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Nonderivative modified gravity: a classification

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Abstract. We analyze the theories of gravity modified by a generic nonderivative potential built from the metric, under the minimal requirement of unbroken spatial rotations. Using the canonical analysis, we classify the potentials V according to the number of degrees of freedom (DoF) that propagate at the nonperturbative level. We then compare the nonperturbative results with the perturbative DoF propagating around Minkowski and FRW backgrounds. A generic V implies 6 propagating DoF at the non-perturbative level, with a ghost on Minkowski background. There exist potentials which propagate 5 DoF, as already studied in previous works. Here, no V with unbroken rotational invariance admitting 4 DoF is found. Theories with 3 DoF turn out to be strongly coupled on Minkowski background. Finally, potentials with only the 2 DoF of a massive graviton exist. Their effect on cosmology is simply equivalent to a cosmological constant. Potentials with 2 or 5 DoF and explicit time dependence appear to be a further viable possibility.

Keywords: modified gravity, gravity, cosmological perturbation theory

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1 Introduction

In the recent years, there has been a substantial progress in understanding possible modifications of Einstein General Relativity at large distances. The quest is on for a theory of gravity which has a massive graviton in the spectrum at the linearized level, thus realizing a full nonlinear theory of gravity modified at large distance.

The main goal of the present investigation is to study systematically the theories of massive gravity obtained by adding a nonderivative potential of the metric components $g_{\mu\nu}$ to the Einstein Hilbert (EH) action¹

$$S = M_{\text{pl}}^2 \int d^3x \sqrt{g} \left(R - m^2 V[g_{\mu\nu}] \right). \quad (1.1)$$

A step forward in taming the zoo of possibilities was made in a series of papers [2–4] through the nonperturbative construction of the most general theories with five propagating degrees of freedom (DoF), characteristic of a massive graviton.² Besides its theoretical interest, the main phenomenological goal is to investigate whether a modification of gravity at large distances and a massive graviton can be realized in a consistent way and in agreement with the wealth of observational tests of gravity, from the smallest (submillimeter) to largest (cosmological) scales. Clearly, one of the crucial tests for a theory of gravity is the existence of a realistic FRW cosmological evolution, which we address later.

¹For the rest of the paper we will set $M_{\text{pl}}^2 = 1$ when not important for our intents.

²See [11, 12] for a alternative analysis using Kuchar’s Hamiltonian formalism.

The key tool for our analysis is the Hamiltonian formalism, which we will use to classify the various potentials V according to the number of degrees of freedom (DoF) that propagate. In GR, where $V = 0$, among the ten components of the metric $g_{\mu\nu}$, diffeomorphism invariance gets rid of eight of them. As soon as we add extra nonderivative terms, diffeomorphism invariance is broken. The invariance can be restored by introducing a set of suitable Stuckelberg fields [6]; of course such procedure does not change the number of DoF. The action we consider can be obtained by choosing a suitable gauge (unitary gauge) where the Stuckelberg fields are trivial; thus, all such a theories have a preferred frame.

On general grounds, once diffeomorphism invariance is broken by nonderivative interactions, one expects six DoF, in contrast with the fact that at the linearized level a massive spin two particle on Minkowski has 5 DoF [14]. Indeed this was the conclusion reached in the past by Boulware and Deser (BD) [13], by studying the nonlinear generalizations of a Lorentz invariant Pauli-Fierz graviton mass term. The mismatch between perturbative and nonperturbative number of DoF is problematic because it is a signal of strong coupling. Moreover, the missing sixth mode is a ghost on a Minkowski background. Nevertheless, there exist particular choices of the potential V where this counting has to be refined and less than six DoF are nonperturbatively present. This helps to get rid of the BD ghost and modifies also the phenomenology of these theories.

According to the Hamiltonian analysis à la Dirac, once we determine all first class (FC) and second class (SC) constraints [15], the number (#) of DoF is given by

$$\# \text{ DoF} = 10 - \frac{1}{2} \# \text{ SC} - \# \text{ FC} . \quad (1.2)$$

The case of five DoF was discussed in detail in [2, 4], together with the phenomenological [3] and cosmological [16] consequences.

Among the various theories, one finds the very special case in which Lorentz symmetry is present around Minkowski background [17], that avoids the BD ghost [18, 19], and is almost unique [2]; see [5] for a complete review on the subject. This theory however is phenomenologically very constrained: denoting with m the graviton mass scale, the “tree level” energy cutoff $\Lambda_3 = (m^2 M_{pl})^{1/3}$ is too low³ as already predicted in [6]; the theory is classically strongly coupled in the solar system [6, 8] and the computation of the static potential in the vicinity of the earth is not an easy task due to quantum corrections [6, 20–25]. Cosmology is also definitely troublesome: spatially flat homogenous Friedmann-Robertson-Walker (FRW) solutions simply do not exist [26] in the unitary gauge and even allowing for open FRW solutions [27] strong coupling [28] and ghostlike instabilities [29] develop. Another issue is the existence of superluminal modes [30, 31]. In the bigravity formulation [32–40] FRW homogenous solutions do exist [41–43], however cosmological perturbations turn out to be strongly coupled [44, 45].

On the other hand, things get better if one gives up Lorentz invariance in the gravitational sector and requires only rotational invariance [3, 9, 10, 46]. Within the general class of theories which propagate five DoF found in [2, 4], in the Lorentz breaking (LB) case most of the theories have a much safer cutoff $\Lambda_2 = (m M_{pl})^{1/2} \gg \Lambda_3$ [3], which is the maximal cutoff that one may obtain. They also avoid all of the phenomenological difficulties mentioned above [3, 16].

³In [7] it has been argued that the cutoff could be higher than Λ_3 to due environmental effects triggered by classical strong coupling.

In the present paper we complete the analysis started in [2, 4] by considering all potentials V which respect rotational invariance, and classify them according to:

- the number of propagating DoF;
- the possibility of a viable FRW cosmology;
- the presence of strong coupling.

The outline of the paper is the following. In section 2, by using Hamiltonian analysis, we find for each V the number of DoF (#DoF). In section 3, we compare the #DoF found by canonical analysis to the #DoF computed using perturbation theory around Minkowski space; as a result we can determine when strong coupling is present. In section 4 we study when a generic V admits a FRW homogeneous solution in the unitary gauge that represents the reference background for our expanding Universe. In appendix B we extend our finding to potentials with an explicit time dependence. Our conclusions are given in section 5.

2 Hamiltonian analysis

The standard Arnowitt-Deser-Misner splitting [47] of spacetime leads to the following parametrization of the metric in terms of lapse N , shifts N^i and spatial metric γ_{ij} :

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^i N^j \gamma_{ij} & \gamma_{ij} N^j \\ \gamma_{ij} N^i & \gamma_{ij} \end{pmatrix}. \quad (2.1)$$

The potential $V(g_{\mu\nu})$ is thus regarded as a function of N , N^i and γ_{ij} .

It is also useful to define

$$\mathcal{V}[N, N^i, \gamma_{ij}] \equiv m^2 N \sqrt{\gamma} V[N, N^i, \gamma_{ij}], \quad (2.2)$$

with $\gamma = \det \gamma_{ij}$, and to write the Hamiltonian as

$$H = \int d^3x \left[\mathcal{H}_A(t, \vec{x}) N^A(t, \vec{x}) + \mathcal{V}(t, \vec{x}) \right], \quad (2.3)$$

where we collected lapse and shifts in $N^A \equiv (N, N^i)$, with $A = 0, 1, 2, 3$, and the first piece is the standard GR Hamiltonian.

Exactly as in GR, the lapse and the shifts appear in the Lagrangian with no time derivatives, so their momenta vanish and lead to the four *primary constraints*

$$\Pi_A = \frac{\partial H}{\partial N^A} \approx 0, \quad A = 0, 1, 2, 3. \quad (2.4)$$

These can be enforced by a set of four Lagrange multipliers λ^A in the *total Hamiltonian*

$$H_T = H + \int d^3x \lambda^A(t, \vec{x}) \Pi_A(t, \vec{x}) \equiv H + \lambda^A \cdot \Pi_A. \quad (2.5)$$

The time evolution of any function F of γ_{ij} , N^A or their momenta is given by the Poisson bracket with H_T

$$\frac{dF(t, \vec{x})}{dt} \equiv \left\{ F(t, \vec{x}), H_T(t) \right\} = \left\{ F(t, \vec{x}), H(t) \right\} + \int d^3y \lambda^A(t, \vec{y}) \left\{ F(t, \vec{x}), \Pi_A(t, \vec{y}) \right\}. \quad (2.6)$$

To avoid excessive cluttering, in the following we will mostly omit the time dependence of the fields. If not stated explicitly, they are evaluated at the same time t .

The conservation in time of the primary constraints leads to four *secondary* constraints

$$\mathcal{S}_A(\vec{x}) = \mathcal{H}_A(\vec{x}) + \mathcal{V}_A(\vec{x}) \approx 0, \quad A = 0, 1, 2, 3, \quad (2.7)$$

where $\mathcal{V}_A \equiv \partial\mathcal{V}/\partial N^A$. Imposing again the conservation of the four secondary constraints, leads to the *tertiary* conditions

$$\mathcal{T}_A(\vec{x}) = \{\mathcal{S}_A(\vec{x}), H\} + \lambda^B(\vec{x}) \mathcal{V}_{AB}(\vec{x}) \approx 0, \quad \mathcal{V}_{AB} \equiv \frac{\partial^2\mathcal{V}}{\partial N^A \partial N^B}. \quad (2.8)$$

The nature of these conditions, i.e. whether they are constraints or determine some of the Lagrange multipliers, depends on the rank of the Hessian of \mathcal{V} with respect to $N^A = (N, N^i)$,

$$r = \text{Rank}|\mathcal{V}_{AB}|. \quad (2.9)$$

The value of r ranges between zero and four, the dimension of spacetime.

2.1 $r = 4$: 6 DoF

If $r = 4$, we can determine all four Lagrangian multipliers from (2.8). All constraints are consistent with the time evolution and the analysis stops here. Thus, we end up with a total number of DoF

$$\#\text{DoF} = \frac{20 - 4(\Pi_A) - 4(\mathcal{S}_A)}{2} = 6. \quad (2.10)$$

In other words, in the general case in which $\det|\mathcal{V}_{AB}| \neq 0$, we have $4(\Pi_A) + 4(\mathcal{S}_A) = 8$ constraints, for a total of 6 propagating DoF. Technically, these 8 constraints are all second class, being the $\text{Rank}\{|\Pi_A, \mathcal{S}_B|\} = \text{Rank}|\mathcal{V}_{AB}| = 4$. As a result, no residual gauge invariance is present. When the action is Lorentz invariant around a Minkowski background, the six DoF must be organized in a massive spin two (5 DoF) representation plus a scalar (1 DoF). This is the Boulware-Deser result, valid for a generic potential. The extra scalar, the so called Boulware-Deser sixth mode [13], turns out to suffer from infinite strong coupling issues and are indeed hardly viable.

It has to be stressed that the ghost can be absent around a FRW background, see section 6 in [48]. As shown in that work, no ghost is present at any momentum if some conditions for the graviton mass terms hold, $m_1^2 > 0$ and $0 < m_0^2 < 6H^2$ (see below section 3 for the notation). Moreover absence of tachyonic instabilities can also be fulfilled by further conditions. The relative constraints on the potential may lead to an interesting scenario and we leave it for a separate complete study.

In any case, a first result is that a necessary condition to have a theory with less than six propagating DoF is that $r \equiv \text{Rank}|\mathcal{V}_{AB}| < 4$ (see also [2]).

2.2 $r < 4$: general analysis

Let us describe here in generality the hamiltonian analysis for $r < 4$, and later specialize to the various cases $r = 3, 2, 1, 0$. For $r < 4$, the matrix \mathcal{V}_{AB} has r non null eigenvectors, denoted by E_n^A with $n = 1, \dots, r$, and $4 - r$ null eigenvectors denoted by χ_α^A ,

$$\mathcal{V}_{AB} \chi_\alpha^B = 0, \quad \alpha = 1, \dots, 4 - r. \quad (2.11)$$

It is useful to decompose the Lagrange multipliers along those eigenvectors,

$$\lambda^A = \sum_{\alpha=1}^{4-r} z_\alpha \chi_\alpha^A + \sum_{n=1}^r d_n E_n^A, \quad (2.12)$$

effectively trading the 4 Lagrange multipliers λ^A for the coefficients z_α and d_n .

Of the four original Lagrange multipliers, the r components along E_n^A are determined by the tertiary condition (2.8):

$$d_n = \frac{E_n^A \{\mathcal{S}_A, H\}}{E_n^A \mathcal{V}_{AB} E_n^B}, \quad n = 1, \dots, r. \quad (2.13)$$

On the other hand, the projection of the conditions (2.8) along the null directions χ_α^A unveils $4 - r$ genuine *tertiary constraints*

$$\mathcal{T}_\alpha \equiv \chi_\alpha^A \{\mathcal{S}_A, H\} \approx 0, \quad \alpha = 1, \dots, 4 - r. \quad (2.14)$$

Indeed, no Lagrange multiplier is involved here.

We have also to impose the conservation in time of these new constraints, which leads to the conditions

$$\mathcal{Q}_\alpha(\vec{x}) = \{\mathcal{T}_\alpha(\vec{x}), H\} + \int d^3y \left[\sum_{n=1}^r d_n(\vec{y}) \{\mathcal{T}_\alpha(\vec{x}), \Pi_A(\vec{y})\} E_n^A(\vec{y}) - \sum_{\beta=1}^{4-r} \theta_{\alpha\beta}(\vec{x}, \vec{y}) z_\beta(\vec{y}) \right] \approx 0, \quad (2.15)$$

where the matrix $\theta_{\alpha\beta}$ is defined as

$$\theta_{\alpha\beta}(\vec{x}, \vec{y}) \equiv \chi_\alpha^A(\vec{x}) \{\mathcal{S}_A(\vec{x}), \mathcal{S}_B(\vec{y})\} \chi_\beta^B(\vec{y}). \quad (2.16)$$

The condition (2.15) consists in $4 - r$ linear equations for the remaining $4 - r$ Lagrange multipliers z_α . Hence, the number of DoF crucially depends on how many of them can be determined, i.e. on the rank of $\theta_{\alpha\beta}$

$$s \equiv \text{Rank} |\theta_{\alpha\beta}|. \quad (2.17)$$

If $s = 4 - r$, then all the remaining Lagrange multipliers are determined and the procedure which enforces the consistency of constraints with time evolution ends. On the other hand if $s < 4 - r$ some of the z_α are not determined and one has $4 - r - s$ *new quaternary constraints* \mathcal{Q}_α , which further reduce the number of DoF.

Altogether so far one has $16 - 2r - s$ constraints, counting $4(\Pi_A) + 4(\mathcal{S}_A) + (4 - r)(\mathcal{T}_\alpha) + (4 - r - s)(\mathcal{Q}_\alpha)$, and the number of DoF is at this point

$$\# \text{DoF} \leq \frac{20 - (16 - 2r - s)}{2} = 2 + r + \frac{s}{2}, \quad 0 \leq r \leq 4, \quad 0 \leq s \leq 4 - r. \quad (2.18)$$

Maximizing s , for fixed r , we have the following upper bound

$$\# \text{DoF} \leq 4 + \frac{r}{2}, \quad 0 \leq r \leq 4. \quad (2.19)$$

Once more, in order to know how far one can go, one has to check the conservation of the quaternary constraints, that reads

$$\mathcal{F}_\alpha(\vec{x}) = \{\mathcal{Q}_\alpha(\vec{x}), H_T\} = \{\mathcal{Q}_\alpha(\vec{x}), H\} + \int d^3y \frac{\partial \mathcal{Q}_\alpha(\vec{x})}{\partial N^A(\vec{y})} \left(\sum_\beta \chi_\beta^A z_\beta + \sum_n E_n^A d_n \right) (\vec{y}) \approx 0. \quad (2.20)$$

Setting

$$A_{\alpha\beta}(\vec{x}, \vec{y}) = \frac{\partial \mathcal{Q}_\alpha(\vec{x})}{\partial N^A(\vec{y})} \chi_\beta^A(\vec{y}), \quad (2.21)$$

if the matrix $A_{\alpha\beta}$ is invertible, then (2.20) does not give rise to new constraints but simply determines the remaining Lagrange multipliers as

$$z_\alpha \propto - \sum_\beta A_{\alpha\beta}^{-1} \left(\{\mathcal{Q}_\beta, H\} + \sum_n d_n E_n^A \frac{\partial \mathcal{Q}_\alpha}{\partial N^A} \right). \quad (2.22)$$

In this case, the procedure ends here and the number of DoF saturates the bound in (2.18). However, again this is only the maximal number. In fact, if some of the z_α are not determined, more steps are necessary and the net effect is to reduce the number of DoF further. In general, also first class constraints may be present corresponding to residual gauge invariances, but again, this implies a further reduction of the number of DoF. In the above discussion we have also ignored the exceptional cases where some constraints are accidentally trivial, e.g. $0 \approx 0$.

It is important to remark that due to the nontrivial dependence on \vec{x} , \vec{y} , the matrix $\theta_{\alpha\beta}(\vec{x}, \vec{y})$ is not necessarily antisymmetric and its rank s is not always even. Thus, for s odd one concludes that an *half integer* number of DoF is present. This is a peculiar phenomenon which arises in classical field theories (infinite dimensional Hamiltonian systems) and it is briefly discussed in general terms in [49] and for Horava-Lifshitz gravity in [50–53]. In general, this problem occurs when some of the would-be constraints contains differential operators acting on Lagrange multipliers; the solutions of the resulting differential equations are in general nontrivial and their form depends strongly on the chosen boundary conditions at spatial infinity. This phenomenon seems affect only LB theories; indeed, in the LI case time derivatives are always paired with the spatial ones. To our knowledge, no general analysis on the nature of such half DoF is present in the literature. We leave the matter for a future investigation. It turns out that the relevant case is when $\#$ DoF is $5 + \frac{1}{2}$, see section 2.3.

We recap the steps that are required to compute the number of propagating DoF for a given deforming potential \mathcal{V} :

1. Compute the rank r of the hessian matrix $\|\mathcal{V}_{AB}\|$ (4×4 matrix).
2. Compute the null eigenvectors χ_α^A of the matrix \mathcal{V}_{AB} .
3. Determine secondary constraints $\mathcal{S}_A = \mathcal{H}_A + \mathcal{V}_A$.
4. Compute the rank s of the matrix $\|\chi_\alpha^A \{\mathcal{S}_A, \mathcal{S}_B\} \chi_\beta^B\|$ ($4 - r \times 4 - r$ matrix).
5. Plug the numbers in the formula $\#$ DoF $\leq 2 + r + s/2$.

In the following sections we discuss separately the cases relative to different values of r and s . The results of this analysis are summarized in table 1, where the maximal number of

$r = \text{Rank} \mathcal{V}_{AB} $	$s = \text{Rank} \theta_{\alpha\beta} $	$\#\text{DoF} \leq$	Rotations?	Realized?
4	0	6	✓	Yes
3	0	5	✓	Yes
3	1	$5 + \frac{1}{2}$	✓	Yes
2	0	4	×	No
2	1	$4 + \frac{1}{2}$	×	No
2	2	5	×	No
1	0	3	✓	Yes
1	1	$3 + \frac{1}{2}$	×	No
1	2	4	×	No
1	3	$4 + \frac{1}{2}$	✓	No
0	0	2	✓	Yes
0	1	$2 + \frac{1}{2}$	✓	No
0	2	3	✓	Yes
0	3	$3 + \frac{1}{2}$	✓	No
0	4	4	✓	No

Table 1. Deforming potentials classified according to the rank r of the Hessian and the rank s of the matrix θ . The number of DoF is obtained from eq. (2.18). The cases consistent with the canonical algebra are highlighted as bold and marked as *Realized*. Notice that a non integer number of DoF can possibly appear with unbroken rotations only in the case $r = 3$ and $s = 1$, namely $5 + \frac{1}{2}$ DoF.

DoF resulting from the canonical analysis is shown for different values of r and s . We also report whether the resulting theory can be built by respecting rotations (fourth column), and whether it can be realized at all with some explicit form of the potential (last column), as we find by direct inspection in the forthcoming sections.

A first outcome of the analysis is that massive deformations of gravity with 5 DoF exist only in two cases: $r = 3, s = 0$ or $r = 2, s = 2$. The first was discussed in full depth in [2–4] where all the rotational invariant potentials of this class were constructed. Concerning the second case, we note that it is not possible to build potentials with $r = 2$ without breaking spatial rotations. Indeed, rotational invariance requires either one or at least three non null eigenvectors of \mathcal{V}_{AB} .

In the following we consider potentials that are at least rotationally invariant on Minkowski space. As a result, we are left only with the cases $r = 4, 3, 1, 0$, with a number of DoF between 6 and 2.

Remarkably, the present analysis shed also some light on the existence of models with 4 DoF, often invoked in the context of massive gravity (see for instance [54]). First, they could exist for $r = 2, s = 0$, but only with broken rotational invariance. The candidates with 4 DoF having $r = 0, s = 4$ are actually not realized, as we will see in section 2.6. Thus, we conclude that the only candidate theories with 4 DoF are to be searched as subcases of the 5 DoF theories with $r = 3$. This is discussed in section 2.4; the high number and complexity of the required constraints makes one doubt that such theories can actually be found.

2.3 The case $r = 3$: massive gravity with 5 DoF

This case was fully analysed in [2–4] (see also [11, 12] for a similar approach). Here for completeness we recollect the main results. The general potential \mathcal{V} of massive gravity theories with five propagating DoF can be parametrized in terms of two arbitrary functions

of specific arguments, $\mathcal{U}[\mathcal{K}^{ij} \equiv \gamma^{ij} - \xi^i \xi^j]$ and $\mathcal{E}[\xi^i, \gamma^{ij}]$,

$$\mathcal{V} \equiv m^2 \sqrt{\gamma} \left(N\mathcal{U} + \mathcal{E} + \mathcal{U}_i \mathcal{Q}^i \right), \quad (2.23)$$

where ξ^i is defined implicitly by the first of the following equations

$$N^i = N \xi^i + \mathcal{Q}^i, \quad \mathcal{Q}^i[\xi^i, \gamma^{ij}] \equiv -\mathcal{U}_{ij}^{-1} \mathcal{E}_j.$$

and where $\mathcal{U}_i = \partial_{\xi^i} \mathcal{U}$ and $\mathcal{U}_{ij} = \partial^2 \mathcal{U} / \partial \xi^i \partial \xi^j$. The use of the variables ξ^i in place of the shifts N^i makes also very transparent the canonical analysis, as recalled in appendix A (see also [4]). The function \mathcal{E} is the bulk on-shell energy (Hamiltonian) density of the system and it has to be non-negative. We remark that, as shown in appendix A, a necessary condition to have 5 DoF, is to have $\mathcal{E} \neq 0$. Potentials with $\mathcal{E} = 0$ have six DoF.

Besides its purely theoretical interest, this result is also relevant from a phenomenological point of view. A large class of massive gravity theories that are ghost free on Minkowski space are uncovered, whereas previously, the only known ghost free theory was the four parameter Lorentz invariant (LI) theory found⁴ in [17–19], which is a special case of our general construction.⁵ When Lorentz symmetry is enforced, the price to be paid is the impossibility of using perturbation theory in many physical important situations like inside our solar system. Moreover, as effective theory, the cutoff is rather low [6], $\Lambda_3 = (m^2 M_{Pl})^{1/3} \sim 10^3$ Km when m is taken to be of order of today’s Hubble scale. As a result, even the static potential between two masses at a distance smaller than 10^3 Km is difficult to compute perturbatively [6, 22–24, 55], in contrast with short distance tests of Newton’s force at submillimeter scale, see for instance [56]. In Lorentz breaking theories we are much better off from a phenomenologically point of view. It ought to be remarked that Lorentz symmetry we are discussing here only concerns the gravitational sector and is not the same symmetry that enters in the formulation of the Einstein’s equivalence principle. As such, it is not subject to strong phenomenological constraints coming from high energy physics. Thus, we conclude that the viability of the theory directly requires us to give up Lorentz symmetry in the gravitational sector, a fact that is testable in the forthcoming gravitational wave experiments.

The concrete phenomenology of the new class of Lorentz breaking theories is also rather promising, as argued in [3]. From a perturbative point of view, exploiting the general expression of \mathcal{V} , there exist remarkable relations among the various Lorentz breaking graviton masses. At the nonperturbative level, besides the absence of ghosts in the spectrum, it is of crucial importance to be able to trust the theory up to the cutoff $\Lambda_2 = (m M_{Pl})^{1/2} \simeq (10^{-3} \text{ mm})^{-1}$, as well as the absence of strong nonlinearities (Vainshtein effect) around macroscopic sources. Such class of Lorentz breaking massive gravity theories is also a natural candidate for dark energy provided its equation of state deviates from -1 [16].

2.4 The subcase of $r = 3$ for massive gravity with 4 DoF

In appendix A we give the further conditions under which a potential with $r = 3$ propagates only four DoF. In comparison with the case of 5 DoF, two extra (differential) conditions on

⁴Also Zumino came up with a similar model, see Brandeis Univ. 1970, Lectures On Elementary Particles And Quantum Field Theory, Vol. 2*, Cambridge, Mass. 1970, 437-500.

⁵For instance, the minimal version of the dRGT LI massive gravity is obtained by taking $\mathcal{U} = (\text{Tr}[\mathcal{K}^{1/2}] - 3)$ and $\mathcal{E} = (1 - \xi^2)^{-1/2}$, that gives rise to their potential $(\text{Tr}[\sqrt{X}] - 3)$ with $X^\mu_\nu = g^{\mu\rho} \eta_{\rho\nu}$. For the other operators in that theory the correspondence is not known explicitly.

the potential have to be imposed. In the Dirac language, they correspond to the requirement that the quinary and the senary constraints are independent from the lapse

$$\partial_N \mathcal{Q} = \partial_N \mathcal{F} = 0. \quad (2.24)$$

These conditions restrict further the dependence on the ADM variables of the functions \mathcal{U} and \mathcal{E} . However, due to their complexity, at present no solution is known, if any exists. In this sense no V with 4 DoF is known. As it will be discussed in section 3, around Minkowski background only two or five DoF can propagate at linearized level; thus, even if a potential with four DoF exists it will lead to strong coupling around flat space.

At linearized level, Lorentz-violating potentials which propagate 4 DoF (two tensor and two vector modes) were analyzed even around a generic FRW background (see ref. [48]). For instance, on de Sitter background if the graviton mass is precisely $m^2 = 2H^2$, a fifth scalar mode disappears from the linearized theory, leading to the so called partially massless (PM) theory [57–59]. The absence of the helicity-0 mode at linearized level is related to the existence of a new scalar gauge symmetry (a special combination of a linearized diff. and a conformal transformation). Unfortunately, the helicity-0 mode reappears non-linearly; so, rather than being free from the scalar mode, the theory is strongly coupled [60, 61].

2.5 The case $r = 1$

When $r = 1$ and rotations are preserved, the only possible form for \mathcal{V} is a function of γ_{ij} and N with nonzero N second derivative:

$$\mathcal{V} \equiv \mathcal{V}[N, \gamma], \quad \text{with} \quad \mathcal{V}_{NN} \neq 0. \quad (2.25)$$

Following the steps of section 2.2, we have

- The secondary constraints in this case are rather simple

$$\mathcal{S}_0 = \mathcal{H} + \mathcal{V}_N \approx 0, \quad \mathcal{S}_i = \mathcal{H}_i \approx 0, \quad i = 1, 2, 3. \quad (2.26)$$

- There are three null eigenvectors that can be chosen to be $\chi_i^A = \delta_i^A$.
- The matrix $\theta_{\alpha\beta}$ of (2.16) now vanishes when the constraints are used, namely

$$\theta_{\alpha\beta}(\vec{x}, \vec{y}) \equiv \theta_{ij}(\vec{x}, \vec{y}) = \{\mathcal{H}_i(\vec{x}), \mathcal{H}_j(\vec{y})\} \propto \mathcal{H}_k \approx 0, \quad (2.27)$$

where GR algebra has been used. Thus $s = 0$.

- # DoF = 3.

The on-shell bulk Hamiltonian is given by

$$H_{\text{on shell}} = \int d^3x (\mathcal{V} - \mathcal{V}_N N)(t, \vec{x}). \quad (2.28)$$

2.6 The case $r = 0$

When $r = 0$ and \mathcal{V} is rotational invariant, the only possibility is that \mathcal{V} is at most a linear function in the lapse, hence we can write

$$\mathcal{V} \equiv m^2 \sqrt{\gamma} \left(N \mathbf{U}[\gamma_{ij}] + \mathbf{E}[\gamma_{ij}] \right). \quad (2.29)$$

The Hamiltonian analysis for specific examples in this class was already given in [62, 63].

Consider first the case of generic $\mathbf{U} \neq 0$:

- The null eigenvectors of the hessian can be chosen to be $\chi_\alpha^A = \delta_\alpha^A$ with $\alpha = 0, 1, 2, 3$.
- The secondary constraints are

$$\mathcal{S}_0 = \mathcal{H} + m^2 \sqrt{\gamma} \mathbf{U} \approx 0, \quad \mathcal{S}_i = \mathcal{H}_i \approx 0, \quad i = 1, 2, 3. \quad (2.30)$$

- We calculate $\theta_{\alpha\beta} = \{\mathcal{S}_\alpha, \mathcal{S}_\beta\}$ by using the same algebra of constraints of GR:

$$\begin{aligned} \theta_{00} &= [\mathcal{H}_i(\vec{x}) + \mathcal{H}_i(\vec{y})] \partial_{x^i} \delta(\vec{x} - \vec{y}) \approx 0, \\ \theta_{0i} &= \sqrt{\gamma} \partial_{x^i} \left(\mathbf{U}_\gamma + \frac{\mathbf{U}}{2} \right) \delta(\vec{x} - \vec{y}) + \sqrt{\gamma} (\gamma_{ia} \partial_{\gamma_{ab}} \mathbf{U}) \partial_{x^b} \delta(\vec{x} - \vec{y}) \neq 0, \\ \theta_{ij} &= (\mathcal{H}_j(\vec{x}) \partial_{x^i} + \mathcal{H}_i(\vec{y}) \partial_{x^j}) \delta(\vec{x} - \vec{y}) \approx 0, \end{aligned} \quad (2.31)$$

so that clearly $s = 2$.

- As a result, $\# \text{DoF} = 3$.

A subcase is also present, which is relevant to our analysis. We note that the equations (2.15) for the Lagrangian multipliers z_α associated with the θ_{0i} given above are linear differential equations in the spatial coordinates, of the form $A \partial_x z + B z = C$. A similar structure is found in Horava-Lifshitz gravity [50–53]. Being

$$A \propto \gamma_{ia} \partial_{\gamma_{ab}} \mathbf{U}, \quad B \propto \partial_{x^i} \left(\mathbf{U}_\gamma + \frac{\mathbf{U}}{2} \right), \quad (2.32)$$

it is easy to show that there is a unique potential \mathbf{U} (function of the rotational invariants $\text{Tr}[\gamma]$, $\text{Tr}[\gamma^2]$, $\text{Tr}[\gamma^3]$) such that A and B vanish automatically. This corresponds to a cosmological constant, i.e. $m^2 \mathbf{U} = \Lambda$,⁶ or

$$\mathcal{V} \equiv \sqrt{\gamma} \left(N \Lambda + m^2 \mathbf{E}[\gamma_{ij}] \right). \quad (2.33)$$

- In this case the Lagrange multipliers are not determined because even θ_{0i} vanish:

$$\theta_{00} \propto \mathcal{H}_i \partial_i \delta \approx 0, \quad \theta_{0i} \propto \partial_i \mathcal{H} \approx 0, \quad \theta_{ij} \propto \mathcal{H}_i \partial_j \delta \approx 0, \quad (2.34)$$

thus $s = 0$.

- As a result, $\# \text{DoF} = 2$.

⁶Such a result is in agreement with the finding of [64, 65] where the requirements of spatial covariance (i.e. closure of the constraints relate to \mathcal{H}_i) singles out GR as the unique theory with two DoF. In other words, the only rotational invariant potential that can be built is a CC.

Note, since still \mathcal{V} contains $m^2 \mathbf{E}[\gamma_{ij}] \neq 0$, the tertiary constraints \mathcal{T}_α do not vanish, and the Lagrange multipliers are determined at the level of quinary conditions \mathcal{F} . This corresponds to a case of 2 DoF but broken diffeomorphisms. This situation contains trivially also the case $\mathbf{U} = 0$. Clearly instead for $\mathbf{E} = 0$ and $\mathcal{V} = \Lambda N \sqrt{\gamma}$ the result is GR with cosmological constant and 2 DoF are present, with unbroken gauge invariance.

In all these cases the on-shell bulk Hamiltonian is given by

$$H_{|\text{on shell}} = m^2 \int d^3x \sqrt{\gamma} \mathbf{E}[\gamma_{ij}](t, \vec{x}). \quad (2.35)$$

3 Perturbations around Minkowski

Consider now the perturbative expansion around flat space. Setting $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, expanding the action (1.1) at the quadratic order in h one gets

$$S = \int d^4x \left[L_{(2)} + \frac{1}{2} \left(m_0^2 h_{00}^2 + 2 m_1^2 h_{0i}^2 - m_2^2 h_{ij}^2 + m_3^2 h_{ii}^2 - 2 m_4^2 h_{00} h_{ii} \right) \right], \quad (3.1)$$

where $L_{(2)}$ is the standard quadratic Lagrangian for a massless spin 2 particle in Minkowski space; for V we have only imposed rotational invariance. The physical consequences of the quadratic action (3.1) were first discussed in [46]. In our case, the various masses can be computed explicitly from the potential. Preliminarily, one has to impose that Minkowski space is a consistent background, i.e. that $g_{\mu\nu} = \eta_{\mu\nu}$ is a solution of the equations of motion; this is equivalent to

$$\bar{\mathcal{V}}_N = 0, \quad \bar{\mathcal{V}}_\gamma = 0. \quad (3.2)$$

The bar indicates that expressions are evaluated on Minkowski space, where we define \mathcal{V}_γ by $\partial\mathcal{V}/\partial\gamma^{ij} \equiv \mathcal{V}_\gamma \gamma_{ij}$. Using the conditions (3.2), we find that

$$m_0^2 = -\frac{1}{4} \frac{\partial^2 \mathcal{V}}{\partial N^2} \Big|_\eta, \quad m_1^2 = -\frac{1}{2} \frac{\partial^2 \mathcal{V}}{\partial N^i \partial N^i} \Big|_\eta. \quad (3.3)$$

The expressions for $m_{2,3,4}^2$ are not particularly illuminating and will be omitted. In general, the following conclusions can be drawn [9, 10, 36, 46, 48]:

- For $m_{0,1}^2 \neq 0$ we have 6 perturbative DoF with one scalar is a ghost around Minkowski background. At most, 6 healthy modes can be obtained around FRW spaces if $m_1^2 > 0$ and $0 < m_0^2 < 6H^2$, plus other conditions to avoid tachyonic instabilities [48].
- For $m_0^2 = 0, m_1^2 \neq 0$ we have 5 perturbative DoF.
- For $m_0^2 \neq 0, m_1^2 = 0$ we have 2 perturbative DoF.
- For $m_{0,1}^2 = 0$ we have 2 perturbative DoF.

In the summary table 2 we compare the number of perturbative DoF around Minkowski space found here with the corresponding number found in the previous section by using the nonperturbative and background independent analysis. If the two numbers differ, the propagation of the missing DoF(s) have to show up at higher orders in the perturbative expansion, or around non-Minkowski backgrounds. In either case, this is a manifestation of strong coupling around Minkowski spacetime.

A comment on the nature of the strong coupling and cutoff scales may be useful. It is straightforward to get by dimensional analysis the classical strong coupling scale around flat space starting from the quadratic action. Indeed, generically scalar fluctuations of the metric behave as [3]

$$\mathcal{L}_{(2)} = M_{Pl}^2 m^2 (\partial \phi)^2 + M_{Pl}^2 m^2 (\partial \phi)^3 + \dots + m^2 \frac{\phi}{|\nabla|} T_{\text{matt}}; \quad (3.4)$$

where T_{matt} is the trace of the matter energy momentum tensor. Thus, in terms of the canonical field $\phi_c \simeq M_{Pl} m \phi$ we get that

$$\mathcal{L}_{(2)} = (\partial \phi_c)^2 + \frac{(\partial \phi_c)^3}{M_{Pl} m} + \dots + \frac{m}{M_{Pl}} \frac{\phi_c}{|\nabla|} T_{\text{matt}}. \quad (3.5)$$

From the above expression we learn two important things. First, the strong coupling scale of our theory is $\Lambda_2 \equiv \sqrt{M_{Pl} m}$, namely the largest possible cut off scale without an explicit Higgs phase dynamics. Second, the matter coupling to scalar modes is suppressed by m and goes like $(m/M_{Pl} \ll 1)$. These two facts together imply the absence of non linearities around heavy sources (and no van Dam-Veltman-Zakharov [66–68] discontinuity) or more precisely, the perturbativity of the weak field expansion down to the Schwarzschild radius as in GR.

Recently in [5, 7] it was pointed out that in general the scale at which perturbative unitarity breaks down (the strong coupling scale) and the cutoff of scale in an effective field theory can be different. It could be that strong coupling effects - like the Vainsthein mechanism - could rise the cutoff scale beyond than the “naive” one. However, when no new strong coupling scale involving m is present, we expect the cutoff scale to be Λ_2 . This is precisely the case when $r = 3$ with 5 DoF for LB potential given by (2.23) and when $r = 1$ with 2 DoF for the potentials given by (2.33). On the contrary, potential with 3 nonperturbative DoF given by eq. (2.25) and eq. (2.29) has an infinite strongly coupling scale as soon as flat space is considered as a background.

4 Cosmology

In this section we analyze the conditions under which the potential \mathcal{V} admits a FRW background solution. We take for FRW metric the following diagonal form

$$g_{00} = -N^2, \quad g_{0i} = 0 \rightarrow N^i = 0, \quad g_{ij} = \gamma_{ij} = a^2 \delta_{ij}. \quad (4.1)$$

Notice that this is the most general ansatz with maximally symmetric $t = \text{const}$ hypersurfaces. Equivalently, the reference frame where the universe is homogenous is the very same frame of the unitary gauge [16]. For simplicity we have also set the spatial curvature to zero.⁷

Due to the diagonal form of the FRW metric, its existence probes the functional dependence of \mathcal{V} with respect to N and γ_{ij} only, no constraints on the N^i dependence can be obtained. The effect of \mathcal{V} is equivalent to the presence in the Einstein equations of an effective energy momentum tensor (EMT) $\mathcal{T}_{\mu\nu}$ defined by

$$\delta \int d^4x \mathcal{V} \equiv \frac{1}{2} \int d^4x \sqrt{g} \mathcal{T}_{\mu\nu} \delta g^{\mu\nu} \quad (4.2)$$

⁷The flat cosmology is by far the most motivate phenomenologically - still, one can check that the conditions for cosmology and the Bianchi equation which we derive below are maintained even in the context of open cosmology, insofar as the physical and auxiliary 3D metrics are assumed to be proportional. The perturbations may display a different behavior instead.

and given by

$$\mathcal{T}_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\partial \mathcal{V}}{\partial g^{\mu\nu}}. \quad (4.3)$$

Specializing to the FRW background, the effective EMT reads

$$\mathcal{T}_{00} = \frac{N^2}{\gamma^{1/2}} \mathcal{V}_N, \quad \mathcal{T}_{0i} = 0, \quad \mathcal{T}_{ij} = \frac{2}{N\gamma^{1/2}} \gamma_{ij} \mathcal{V}_\gamma, \quad (4.4)$$

where we denote $\mathcal{V}_N \equiv \partial \mathcal{V} / \partial N$ and we have used the fact that on FRW background $\partial \mathcal{V} / \partial \gamma^{ij}$ is proportional to γ_{ij} by defining \mathcal{V}_γ through $\partial \mathcal{V} / \partial \gamma^{ij} \equiv \mathcal{V}_\gamma \gamma_{ij}$. (For instance, for $\mathcal{V} = \gamma^n$, with $\gamma = \text{Det}[\gamma_{ij}]$, we have $\mathcal{V}_\gamma = -n\gamma^n$). We retain here the explicit N dependence of $g_{\mu\nu}$; indeed, besides being instrumental in exploiting constraints on the functional dependence of \mathcal{V} on N and $\gamma_{ij} \propto a$, in general it cannot be gauged away.

The gravitational fluid has energy and pressure densities and effective equation of state given by

$$\rho_{\text{eff}} = \frac{\mathcal{V}_N}{\gamma^{1/2}}, \quad p_{\text{eff}} = \frac{2\mathcal{V}_\gamma}{N\gamma^{1/2}}, \quad w_{\text{eff}} = \frac{2\mathcal{V}_\gamma}{N\mathcal{V}_N}. \quad (4.5)$$

Because of the Bianchi identities, $\mathcal{T}_{\mu\nu}$ must be covariantly conserved. This requires $\partial_t \rho_{\text{eff}} = -3 \frac{\dot{a}}{a} (\rho_{\text{eff}} + p_{\text{eff}})$, which leads to the condition

$$\dot{N} \mathcal{V}_{NN} - 6 \frac{\dot{a}}{a} \left(\mathcal{V}_{N\gamma} - \frac{\mathcal{V}_\gamma}{N} \right) = 0. \quad (4.6)$$

Notice that $\partial_t \mathcal{V} = \gamma_{ij} \dot{\gamma}^{ij} \mathcal{V}_\gamma + \dot{N} \mathcal{V}_N = -\frac{6\dot{a}}{a} \mathcal{V}_\gamma + \dot{N} \mathcal{V}_N$. In general eq. (4.6) is a differential equation which dictates the dynamics of N , as can be seen by solving for \dot{N} . In this case N is dynamically determined (and cannot be eliminated by a choice of time). Then, the Friedmann equation determines the time dependence of the Hubble parameter, and results in a well-behaved cosmology in the presence of the effective fluid 4.5.

However, looking at the classification of admissible potentials from section 2, we see that in most cases the situation is crucially different. Except in the case $r = 1$, in all cases ($r = 3$ or 0) on FRW background where one has $N^i = \xi^i = 0$, the potential is a linear function of the lapse N . Thus $\mathcal{V}_{NN} = 0$ and eq. (4.6) has to hold in the form (we consider $\dot{a} \neq 0$ for a realistic cosmology)

$$\mathcal{V}_{NN} = 0, \quad \left(\mathcal{V}_{N\gamma} - \frac{\mathcal{V}_\gamma}{N} \right) = 0. \quad (4.7)$$

Parametrizing \mathcal{V} , we find that the Bianchi condition constrains only the N -independent part:

$$\mathcal{V} \equiv m^2 \sqrt{\gamma} \left(N\mathcal{A} + \mathcal{B} \right) \quad \Rightarrow \quad \left(\sqrt{\gamma} \mathcal{B} \right)_\gamma = \sqrt{\gamma} \left(\mathcal{B}_\gamma - \frac{\mathcal{B}}{2} \right) \Big|_{FRW} = 0, \quad (4.8)$$

where, as in the cases of section 2, \mathcal{A} and \mathcal{B} are generic functions (of the spatial metric, on FRW). Note that \mathcal{A} as well as N drop out of the Bianchi condition, so that N is now left undetermined by the background equations. More importantly, we see that in general the Bianchi condition ends up in an algebraic constraint on the scale factor a , which is incompatible with a realistic cosmology. Thus, the only possibility is that some specific form of \mathcal{B} is chosen so that $\mathcal{B}_\gamma - \frac{\mathcal{B}}{2} = 0$ identically on FRW. As an example, $\mathcal{B} = 3 \text{Tr}[\gamma^2] - \text{Tr}[\gamma]^2$ has this property. This is the condition to have a realistic FRW cosmology, and will apply to the cases of sections 2.3, 2.4 and 2.6.

Density pressure and equation of state of the gravitational fluid now take the form

$$\rho_{\text{eff}} = m^2 \mathcal{A}, \quad p_{\text{eff}} = 2m^2 \left(\mathcal{A}_\gamma - \frac{1}{2} \mathcal{A} \right) \quad \Rightarrow \quad w_{\text{eff}} = -1 + \frac{2\mathcal{A}_\gamma}{\mathcal{A}}. \quad (4.9)$$

We note that they do not depend on the function \mathcal{B} .⁸ This can be explained by observing that only the function \mathcal{B} breaks time reparametrizations, in the potential. The function \mathcal{A} appears in the combination $N\mathcal{A}$ and has the same structure of the Hamiltonian constraint in GR. Thus it cancels out from the Bianchi condition (4.6) which is exactly the constraint related to the (breaking of) time reparametrizations. In other words, the part of $\mathcal{T}_{\mu\nu}$ containing \mathcal{A} is automatically conserved, while the remaining part containing \mathcal{B} has to be conserved by itself. We stress that from the existence of a FRW background nothing can be said on the N^i dependence of \mathcal{V} . Thus, a FRW solution would exist when the potential has the general structure

$$\mathcal{V} = m^2 \sqrt{\gamma} \left(N\mathcal{A}[\gamma, N^i f_1[\gamma, N, N^k]] + \mathcal{B}[\gamma, N^i f_2[\gamma, N, N^k]] + N^i \gamma_{ij} N^j f_3[\gamma, N, N^k] \right), \quad (4.10)$$

with f_i generic functions and where \mathcal{B} must again be chosen such that $(\sqrt{\gamma} \mathcal{B})_\gamma = 0$ on FRW.

We can now put together the results above with the analysis of the previous sections, and spell out the potentials which exist and admit a consistent FRW background:

- (a) 6 DoF with $r = 4$. Two tensors, two vectors and two scalar modes are present, of which one is a ghost around Minkowski spacetime. As recalled in section 3, the ghost can be absent around FRW backgrounds if $H' < 0$ [16, 48].
- (b) 5 DoF with $r = 3, s = 0$. Here we have

$$\mathcal{V} = m^2 \sqrt{\gamma} \left(N\mathcal{U}[\gamma^{ij} - \xi^i \xi^j] + \mathcal{E}[\gamma^{ij}, \xi^i] + \mathcal{U}_{\xi^i} \mathcal{Q}^i \right) \quad \text{and} \quad w_{\text{eff}} = -1 + \frac{2\mathcal{U}_\gamma}{\mathcal{U}}, \quad (4.11)$$

where we recall that $N^i = N \xi^i + \mathcal{Q}^i$ and $\mathcal{Q}^i = -\|\mathcal{U}_{\xi^i \xi^j}\|^{-1} \mathcal{E}_{\xi^j}$.

The potential is such that (4.6) is solved in the form (4.7). The existence of a nontrivial FRW solution requires $\mathcal{E}_\gamma = \mathcal{E}/2$. This, if combined with the condition for the existence of a strict Minkowski background (3.2), predicts that the m_1^2 in (3.1) is zero. This leads to strong coupling in the vector and scalar sector of perturbations around the strict Minkowski background. The same conclusion is reached for a strict de Sitter space, and the only healthy possibility is to deviate from a de Sitter phase [16].

- (c) 3 DoF with $r = 1, s = 0$. In this case, generically (4.6) can be solved for \dot{N} and a FRW solution exists. The dependence of N on a strongly depends on the explicit form of \mathcal{V} . On Minkowski spacetime strong coupling of gravitational perturbations is present, see table 2. Instead, by using the results of [48], we see that around FRW 3 DoF are present in the linearized theory, thus in agreement with the 3 nonperturbative DoF as predicted by the canonical analysis.

- (d) 3 DoF with $r = 0, s = 2$. Here

$$\mathcal{V} = m^2 \sqrt{\gamma} \left(N \mathbf{U}[\gamma] + \mathbf{E}[\gamma] \right) \quad \text{and} \quad w_{\text{eff}} = -1 + \frac{2\mathbf{U}_\gamma}{\mathbf{U}}. \quad (4.12)$$

⁸As a check, for a cosmological constant, $\mathcal{V} \propto N\gamma^{1/2}$, we have $\mathcal{A} = 1$ and $\mathcal{A}_\gamma = 0$ giving exactly $w_{\text{eff}} = -1$.

Potential	Nonpert. #DoF	LB Masses	Pert. #DoF	FRW Cosmo
$\mathcal{V}[N^A, \gamma]$	6	$m_{0,\dots,4}^2 \neq 0$	$6=5+\text{ghost}^\dagger$	\checkmark
$\sqrt{\gamma}(N\mathcal{U}[\mathcal{K}] + \mathcal{E}[\gamma, \xi] + \mathcal{U}_i \mathcal{Q}^i)$	5	$m_0^2 = 0$	5	\checkmark^*
As above + Lorentz Invariance	5	$m_0^2 = 0$	5	no
$\mathcal{V}[N, \gamma]$	3	$m_1^2 = 0$	2	\checkmark
$\sqrt{\gamma}(N\mathbf{U}[\gamma] + \mathbf{E}[\gamma])$	3	$m_{0,1}^2 = 0$	2	\checkmark^*
$\sqrt{\gamma}(\Lambda N + \mathbf{E}[\gamma])$	2	$m_{0,1,4}^2 = 0$	2	$\checkmark^*(\text{CC})$

Table 2. The allowed potentials supporting spatial rotations, and the number of perturbative and nonperturbative DoF. For perturbative DoF the reference background is Minkowski space. Whether a realistic spatially flat FRW cosmology is admitted is also shown. The symbol * denotes that a further tuning of the functional form of V is required (see condition (4.8)). This tuning is not necessary for time-dependent potentials (see appendix B and [69]). (\dagger) the scalar ghost state can become safe on FRW backgrounds (only) [48].

The Bianchi condition is realised in the form (4.7), thus $\mathbf{E}_\gamma - \mathbf{E}/2|_{FRW} = 0$ must be satisfied. For the perturbations, the same considerations hold as in case (c) .

(e) 2 DoF with $r = 0$, $s = 0$. Here

$$\mathcal{V} = \sqrt{\gamma} \left(N \Lambda + m^2 \mathbf{E}[\gamma] \right) \quad \text{and} \quad w_{\text{eff}} = -1. \quad (4.13)$$

Even in this case the conservation of the effective EMT is realised in the form (4.7) and thus one needs $\mathbf{E}_\gamma - \mathbf{E}/2|_{FRW} = 0$. However $\rho_{\text{eff}} = -p_{\text{eff}} = \Lambda$, hence the effect on cosmology of this class of potentials is indistinguishable from a plain cosmological constant. Differences with GR may appear in spherically symmetric Schwarzschild-like solutions. While the matter is beyond the scope of the present analysis, following [48] we can anticipate that no vDVZ discontinuity is found at the linearized order, both on Minkowski and on FRW backgrounds. Thus, GR is recovered smoothly in the limit of small graviton mass. The absence of discontinuity implies also the absence of Vainshtein spatial strong coupling. As a result, these models could represent a new interesting class of massive gravity theories.

The results are summarized in table 2.

5 Conclusion

We analyzed the Hamiltonian structure of modified gravity theories obtained by adding a nonderivative function of the ADM variables $\mathcal{V}(N, N^i, \gamma_{ij})$ to the Einstein-Hilbert action, and under the minimal requirement of unbroken rotational invariance, thus encompassing Lorentz-invariant and Lorentz-breaking theories. The classification of the various potentials according to the number of propagating DoF in the perturbative and nonperturbative regime was given in table 1. Further restrictions were obtained by requiring the existence of a realistic FRW cosmology. The results are summarized in table 2.

The simplest deformation, which turns out to propagate 2 DoF, corresponds to a potential of the form $\sqrt{g} \Lambda + m^2 \sqrt{\gamma} \mathbf{E}[\gamma]$, i.e. a function of the sole 3d metric, besides a standard cosmological constant. At the level of FRW cosmological background it is indistinguishable

from GR. Nevertheless, it could lead to possible modifications of gravity in static solutions. The investigation of the relative phenomenology, for instance of Schwarzschild-like solutions, is beyond the scope of the present work and will be presented elsewhere.

Potentials that depend on the lapse and the $3d$ metric, $\mathcal{V}[N, \gamma]$, propagate 3 DoF at non-perturbative level and they also support FRW solutions where 3 DoF propagate at linear level (see [48]). Unfortunately, only 2 DoF can propagate at linearized order around Minkowski background, indicating strong coupling in the scalar sector.

No potential with 4 DoF is found. In fact, it seems very difficult if not impossible to construct $SO(3)$ invariant deforming potential \mathcal{V} with *four* DoF, and so far no nonlinear realization of partially-massless gravity has been found [60, 61]. Here we showed that, if any such theory exists, it will appear as a subclass of the Lorentz-breaking potentials with 5 DoF.

The case with 5 DoF was discussed in depth in [2, 4, 16] and appears to be promising from a phenomenological point of view, being that the cutoff of the theory is of the order of $\Lambda_2 \sim (mM_{Pl})^{1/2}$ and no vDVZ discontinuity is present. Although the theory is weakly coupled with 5 DoF on either Minkowski or FRW backgrounds, the background equations result incompatible with the requirement of a weakly coupled spectrum on both spaces. Choosing the existence of FRW spacetime as physical request, one has strong coupling around exact Minkowski and de Sitter space, with progressively safer cutoff as long as w_{eff} deviates from -1 . Thus, there is a connection between the infrared behaviour of the theory (cosmological scales) with the short distance behaviour (possible short distance strong coupling). In fact, such a behaviour can set the scale for possible deviations from GR, that may be just around the corner, provided $w_{\text{eff}} \neq -1$ [16]. It is important to remark that Lorentz breaking theories with 5 DoF are immune from issues of spatial (Vainshtein) strong coupling, and thus constitute the first modified gravity theories for which a weak field expansion is possible.

As is known, the most general potentials, which propagate 6 DoF, contain the Boulware-Deser ghost around Minkowski background. Nevertheless, they can support 6 healthy states around FRW backgrounds [48] (see section 2.1 and 3) and we intend to analyze in detail the viability of this scenario in a forthcoming work.

We further found technically possible cases with $5 + \frac{1}{2}$ DoF, but whether or not one has to *add or subtract* a half DoF, and under which conditions this has to be considered, is still an open question [50–53]. We leave these cases for further investigation.

Finally, our results can be extended to the case of explicitly time-dependent potentials, as realized if for instance the reference metric is explicitly time dependent. In this case many issues disappear or are less dangerous (see appendix B): mainly, the strong coupling around de Sitter of the models with 5 DoF disappears [69], or, in the case of 2 DoF, the tuning of the potentials required to support a FRW background is no longer needed.

A Less than five DoF

We briefly review first (see appendix A in [4]) the analysis of the 5 DoF potentials working with the simpler canonical variables N, ξ^i, γ^{ij} where the transformation from $N^i \rightarrow \xi^i$ is given by

$$N^i = N \xi^i + Q^i[\xi, \gamma] \quad \text{with} \quad Q^i = - \left(\partial_{\xi^i \xi^j}^2 \mathcal{U} \right)^{-1} \partial_{\xi^j} \mathcal{E}. \quad (\text{A.1})$$

As described in the text the potential is of the form

$$\mathcal{V} = m^2 \sqrt{\gamma} \left(N \mathcal{U} + \partial_{\xi^i} \mathcal{U} Q^i + \mathcal{E} \right), \quad (\text{A.2})$$

and implies the relations $\partial_N \mathcal{V} = m^2 \sqrt{\gamma} \mathcal{U}$, $\partial_{N^i} \mathcal{V} = m^2 \sqrt{\gamma} \partial_{\xi^i} \mathcal{U}$.

The total Hamiltonian in the new variables is

$$H_T = \int d^3x \left[\mathcal{H}_A N^A + \mathcal{V} + \lambda^A \Pi_A \right] = \int d^3x \left[(\mathcal{H}_0 + \mathcal{H}_i \xi^i) N + \mathcal{H}_i Q^i + \mathcal{V} + \lambda^A \Pi_A \right], \quad (\text{A.3})$$

where the momenta Π_0 and Π_i , relative to the variables N and ξ^i , are now the primary constraints. The secondary and tertiary conditions are given by

$$\mathcal{S}_0 = (\mathcal{H}_0 + \mathcal{H}_i \xi^i) + m^2 \sqrt{\gamma} \mathcal{U}, \quad \mathcal{S}_i = (N \delta_i^j + Q_i^j) \bar{\mathcal{S}}_j, \quad (\text{A.4})$$

$$\begin{aligned} \mathcal{T}(\vec{x}) \equiv \mathcal{T}_0 = \{\mathcal{S}_0(\vec{x}), H\}, & \quad \mathcal{T}_i(\vec{x}) = \{\bar{\mathcal{S}}_i(\vec{x}), H\} + m^2 \sqrt{\gamma} \mathcal{U}_{ij}(\vec{x}) \lambda^j(\vec{x}) \\ \Rightarrow \lambda^0 \text{ (undetermined),} & \quad \lambda^i \text{ (determined),} \end{aligned} \quad (\text{A.5})$$

with $\bar{\mathcal{S}}_j = (\mathcal{H}_j + m^2 \sqrt{\gamma} \mathcal{U}_j)$, which is assumed to vanish to enforce the constraint $\mathcal{S}_i \approx 0$.⁹

The quaternary condition is then

$$\mathcal{Q}(\vec{x}) = \{\{\mathcal{S}_0(\vec{x}), H\}, H\} + \int d^3y \left(\lambda^0(\vec{y}) \{\mathcal{S}_0(\vec{x}), \mathcal{S}_0(\vec{y})\} + \lambda^i(\vec{y}) \partial_{\xi^i(\vec{y})} \mathcal{T}(\vec{x}) \right) \quad (\text{A.6})$$

and the last lagrange multiplier λ^0 is again not determined, completing the elimination of the sixth mode, provided that

$$\{\mathcal{S}_0(\vec{x}), \mathcal{S}_0(\vec{y})\} = 0. \quad (\text{A.7})$$

This leads to a simple partial differential equation for the potential \mathcal{U} , which is solved [4] by the requirement that \mathcal{U} is a function of the combination $\mathcal{K}^{ij} = \gamma^{ij} - \xi^i \xi^j$. Using the expressions for the secondary constraints we can write the Hamiltonian as

$$H = \int d^3x \left[\mathcal{S}_0 N + \bar{\mathcal{S}}_i Q^i + m^2 \sqrt{\gamma} \mathcal{E} \right], \quad (\text{A.8})$$

and we note incidentally that if $\mathcal{E} \equiv 0$ then $Q^i \equiv 0$ and $H = \int d^3x \mathcal{S}_0 N$. Thus if $\{\mathcal{S}_0, \mathcal{S}_0\} = 0$ then also the tertiary constraint is actually identically zero: $\mathcal{T} = \int d^3y N(\vec{y}) \{\mathcal{S}_0(\vec{x}), \mathcal{S}_0(\vec{y})\} \equiv 0$. In this case the Dirac analysis stops at the level of secondary constraints, and one is left with 6 DoF instead of 5. Therefore, $\mathcal{E} \neq 0$ is a necessary conditions to have 5 DoF.

We can now find the further conditions under which a potential of the form (A.2) propagates only 4 DoF. One must require that even the quinary and the senary conditions

$$\mathcal{F} = \{\mathcal{Q}, H\} + \lambda^i \partial_{\xi^i} \mathcal{Q} + \lambda^0 \partial_N \mathcal{Q}, \quad (\text{A.9})$$

$$\mathcal{G} = \{\mathcal{F}, H\} + \lambda^i \partial_{\xi^i} \mathcal{F} + \lambda^0 \partial_N \mathcal{F} \quad (\text{A.10})$$

do not determine the last lagrange multiplier λ^0 . This implies the following partial differential conditions in field space

$$\partial_N \mathcal{Q} = 0, \quad \text{and} \quad \partial_N \mathcal{F} = 0. \quad (\text{A.11})$$

The explicit expressions consist in rather complicated equations for \mathcal{U} and \mathcal{E} ; no solution, if any exists, is presently known.

The last Lagrange multiplier ought to be determined at the next (septenary) step. However as usual, even less DoF may be present. For instance, gauge invariance could be present, if the lagrange multiplier is not determined even by the septenary and octonary conditions and the procedure stops there. The number of DoF would in this case be 3, with both second and first class constraints.

⁹To be rigorous, there exists also a branch of solutions to (A.4) corresponding to $(N \delta_i^j + Q_i^j) \approx 0$. For a simple solvable case where $Q^i \equiv \zeta[\gamma] \xi^i$ (see [4]) this branches gives 6 DoF and is thus unviable, while 5 DoF are present in the branch $\bar{\mathcal{S}}_i \approx 0$, that we chose.

B Explicitly time dependent potentials

Here we analyze the case where the potential \mathcal{V} has an explicit time dependence allowed by spatial rotational SO(3) symmetry (see [69] where the class of potentials with 5 DoF was considered). For what concerns the existence of a FRW background, the Bianchi conservation equation (4.6) acquires an extra term, namely

$$\dot{N} \mathcal{V}_{NN} - 6 \frac{\dot{a}}{a} \left(\mathcal{V}_{N\gamma} - \frac{\mathcal{V}_\gamma}{N} \right) + N \partial_t \mathcal{V}_N = 0. \quad (\text{B.1})$$

For the potentials of the form $\mathcal{V} = \sqrt{\gamma} (N\hat{\mathcal{A}} + \hat{\mathcal{B}})$, with $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ explicitly time dependent, equation (B.1) becomes an algebraic equation for the lapse N (see [69])

$$6 \frac{\dot{a}}{a} \left(\hat{\mathcal{B}}_\gamma - \frac{1}{2} \hat{\mathcal{B}} \right) + N \partial_t \hat{\mathcal{A}} = 0. \quad (\text{B.2})$$

The last term can be understood by the fact that the explicit time dependence of $\hat{\mathcal{A}}$ is also a source of breaking of time reparametrization. In any case, now N will be determined, and from the 00 component of the Einstein equations, $3\left(\frac{\dot{a}}{a}\right)^2 = H^2 = N^2 \rho_{\text{eff}}$, one can determine the scale factor a , leading to a sensible cosmology.

An straightforward construction leading to explicit time dependence is provided by a nontrivial spatial reference metric, if one replaces δ_{ij} with $b(t)\delta_{ij}$, see [69]. All invariants are built from the spatial tensor $\gamma^{ik} \delta_{kj}$ and the net effect is to replace γ^{ij} by $b(t) \gamma^{ij} \equiv \tilde{\gamma}^{ij}$. In this case, we can write eq. (B.2) as:

$$N = 2 \frac{\dot{a}/a}{\dot{b}/b} \frac{\hat{\mathcal{B}}_\gamma - \hat{\mathcal{B}}/2}{\hat{\mathcal{A}}_\gamma}, \quad (\text{B.3})$$

where $\partial_t \hat{\mathcal{A}} = \partial_t \tilde{\gamma}^{ij} \partial_{\tilde{\gamma}^{ij}} \hat{\mathcal{A}}$ and $\partial_{\tilde{\gamma}^{ij}} \hat{\mathcal{A}} \equiv \hat{\mathcal{A}}_\gamma \tilde{\gamma}_{ij}$ and $\partial_t \tilde{\gamma}^{ij} = \dot{b} \gamma^{ij}$. Using comoving time τ (i.e. $N dt = d\tau$) equation (B.2) remains the same (and determines $N(\tau)$) while the 00 Einstein eq. becomes $3\frac{a'(\tau)^2}{a(\tau)^2} = 8\pi G \rho(\tau) + \rho_{\text{eff}}(\tau)$. Note that Minkowski space is not solution of the modified Einstein equations. Also, note that the limit of static b is singular, and one turns back to the constraint $\mathcal{B}_\gamma - \mathcal{B}/2 = 0$ as in the text.

The analysis reported in the present work, extended to time dependent potentials, gives basically the same results summarized in table 2, but, despite the need to introduce an arbitrary time-dependent function, it represents an interesting possibility, because it avoids the tuning condition for the functional form of \mathcal{V} required to have a FRW background. Moreover, for class of potentials with 5 DoF, the theory is weakly coupled also near de Sitter backgrounds [69] and thus represent a viable massive gravity with 5 DoF.

We finally carried out a check that all the nonperturbative results mentioned in this work can be extended to the case of an explicit time dependence, by repeating the analysis of constraints for the case of 5 DoF. It is straightforward to check that primary (Π_A) and secondary constraints (\mathcal{S}_A) are not affected by the explicit time dependence. This is the main reason why our results can be safely extended. Tertiary constraints are modified according to

$$\hat{\mathcal{T}} = \mathcal{T} + m^2 \partial_t(\sqrt{\gamma}\mathcal{U}), \quad \hat{\mathcal{T}}_i = \mathcal{T}_i + \partial_t \bar{\mathcal{S}}_i; \quad (\text{B.4})$$

where we have denoted with $\hat{}$ the corresponding constraints in the case of explicit time dependence, see appendix A. The quaternary constraint becomes

$$\hat{\mathcal{Q}} = \mathcal{Q} + m^2 \{ \partial_t(\sqrt{\gamma}\mathcal{U}), H \} + m^2 \partial_t^2(\sqrt{\gamma}\mathcal{U}). \quad (\text{B.5})$$

The lagrange multiplier λ^0 is again not determined at this stage if $\{\mathcal{S}_0, \mathcal{S}_0\} = 0$, which is the same partial differential equation for the potential \mathcal{U} as in the time-independent case. Thus, we again find that the potentials of the form

$$\mathcal{V} \equiv m^2 \sqrt{\gamma} \left(N \hat{\mathcal{U}} + \partial_\xi \hat{\mathcal{U}} \hat{\mathcal{Q}}^i + \hat{\mathcal{E}} \right), \quad (\text{B.6})$$

with $\hat{\mathcal{U}} = \mathcal{U}[\mathcal{K}^{ij}, t]$ and $\hat{\mathcal{E}} = \mathcal{E}[\gamma^{ij}, t]$, propagate nonperturbatively 5 DoF. For the cases with less than 5 DoF, all the results can be extended along the same lines.

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