{2}t_1+\frac{1}{2}t_2\right)\right) \le k_2 g\left(x, \frac{1}{2}t_1+\frac{1}{2}t_2\right) \le \frac{k_2}{2}\big(g(x,t_1)+g(x,t_2)\big).$$

Now we need the following inequality: Let $h, f : I \subset \mathbb{R} \to [0, +\infty)$ be increasing, then

$$h(t)f(s) \le h(t)f(t) + h(s)f(s) \quad \forall t, s \in I. \tag{40}$$

Let us apply (40) with $h(t) = \frac{\partial g}{\partial t}(x, t)$ and $f(s) = s$, so that, for $t_1, t_2 \in [0, +\infty)$, we have

$$0 \le \frac{\partial g}{\partial t}(x, t_1)t_2 \le \frac{\partial g}{\partial t}(x, t_1)t_1 + \frac{\partial g}{\partial t}(x, t_2)t_2.$$

Moreover, (39) allows us to write

$$\frac{\partial g}{\partial t}(x, t_1)t_1 + \frac{\partial g}{\partial t}(x, t_2)t_2 \le (k_2 - 1)(g(x, t_1) + g(x, t_2)).$$

4.2. Example 1. Let us define

$$f(x, z) = g(x, |z|)$$

where $g : \Omega \times [0, +\infty) \to [0, +\infty)$ satisfies (33), (34) and (36). We obtain

$$\frac{\partial f}{\partial z_i^\alpha}(x, z) = \begin{cases} \dfrac{\partial g}{\partial t}(x, |z|)\dfrac{z_i^\alpha}{|z|} & \text{if } z \ne 0, \\ 0 & \text{if } z = 0, \end{cases}$$

so that, if $z \neq 0$,

$$\sum_{i=1}^{n}\sum_{\alpha=1}^{N}\frac{\partial f}{\partial z_i^\alpha}(x, z)z_i^\alpha = \sum_{i=1}^{n}\sum_{\alpha=1}^{N}\frac{\partial g}{\partial t}(x, |z|)\frac{z_i^\alpha}{|z|}z_i^\alpha = \frac{\partial g}{\partial t}(x, |z|)|z| \ge g(x, |z|) = f(x, z)$$

where we used (38) in the inequality. If $z = 0$ then $\frac{\partial f}{\partial z_i^\alpha}(x, z) = 0 = g(x, 0) = f(x, z)$. Then (10) holds true with $\nu = 1$. In order to verify (11), assume that $z = v + tw \neq 0$. By means of properties of g, $|z| \le |v| + |w|$, provided $\epsilon \in (0, 1]$, we have

$$
\begin{aligned}
&\left|\sum_{i=1}^{n}\sum_{\alpha=1}^{N}\frac{\partial f}{\partial z_i^\alpha}(x,z)w_i^\alpha\right| \\
&= \frac{\partial g}{\partial t}(x,|z|)\frac{1}{|z|}\left|\sum_{i=1}^{n}\sum_{\alpha=1}^{N} z_i^\alpha w_i^\alpha\right| \\
&\leq \frac{\partial g}{\partial t}(x,|z|)|w| \\
&= \epsilon\frac{\partial g}{\partial t}(x,|z|)\frac{|w|}{\epsilon} \\
&\leq \epsilon(k_2-1)\left[g(x,|z|)+g\left(x,\frac{|w|}{\epsilon}\right)\right] \qquad (41)\\
&\leq \epsilon(k_2-1)\left[g(x,|v|+|w|)+g\left(x,\frac{|w|}{\epsilon}\right)\right] \\
&\leq \epsilon(k_2-1)\left[\frac{k_2}{2}g(x,|v|)+\frac{k_2}{2}g(x,|w|)+k_2\left(\frac{1}{\epsilon}\right)^{\frac{\ln(k_2)}{\ln(2)}}g(x,|w|)\right] \\
&= \epsilon(k_2-1)\frac{k_2}{2}\left[f(x,v)+\left(1+2\left(\frac{1}{\epsilon}\right)^{\frac{\ln(k_2)}{\ln(2)}}\right)f(x,w)\right].
\end{aligned}
$$

Since $k_2 > 2$ we take $\epsilon = \frac{1}{(k_2-1)k_2} \in \left(0, \frac{1}{2}\right)$ and (41) becomes

$$
\left|\sum_{i=1}^{n}\sum_{\alpha=1}^{N}\frac{\partial f}{\partial z_i^\alpha}(x,z)w_i^\alpha\right| \leq \frac{1}{2}\left[f(x,v)+\left(1+2k_2^{\frac{2\ln(k_2)}{\ln(2)}}\right)f(x,w)\right]. \tag{42}
$$

When $z = v + tw = 0$ easily (42) holds true. Then we checked (11) with $L = \frac{1}{2}\left(1+2k_2^{\frac{2\ln(k_2)}{\ln(2)}}\right)$. Inequality (13) follows easily. Indeed, if $z \neq 0$ we have

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{\alpha=1}^{N}\frac{\partial f}{\partial z_i^\alpha}(x,z)y^\alpha\sum_{\beta=1}^{N}y^\beta z_i^\beta &= \sum_{i=1}^{n}\sum_{\alpha=1}^{N}\frac{\partial g}{\partial t}(x,|z|)\frac{z_i^\alpha}{|z|}y^\alpha\sum_{\beta=1}^{N}y^\beta z_i^\beta \\
&= \frac{\partial g}{\partial t}(x,|z|)\frac{1}{|z|}\sum_{i=1}^{n}\sum_{\alpha=1}^{N}z_i^\alpha y^\alpha\sum_{\beta=1}^{N}y^\beta z_i^\beta \\
&= \frac{\partial g}{\partial t}(x,|z|)\frac{1}{|z|}\sum_{i=1}^{n}(\langle z_i; y\rangle)^2 \\
&\geq 0.
\end{aligned}
$$

Now we are going to verify (12). If $|z_i| \leq |w_i|$ for every i, then $|z| \leq |w|$. Since $t \to g(x,t)$ is increasing, we get $f(x,z) = g(x,|z|) \leq g(x,|w|) = f(x,w)$.

Thus (12) holds true with $H = 1$. Note that (8) is verified. If g satisfies also (35) then (9) is satisfied, too.

4.3. Example 2. Define

$$f(x, z) = \sum_{j=1}^{n} g_j(x, |z_j|)$$

where every $g_j : \Omega \times [0, +\infty) \to [0, +\infty)$ satisfies (33), (34) and (36). Note that Δ_2-property (36) holds true with the same constant k_2 for every g_j. Then

$$\frac{\partial f}{\partial z_i^\alpha}(x, z) = \begin{cases} \dfrac{\partial g_i}{\partial t}(x, |z_i|)\dfrac{z_i^\alpha}{|z_i|} & \text{if } z_i \neq 0, \\ 0 & \text{if } z_i = 0. \end{cases}$$

Similar arguments to those performed in the above Example 1 on each g_j allow us to check (10) with $\nu = 1$, (11) with $L = \frac{1}{2}\left(1 + 2k_2^{\frac{2\ln(k_2)}{\ln(2)}}\right)$, (13) and (12) with $H = 1$. Note that (8) is verified. If, in addition, every g_j satisfies also (35) then (9) is satisfied, too.

4.4. Example 3. We take

$$f(x, z) = a(x, |z_*|) + b(x, |z^*|)$$

where $a, b : \Omega \times [0, +\infty) \to [0, +\infty)$ satisfy (33), (34) and (36). Note that the Δ_2-property (36) holds true for a and b with the same constant k_2. Moreover, I_* and I^* are not empty subsets of $\{1, \ldots, n\}$ with $I_* \cap I^* = \emptyset$ and $I_* \cup I^* = \{1, \ldots, n\}$.

$$z_* = \{z_i^\alpha : i \in I_* \text{ and } \alpha = 1, \ldots, N\}$$

and

$$z^* = \{z_i^\alpha : i \in I^* \text{ and } \alpha = 1, \ldots, N\}.$$

We get

$$\frac{\partial f}{\partial z_i^\alpha}(x, z) = \begin{cases} \dfrac{\partial a}{\partial t}(x, |z_*|)\dfrac{z_i^\alpha}{|z_*|} & \text{if } i \in I_* \text{ and } z_* \neq 0, \\ 0 & \text{if } i \in I_* \text{ and } z_* = 0, \\ \dfrac{\partial b}{\partial t}(x, |z^*|)\dfrac{z_i^\alpha}{|z^*|} & \text{if } i \in I^* \text{ and } z^* \neq 0, \\ 0 & \text{if } i \in I^* \text{ and } z^* = 0. \end{cases}$$

By proceeding as in Example 1, separately on a and b, we obtain (10) with $\nu = 1$, (11) with $L = \frac{1}{2}\left(1 + 2k_2^{\frac{2\ln(k_2)}{\ln(2)}}\right)$, (13) and (12) with $H = 1$. Note that (8) is verified. When a and b satisfy also (35) then (9) holds true.

Remark 4.1. Now we show a "negative" example in which sign condition (13) is not fulfilled. When $N = n$ we take

$$f(x,z) = |z|^2 + (tr(z))^2 = \sum_{r,s=1}^{n} (z_r^s)^2 + \left(\sum_{r=1}^{n} z_r^r\right)^2.$$

Then $\frac{\partial f}{\partial z_i^\alpha}(z) = 2z_i^\alpha + 2\left(\sum_{r=1}^n z_r^r\right)\delta_{i\alpha}$ where $\delta_{i\alpha} = 1$ when $i = \alpha$ and $\delta_{i\alpha} = 0$ when $i \neq \alpha$. We take z to be a diagonal matrix and y to be the unit vector in the first direction: $z_i^\alpha = t_i\delta_{i\alpha}$ for suitable constants $t_1, \ldots, t_n$ and $y^\alpha = \delta_{1\alpha}$. Then we have

$$\sum_{i,\alpha} \frac{\partial f}{\partial z_i^\alpha}(z) y^\alpha \sum_\beta y^\beta z_i^\beta = 2t_1 \left[t_1 + \sum_{r=1}^n t_r\right] < 0$$

provided $t_1 = 1$, $t_2 < -2$ and $t_r = 0$ for $r = 3, \ldots, n$.

5. Proof of Theorem 2.1

Let u be a minimizer of (1). We split the proof into several steps.

Step 1. We construct a suitable test function v to be inserted into Euler system (30). Let $\phi : [0, +\infty) \to [0, +\infty)$ be increasing and $C^1([0, +\infty))$. Moreover we assume that there exists a constant $\tilde{c} \in [1, +\infty)$ such that

$$0 \le \phi(t) \le \tilde{c} \quad \forall\, t \in [0, +\infty) \tag{43}$$
$$0 \le \phi'(t) \le \tilde{c} \quad \forall\, t \in [0, +\infty) \tag{44}$$
$$0 \le \phi'(t)t \le \tilde{c} \quad \forall\, t \in [0, +\infty). \tag{45}$$

Let $B_\rho = B(x_0, \rho)$ and $B_R = B(x_0, R)$ be open balls with the same center x_0 and radii $0 < \rho < R \le 1$, with $\overline{B_R} \subset \Omega$. We assume that $\eta : \mathbb{R}^n \to \mathbb{R}$, $\eta \in C_0^1(B_R)$ with $0 \le \eta \le 1$ in $\mathbb{R}^n$, $\eta = 1$ on B_ρ, $|D\eta| \le \frac{4}{R-\rho}$ in $\mathbb{R}^n$. Note that $0 < R - \rho < R \le 1$ so $\frac{4}{R-\rho} > 4$. Let $m > 1$. We consider the test function $v = (v^1, \ldots, v^N)$ defined as follows

$$v^\alpha = \phi(|u|)u^\alpha\eta^m. \tag{46}$$

It results that $v^\alpha \in W_0^{1,1}(B_R) \subset W_0^{1,1}(\Omega)$ and

$$D_i v^\alpha = \eta^m \left[\phi'(|u|)1_{\{|u|>0\}} \sum_{\beta=1}^N \frac{u^\beta}{|u|}(D_i u^\beta)u^\alpha + \phi(|u|)D_i u^\alpha\right] + [\phi(|u|)u^\alpha]D_i(\eta^m)$$

where $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$. We claim that $x \to f(x, Dv(x)) \in L^1(\Omega)$. Indeed, (45) gives

$$\sum_{\alpha=1}^{N} \left| \phi'(|u|) 1_{\{|u|>0\}} \sum_{\beta=1}^{N} \frac{u^\beta}{|u|} (D_i u^\beta) u^\alpha \eta^m \right|^2 \leq (\tilde{c})^2 |D_i u|^2. \tag{47}$$

Let us set

$$z_i^\alpha = \phi'(|u|) 1_{\{|u|>0\}} \sum_{\beta=1}^{N} \frac{u^\beta}{|u|} (D_i u^\beta) u^\alpha \eta^m \quad \text{and} \quad w_i^\alpha = \tilde{c} D_i u^\alpha.$$

Since inequality (47) gives $|z_i| \leq |w_i|$, by assumption (12) and property (20) with $\tilde{c} \leq k \in \mathbb{N}$ we get:

$$\begin{aligned} f\Big(x, \phi'(|u|) 1_{\{|u|>0\}} \sum_{\beta=1}^{N} \frac{u^\beta}{|u|} [(Du^\beta) \times u] \eta^m \Big) &\leq H f(x, \tilde{c} Du) \\ &\leq 2H f(x, Du) \sum_{i=1}^{k+1} (\tilde{L})^i. \end{aligned} \tag{48}$$

Since u has finite energy (2), the positivity of f and inequality (48) ensure that

$$x \to f\Big(x, \phi'(|u(x)|) 1_{\{|u|>0\}}(x) \sum_{\beta=1}^{N} \frac{u^\beta(x)}{|u(x)|} [(Du^\beta(x)) \times u(x)] \eta^m(x)\Big) \in L^1(\Omega) \tag{49}$$

Moreover, (43) and properties of η give $0 \leq \phi(|u|)\eta^m \leq \tilde{c} \leq k$ for a suitable $k \in \mathbb{N}$. Then (20) implies $f(x, \phi(|u|)\eta^m Du) \leq 2 f(x, Du) \sum_{i=1}^{k+1} (\tilde{L})^i$ and then

$$x \to f(x, \phi(|u(x)|)\eta^m(x) Du(x)) \in L^1(\Omega). \tag{50}$$

Finally, again by (43) and (20) we get $f(x, \phi(|u|) u \times D(\eta^m)) \leq 2 f(x, u \times D(\eta^m)) \sum_{i=1}^{k+1} (\tilde{L})^i$. Since u has finite energy (2), the left hand side of (14) guarantees that $Du \in L^p(\Omega)$. Sobolev embedding and (15) give us $u \in L^{p^*}(B_R) \subset L^q(B_R)$. We recall that $\eta = 0$ outside B_R. Since $f(x, 0) = 0$, then

$$f(x, u \times D(\eta^m)) = f(x, u \times D(\eta^m)) 1_{B_R}.$$

Now we use the right hand side of (14) and the estimate for $|D\eta|$:

$$f(x, u \times D(\eta^m)) 1_{B_R} \leq (c_3 |u \times D(\eta^m)|^q + c_4) 1_{B_R} \leq \Big(c_3 m^q \Big(\frac{4}{R-\rho}\Big)^q |u|^q + c_4\Big) 1_{B_R}.$$

Since $q < p^*$, we have $u \in L^q(B_R)$ and

$$x \to f(x, \phi(|u(x)|) u(x) \times D(\eta^m(x))) \in L^1(\Omega). \tag{51}$$

Inequality (19) and (49), (50), (51) give $x \to f(x, Dv(x)) \in L^1(\Omega)$.

Step 2. For ϕ and η as in the previous step we prove that

$$\begin{aligned}\int_{B_R} |Du|^p \phi(|u|)\eta^m \, dx \leq \frac{2Lc_3}{\nu c_1}\left(\frac{4m}{R-\rho}\right)^q \int_{B_R} |u|^q \phi(|u|)\, dx \\ + \left(\frac{2Lc_4}{\nu c_1} + \frac{c_2}{c_1}\right) \int_{B_R} \phi(|u|)\, dx.\end{aligned} \tag{52}$$

By inserting $v = \phi(|u|)u\eta^m$ into Euler System (30), we get

$$\begin{aligned}0 &= \int_\Omega \sum_{i=1}^n \sum_{\alpha=1}^N \frac{\partial f}{\partial z_i^\alpha}(x, Du) D_i v^\alpha \, dx \\ &= \int_\Omega \sum_{i=1}^n \sum_{\alpha=1}^N \frac{\partial f}{\partial z_i^\alpha}(x, Du)\phi'(|u|)1_{\{|u|>0\}} \sum_{\beta=1}^N \frac{u^\beta}{|u|}(D_i u^\beta) u^\alpha \eta^m \, dx \\ &+ \int_\Omega \sum_{i=1}^n \sum_{\alpha=1}^N \frac{\partial f}{\partial z_i^\alpha}(x, Du)\phi(|u|)(D_i u^\alpha)\eta^m \, dx \\ &+ \int_\Omega \sum_{i=1}^n \sum_{\alpha=1}^N \frac{\partial f}{\partial z_i^\alpha}(x, Du)\phi(|u|) u^\alpha D_i(\eta^m)\, dx \\ &= (A_1) + (A_2) + (A_3).\end{aligned}$$

Thus

$$(A_1) + (A_2) = -(A_3). \tag{53}$$

We can use assumption (13) with $z = Du(x)$ and $y = u(x)$ in such a way that $0 \leq (A_1)$. Coercivity assumption (10) with $z = Du(x)$ gives:

$$\nu \int_\Omega f(x, Du)\phi(|u|)\eta^m dx \leq (A_2).$$

We apply (11) with $v = Du(x)$, $t = 0$ and $w = [u(x) \times D\eta(x)]m\eta^{-1}(x)$ as follows

$$\begin{aligned}-(A_3) &= \int_{\{\eta>0\}} -\sum_{i=1}^n \sum_{\alpha=1}^N \frac{\partial f}{\partial z_i^\alpha}(x, Du) u^\alpha (D_i\eta)\eta^{-1} m\phi(|u|)\eta^m dx \\ &\leq \frac{\nu}{2}\int_\Omega f(x, Du)\phi(|u|)\eta^m dx + L\int_{\{\eta>0\}} f(x, [u \times D\eta]m\eta^{-1})\phi(|u|)\eta^m dx.\end{aligned}$$

These inequalities can be inserted into (53) and we get the following Caccioppoli estimate

$$\frac{\nu}{2}\int_\Omega f(x, Du)\phi(|u|)\eta^m \, dx \leq L\int_{\{\eta>0\}} f(x, [u \times D\eta]m\eta^{-1})\phi(|u|)\eta^m \, dx. \tag{54}$$

The right hand side of growth assumption (14) allows us to write

$$
\begin{aligned}
&\int_{\{\eta>0\}} f(x,[u\times D\eta]m\eta^{-1})\phi(|u|)\eta^m\,dx\\
&\leq \int_{\{\eta>0\}} [c_3(|u\times D\eta|m\eta^{-1})^q + c_4]\phi(|u|)\eta^m\,dx\\
&= \int_{\{\eta>0\}} [c_3(|u|^q|D\eta|^q m^q\eta^{-q+m}\phi(|u|) + c_4\phi(|u|)\eta^m]\,dx\\
&= (A_4).
\end{aligned}
$$

By choosing $m = q+1$, since $0 \leq \eta \leq 1$, we have

$$
(A_4) \leq \int_\Omega [c_3(|u|^q|D\eta|^q m^q\phi(|u|) + c_4\phi(|u|)\eta^m]\,dx.
$$

The left hand side of growth assumption (14) allows us to get

$$
\int_\Omega [c_1|Du|^p - c_2]\phi(|u|)\eta^m dx \leq \int_\Omega f(x,Du)\phi(|u|)\eta^m\,dx.
$$

Thus Caccioppoli inequality (54) gives

$$
\frac{\nu}{2}\int_\Omega [c_1|Du|^p - c_2]\phi(|u|)\eta^m dx \leq L\int_\Omega [c_3(|u|^q|D\eta|^q m^q\phi(|u|) + c_4\phi(|u|)\eta^m]dx
$$

so that

$$
\int_\Omega |Du|^p\phi(|u|)\eta^m dx \leq \frac{2Lc_3m^q}{\nu c_1}\int_\Omega |u|^q|D\eta|^q\phi(|u|)dx + \left(\frac{2Lc_4}{\nu c_1} + \frac{c_2}{c_1}\right)\int_\Omega \phi(|u|)\eta^m dx.
$$

By the properties of η and $|D\eta|$, we get (52).

Step 3. Let $\beta \in (1,+\infty)$ and assume that

$$
|u| \in L^{q+p(\beta-1)}(B_R). \tag{55}
$$

With a suitable choice of ϕ we are going to show that

$$
\int_{B_R} |Du|^p\beta^p|u|^{p(\beta-1)}\eta^m\,dx \leq c_5\left(\frac{4m}{R-\rho}\right)^q\beta^p\int_{B_R}(1+|u|^{q+p(\beta-1)})\,dx, \tag{56}
$$

where $c_5 = \frac{2L(c_2+c_3+c_4)}{\nu c_1}$. Indeed, for every $k \in \mathbb{N}$, we consider $\phi_k : [0,+\infty) \to [0,+\infty)$ in $C^1([0,+\infty))$ such that there exists $\tilde{c}_k \in [1,+\infty)$ for which the following properties hold true:

$$
\phi_k(t), \phi_k'(t), \phi_k'(t)t \in [0,\tilde{c}_k] \quad \forall\, t \in [0,+\infty), \tag{57}
$$

$$
0 \leq \phi_k(t) \leq (\beta t^{\beta-1})^p \qquad \forall\, t \in [0,+\infty), \tag{58}
$$

$$\lim_{k\to+\infty} \phi_k(t) = (\beta t^{\beta-1})^p \quad \forall t \in [0,+\infty). \tag{59}$$

For instance, the construction of ϕ_k can be done as follows. We consider

$$\tilde{\phi}(t) = ct^\alpha$$

where $c = \beta^p$ and $\alpha = (\beta-1)p$. Since $\tilde{\phi}'(t) = c\alpha t^{\alpha-1}$ and $\tilde{\phi}''(t) = c\alpha(\alpha-1)t^{\alpha-2}$, we have to distinguish the case $0 < \alpha < 1$ from $1 \leq \alpha$. Indeed, when $0 < \alpha < 1$, we see that $\tilde{\phi}'$ is decreasing and $\lim_{t\to0^+} \tilde{\phi}'(t) = +\infty$. On the other hand, when $1 \leq \alpha$, then ϕ' is increasing and $\lim_{t\to0^+} \phi'(t) \in \mathbb{R}$. Thus, when $0 < \alpha < 1$ we consider

$$\theta_k(t) = \begin{cases} \tilde{\phi}'\left(\dfrac{1}{k}\right) & \text{for } t \in \left[0, \dfrac{1}{k}\right) \\ \tilde{\phi}'(t) & \text{for } t \in \left[\dfrac{1}{k}, k\right] \\ \tilde{\phi}'(k)(k+1-t) & \text{for } t \in (k, k+1) \\ 0 & \text{for } t \in [k+1, +\infty). \end{cases}$$

When $1 \leq \alpha$ it is not necessary to modify $\tilde{\phi}'(t)$ for small t and we can consider

$$\theta_k(t) = \begin{cases} \tilde{\phi}'(t) & \text{for } t \in [0, k] \\ \tilde{\phi}'(k)(k+1-t) & \text{for } t \in (k, k+1) \\ 0 & \text{for } t \in [k+1, +\infty). \end{cases}$$

We set $\phi_k(s) = \int_0^s \theta_k(t)dt$ and all the required properties are verified. Consider (52) with ϕ replaced by ϕ_k. Assumption (55) and property (58) allow us to write

$$\begin{aligned} 0 \leq \phi_k(|u|) \quad &\leq \beta^p |u|^{p(\beta-1)} \quad \in L^1(B_R), \\ 0 \leq |u|^q \phi_k(|u|) &\leq \beta^p |u|^{q+p(\beta-1)} \in L^1(B_R). \end{aligned}$$

So (52) becomes

$$\begin{aligned} &\int_{B_R} |Du|^p \phi_k(|u|)\eta^m \, dx \\ &\leq \frac{2Lc_3}{\nu c_1}\left(\frac{4m}{R-\rho}\right)^q \int_{B_R} \beta^p |u|^{q+p(\beta-1)} \, dx + \left(\frac{2Lc_4}{\nu c_1} + \frac{c_2}{c_1}\right) \int_{B_R} \beta^p |u|^{p(\beta-1)} \, dx \\ &\leq \frac{2L(c_2+c_3+c_4)}{\nu c_1}\left(\frac{4m}{R-\rho}\right)^q \beta^p \int_{B_R} (1 + |u|^{q+p(\beta-1)}) \, dx \end{aligned}$$

since $\frac{4m}{R-\rho} > 4m > 4$ and (29) implies $\frac{L}{\nu} \geq 1$. We set $c_5 = \frac{2L(c_2+c_3+c_4)}{\nu c_1}$ and get

$$\int_{B_R} |Du|^p \phi_k(|u|)\eta^m \, dx \leq c_5 \left(\frac{4m}{R-\rho}\right)^q \beta^p \int_{B_R} (1 + |u|^{q+p(\beta-1)}) \, dx.$$

Fatou lemma and (59) allow us to let k go to ∞ and (56) follows.

Step 4. Now we prove that

$$u \in L^{q+p(\beta-1)}(B_R) \quad \text{for some } \beta > 1 \quad \Longrightarrow \quad u \in L^{\beta p^*}(B_\rho) \tag{60}$$

and the following estimate holds true

$$\int_{B_\rho} (1+|u|^{\beta p^*})\,dx \le c_8 \beta^{p^*} \left(\frac{8m}{R-\rho}\right)^{q\frac{p^*}{p}} \left(\int_{B_R} (1+|u|^{q+p(\beta-1)})\,dx\right)^{\frac{p^*}{p}} \tag{61}$$

where $c_8 = 2\left((1+|B_1|^{-\frac{p}{n}}) + \frac{2L(c_1+c_2+c_3+c_4)}{\nu c_1}\left(\frac{p(n-1)}{n-p}\right)^p\right)^{\frac{p^*}{p}} \in (1,+\infty)$. Indeed, assumption (55) and Caccioppoli inequality (56) allow us to check that the function $w = |u|^\beta \eta^m$ is in $W_0^{1,p}(B_R)$ with

$$|Dw| \le \beta |u|^{\beta-1}|Du|\eta^m + |u|^\beta m \eta^{m-1}|D\eta|$$

and

$$\begin{aligned}
\int_{B_R} |Dw|^p\,dx &\le 2^p \int_{B_R} |Du|^p \beta^p |u|^{p(\beta-1)}\eta^m\,dx \\
&\quad + 2^p \left(\frac{4m}{R-\rho}\right)^p \int_{B_R} (1+|u|^{q+p(\beta-1)})\,dx \\
&\le 2^p c_5 \left(\frac{4m}{R-\rho}\right)^q \beta^p \int_{B_R} (1+|u|^{q+p(\beta-1)})\,dx \\
&\quad + 2^p \left(\frac{4m}{R-\rho}\right)^p \int_{B_R} (1+|u|^{q+p(\beta-1)})\,dx.
\end{aligned}$$

Then $\int_{B_R} |Dw|^p\,dx \le (1+c_5)\left(\frac{8m}{R-\rho}\right)^q \beta^p \int_{B_R}(1+|u|^{q+p(\beta-1)})\,dx$. Since $p<n$, we can use Sobolev embedding theorem and we get

$$\begin{aligned}
\left(\int_{B_R} |w|^{p^*}\,dx\right)^{\frac{p}{p^*}} &\le \left(\frac{p(n-1)}{n-p}\right)^p \int_{B_R} |Dw|^p\,dx \\
&\le \left(\frac{p(n-1)}{n-p}\right)^p (1+c_5)\left(\frac{8m}{R-\rho}\right)^q \beta^p \int_{B_R} (1+|u|^{q+p(\beta-1)})\,dx
\end{aligned}$$

so that

$$\left(\int_{B_R} (|u|^\beta \eta^m)^{p^*}\,dx\right)^{\frac{p}{p^*}} \le c_6 \beta^p \left(\frac{8m}{R-\rho}\right)^q \int_{B_R} (1+|u|^{q+(\beta-1)p})\,dx$$

where $c_6 = \frac{2L(c_1+c_2+c_3+c_4)}{\nu c_1}\left(\frac{p(n-1)}{n-p}\right)^p \in (1,+\infty)$ since $1 = \frac{c_1}{c_1} \le \frac{2Lc_1}{\nu c_1}$. Note that

$$\begin{aligned}\left(\int_{B_R} 1\,dx\right)^{\frac{p}{p^*}} &= \left(\int_{B_R} 1\,dx\right)(|B_1|R^n)^{-\frac{p}{n}}\\ &\le (1+|B_1|^{-\frac{p}{n}})\frac{1}{(R-\rho)^p}\int_{B_R} 1\,dx\\ &\le (1+|B_1|^{-\frac{p}{n}})\beta^p\left(\frac{8m}{R-\rho}\right)^q\int_{B_R}(1+|u|^{q+p(\beta-1)})\,dx.\end{aligned}$$

Then we obtain

$$\begin{aligned}\left(\int_{B_R}(1+(|u|^\beta\eta^m)^{p^*})\,dx\right)^{\frac{p}{p^*}} &\le 2^{\frac{p}{p^*}}(1+|B_1|^{-\frac{p}{n}})\beta^p\left(\frac{8m}{R-\rho}\right)^q\int_{B_R}(1+|u|^{q+p(\beta-1)})\,dx\\ &\quad + 2^{\frac{p}{p^*}}c_6\beta^p\left(\frac{8m}{R-\rho}\right)^q\int_{B_R}(1+|u|^{q+p(\beta-1)})\,dx\\ &= c_7\beta^p\left(\frac{8m}{R-\rho}\right)^q\int_{B_R}(1+|u|^{q+p(\beta-1)})\,dx\end{aligned}$$

where $c_7 = 2^{\frac{p}{p^*}}((1+|B_1|^{-\frac{p}{n}})+c_6) \in (1,+\infty)$. Since $\eta = 1$ on B_ρ and $0 \le \eta$, we have $\left(\int_{B_\rho}(1+|u|^{\beta p^*})\,dx\right)^{\frac{p}{p^*}} \le c_7\beta^p\left(\frac{8m}{R-\rho}\right)^q\int_{B_R}(1+|u|^{q+p(\beta-1)})\,dx$ and (61) follows.

Step 5. Now we use Moser's iteration. Let us recall assumption (15): $q < p^*$. Then

$$q + p(\beta-1) < \beta p^*.$$

Let us define β_1 such that $q+p(\beta_1-1) = p^*$. It turns out that $\beta_1 = 1+(p^*-q)/p$. Since $q < p^*$, then $\beta_1 > 1$ and (60) gives higher integrabilty. We iterate this procedure as follows. Let B_σ be the open ball with radius $\sigma \le 1$, centered at x_0, with $\overline{B_\sigma} \subset \Omega$. We define the radii ρ_k in this way

$$\rho_1 = \sigma - \frac{\sigma}{2^{1+1}} \quad \text{and} \quad \rho_{j+1} = \rho_j - \frac{\sigma}{2^{1+j+1}} \quad \text{for } j \in \mathbb{N}.$$

Then $\frac{1}{2}\sigma < \rho_k \le \frac{3}{4}\sigma$. We define R_k as follows

$$R_1 = \sigma \quad \text{and} \quad R_{j+1} = \rho_j \quad \text{for } j \in \mathbb{N}.$$

Then $R_k - \rho_k = \frac{\sigma}{2^{1+k}}$. We define exponents β_k as follows

$$q + p(\beta_1 - 1) = p^* \quad \text{and} \quad q + p(\beta_{j+1} - 1) = p^*\beta_j \quad \text{for } j \in \mathbb{N}.$$

It results that $\beta_j \in (1, +\infty)$ and

$$\beta_j = \left(\frac{p^*}{p}\right)^j \frac{p^* - q}{p^* - p} + \frac{q-p}{p^* - p}.$$

We iterate (61) and, for every $j \in \mathbb{N}$, we get

$$\int_{B_{\rho_j}} (1+|u|^{p^*\beta_j})\, dx \le (c_8)^{\sum_{k=0}^{j-1}\left(\frac{p^*}{p}\right)^k} \left(\Pi_{k=1}^{j} (\beta_k)^{p^*\left(\frac{p^*}{p}\right)^{j-k}}\right)$$
$$\times \left(\Pi_{h=1}^{j} \left(\frac{8m}{\sigma} 2^{1+h}\right)^{q\left(\frac{p^*}{p}\right)^{1+j-h}}\right) \left(\int_{B_\sigma} (1+|u|^{p^*})\, dx\right)^{\left(\frac{p^*}{p}\right)^j}$$

where all balls have the same center x_0. Since $\frac{\sigma}{2} < \rho_k$, taking the power of both sides with exponent $\frac{1}{p^*\beta_j}$ we obtain

$$\left(\int_{B_{\frac{\sigma}{2}}} |u|^{p^*\beta_j}\, dx\right)^{\frac{1}{p^*\beta_j}}$$
$$\le (c_8)^{\frac{1}{p^*\beta_j}\sum_{k=0}^{j-1}\left(\frac{p^*}{p}\right)^k} \left(\Pi_{k=1}^{j} (\beta_k)^{\left(\frac{p^*}{p}\right)^{j-k}\frac{1}{\beta_j}}\right) \tag{62}$$
$$\times \left(\Pi_{h=1}^{j} \left(\frac{8m}{\sigma} 2^{1+h}\right)^{\frac{q}{p^*}\left(\frac{p^*}{p}\right)^{1+j-h}\frac{1}{\beta_j}}\right) \left(\int_{B_\sigma} (1+|u|^{p^*})\, dx\right)^{\left(\frac{p^*}{p}\right)^j \frac{1}{p^*\beta_j}}.$$

Note that for every $j \in \mathbb{N}$ we have $1 \le \frac{\left(\frac{p^*}{p}\right)^j}{\beta_j} \le \frac{p^*-p}{p^*-q}$,

$$(c_8)^{\frac{1}{p^*\beta_j}\sum_{k=0}^{j-1}\left(\frac{p^*}{p}\right)^k} < (c_8)^{\frac{p}{p^*(p^*-q)}} \tag{63}$$

and $\left(\int_{B_\sigma} (1+|u|^{p^*})\, dx\right)^{\left(\frac{p^*}{p}\right)^j \frac{1}{p^*\beta_j}} \le \left(\int_{B_\sigma} (1+|u|^{p^*})\, dx\right)^{\frac{1}{p^*}} + \left(\int_{B_\sigma} (1+|u|^{p^*})\, dx\right)^{\frac{p^*-p}{p^*(p^*-q)}}$.
Moreover

$$\Pi_{k=1}^{j} (\beta_k)^{\left(\frac{p^*}{p}\right)^{j-k}\frac{1}{\beta_j}} < e^{\frac{p^*-p}{p^*-q}\left(\ln\left(\frac{p^*}{p}\right)\right)\sum_{k=1}^{+\infty} k\left(\frac{p}{p^*}\right)^k} \tag{64}$$

and

$$\Pi_{h=1}^{j} \left(\frac{8m}{\sigma} 2^{1+h}\right)^{\frac{q}{p^*}\left(\frac{p^*}{p}\right)^{1+j-h}\frac{1}{\beta_j}} < e^{\frac{q}{p}\frac{p^*-p}{p^*-q}\left(\ln\left(\frac{32m}{\sigma}\right)\right)\sum_{h=1}^{+\infty}\left(\frac{p}{p^*}\right)^h h}. \tag{65}$$

We insert the previous estimates (63), (64) and (65) into (62). For every $j \in \mathbb{N}$ we obtain

$$\left(\int_{B_{\frac{\sigma}{2}}} |u|^{p^*\beta_j}\, dx\right)^{\frac{1}{p^*\beta_j}} \le (c_8)^{\frac{p}{p^*(p^*-q)}} e^{\frac{p^*-p}{p^*-q}\left(\ln\left(\frac{p^*}{p}\right)\right)\sum_{k=1}^{+\infty} k\left(\frac{p}{p^*}\right)^k}$$
$$\times e^{\frac{q}{p}\frac{p^*-p}{p^*-q}\left(\ln\left(\frac{32m}{\sigma}\right)\right)\sum_{h=1}^{+\infty}\left(\frac{p}{p^*}\right)^h h} \left(\int_{B_\sigma} (1+|u|^{p^*}) dx\right)^{\left(\frac{p^*}{p}\right)^j \frac{1}{p^*\beta_j}}. \tag{66}$$

Again by (15), $q < p^*$, we get

$$\lim_{j \to +\infty} \beta_j = +\infty \quad \text{and} \quad \lim_{j \to +\infty} \left(\frac{p^*}{p}\right)^j \frac{1}{p^* \beta_j} = \frac{p^* - p}{p^*(p^* - q)}.$$

So, taking the limit as $j \to +\infty$ in (66), we get

$$||u||_{L^\infty(B_{\frac{\sigma}{2}})} \leq (c_8)^{\frac{p}{p^*(p^*-q)}} e^{\frac{p^*-p}{p^*-q}\left(\ln\left(\frac{p^*}{p}\right)\right) \sum_{k=1}^{+\infty} k\left(\frac{p}{p^*}\right)^k}$$
$$\times e^{\frac{q}{p}\frac{p^*-p}{p^*-q}\left(\ln\left(\frac{32m}{\sigma}\right)\right) \sum_{h=1}^{+\infty} \left(\frac{p}{p^*}\right)^h h} \left(\int_{B_\sigma} (1 + |u|^{p^*})\, dx\right)^{\frac{p^*-p}{p^*(p^*-q)}}.$$

This ends the proof. □

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