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PROFINITE GROUPS WITH FINITE VIRTUAL LENGTH

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Abstract

In this paper we introduce the notion of finite virtual length for profinite groups (that is, every series has a bounded number of infinite factors) and we prove a Jordan–Hölder type theorem for profinite groups with finite virtual length. More structural results are provided in the pronilpotent and p-adic analytic cases.

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1. Introduction

It has been observed that hereditarily just infinite pro-p groups play, in the theory of pro-p groups, a role which is analogous to that played by simple groups in finite group theory (see [7]). In this paper we produce further support to this parallel, establishing a Jordan–Hölder type theory for profinite groups, which in the case of pronilpotent groups identifies hereditarily just infinite pro-p groups as the building blocks. On the way we state and prove every result in maximum generality as follows: in Section 2 we discuss some general theory of profinite groups with operators; in Section 3 we introduce the notion of virtual length and prove our version of the Jordan–Hölder theorem for profinite groups. In Section 4 we focus on the pronilpotent case and give several structural results: assuming finite virtual length we prove that every subnormal subgroup is finitely generated and that the maximal condition is satisfied by families of subnormal subgroups of bounded defect. Finally, in Section 5 we compare virtual length with other well-known invariants, like the dimension of a p-adic analytic group.

2. Profinite groups with operators

A *topological group with operators* is a topological group G carrying the structure of Ω-group for a (possibly empty) set Ω of operators with the additional requirement

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that every element of Ω acts continuously on G. We then simply say that G is a topological Ω-group. Our notation is standard: g^ω denotes the result of the action of the element ω of Ω on the element g of G; an Ω-subgroup of G is a subgroup H such that $h^\omega \in H$ for every $h \in H$ and every $\omega \in \Omega$. Unless otherwise explicitly stated, subgroups of topological groups are assumed to be closed. A homomorphism from a topological group with operators (G_1, Ω_1) to another topological group with operators (G_2, Ω_2) is a pair (φ, Γ), where φ is a continuous homomorphism from G_1 into G_2 and Γ is a mapping from Ω_1 into Ω_2 such that $(g^\omega)^\varphi = (g^\varphi)^{(\omega^\Gamma)}$ for every g in G_1 and ω in Ω_1. When $\Omega_1 = \Omega_2 = \Omega$ and Γ is the identity map we simply say that φ is an Ω-homomorphism from G_1 to G_2 (the terms Ω-isomorphism and Ω-isomorphic are used consistently). Given an inverse system of topological groups with operator $\{(G_i, \Omega_i), (\varphi_{ij}, \Gamma_{ij})\}$ indexed by a directed poset I, it is possible to define its inverse limit (G, Ω). It is not difficult to show that G is the inverse limit (both as an abstract and a topological group) of $\{G_i, \varphi_{ij}\}$ and Ω is the inverse limit (as a set) of $\{\Omega_i, \Gamma_{ij}\}$ (see [3, Ch. 1, Section 10]). This leads us to the following definition.

Definition 2.1. Let I be a directed poset and let $\{(G_i, \Omega_i), (\varphi_{ij}, \Gamma_{ij})\}$ be an inverse system of topological groups with operators, where every G_i is a finite group endowed with the discrete topology. If (G, Ω) is the inverse limit of the given inverse system we say that G is a *profinite Ω-group*.

The notion of profinite Ω-group generalizes the notion of profinite group: we state without proof a number of results that can be easily adapted from [9, Ch. 2 and 3].

Proposition 2.2. *Let G be a profinite Ω-group. Then:*

(1) *the open normal Ω-subgroups form a base of neighborhoods of* 1*;*
(2) *every Ω-subgroup of G is the intersection of the open Ω-subgroups of G containing it;*
(3) *if H is an Ω-subgroup of G then H is a profinite Ω-group;*
(4) *if N is a normal Ω-subgroup of G then G/N is a profinite Ω-group.*

A profinite group G which is a topological Ω-group for some set Ω is not necessarily a profinite Ω-group according to our definition.

Example 2.3. Let $G := \prod_I C$ be the cartesian product of infinitely many copies of a nontrivial finite group C. Let Ω be the set of the continuous automorphisms of G, so Ω-subgroups are just topologically characteristic subgroups. Fix an index i in I and let H be the open subgroup formed by those elements whose ith component is trivial. Let g be a nontrivial element of H and let j be an index (necessarily different from i) such that the jth component of g is nontrivial. The automorphism induced by the transposition of i and j is continuous and sends g to an element which does not belong to H. As a consequence, the only topologically characteristic subgroup contained in the open subgroup H is the trivial one: this means that G does not satisfy statement (1) of Proposition 2.2 and then it is not a profinite Ω-group.

Statement (1) of Proposition 2.2 can be reverted.

Proposition 2.4. *Let G be a profinite group and let Ω be a set operating on the abstract group G. If G has a base of neighborhoods of 1 formed by open normal Ω-subgroups then G is a profinite Ω-group.*

Proof. The topological Ω-group G is just the inverse limit of the finite Ω-groups G/N as N varies in the set of open normal Ω-subgroups of G. The details are left to the reader. □

Remark 2.5. In the statement of Proposition 2.4 no explicit requirement that the elements of Ω act continuously on G is done: this follows from the proof.

Remark 2.6. In the definition of a profinite Ω-group as an inverse limit of finite groups with operators, different sets of operators are allowed. However, statement (1) of Propositions 2.2 and 2.4 permit to regard a profinite Ω-group as an inverse limit of finite Ω-groups.

We now give several examples of profinite Ω-groups. We will extensively use the characterization given by Proposition 2.4 without any further notice.

Example 2.7. Let G be a profinite group and let Ω be a subset of G acting on G by conjugation. Since a profinite group has a base of neighborhoods of 1 formed by open normal subgroups, G is a profinite Ω-group.

Example 2.8. Let G be a finitely generated profinite group and let Ω be a set formed by automorphisms of G. The elements of Ω are continuous (see [8, Theorem 1.1]). Since G has a base of neighborhoods of 1 formed by open topologically characteristic subgroups (see [9, Proposition 2.5.1]) G is a profinite Ω-group.

Example 2.9. Let G be a finitely generated pro-p group and let Ω be a set formed by endomorphisms of G. The elements of Ω are continuous (see [5, Corollary 1.21]). Since G has a base of neighborhoods of 1 formed by open topologically fully invariant subgroups (namely, the Frattini series, see [9, Proposition 2.8.13]), G is a profinite Ω-group.

We recall that, for a given prime p, a p-Sylow subgroup of a profinite group is a pro-p subgroup which is maximal in the family of pro-p subgroups. If G is a pronilpotent group then it has, for every prime p, a unique p-Sylow subgroup (which is then topologically fully invariant) and G is the cartesian product of its Sylow subgroups (see [9, Proposition 2.3.8]).

Example 2.10. Let G be a pronilpotent group. If all the Sylow subgroups of G are finitely generated, then, using the Frattini series of the Sylow subgroups, it is easy to construct a base of neighborhoods of 1 formed by open topologically fully invariant subgroups of G. Therefore, G turns out to be a profinite Ω-group for every set Ω of continuous endomorphisms of G.

By Proposition 2.2 every Ω-subgroup of a profinite Ω-group G is the intersection of open Ω-subgroups. As a consequence, every proper Ω-subgroup is contained in a maximal (necessarily open) Ω-subgroup.

Definition 2.11. The Ω-*Frattini subgroup* of a profinite Ω-group G is the intersection $\Phi^{\Omega}(G)$ of the maximal Ω-subgroups of G (for the sake of completeness we set $\Phi^{\Omega}(1) := 1$).

The Ω-Frattini subgroup is not necessarily normal in G (we will however observe in Section 4 that this is true provided that G is pronilpotent). Consider the following example.

Example 2.12. Let G be a finite simple group which is not cyclic of prime order and let H be a maximal subgroup of G. We regard G as a profinite (actually, finite) H-group with the elements of H operating on G by conjugation. The H-subgroups of G are those subgroups of G that are normalized by H. If K is an H-subgroup of G which is not contained in H, then $N_G(K)$ contains K and H. Thus, by the maximality of H, we have $N_G(K) = G$ and the simplicity of G gives $K = G$. Therefore, the proper H-subgroups of G are contained in H. Since H is itself an H-subgroup, it is the only maximal H-subgroup and so $\Phi^H(G) = H$ is not normal in G.

We recall that if X is a subset of a profinite group, the subgroup generated by X is the intersection of the (closed) subgroups containing X: similarly the Ω-*subgroup* Ω-*generated* by X is the intersection of the (closed) Ω-subgroups containing X. By saying that a profinite Ω-group G is finitely Ω-generated we then mean that there exists a finite subset X such that G is the smallest Ω-subgroup of G containing X. It is not difficult to prove the following proposition.

Proposition 2.13. *A profinite Ω-group G is Ω-generated by a subset X if and only if it is Ω-generated by $X \cup \Phi^{\Omega}(G)$.*

The previous result shows, in particular, that if $\Phi^{\Omega}(G)$ has finite index in G then G is finitely Ω-generated. It is well known that, if we restrict ourselves to consider profinite groups without operators, the converse is also true for pro-p groups (or, more generally, for prosupersoluble groups with additional properties, see [9, Proposition 2.8.11]). This is no longer true for a generic set Ω:

Example 2.14. Let $G := \prod_{\Omega} C$ be the cartesian product of infinitely many copies (indexed by an infinite set Ω) of the cyclic group C of order p. For every ω in Ω we define the action of ω on G to be the canonical projection of G over $H_\omega := \prod_{\Omega\setminus\{\omega\}} C$. For every finite subset Ψ of Ω we consider the subgroup of G formed by those elements whose components relative to the indices in Ψ are 1. These subgroups form a base of neighborhoods of 1 and are clearly Ω-subgroups, so G is a pro-p Ω-group. The subgroups H_ω are maximal Ω-subgroups and their intersection is trivial, so $\Phi^{\Omega}(G) = 1$ has infinite index. Let c be a generator of C and let g be the element of G with every component equal to c. The Ω-subgroup Ω-generated by g contains, for every ω in Ω, the element $(g^{\omega})^{-1}g$ which generates the copy of C of index ω: therefore, G is Ω-generated by g.

3. Series

We use standard notation for Ω-series in Ω-groups (see [10, Ch. 3]).

DEFINITION 3.1. Given an Ω-series $\mathbf{S}\colon 1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ of a profinite Ω-group G, we denote by $\mathrm{L}(\mathbf{S})$ the number of indices i such that $|G_i : G_{i-1}|$ is infinite. The *virtual* Ω*-composition length* of G, denoted by $\mathrm{vl}_\Omega(G)$, is the (possibly infinite) supremum of $\mathrm{L}(\mathbf{S})$ as $\mathbf{S}$ varies among the Ω-series of G. When Ω is empty we will simply speak of the *virtual composition length* $\mathrm{vl}(G)$.

DEFINITION 3.2. An Ω-series $\mathbf{S}$ such that $\mathrm{L}(\mathbf{S}) = \mathrm{vl}_\Omega(G)$ is said to be a *virtual* Ω*-composition series*.

REMARK 3.3. Clearly, $\mathrm{vl}_\Omega(G) = 0$ if and only if G is finite.

DEFINITION 3.4. An infinite profinite Ω-group is said to be *just infinite* if every nontrivial normal Ω-subgroup has finite index. A profinite Ω-group is said to be *hereditarily just infinite* if every open Ω-subgroup is just infinite.

REMARK 3.5. Let $\mathbf{S}\colon 1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ be an Ω-series of a hereditarily just infinite profinite Ω-group G. If i is the maximum integer such that G_i has infinite index in G, then G_{i+1} is open in G and it is then just infinite. Therefore G_i, being normal in G_{i+1}, is trivial and $\mathrm{L}(\mathbf{S}) = 1$. As a consequence, $\mathrm{vl}_\Omega(G) = 1$.

REMARK 3.6. If $\mathbf{S}$ is an Ω-series of a profinite Ω-group G, and $\mathbf{T}$ is an Ω-series of G refining $\mathbf{S}$, then $\mathrm{L}(\mathbf{S}) \le \mathrm{L}(\mathbf{T})$. By the Schreier refinement theorem (see, for example, [10, 3.1.2]) it is then possible to refine $\mathbf{S}$ to a virtual Ω-composition series if $\mathrm{vl}_\Omega(G)$ is finite and to an Ω-series $\mathbf{U}$ such that $\mathrm{L}(\mathbf{U})$ is arbitrarily large if $\mathrm{vl}_\Omega(G)$ is infinite.

The following proposition will be used throughout without any further notice.

PROPOSITION 3.7. *If* $\mathbf{S}\colon 1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ *is an* Ω*-series of a profinite* Ω*-group* G *then*

$$\mathrm{vl}_\Omega(G) = \sum_{i=1}^{n} \mathrm{vl}_\Omega(G_i/G_{i-1}),$$

where, as usual, we agree that the right-hand side is infinite if at least one of the summands is infinite.

PROOF. Remark 3.6 implies that in calculating $\mathrm{vl}_\Omega(G)$ we may limit ourselves to consider refinements of $\mathbf{S}$. Such a refinement can be thought of as a 'juxtaposition' of Ω-series of the quotients G_i/G_{i-1}. The claim follows. □

We briefly recall some results about subnormal subgroups in profinite groups: for an account, see [11, Section 2]. If H is a (closed) subgroup of a profinite group G then the normal closure of H in G regarded as a profinite group is just the topological closure of the normal closure of H in G regarded as an abstract group. So, H is subnormal in G regarded as an abstract group if and only H is subnormal in G regarded as a profinite group and the subnormality defect in the two contexts actually coincide. If G is a profinite Ω-group and H is a (closed) Ω-subgroup of G then the abstract normal closure of H in G is clearly a (nonnecessarily closed) Ω-subgroup of G. Since the topological closure of an Ω-subgroup is an Ω-subgroup (see [4, Ch. 1, Section 6.1, Proposition 1]) we have the following proposition.

Proposition 3.8. *If H is an Ω-subgroup of a profinite Ω-group, then the normal closure H^G of H in G regarded as a profinite group is an Ω-subgroup. Moreover, if H is subnormal of defect at most n in G regarded as an abstract group then there exists a sequence*

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G$$

such that every H_i is an Ω-subgroup.

Thus, by Proposition 3.7, we have the following corollary.

Corollary 3.9. *If H is a subnormal Ω-subgroup of a profinite Ω-group G then* $\mathrm{vl}_\Omega(H) \leq \mathrm{vl}_\Omega(G)$.

Here is another direct consequence of Proposition 3.7.

Corollary 3.10. *Let G be a profinite Ω-group. If $G = \prod_{i=1}^n G_i$, where every G_i is an Ω-subgroup, then*

$$\mathrm{vl}_\Omega(G) = \sum_{i=1}^{n} \mathrm{vl}_\Omega(G_i).$$

Lemma 3.11. *Let G be a profinite Ω-group which is the cartesian product of infinitely many nontrivial Ω-subgroups. Then* $\mathrm{vl}_\Omega(G) = \infty$.

Proof. Let $G = \prod_{i \in I} H_i$, where every H_i is a nontrivial Ω-subgroup and I is infinite. Let J be an infinite subset of I such that $I \setminus J$ is infinite too. The cartesian product $\prod_{i \in J} H_i$ is then an infinite normal Ω-subgroup H with infinite index in G. Iteration of this process provides an Ω-series with arbitrarily many infinite sections. □

Example 3.12. There are two inequivalent definitions of a profinite *branch* group (see, for example, [6] or [2]): one involving a group action on a tree, and another purely algebraic one which includes the previous one as a particular case. According to the latter definition, a profinite branch group is infinite and contains, for each natural number n, an open normal subgroup H_n which can be expressed as the direct product of k_n copies of a subgroup L_n, where $\{k_n\}$ is a strictly increasing sequence of natural numbers. These are not all the requirements of the definition, but they are enough to show that $\mathrm{vl}(G) = \infty$ for every profinite branch group G. Since H_n has finite index, L_n is infinite: in particular, $\mathrm{vl}(L_n) \geq 1$. By Corollary 3.10 we then get $\mathrm{vl}(G) = \mathrm{vl}(H_n) = k_n\, \mathrm{vl}(L_n) \geq k_n$. This is true for every n, so $\mathrm{vl}(G) = \infty$.

Remark 3.13. According to Wilson's dichotomy (see, for example, [6, Section 6, Theorem 3], if G is a profinite just infinite group then either G is a profinite branch group and therefore $\mathrm{vl}(G) = \infty$, or G contains an open normal subgroup which is the direct product of a finite number n of copies of some hereditarily just infinite profinite group and therefore $\mathrm{vl}(G) = n$.

Definition 3.14. Two profinite Ω-groups G and H are *Ω-commensurable* if they contain open Ω-isomorphic Ω-subgroups.

Definition 3.15. Let G be a profinite Ω-group. Two Ω-series $\mathbf{S}$ and $\mathbf{T}$ of G are *Ω-commensurable* if $\mathrm{L}(\mathbf{S}) = \mathrm{L}(\mathbf{T})$ and there exists a bijection between the set of infinite sections of $\mathbf{S}$ and the set of infinite sections of $\mathbf{T}$ such that corresponding sections are Ω-commensurable.

Remark 3.16. We recall that two Ω-series $\mathbf{S}$ and $\mathbf{T}$ of an Ω-group are said to be Ω-isomorphic if there exists a bijection between the set of sections of $\mathbf{S}$ and the set of sections of $\mathbf{T}$ such that corresponding factors are Ω-isomorphic. Clearly, Ω-isomorphic series of a profinite Ω-group are Ω-commensurable.

Lemma 3.17. *Let $\mathbf{S}$ be an Ω-series of a profinite Ω-group G. If $\mathbf{T}$ is a refinement of $\mathbf{S}$ and $\mathrm{L}(\mathbf{S}) = \mathrm{L}(\mathbf{T})$ then $\mathbf{S}$ and $\mathbf{T}$ are Ω-commensurable.*

Proof. We may just consider the case in which $\mathbf{T}$ is obtained by inserting a single new term in $\mathbf{S}$ and then arguing by induction. So let us insert N between the terms G_i and G_{i+1} of $\mathbf{S}$, that is, $G_i \trianglelefteq N \trianglelefteq G_{i+1}$. All the sections of $\mathbf{S}$ but G_{i+1}/G_i are left untouched so we may consider just this section. If G_{i+1}/G_i is finite then there is nothing to prove. If G_{i+1}/G_i is infinite then the equality $\mathrm{L}(\mathbf{T}) = \mathrm{L}(\mathbf{S})$ yields that either G_{i+1}/N or N/G_i is finite. In the former case, N/G_i is an open Ω-subgroup of G_{i+1}/G_i, and hence they are obviously Ω-commensurable. In the latter case, by Proposition 2.2 there exists an open normal Ω-subgroup H/G_i of G_{i+1}/G_i such that $H \cap N = G_i$. Therefore, H/G_i and HN/N are Ω-isomorphic open Ω-subgroups of G_{i+1}/G_i and G_{i+1}/N respectively, as required. □

By combining Remark 3.6 and Lemma 3.17 we immediately get the following version of the Jordan–Hölder theorem.

Theorem 3.18. *Let G be a profinite Ω-group with finite virtual Ω-composition length. If $\mathbf{S}$ and $\mathbf{T}$ are two virtual Ω-composition series then $\mathbf{S}$ and $\mathbf{T}$ are Ω-commensurable.*

4. Pronilpotent Ω-groups

Proposition 4.1. *If G is a pronilpotent Ω-group and M is a maximal Ω-group then M is normal in G and G/M is an elementary abelian finite p-group for some prime p. In particular, $\Phi^{\Omega}(G)$ is a normal subgroup of G and $G/\Phi^{\Omega}(G)$ is abelian.*

Proof. By Proposition 3.8 we know that the normal closure of an Ω-subgroup is an Ω-subgroup. The normal closure of a proper subgroup in a pronilpotent group is a proper subgroup, so M^G necessarily coincides with M, that is, M is normal. We already observed that maximal Ω-subgroups of a profinite Ω-group are open, so $K := G/M$ is a finite nilpotent Ω-group. The derived subgroup of K is then a proper fully invariant subgroup of K: in particular it is a proper Ω-subgroup of K. By the maximality of M it follows that K is abelian. The same argument applied to K^p, where p is a prime dividing the order of K, shows that K has exponent p. □

Corollary 4.2. *If L is an Ω-simple pronilpotent Ω-group then L is an elementary abelian finite p-group for some prime p.*

Lemma 4.3. *Let G be a profinite Ω-group and let n be a positive integer. If $|H : \Phi^{\Omega}(H)|$ is finite for every subnormal Ω-subgroup H of defect at most n then every nonempty family of subnormal Ω-subgroups of defect at most n admits a maximal element.*

Proof. The proof is by induction over n.

We start by considering the case $n = 1$. Let $H_1 \leq H_2 \leq \cdots \leq H_i \leq \cdots$ be an ascending chain of normal Ω-subgroups of G. The topological closure H of $\bigcup_i H_i$ is a normal subgroup of G: since the elements of Ω act continuously, H is a normal Ω-subgroup and $|H : \Phi^{\Omega}(H)|$ is finite by hypothesis, so H has only finitely many maximal Ω-subgroups $M_1, \ldots, M_r$. None of them can contain all the H_i, for otherwise it would contain H. For every $j = 1, \ldots, r$ we may then choose an integer k_j such that $H_{k_j} \nleq M_j$. By setting $k := \max_j k_j$, we see that H_k is not contained in any of the maximal Ω-subgroups of H. As a consequence, $H_k = H$, and thus the ascending chain of the H_i becomes stationary. This establishes the inductive basis.

Assume now $n > 1$. Let $\mathcal{S}$ be a nonempty family of subnormal Ω-subgroups of G of defect bounded by n. We consider the family C of the normal closures in G of the elements of $\mathcal{S}$. By Proposition 3.8 the elements of C are normal Ω-subgroups, so inductive basis implies that there exists a maximal element N of C. Consider the family $\mathcal{H}$ formed by the elements of $\mathcal{S}$ whose normal closure is N. These elements form a nonempty family of subnormal Ω-subgroups of N of defect at most $n - 1$. Every subnormal Ω-subgroup of defect at most $n - 1$ of N is a subnormal Ω-subgroup of defect at most n of G: by inductive hypothesis there exists a maximal element H of $\mathcal{H}$. We claim that H is maximal in $\mathcal{S}$: if K is an element of $\mathcal{S}$ containing H, then $N = H^G \leq K^G$. The maximality of N in C implies that $K^G = N$, that is, K belongs to $\mathcal{H}$ and thus coincides with H. □

If G is a pronilpotent Ω-group and p is a prime number, we denote by G_p the unique p-Sylow subgroup of G: as G_p is fully invariant, it is an Ω-subgroup.

Theorem 4.4. *Let G be a pronilpotent Ω-group. If $\mathrm{vl}_{\Omega}(G)$ is finite then:*

(1) *$G_p = 1$ for all but finitely many primes p;*
(2) *$G = \prod_{i=1}^{n} G_{p_i}$, where $p_1, \ldots, p_n$ are the primes whose corresponding Sylow subgroups of G are nontrivial and $\mathrm{vl}_{\Omega}(G) = \sum_{i=1}^{n} \mathrm{vl}_{\Omega}(G_{p_i})$;*

(3) $|H : \Phi^\Omega(H)|$ *is finite for every subnormal* Ω*-subgroup* H*;*
(4) *every subnormal* Ω*-subgroup is finitely* Ω*-generated;*
(5) *every nonempty family of subnormal* Ω*-subgroups of* G *with bounded defect admits a maximal element.*

PROOF. By [9, Proposition 2.3.8], the group G is the cartesian product of its Sylow subgroups, which are normal Ω-subgroups. Lemma 3.11 then gives statement (1), while Corollary 3.10 yields statement (2).

By Corollary 3.9 we know that $\mathrm{vl}_\Omega(H) \le \mathrm{vl}_\Omega(G)$ for every subnormal Ω-subgroup H of G, so to prove statement (3) it is enough to show that $|G : \Phi^\Omega(G)|$ is finite. If it were false then there would exist a sequence $\{M_i\}_{i=1}^\infty$ of maximal Ω-subgroups of G such that M_{i+1} does not contain $\bigcap_{j=1}^i M_j$. By Proposition 4.1 each M_i is normal in G, so we may consider the inverse system of finite Ω-groups $H_i := \prod_{j=1}^i G/M_j$, where, for $i \ge l$, the Ω-homomorphism from H_i into H_l is the canonical projection. The inverse limit H of this inverse system is the cartesian product $\prod_{j=1}^\infty G/M_j$, so $\mathrm{vl}_\Omega(H) = \infty$ by Lemma 3.11. For every i let φ_i be the Ω-homomorphism from G into H_i defined by $g^{\varphi_i} := (gM_1, gM_2, \ldots, gM_i)$. This family of Ω-homomorphisms is compatible with the inverse system so it induces an Ω-homomorphism φ from G into H. The kernel of φ_i is $K_i := \bigcap_{j=1}^i M_j$. We claim that φ_i is surjective for every i: this is equivalent to proving that $|G : K_i| = \prod_{j=1}^i |G/M_j|$ for every i. We proceed by induction over i, the case $i = 1$ being trivial. Assuming that $|G : K_i| = \prod_{j=1}^i |G/M_j|$, to prove that $|G : K_{i+1}| = \prod_{j=1}^{i+1} |G/M_j|$ it suffices to show that $|K_i : K_{i+1}| = |G/M_{i+1}|$. Since M_{i+1} is maximal and it does not contain K_i we have $M_{i+1}K_i = G$. Therefore,

$$|K_i : K_{i+1}| = |K_i : K_i \cap M_{i+i}| = |M_{i+1}K_i : M_{i+1}| = |G : M_{i+1}|,$$

as claimed. Since every φ_i is surjective, the homomorphism φ is surjective too (see [9, Corollary 1.1.6]). This would imply that $\mathrm{vl}_\Omega(K) \le \mathrm{vl}_\Omega(G)$, contradicting the fact that $\mathrm{vl}_\Omega(K) = \infty$. Statement (3) is then proved. Statement (4) follows immediately while statement (5) is a consequence of Lemma 4.3. □

If we drop the hypothesis that G is pronilpotent then we cannot ensure that G is finitely Ω-generated. Indeed, Wilson [12] provided examples of hereditarily just infinite prosoluble groups (hence with virtual composition length 1) which are not finitely generated.

COROLLARY 4.5. *Let* G *be a pronilpotent* Ω*-group with* $\mathrm{vl}_\Omega(G)$ *finite. If* $\mathcal{F}$ *is a nonempty family of subnormal* Ω*-subgroups of bounded defect closed under taking joins, then* $\mathcal{F}$ *has a greatest element. In particular, there exists a greatest finite normal* Ω*-subgroup of* G*.*

REMARK 4.6. Statement (5) of Theorem 4.4 has several other consequences. For instance, if G is a pronilpotent Ω-group with $\mathrm{vl}_\Omega(G)$ finite and every term of the upper central series is an Ω-subgroup (which is always the case if every element of Ω acts bijectively on G) then the upper central series stabilizes, that is, there exists an integer n such that $Z_n(G) = Z_\infty(G)$.

Remark 4.7. Let G be a just infinite pronilpotent Ω-group. A nontrivial Sylow subgroup of G, being a normal Ω-subgroup, has finite index in G (in particular, it is infinite). As a consequence G has at most one nontrivial Sylow subgroup, that is, G is a pro-p group for some prime p.

Proposition 4.8. *Let G be a pronilpotent Ω-group. The following conditions are equivalent:*

(i) *there exists a finite normal Ω-subgroup N such that G/N is a hereditarily just infinite pro-p Ω-group for some prime p;*
(ii) $\mathrm{vl}_\Omega(G) = 1$.

Proof. If condition (i) holds then $\mathrm{vl}_\Omega(G) = \mathrm{vl}_\Omega(G/N) + \mathrm{vl}_\Omega(N)$: condition (ii) then follows from Remarks 3.3 and 3.5.

Conversely, assume that condition (ii) holds. By Corollary 4.5 there exists a greatest finite normal Ω-subgroup N, so $K := G/N$ has no nontrivial finite normal Ω-subgroups and $\mathrm{vl}_\Omega(K) = 1$. We now prove that K is hereditarily just infinite: the fact that it is a pro-p group will then follow readily from Remark 4.7. Let H be an open Ω-subgroup of K and let L be a normal Ω-subgroup of H with infinite index. We need to prove that $L = 1$. Since K is pronilpotent, H is subnormal in K and by Proposition 2.2 there exists an Ω-series

$$1 \trianglelefteq L \trianglelefteq H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = K.$$

As $\mathrm{vl}_\Omega(K) = 1$ and $|H : L| = \infty$, we see that L is finite. We claim, by induction over n, that the normal closure of L in K is finite too. This is obvious for $n = 0$. If $n > 0$, the conjugates of L in H_1 are normal in H and they are at most $|H_1 : H|$ (which is finite). Therefore, L^{H_1} is a finite normal Ω-subgroup of H_1. By inductive hypothesis L^K is finite. Since K has no nontrivial finite normal Ω-subgroups, $L = 1$ as claimed. □

Theorem 4.9. *Let G be a pronilpotent Ω-group. If $\mathrm{vl}_\Omega(G)$ is finite then there exists a virtual Ω-composition series $\mathbf{S}$: $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ such that each quotient G_i/G_{i-1} is either:*

- *a hereditarily just infinite pro-p Ω-group for some prime number p; or*
- *an elementary abelian finite p-group for some prime number p.*

Proof. Let $\mathbf{T}$ be a virtual Ω-composition series of G. We will obtain the required Ω-series by refining $\mathbf{T}$. Since the infinite sections of $\mathbf{T}$ have virtual Ω-composition length 1, a repeated application of Proposition 4.8 allows to refine $\mathbf{T}$ to a virtual Ω-composition series such that every infinite section is a hereditarily just infinite pro-p Ω-group for some prime number p.

We now use Corollary 4.2 and we refine the finite sections of the Ω-series until we get a series such that all the finite sections are elementary abelian finite p-groups for some prime p. □

5. Other finiteness conditions

We recall that the dimension of a pro-p group G of finite rank is the number of generators of an arbitrary open uniform subgroup of G (see [5, Definition 4.7]).

Proposition 5.1. *If G is a pro-p group of finite rank then* $\mathrm{vl}(G)$ *is finite. More precisely,* $\mathrm{vl}(G) \leq \dim G$ *and the equality holds if and only if G is soluble.*

Proof. Given a series $\mathbf{S}\colon 1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$, we write $\mathrm{vl}_i(\mathbf{S})$ for $\mathrm{vl}(G_i/G_{i-1})$ and $\mathrm{d}_i(\mathbf{S})$ for $\dim(G_i/G_{i-1})$, so Proposition 3.7 and [5, Theorem 4.8] give

$$\mathrm{vl}(G) = \sum_{i=1}^{n} \mathrm{vl}_i(\mathbf{S}) \quad \text{and} \quad \dim G = \sum_{i=1}^{n} \mathrm{d}_i(\mathbf{S}). \tag{5.1}$$

Therefore, to prove that $\mathrm{vl}(G) \leq \dim G$ (respectively, $\mathrm{vl}(G) = \dim G$) it suffices to find a series $\mathbf{S}$ such that $\mathrm{vl}_i(\mathbf{S}) \leq \mathrm{d}_i(\mathbf{S})$ (respectively, $\mathrm{vl}_i(\mathbf{S}) = \mathrm{d}_i(\mathbf{S})$) for every i.

We prove that $\mathrm{vl}(G) \leq \dim G$ by induction over $\mathrm{vl}(G)$. The conditions that $\dim G = 0$, that G is finite and that $\mathrm{vl}(G) = 0$ are equivalent: the inequality then holds if $\mathrm{vl}(G) \leq 1$. If $\mathrm{vl}(G) \geq 2$ we take a series $\mathbf{S}$ with at least two infinite sections, so $\mathrm{vl}_i(\mathbf{S}) > 0$ for at least two integers. By Equation (5.1) we have $\mathrm{vl}_i(\mathbf{S}) < \mathrm{vl}(G)$ for every i: the inductive hypothesis implies that $\mathrm{vl}_i(\mathbf{S}) \leq \mathrm{d}_i(\mathbf{S})$ for every i and the claim follows.

Suppose now that G is soluble and let $\mathbf{S}$ be a series of G with abelian sections. Since G has finite rank, a generic section G_i/G_{i-1} of $\mathbf{S}$ is a finitely generated abelian pro-p group and it is then isomorphic to $\mathbb{Z}_p^r \oplus F$ for some integer r and some finite group F: it is then easy to check that $\mathrm{vl}_i(\mathbf{S}) = \mathrm{d}_i(\mathbf{S}) = r$. Henceforth, $\mathrm{vl}(G) = \dim G$.

Conversely, suppose that $\mathrm{vl}(G) = \dim G$ and let $\mathbf{S}$ be a virtual Ω-composition series, so $\mathrm{vl}_i(\mathbf{S}) \leq 1$ for every i. As every factor in $\mathbf{S}$ is itself a pro-p group of finite rank we have $\mathrm{vl}_i(\mathbf{S}) \leq \mathrm{d}_i(\mathbf{S})$ for every i: Equation (5.1) then implies that $\mathrm{vl}_i(\mathbf{S}) = \mathrm{d}_i(\mathbf{S})$ for every i. If $\mathrm{d}_i(\mathbf{S}) = 0$ then G_i/G_{i-1} is a finite p-group, hence soluble; if $\mathrm{d}_i(\mathbf{S}) = 1$ then G_i/G_{i-1} contains a copy of $\mathbb{Z}_p$ as an open subgroup and it is then soluble. Since every section of $\mathbf{S}$ is soluble, the group G is soluble. □

The rank of a pro-p group can be equivalently defined as the supremum of the number of generators of open subgroups or as the supremum of the number of generators of (closed) subgroups [5, Definition 3.12]. Since every open subgroup in a pro-p group is subnormal, Proposition 5.1 can be rephrased by saying that a pro-p group such that there exists an upper bound for the number of generators of (closed) subnormal subgroup has finite virtual length. On the other hand, Theorem 4.4 states that in a pro-p group of finite virtual length every (closed) subnormal subgroup is finitely generated. This leads naturally to the following problem.

Problem. Let G be a pro-p group (or, more generally, a pronilpotent group). If every (closed) subnormal subgroup of G is finitely generated, is it true that G has finite virtual length?

A pro-p group G is said to have *constant normal subgroup growth* (for short CNSG) if there exists an upper bound for the number of normal subgroups of index n. If G is a pro-p group G with CNSG then there exists a finite normal subgroup N of G such that G/N is just infinite, it has CNSG and it is not a branch group (see [1, Corollaries 23 and 60]). By Remark 3.13, $\mathrm{vl}(G/H)$ is finite and by Proposition 3.7 we have that $\mathrm{vl}(G)$ is finite too. We have then proved the following proposition.

PROPOSITION 5.2. *A pro-p group with CNSG has finite virtual length.*

In [1] several classes of pro-p groups with CNSG are given, thus providing examples of pro-p groups with finite virtual length. In particular, pro-p groups of finite coclass have finite virtual length.

REMARK 5.3. A pro-p groups with finite virtual length need not have CNSG. Take, for instance, two infinite pro-p groups with finite virtual length: their direct product has finite virtual length by Corollary 3.10 but it is not difficult to show that it does not have CNSG.

References

[1] Y. Barnea, N. Gavioli, A. Jaikin-Zapirain, V. Monti and C. M. Scoppola, 'Pro-p groups with few normal subgroups', *J. Algebra* (2) **321** (2009), 429–449.
[2] L. Bartholdi, R. I. Grigorchuk and Z. Šunić, 'Branch groups', in: *Handbook of Algebra,* Vol. 3 (North-Holland, Amsterdam, 2003), 989–1112.
[3] N. Bourbaki, *Algebra I. Ch. 1–3, Elements of Mathematics (Berlin)* (Springer, Berlin, 1998), translated from the French, reprint of the 1989 English translation.
[4] N. Bourbaki, *General Topology. Ch. 1–4, Elements of Mathematics (Berlin)* (Springer, Berlin, 1998), translated from the French, reprint of the 1989 English translation.
[5] J. D. Dixon and M. P. F. du Sautoy, 'Avinoam Mann and Dan Segal', in: *Analytic pro-p Groups*, 2nd edn, Cambridge Studies in Advanced Mathematics, 61 (Cambridge University Press, Cambridge, 1999).
[6] R. I. Grigorchuk, 'Just infinite branch groups', in: *New Horizons in pro-p Groups* (eds. du Sautoy *et al.*) (Birkhäuser, Boston, MA, 2000), 121–179.
[7] C. R. Leedham-Green and S. McKay, *The Structure of Groups of Prime Power Order*, London Mathematical Society Monographs, New Series, 27 (Oxford University Press, Oxford, 2002).
[8] N. Nikolov and D. Segal, 'On finitely generated profinite groups. I. Strong completeness and uniform bounds', *Ann. of Math.* (2) **165**(1) (2007), 171–238.
[9] L. Ribes and P. Zalesskii, *Profinite Groups*, 2nd edn, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge (Results in Mathematics and Related Areas, 3rd Series). A Series of Modern Surveys in Mathematics, 40 (Springer-Verlag, Berlin, 2010).
[10] D. J. S. Robinson, *A Course in the Theory of Groups*, 2nd edn, Graduate Texts in Mathematics, 80 (Springer, New York, 1996).
[11] J. S. Wilson, 'On just infinite abstract and profinite groups', in: *New Horizons in pro-p Groups* (eds. du Sautoy *et al.*) (Birkhäuser, Boston, MA, 2000), 181–203.
[12] J. S. Wilson, 'Large hereditarily just infinite groups', *J. Algebra* (2) **324** (2010), 248–255.

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