The role of interface DoFs in decoupling of substructures based on the dual domain decomposition

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Abstract
The paper considers the decoupling problem, i.e. the identification of the dynamic behaviour of a structural subsystem, starting from the known dynamic behaviour of the coupled system, and from information about the remaining part of the structural system (residual subsystem). Typically, the FRF matrix of the coupled system is assumed to be known at the coupling DoFs (standard interface). To circumvent ill-conditioning around particular frequencies, some authors suggest the use of FRFs at some internal DoFs of the residual subsystem. In this paper, the decoupling problem is revisited in the general framework of Frequency Based Substructuring. Specifically, the dual domain decomposition is used by adding a fictitious subsystem, which is the negative of the residual subsystem, to the coupled system. In this framework, the use of internal DoFs of the residual subsystem, in addition to coupling DoFs, appears quite natural (extended interface). The effects of using an extended interface are widely discussed: the main drawback is that the problem becomes singular at any frequency. However, this singularity is easily removed by using standard smart inversion techniques. The approach is tested on a discrete system describing a two-speed transmission, using simulated data polluted by noise. Results are compared with those obtained from existing approaches.

1 Introduction

The paper considers the decoupling problem, i.e. the identification of the dynamic behaviour of a structural subsystem, starting from information about the remaining part of the structural system (residual subsystem) and from the known dynamic behaviour of the complete system. A trivial application of decoupling is mass cancellation, to get rid of the effect of the accelerometer mass on FRF measurements. Another application is joint identification, which is sometimes approached using specific techniques [1].

The decoupling problem can be seen as the reverse of the substructuring problem or as a structural modification problem with negative modification [2]. Due to modal truncation problems, in experimental dynamic substructuring, the use of FRFs (Frequency Based Substructuring) is preferred with respect to the use of modal parameters. The main algorithm for frequency based substructuring is the improved impedance coupling [3] that involves just one matrix inversion with respect to the classical impedance coupling technique that requires three inversions. A general framework for dynamic substructuring is provided in [4, 5]: an interesting formulation is the so called dual domain decomposition that allows to retain the full set of global DoFs by ensuring equilibrium at the interface between substructures.

Reliable solutions of the decoupling problem could lead to promising developments, both in the field of diagnostics (i.e. monitoring the dynamic behaviour of a critical subsystem which can not be removed or accessed easily) and in the field of vibration control (i.e. identifying the dynamic behaviour of a coupled subsystem that can be changed to affect the dynamic behaviour of the complete system).

In view of such promising applications, several approaches have been proposed in the literature to tackle the decoupling problem: a state space approach [6] including a sensitivity analysis showing possible ill-conditioning due to inertia ratios at the interface; a modal based approach [7] exhibiting modal truncation problems; FRF based approaches [8, 9] showing ill-conditioning troubles. Whatever be the used approach, lack of information on rotational coupling DoFs is always a problem. There are several different reasons leading to ill-conditioning: inertia ratios at interface [6], different stiffnesses at interface [8], internal resonances of the residual subsystem with fixed interface [10, 11]. In [10], two FRF based approaches are considered: an impedance based approach and a mobility based approach. The latter is equivalent to the approach presented in [11].

In this paper, an approach derived through the dual formulation, within the general framework of Frequency Based Substructuring, is developed and discussed. Typically, the FRF matrix of the coupled system is assumed to be known at the coupling DoFs (standard interface). To circumvent ill-conditioning due to internal resonances of the residual subsystems with fixed interface, the use of FRFs at some internal DoFs of the residual subsystem is suggested [10, 11]. By assuming that FRFs are curve-fitted from experimental tests, errors due to measurement noise and to identification can be expected. Information about the residual subsystem can consist either of measured FRFs or of a physical model. Here, the second assumption is considered because it seems unlikely to be able to perform experimental tests on the residual subsystem. The dual domain decomposition is used by adding a fictitious subsystem, which is the negative of the residual subsystem, to the coupled system. In this framework, the use of internal DoFs of the residual subsystem, in addition to coupling DoFs, appears quite natural (extended interface). The effects of using an extended interface are widely discussed: the main drawback is that the problem becomes singular at any frequency. However, this singularity is easily removed by using standard smart inversion techniques.

The paper is organised as follows: after a short reminder on frequency based substructuring, the decoupling problem, based on dual domain decomposition, is presented and possible options for the choice of interface DoFs are analysed. The approach is tested on a discrete system describing a two-speed transmission, using simulated data polluted by noise. Results are compared with those obtained from existing approaches.

2 Reminder on dynamic substructuring in the frequency domain

Let us consider a structural system consisting of n coupled subsystems. In the frequency domain, the equation of motion of a linear time-invariant subsystem r may be written as:

$$\left[Z^{(r)}(\omega)\right]\left\{u^{(r)}(\omega)\right\} = \left\{f^{(r)}(\omega)\right\} + \left\{g^{(r)}(\omega)\right\} \tag{1}$$

where:

$[Z^{(r)}]$ is the dynamic stiffness matrix of subsystem r;
$\{u^{(r)}\}$ is the vector of degrees of freedom of subsystem r;
$\{f^{(r)}\}$ is the external force vector;
$\{g^{(r)}\}$ is the vector of connecting forces with other subsystems (constraint forces associated with compatibility conditions).

For the sake of simplicity, the explicit frequency dependence will be omitted.

The equation of motion of the n subsystems to be coupled can be written in a block diagonal format as:

$$[Z]\{u\} = \{f\} + \{g\} \tag{2}$$

with

$$[Z] = \begin{bmatrix} [Z^{(1)}] & & \\ & \ddots & \\ & & [Z^{(n)}] \end{bmatrix}, \quad \{u\} = \begin{Bmatrix} \{u^{(1)}\} \\ \vdots \\ \{u^{(n)}\} \end{Bmatrix}, \quad \{f\} = \begin{Bmatrix} \{f^{(1)}\} \\ \vdots \\ \{f^{(n)}\} \end{Bmatrix}, \quad \{g\} = \begin{Bmatrix} \{g^{(1)}\} \\ \vdots \\ \{g^{(n)}\} \end{Bmatrix}$$

The compatibility condition at the interface DoFs implies that any pair of matching DoFs $u_l^{(r)}$ and $u_m^{(s)}$, i.e. DoF l on subsystem r and DoF m on subsystem s must have the same displacement, that is $u_l^{(r)} - u_m^{(s)} = 0$. This condition can be generally expressed as:

$$[B]\{u\} = \{0\} \tag{3}$$

where each row of $[B]$ corresponds to a pair of matching DoFs. Note that $[B]$ is, in most cases, a signed Boolean matrix and it can be written by distinguishing the contribution of the different subsystems:

$$[B] = \left[\left[B^{(1)}\right] \quad \cdots \quad \left[B^{(n)}\right] \right]$$

The equilibrium condition for constraint forces associated with the compatibility conditions implies that, when the connecting forces are added for a pair of matching DoFs, their sum must be zero, i.e. $g_l^{(r)} + g_m^{(s)} = 0$: this holds for any pair of matching DoFs. Furthermore, if DoF k on subsystem q is not a connecting DoF, it must be $g_k^{(q)} = 0$: this holds for any non-interface DoF.

Overall, the above conditions can be expressed as:

$$[L]^T \{g\} = \{0\} \tag{4}$$

where the matrix $[L]$ is a Boolean localisation matrix. Note that the number of rows of $[L]^T$ is equal to the number of non-interface DoFs plus the number of pairs of interface DoFs.

Eqs. (2-4) can be put together to obtain the total system describing the coupling between any number of substructures:

$$\begin{cases} [Z]\{u\} = \{f\} + \{g\} \\ [B]\{u\} = \{0\} \\ [L]^T \{g\} = \{0\} \end{cases} \tag{5}$$

2.1 Primal formulation in the frequency domain

In the primal formulation, a unique set of interface DoFs is defined and the interface forces are eliminated by automatically satisfying the interface equilibrium. This is obtained by stating that:

$$\{u\} = [L]\{q\} \tag{6}$$

where $\{q\}$ is the unique set of DoFs, including also non-interface DoFs, and $[L]$ is the localisation matrix introduced previously. Since Eq. (6) states that the DoFs of all subsystems are obtained from the unique set $\{q\}$, the compatibility condition holds for any set $\{q\}$, i.e.

$$[B]\{u\} = [B][L]\{q\} = \{0\} \qquad \forall \{q\} \tag{7}$$

Hence, $[L]$ represents the nullspace of $[B]$ or viceversa:

$$\begin{cases} [L] = \text{null}\left([B]\right) \\ [B]^T = \text{null}\left(\left[L^T\right]\right) \end{cases} \tag{8}$$

Since the compatibility condition in Eq. (5) is satisfied by the choice of the unique set $\{q\}$, the system of equations is:

$$\begin{cases} [Z]\,[L]\,\{q\} = \{f\} + \{g\} \\ [L]^T\,\{g\} = \{0\} \end{cases} \tag{9}$$

Pre-multiplying the dynamic equilibrium equation by $[L]^T$ and noting that $[L]^T\{g\} = \{0\}$, the assembled system reduces to:

$$[L]^T\,[Z]\,[L]\,\{q\} = [L]^T\,\{f\} \tag{10}$$

2.2 Dual formulation in the frequency domain

In the dual formulation, the total set of DoFs is retained, i.e. each interface DoF is present as many times as there are substructures connected through that DoF. The equilibrium condition $g_l^{(r)} + g_m^{(s)} = 0$ at a pair of interface DoFs is ensured by choosing, for instance, $g_l^{(r)} = -\lambda$ and $g_m^{(s)} = \lambda$. Due to the construction of $[B]$, the overall interface equilibrium can be ensured by writing the connecting forces in the form:

$$\{g\} = -\,[B]^T\,\{\lambda\} \tag{11}$$

where $\{\lambda\}$ are Lagrange multipliers corresponding to connecting force intensities.

The interface equilibrium condition (4) is thus written:

$$[L]^T\,\{g\} = -\,[L]^T\,[B]^T\,\{\lambda\} = \{0\} \tag{12}$$

Because $[B]^T$ is the nullspace of $[L]^T$, Eq. (12) is always satisfied and the system of equations (5) becomes:

$$\begin{cases} [Z]\,\{u\} + [B]^T\,\{\lambda\} = \{f\} \\ [B]\,\{u\} = \{0\} \end{cases} \tag{13}$$

In matrix notation:

$$\begin{bmatrix} [Z] & [B]^T \\ [B] & [0] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{\lambda\} \end{Bmatrix} = \begin{Bmatrix} \{f\} \\ \{0\} \end{Bmatrix} \tag{14}$$

By distinguishing the contribution of the different subsystems, the previous equation can be rewritten as:

$$\begin{bmatrix} \left[Z^{(1)}\right] & & & \left[B^{(1)}\right]^T \\ & \ddots & & \vdots \\ & & \left[Z^{(n)}\right] & \left[B^{(n)}\right]^T \\ \left[B^{(1)}\right] & \cdots & \left[B^{(n)}\right] & [0] \end{bmatrix} \begin{Bmatrix} \left\{u^{(1)}\right\} \\ \vdots \\ \left\{u^{(n)}\right\} \\ \{\lambda\} \end{Bmatrix} = \begin{Bmatrix} \left\{f^{(1)}\right\} \\ \vdots \\ \left\{f^{(n)}\right\} \\ \{0\} \end{Bmatrix} \tag{15}$$

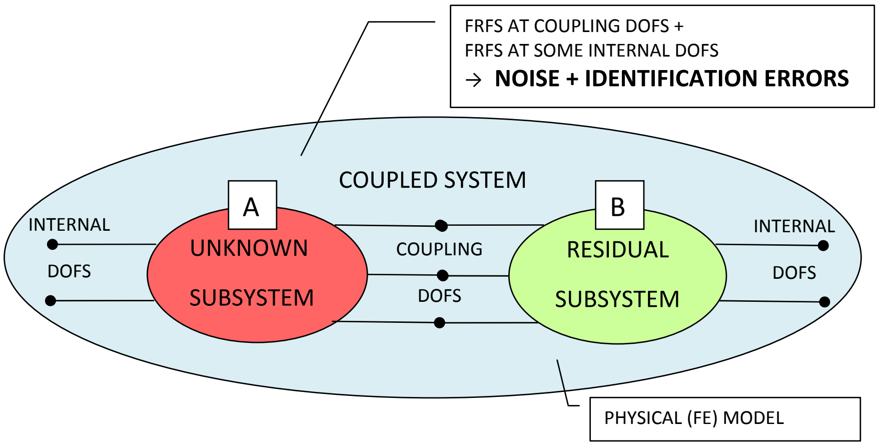

Figure 1: Scheme of the decoupling problem

To eliminate $\{\lambda\}$, from the first of Eq. (13) it can be written:

$$\{u\} = -[Z]^{-1}[B]^T\{\lambda\} + [Z]^{-1}\{f\}$$

which substituted in the second of Eq. (13) gives:

$$[B][Z]^{-1}[B]^T\{\lambda\} = [B][Z]^{-1}\{f\} \quad \Rightarrow \quad \{\lambda\} = \left([B][Z]^{-1}[B]^T\right)^{-1}[B][Z]^{-1}\{f\}$$

Substituting $\{\lambda\}$ in the first of Eq. (13), it is obtained:

$$\begin{aligned} & [Z]\{u\} + [B]^T\left([B][Z]^{-1}[B]^T\right)^{-1}[B][Z]^{-1}\{f\} = \{f\} \\ \Rightarrow \quad & \{u\} = \left([Z]^{-1} - [Z]^{-1}[B]^T\left([B][Z]^{-1}[B]^T\right)^{-1}[B][Z]^{-1}\right)\{f\} \end{aligned}$$

3 The decoupling problem based on dual domain decomposition

The coupled structural system is assumed to be made by an unknown subsystem (A) and a residual subsystem (B) joined through a number of couplings (see fig. 1). The residual subsystem (B) can be made by one or more substructures. The degrees of freedom (DoFs) of the coupled system can be partitioned into internal DoFs (not belonging to the couplings) of subsystem A (a), internal DoFs of subsystem B (b), and coupling DoFs (c).

It is required to find the FRF of the unknown substructure A starting from the FRF of the coupled system AB. The subsystem A can be extracted from the coupled system AB by cancelling the dynamic effect of the residual subsystem B. This can be accomplished by adding to the coupled system AB a fictitious subsystem with a dynamic stiffness opposite to that of the residual subsystem B and satisfying compatibility and equilibrium conditions. According to this point of view, the interface between the coupled system AB and the fictitious subsystem should not only include the coupling DoFs between subsystems A and B, but should as well include the internal DoFs of subsystem B. However, by taking into account that most substructuring techniques consider only coupling DoFs, two options for interface DoFs can be considered:

- standard interface, including only the coupling DoFs (c) between subsystems A and B;
- extended interface, including also some internal DoFs ($i \subseteq b$) of the residual substructure.

In the framework of the dual formulation in the frequency domain [4, 12], the union between the coupled system AB and the fictitious subsystem can be written (see Eq. 15) as:

$$\begin{bmatrix} [Z^{AB}] & [0] & [B^{AB}]^T \\ [0] & -[Z^B] & [B^B]^T \\ [B^{AB}] & [B^B] & [0] \end{bmatrix} \begin{Bmatrix} \{u^{AB}\} \\ \{u^B\} \\ \{\lambda\} \end{Bmatrix} = \begin{Bmatrix} \{f^{AB}\} \\ \{f^B\} \\ \{0\} \end{Bmatrix} \tag{16}$$

Note that $[B^{AB}]$ and $[B^B]$ extract the (standard or extended) interface DoFs among the full set of DoFs.

By eliminating $\{\lambda\}$, it is possible to obtain a relation in the form $\{u\} = [H]\{f\}$, which provides the FRF of the unknown subsystem A:

$$\begin{aligned} \begin{Bmatrix} \{u^{AB}\} \\ \{u^B\} \end{Bmatrix} = &\begin{bmatrix} [Z^{AB}] & [0] \\ [0] & -[Z^B] \end{bmatrix}^{-1} \left(\begin{bmatrix} [I^{AB}] & [0] \\ [0] & [I^B] \end{bmatrix} - \begin{bmatrix} [B^{AB}]^T \\ [B^B]^T \end{bmatrix} \times \right. \\ &\times \left(\begin{bmatrix} [B^{AB}] & [B^B] \end{bmatrix} \begin{bmatrix} [Z^{AB}] & [0] \\ [0] & -[Z^B] \end{bmatrix}^{-1} \begin{bmatrix} [B^{AB}]^T \\ [B^B]^T \end{bmatrix} \right)^{-1} \times \\ &\left. \times \begin{bmatrix} [B^{AB}] & [B^B] \end{bmatrix} \begin{bmatrix} [Z^{AB}] & [0] \\ [0] & -[Z^B] \end{bmatrix}^{-1} \right) \begin{Bmatrix} \{f^{AB}\} \\ \{f^B\} \end{Bmatrix} \end{aligned} \tag{17}$$

i.e., by introducing the FRFs $[H^{AB}]$ and $[H^B]$ at the full set of DoFs instead of $[Z^{AB}]^{-1}$ and $[Z^B]^{-1}$:

$$\begin{aligned} [H^A] = &\begin{bmatrix} [H^{AB}] & [0] \\ [0] & -[H^B] \end{bmatrix} - \begin{bmatrix} [H^{AB}] & [0] \\ [0] & -[H^B] \end{bmatrix} \begin{bmatrix} [B^{AB}]^T \\ [B^B]^T \end{bmatrix} \times \\ &\times \left(\begin{bmatrix} [B^{AB}] & [B^B] \end{bmatrix} \begin{bmatrix} [H^{AB}] & [0] \\ [0] & -[H^B] \end{bmatrix} \begin{bmatrix} [B^{AB}]^T \\ [B^B]^T \end{bmatrix} \right)^{-1} \begin{bmatrix} [B^{AB}] & [B^B] \end{bmatrix} \begin{bmatrix} [H^{AB}] & [0] \\ [0] & -[H^B] \end{bmatrix} \end{aligned} \tag{18}$$

With the dual formulation, when using an extended interface, $[H^A]$ contains some meaningless rows and columns: those corresponding to the internal DoFs of the residual substructure B. Furthermore, the rows and columns corresponding to the coupling DoFs appear twice. Obviously, only meaningful and independent entries are retained.

In Eq. (18), the product of the three matrices to be inverted becomes:

$$\begin{aligned} &\left(\begin{bmatrix} [B^{AB}] & [B^B] \end{bmatrix} \begin{bmatrix} [H^{AB}] & [0] \\ [0] & -[H^B] \end{bmatrix} \begin{bmatrix} [B^{AB}]^T \\ [B^B]^T \end{bmatrix} \right)^{-1} = \\ &= \left([B^{AB}] \, [H^{AB}] \, [B^{AB}]^T - [B^B] \, [H^B] \, [B^B]^T \right)^{-1} \end{aligned} \tag{19}$$

It can be noticed that

$$[B^{AB}] \, [H^{AB}] \, [B^{AB}]^T = \left[\hat{H}^{AB} \right]$$

where $[\hat{H}^{AB}]$ represents the FRF of the coupled structure at the interface DoFs. Similarly,

$$\left[B^B\right]\left[H^B\right]\left[B^B\right]^T = \left[\hat{H}^B\right]$$

where $[\hat{H}^B]$ represents the FRF of the residual structure at the interface DoFs.

Therefore Eq. (19) becomes:

$$\left(\left[B^{AB}\right]\left[H^{AB}\right]\left[B^{AB}\right]^T - \left[B^B\right]\left[H^B\right]\left[B^B\right]^T\right)^{-1} = \left(\left[\hat{H}^{AB}\right] - \left[\hat{H}^B\right]\right)^{-1} \tag{20}$$

Note that $[\hat{H}^{AB}]$ and $[\hat{H}^B]$ can be seen as the inverse of the condensed dynamic stiffness matrices of the coupled structure $[\hat{Z}^{AB}]$ and the residual structure $[\hat{Z}^B]$, respectively.

In the following, the influence of the choice of interface DoFs on singularity and ill-conditioning of $([\hat{H}^{AB}] - [\hat{H}^B])$ will be analysed.

3.1 Possible choices of interface DoFs

An expression of the condensed dynamic stiffness matrix $[\hat{Z}^{(r)}]$ of a substructure (r) is shown in Appendix A. Substructure (r) can be identified either with the coupled system AB or with the residual subsystem B. Master (interface) DoFs M and slave (unmeasured) DoFs S are defined as follows.

a) When a standard interface is considered:

- $M \equiv c$, that is interface DoFs are just the coupling DoFs;
- for the residual subsystem B, $S \equiv b$, that is the slave DoFs are all its internal DoFs;
- for the coupled system AB, $S \equiv a \cup b$, that is the slave DoFs are all its internal DoFs.

b) When an extended interface is considered, also some internal DoFs i of the residual substructure are included among master DoFs:

- $M \equiv c \cup i$, as stated previously;
- for the residual subsystem B, $S \equiv b \backslash i \equiv u$, that is the slave DoFs are given by the set difference between its internal DoFs and the internal DoFs included in the interface;
- for the coupled system AB, $S \equiv a \cup u$.

b1) A special case occurs when $i \equiv b$. In this case, u is an empty set.

The dynamic stiffness matrices of the residual subsystem B and of the coupled system AB can be partitioned as follows:

$$\left[Z^B\right] = \begin{bmatrix} \left[Z^B\right]_{MM} & \left[Z^B\right]_{MS} \\ \left[Z^B\right]_{SM} & \left[Z^B\right]_{SS} \end{bmatrix} = \begin{bmatrix} \left[Z^B\right]_{cc} & \left[Z^B\right]_{ci} & \left[Z^B\right]_{cu} \\ \left[Z^B\right]_{ic} & \left[Z^B\right]_{ii} & \left[Z^B\right]_{iu} \\ \left[Z^B\right]_{uc} & \left[Z^B\right]_{ui} & \left[Z^B\right]_{uu} \end{bmatrix} \tag{21}$$

$$\left[Z^{AB}\right] = \begin{bmatrix} \left[Z^{AB}\right]_{MM} & \left[Z^{AB}\right]_{MS} \\ \left[Z^{AB}\right]_{SM} & \left[Z^{AB}\right]_{SS} \end{bmatrix} = \begin{bmatrix} \left[Z^A\right]_{cc} + \left[Z^B\right]_{cc} & \left[Z^B\right]_{ci} & \left[Z^B\right]_{cu} & \left[Z^A\right]_{ca} \\ \left[Z^B\right]_{ic} & \left[Z^B\right]_{ii} & \left[Z^B\right]_{iu} & [0]_{ia} \\ \left[Z^B\right]_{uc} & \left[Z^B\right]_{ui} & \left[Z^B\right]_{uu} & [0]_{ua} \\ \left[Z^A\right]_{ac} & [0]_{ai} & [0]_{au} & \left[Z^A\right]_{aa} \end{bmatrix} \tag{22}$$

where the zero blocks arise from the fact that the unknown subsystem A is joined to the residual subsystem B only through the coupling DoFs c. Note that, when a standard interface is considered, i is an empty set, and $u \equiv b$.

According to Eq. (35), the condensed dynamic stiffness matrix of the residual subsystem B is:

$$\begin{aligned}\left[\hat{Z}^B\right] &= \begin{bmatrix}[Z^B]_{cc} & [Z^B]_{ci}\\ [Z^B]_{ic} & [Z^B]_{ii}\end{bmatrix} - \begin{bmatrix}[Z^B]_{cu}\\ [Z^B]_{iu}\end{bmatrix}[Z^B]_{uu}^{-1}\begin{bmatrix}[Z^B]_{uc} & [Z^B]_{ui}\end{bmatrix} = \\ &= \begin{bmatrix}[Z^B]_{cc} & [Z^B]_{ci}\\ [Z^B]_{ic} & [Z^B]_{ii}\end{bmatrix} - \begin{bmatrix}[Z^B]_{cu}[Z^B]_{uu}^{-1}[Z^B]_{uc} & [Z^B]_{cu}[Z^B]_{uu}^{-1}[Z^B]_{ui}\\ [Z^B]_{iu}[Z^B]_{uu}^{-1}[Z^B]_{uc} & [Z^B]_{iu}[Z^B]_{uu}^{-1}[Z^B]_{ui}\end{bmatrix}\end{aligned} \tag{23}$$

and the condensed dynamic stiffness matrix of the coupled subsystem AB is:

$$\begin{aligned}\left[\hat{Z}^{AB}\right] &= \begin{bmatrix}[Z^A]_{cc}+[Z^B]_{cc} & [Z^B]_{ci}\\ [Z^B]_{ic} & [Z^B]_{ii}\end{bmatrix} - \begin{bmatrix}[Z^B]_{cu} & [Z^A]_{ca}\\ [Z^B]_{iu} & [0]_{ia}\end{bmatrix}\begin{bmatrix}[Z^B]_{uu} & [0]_{ua}\\ [0]_{au} & [Z^A]_{aa}\end{bmatrix}^{-1}\begin{bmatrix}[Z^B]_{uc} & [Z^B]_{ui}\\ [Z^A]_{ac} & [0]_{ai}\end{bmatrix}\\ &= \begin{bmatrix}[Z^A]_{cc}+[Z^B]_{cc} & [Z^B]_{ci}\\ [Z^B]_{ic} & [Z^B]_{ii}\end{bmatrix} - \\ &- \begin{bmatrix}[Z^B]_{cu}[Z^B]_{uu}^{-1}[Z^B]_{uc}+[Z^A]_{ca}[Z^A]_{aa}^{-1}[Z^A]_{ac} & [Z^B]_{cu}[Z^B]_{uu}^{-1}[Z^B]_{ui}\\ [Z^B]_{iu}[Z^B]_{uu}^{-1}[Z^B]_{uc} & [Z^B]_{iu}[Z^B]_{uu}^{-1}[Z^B]_{ui}\end{bmatrix}\end{aligned} \tag{24}$$

It can be noticed that $\left[\hat{Z}^B\right]$ and $\left[\hat{Z}^{AB}\right]$ differ only in the upper left cc block, i.e. that relative to the coupling DoFs, and they can be conveniently written as:

$$\left[\hat{Z}^{AB}\right] = \begin{bmatrix}\left[\hat{Z}^{AB}\right]_{cc} & \left[\hat{Z}^{B}\right]_{ci}\\ \left[\hat{Z}^{B}\right]_{ic} & \left[\hat{Z}^{B}\right]_{ii}\end{bmatrix} \qquad \left[\hat{Z}^{B}\right] = \begin{bmatrix}\left[\hat{Z}^{B}\right]_{cc} & \left[\hat{Z}^{B}\right]_{ci}\\ \left[\hat{Z}^{B}\right]_{ic} & \left[\hat{Z}^{B}\right]_{ii}\end{bmatrix} \tag{25}$$

Note that, when a standard interface is considered, only the upper left cc block exists because i is an empty set.

3.1.1 Singularity due to unmeasured internal DoFs

This kind of singularity occurs, at some discrete frequencies, in all cases except for case b1 (extended interface including all internal DoFs of the residual subsystem, $i \equiv b$)

By looking at Eq. (23), it can be noticed that $[Z^B]_{uu}$ must be inverted. $[Z^B]_{uu}$ is the dynamic stiffness matrix of the residual subsystem B with master (interface) DoFs grounded, and it is singular at its own resonant frequencies. Therefore, det($[\hat{Z}^B]$) tends to infinity at the resonant frequencies of $[Z^B]_{uu}$: at those frequencies, det($[\hat{H}^B]$) tends to zero and $[\hat{H}^B]$ is singular . Similarly, det($[\hat{Z}^{AB}]$) tends to infinity at the resonant frequencies of the coupled structure AB with master (interface) DoFs grounded: at those frequencies, $[\hat{H}^{AB}]$ is singular. By looking at Eq. (24), where the matrix to be inverted is a block diagonal matrix including $[Z^B]_{uu}$ and $[Z^A]_{aa}$, it can be noticed that the resonant frequencies of the residual substructure B, with interface DoFs grounded $[Z^B]_{uu}$, are a subset of the resonant frequencies of the coupled structure AB with interface DoFs grounded. This is also apparent from Fig. 2, showing that the unknown subsystem A and the residual subsystem B are independent one of another when interface DoFs are grounded.

It can be shown that also $\left([\hat{H}^{AB}] - [\hat{H}^B]\right)$ is singular at the resonant frequencies of the residual substructure B with coupling DoFs grounded. In fact:

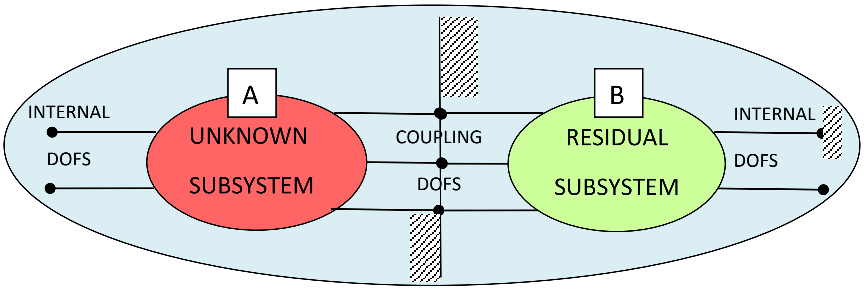

Figure 2: Structure with interface DoFs grounded

- both det $\left([\hat{H}^{AB}]\right)$ and det $\left([\hat{H}^{B}]\right)$ tend to zero at those frequencies;
- the two matrices $[\hat{H}^{AB}]$ and $[\hat{H}^{B}]$ share the same one-dimensional nullspace (see proof in Appendix B), i.e.

$$\left[\hat{H}^{AB}\right]\left\{\hat{f}^{AB}\right\}^{*}=0 \qquad \text{and} \qquad \left[\hat{H}^{B}\right]\left\{\hat{f}^{B}\right\}^{*}=0 \qquad \text{with} \qquad \left\{\hat{f}^{AB}\right\}^{*}=\alpha\left\{\hat{f}^{B}\right\}^{*}$$

where α is any scalar constant.

Therefore:

$$\left[\hat{H}^{AB}\right]\left\{\hat{f}^{AB}\right\}^{*}+\left[\hat{H}^{B}\right]\left\{\hat{f}^{B}\right\}^{*}=\left(\alpha\left[\hat{H}^{AB}\right]+\left[\hat{H}^{B}\right]\right)\left\{\hat{f}^{B}\right\}^{*}=0$$

from which, by taking $\alpha=-1$, it follows that $([\hat{H}^{AB}]-[\hat{H}^{B}])$ is singular at the resonant frequencies of the residual substructure B with coupling DoFs grounded.

If data are known without errors or noise, the use of smart inversion techniques, e.g. the truncated SVD, can remove the singularity. However, if noise is present as usual, the problem becomes ill conditioned.

3.1.2 Singularity due to extended interface

This kind of singularity occurs at all frequencies whenever an extended interface is considered. It can be shown that, when using an extended interface, the matrix $([\hat{H}^{AB}]-[\hat{H}^{B}])$ is always singular. In fact:

$$\begin{aligned}\left[\hat{H}^{AB}\right]-\left[\hat{H}^{B}\right]&=\left[\hat{Z}^{AB}\right]^{-1}\left[\hat{Z}^{AB}\right]\left(\left[\hat{H}^{AB}\right]-\left[\hat{H}^{B}\right]\right)\left[\hat{Z}^{B}\right]\left[\hat{Z}^{B}\right]^{-1}=\\&=\left[\hat{Z}^{AB}\right]^{-1}\left(\left[\hat{Z}^{B}\right]-\left[\hat{Z}^{AB}\right]\right)\left[\hat{Z}^{B}\right]^{-1}=\left[\hat{H}^{AB}\right]\left(\left[\hat{Z}^{B}\right]-\left[\hat{Z}^{AB}\right]\right)\left[\hat{H}^{B}\right]\end{aligned} \tag{26}$$

and because of Eq. (25), $([\hat{Z}^{B}]-[\hat{Z}^{AB}])$ is singular. Therefore, since the determinant of a matrix product equals the product of the determinants, the matrix $([\hat{H}^{AB}]-[\hat{H}^{B}])$ is singular too, except from some limit cases which will be discussed afterwards. Note that the matrix is truly singular only when all data are known without errors or noise. In any case, the use of smart inversion techniques, e.g. the truncated SVD, allows to deal with the problem.

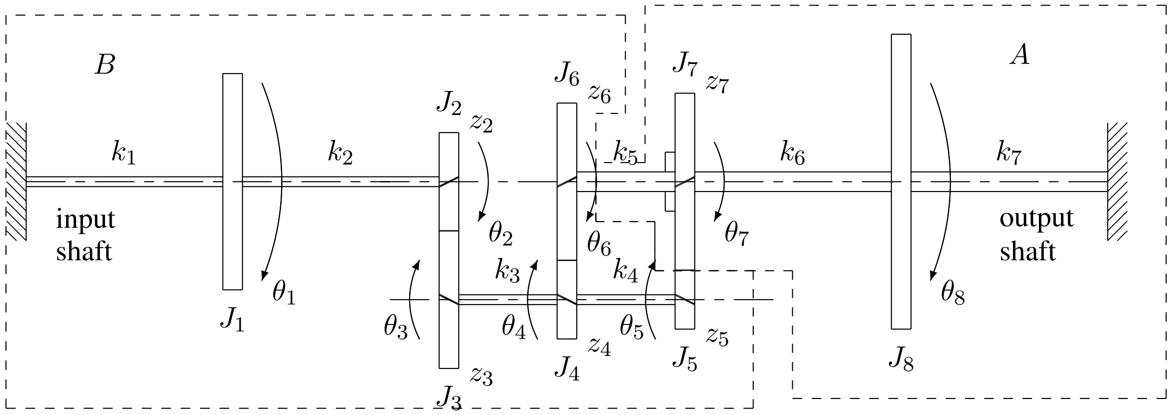

Figure 3: Sketch of the test system: a two speed transmission in first gear

4 Application

A relatively simple application is considered on a torsional system that represents a model of a two-speed transmission. The complete system consists of three shafts: an input shaft, a layshaft and an output shaft (see Fig. 3). The layshaft is coupled both to the input shaft and to the output shaft by helical gears. Power flows through the gear that is locked to the output shaft by the shift collar (e.g. through gears 5 and 7 in Fig. 3). Input and output shafts are assumed to be fixed at the outer ends. Such a boundary condition is a good approximation whenever the mass moments of inertia upstream and downstream the transmission are very large compared to those within the transmission.

Rotational inertias and torsional stiffnesses, as well as torsional dampings, are shown in Table 1 together with the number of teeth z of each gear.

Table 1: Physical parameter values and number of teeth

Item	J [kg m^2]	c [Nm s/rad]	k [Nm/rad]	z
1	$3 \cdot 10^{-3}$	$8.25 \cdot 10^{-4}$	82.5	–
2	$1.98 \cdot 10^{-4}$	$8.25 \cdot 10^{-4}$	82.5	30
3	$7.62 \cdot 10^{-4}$	$1.65 \cdot 10^{-3}$	165	42
4	$8.13 \cdot 10^{-5}$	$1.65 \cdot 10^{-3}$	165	24
5	$2.57 \cdot 10^{-5}$	$6.875 \cdot 10^{-3}$	687.5	18
6	$1.3 \cdot 10^{-3}$	$3.435 \cdot 10^{-3}$	343.5	48
7	$2.08 \cdot 10^{-3}$	$3.435 \cdot 10^{-3}$	343.5	54
8	$1 \cdot 10^{-2}$	–	–	–

The rotations of meshing gear pairs are related by their gear ratio, e.g. the rotations θ_2, θ_6 and θ_7 are related to rotations θ_3, θ_4 and θ_5, respectively. The whole set of rotations $\theta_1 \cdots \theta_8$ can be expressed through a reduced set of five independent rotations, as shown by Eq. (27):

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \\ \theta_7 \\ \theta_8 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & z_3/z_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & z_4/z_6 & 0 & 0 \\ 0 & 0 & 0 & z_5/z_7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_8 \end{Bmatrix} \quad \Leftrightarrow \quad \{u\} = [L]\,\{q\} \tag{27}$$

Therefore, the coupled system can be modelled as a 5 DoFs lumped parameter system. (This can be seen as an application of the primal formulation discussed in Section 2.1).

As outlined in Fig. 3, the unknown subsystem A is a 2-DoFs system made by the output shaft to which gear 7 and flywheel 8 are locked. Note that gear 6 is not locked to the output shaft so that it can be considered as belonging to the residual subsystem B.

For the residual subsystem B, rotations $\theta_1^B \cdots \theta_6^B$ can be expressed through a reduced set of four independent rotations, as shown by Eq. (28):

$$\begin{Bmatrix} \theta_1^B \\ \theta_2^B \\ \theta_3^B \\ \theta_4^B \\ \theta_5^B \\ \theta_6^B \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & z_3/z_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & z_4/z_6 & 0 \end{bmatrix} \begin{Bmatrix} \theta_1^B \\ \theta_3^B \\ \theta_4^B \\ \theta_5^B \end{Bmatrix} \tag{28}$$

Therefore, subsystem B can be modelled as a 4 DoFs lumped parameter system.

To have an idea of the dynamic behaviour of the torsional system, the natural frequencies of the subsystems A and B, and of the coupled system AB are shown in Table 2.

Table 2: Natural frequencies of the systems [Hz]

	Mode				
System	1	2	3	4	5
A	26.3881	72.2981	–	–	–
B	22.7742	56.3484	120.4927	416.6248	–
A+B	23.3477	37.5007	59.7304	106.7501	184.4634

4.1 Decoupling

It is assumed that the rotational FRFs (mobilities) describing the angular velocity/torque relationship of the coupled system AB, and the mechanical impedance of the residual subsystem B are known. It is desired to determine the rotational mobility of subsystem A. The exact FRFs $\hat{H}_{ij}$ of the coupled system AB and the impedances of subsystem B are computed starting from the physical parameters shown in Table 1. To simulate the effect of noise on the FRFs of the coupled system, a complex random perturbation is added to the true FRFs:

$$H_{ij}(\omega_k) = \hat{H}_{ij}(\omega_k) + m_{ij,k} + i\, n_{ij,k} \tag{29}$$

where $m_{ij,k}$ and $n_{ij,k}$ are independent random variables with gaussian distribution, zero mean and a standard deviation of 0.1 rad/sNm. The effect of such perturbation on the drive point rotational mobility at the coupling DoF is shown in Fig. 4 together with the FRF obtained after curve-fitting.

The rotational mobility of subsystem A can be determined by using the procedure described in section 3. The compatibility condition is written generally as:

$$\left[B^{AB}\right]\left\{u^{AB}\right\} + \left[B^{B}\right]\left\{u^{B}\right\} = \{0\}$$

where

$$\left\{u^{AB}\right\}^T = \begin{bmatrix} \theta_1 & \theta_3 & \theta_4 & \theta_5 & \theta_8 \end{bmatrix} \qquad \left\{u^{B}\right\}^T = \begin{bmatrix} \theta_1^B & \theta_3^B & \theta_4^B & \theta_5^B \end{bmatrix}$$

and $[B^{AB}]$ and $[B^B]$ built as detailed afterwards. In fact, the rotational mobility can be assumed to be known:

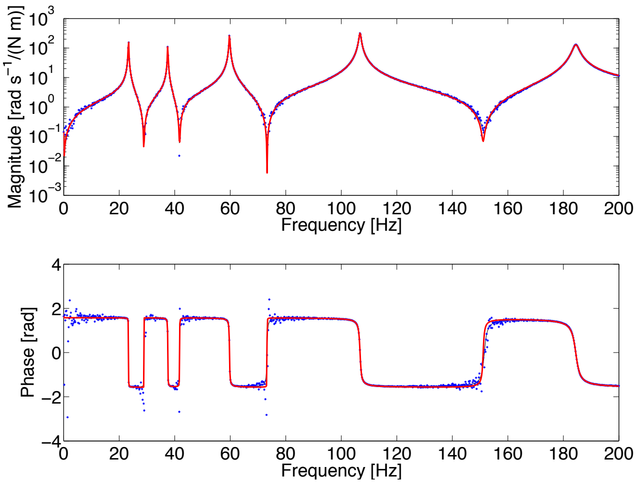

Figure 4: Drive point rotational mobility of the complete system at the coupling DoF: true (—), perturbed by noise (∗∗∗) and fitted (—).

- only at the coupling DoFs (standard interface). In this case, it is:

$$
[B] = \begin{array}{c} \begin{array}{ccccc|cccc} \theta_1 & \theta_3 & \theta_4 & \theta_5 & \theta_8 & \theta_1^B & \theta_3^B & \theta_4^B & \theta_5^B \end{array} \\ \left[\begin{array}{ccccc|cccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \end{array}\right] \\ \qquad [B^{AB}] \qquad\qquad\qquad [B^B] \end{array}
$$

- also at some internal DoFs of the residual subsystem B (extended interface). By adding only one internal DoF, it is:
  - using θ_1 as additional internal DoF:

$$
[B] = \begin{array}{c} \begin{array}{ccccc|cccc} \theta_1 & \theta_3 & \theta_4 & \theta_5 & \theta_8 & \theta_1^B & \theta_3^B & \theta_4^B & \theta_5^B \end{array} \\ \left[\begin{array}{ccccc|cccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{array}\right] \\ \qquad [B^{AB}] \qquad\qquad\qquad [B^B] \end{array}
$$

  - using θ_3 as additional internal DoF:

$$
[B] = \begin{array}{c} \begin{array}{ccccc|cccc} \theta_1 & \theta_3 & \theta_4 & \theta_5 & \theta_8 & \theta_1^B & \theta_3^B & \theta_4^B & \theta_5^B \end{array} \\ \left[\begin{array}{ccccc|cccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right] \\ \qquad [B^{AB}] \qquad\qquad\qquad [B^B] \end{array}
$$

  - using θ_4 as additional internal DoF:

$$
[B] = \begin{array}{c} \begin{array}{ccccc|cccc} \theta_1 & \theta_3 & \theta_4 & \theta_5 & \theta_8 & \theta_1^B & \theta_3^B & \theta_4^B & \theta_5^B \end{array} \\ \left[\begin{array}{ccccc|cccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right] \\ \qquad [B^{AB}] \qquad\qquad\qquad [B^B] \end{array}
$$

In the sequel, only FRFs perturbed by simulated noise will be considered. In fact, if noise-free FRFs of the coupled system are used, the FRF of the unknown subsystem is always predicted without errors (although the problem may be singular for several reasons, as stated in sections 3.1.1 and 3.1.2, the use of smart inversion techniques removes the singularity). Furthermore, perturbed FRFs are not used in raw form but are regenerated after a curve fitting procedure.

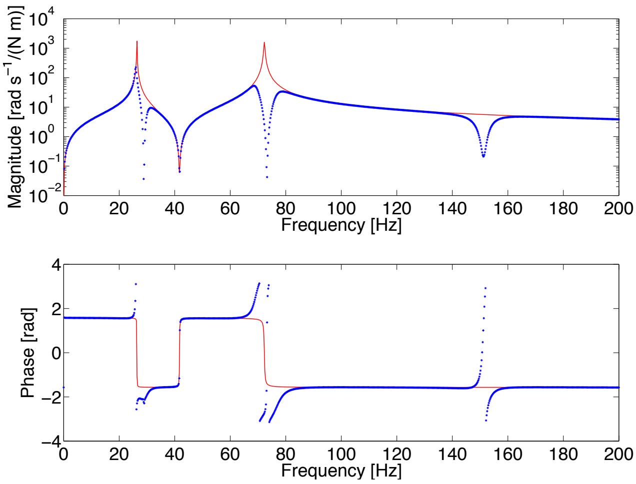

Figure 5: Rotational mobility at the coupling DoF of subsystem A: true (—), computed from fitted perturbed FRF (∗∗∗) without additional internal DoFs.

First of all, the case of standard interface (rotational mobility only at the coupling DoF θ_5) is considered. In this case, the present approach is equivalent to the impedance based and the mobility based approaches presented in previous papers [10]. In Fig. 5, the true drive point rotational mobility at the coupling DoF of subsystem A is compared with the corresponding FRF computed through Eq. (17), starting from the fitted perturbed FRF of the coupled system. Results obtained starting from raw perturbed FRFs, not shown in the paper, are always much worse than those obtained starting from fitted perturbed FRFs.

It can be noticed that the predicted rotational mobility of subsystem A is badly identified at frequencies around 30, 70 and 150 Hz. This depends on ill conditioning due to unmeasured internal DoFs, as explained in section 3.1.1. In fact, the coupled system AB and the residual subsystem B, with the "measured" coupling DoF θ_3 grounded, share three resonance frequencies, namely $f_{n1} = 28.94$ Hz, $f_{n2} = 72.23$ Hz and $f_{n3} = 151.5$ Hz. Around these frequencies, $[\hat{H}^{AB}] - [\hat{H}^{B}]$ is ill-conditioned and noise is greatly amplified.

A way to circumvent this problem is to use an extended interface, i.e. to assume that the FRF matrix of the coupled system is known not only at the coupling DoF but also at some of the three internal DoFs (θ_1, θ_3, θ_4) of the residual subsystem B. The resonant frequencies of the residual subsystem B, with the measured DoFs grounded (the coupling DoF and one additional internal DoF), are shown in Table 3 versus the added internal DoF.

Table 3: Resonant frequencies of the residual subsystem B with extended interface DoFs grounded (the coupling DoF θ_5 and one additional internal DoF)

Added DoF	$f_{n1}[Hz]$	$f_{n2}[Hz]$
θ_1	69.4	151.5
θ_3	37.3	143.4
θ_4	31.5	87.15

Figures 6, 7 and 8 show the true drive point rotational mobility at the coupling DoF of subsystem A, together with the corresponding FRFs predicted using either the mobility based approach, [10], or the approach developed in this paper, Eq. (17), starting from the fitted perturbed FRFs of the complete system and using θ_1, θ_3, or θ_4 as additional internal DoF, respectively.

From Fig. 6, it can be noticed that the predicted rotational mobility of subsystem A is badly identified at frequencies close to those shown in Table 3, around which the inversion of an ill conditioned matrix is performed. This is also true for Fig. 8, although only in the phase plot.

The best results are those in Fig. 7, when the additional internal DoF is θ_3, because in this case the ill-

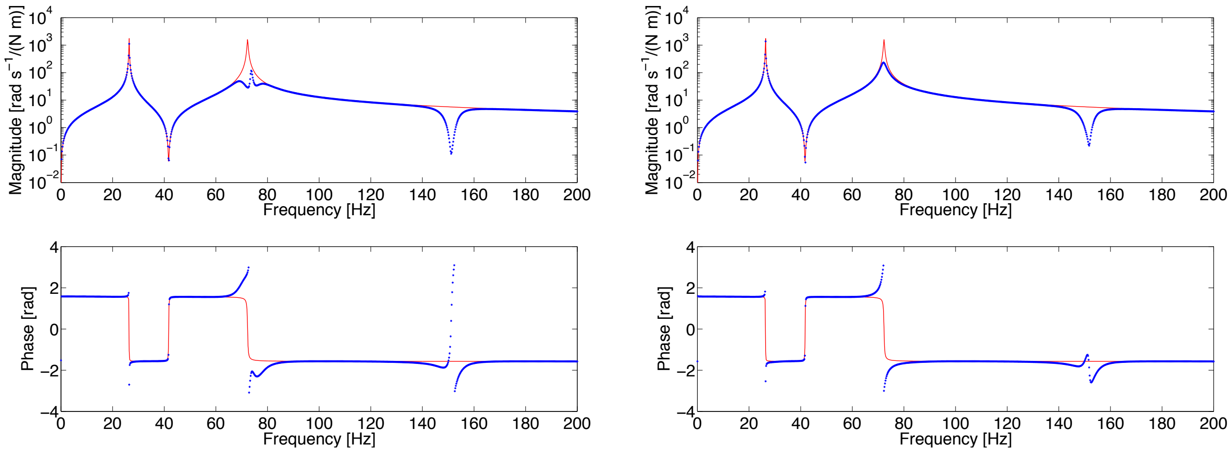

Figure 6: Rotational mobility at the coupling DoF of subsystem A: true (—), predicted using mobility based approach (left) and dual formulation (right) from fitted perturbed FRF (∗∗∗), with θ_1 as additional internal DoF.

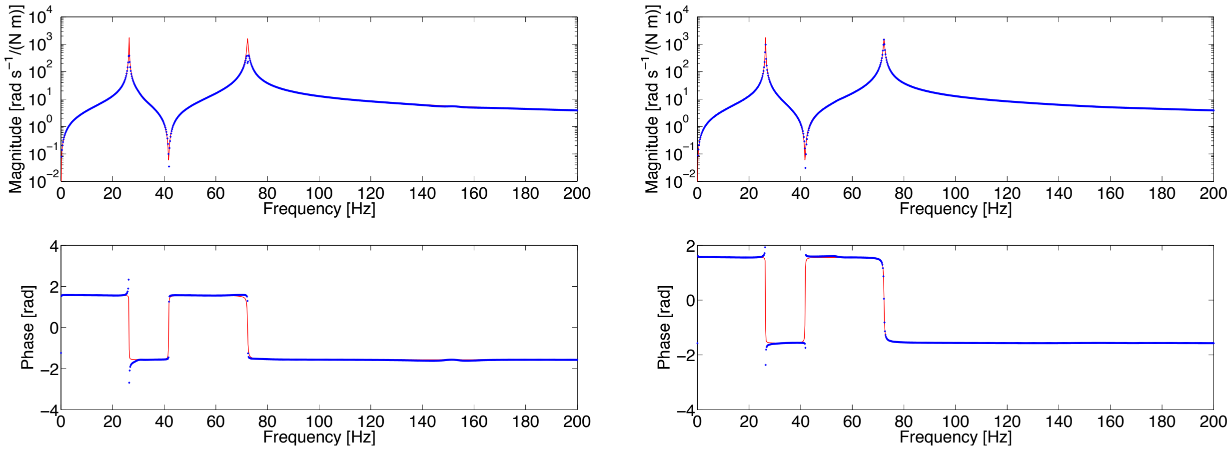

Figure 7: Rotational mobility at the coupling DoF of subsystem A: true (—), predicted using mobility based approach (left) and dual formulation (right) from fitted perturbed FRF (∗∗∗), with θ_3 as additional internal DoF.

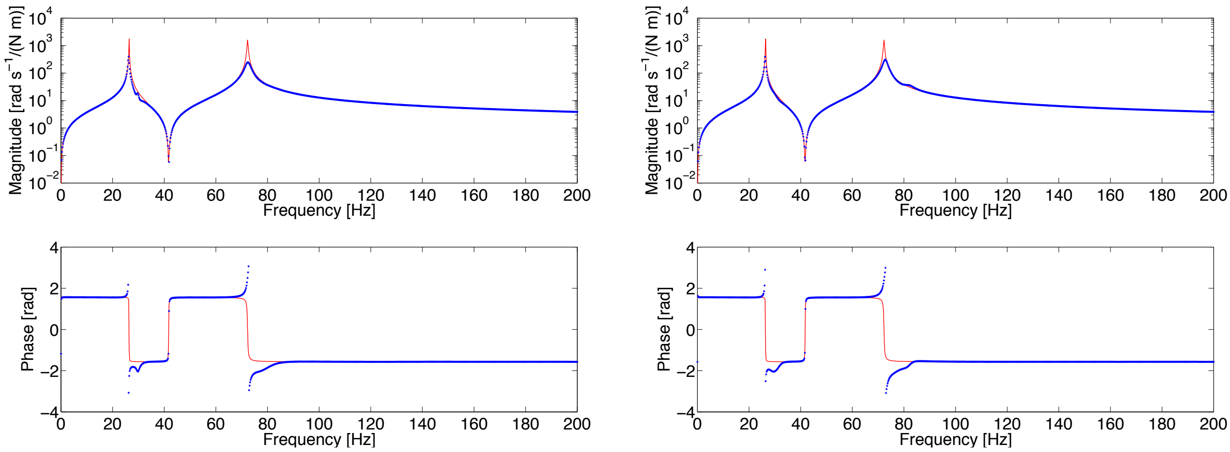

Figure 8: Rotational mobility at the coupling DoF of subsystem A: true (—), predicted using mobility based approach (left) and dual formulation (right) from fitted perturbed FRF (∗∗∗), with θ_4 as additional internal DoF.

conditioned frequencies are not close to the natural frequencies of the unknown subsystem A. However, it would not have been possible to decide a priori about θ_3 as the best additional DoF, because natural frequencies of the unknown subsystem A are not available in advance.

By looking at Table 3, which represents the only piece of information available in advance, it could be concluded that θ_1 is a good choice for the additional internal DoF, because with this choice the ill-conditioned frequencies are higher than in other cases. Unfortunately, this criterion is denied by the observed results. In practice, it is advisable to test several alternatives for the additional internal DoFs to get acceptable results.

Finally, it can be noticed that, in all cases, the approach presented in this paper provides better results than the mobility based approach.

5 Conclusion

In this paper, the role of interface DoFs in decoupling of substructures based on the dual domain decomposition is discussed.

Two options for interface DoFs are considered:

- standard interface, including only the coupling DoFs between the unknown and the residual subsystems. The problem is ill-conditioned in the neighbourhood of the natural frequencies of the residual subsystem with coupling DoFs grounded;
- extended interface, including also some internal DoFs of the residual substructure. In this case, other questions arise:
  - assuming that all data are known without errors or noise, the problem becomes singular at all frequencies, although the use of smart inversion techniques, e.g. the truncated SVD, provides good results;
  - the presence of noise in the FRF data of the coupled system removes the singularity but affects the results;

  Therefore, the use of an extended interface could be limited to the neighbourhood of the singular frequencies related to the standard interface.

Results are compared with those obtained through the best performer among the previously developed approaches i.e. the mobility based approach. In all the considered cases, the present approach provides better results than the mobility based approach.

Acknowledgements

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A The dynamic condensation

The dynamic behaviour of a substructure (r) is given by:

$$\left[Z^{(r)}\right]\left\{u^{(r)}\right\}=\left\{f^{(r)}\right\} \tag{30}$$

where $[Z^{(r)}]$ is the $(N \times N)$ dynamic stiffness matrix, $\{u^{(r)}\}$ is the $(N \times 1)$ full set of displacements and $\{f^{(r)}\}$ is the $(N \times 1)$ set of applied forces.

The degrees of freedom can be partitioned into retained (master) DoFs $\{\hat{u}^{(r)}\}$ $(N_m \times 1)$ and discarded (slave) DoFs $\{\tilde{u}^{(r)}\}$, where

$$\left\{\hat{u}^{(r)}\right\}=\left[P_M^{(r)}\right]\left\{u^{(r)}\right\} \tag{31}$$

where $[P_M^{(r)}]$ is a $(N_m \times N)$ Boolean matrix, that extracts the master DoFs from the full set of DoFs, and

$$\left\{\tilde{u}^{(r)}\right\}=\left[P_S^{(r)}\right]\left\{u^{(r)}\right\} \tag{32}$$

where $[P_S^{(r)}]$ is a $(N-N_m) \times N$ Boolean matrix, that extracts the slave DoFs from the full set of DoFs.

By assuming that no external force is applied at the slave DoFs, the equation (30) can be rewritten in partitioned block form:

$$\begin{bmatrix} [Z^{(r)}]_{MM} & [Z^{(r)}]_{MS} \\ [Z^{(r)}]_{SM} & [Z^{(r)}]_{SS} \end{bmatrix} \begin{Bmatrix} \{\hat{u}^{(r)}\} \\ \{\tilde{u}^{(r)}\} \end{Bmatrix} = \begin{Bmatrix} \left\{\hat{f}^{(r)}\right\} \\ \{0\} \end{Bmatrix} \tag{33}$$

By eliminating the slave DoFs $\{\tilde{u}^{(r)}\}$, it is obtained:

$$\left(\left[Z^{(r)}\right]_{MM} - \left[Z^{(r)}\right]_{MS} (\left[Z^{(r)}\right]_{SS})^{-1} \left[Z^{(r)}\right]_{SM}\right) \left\{\hat{u}^{(r)}\right\} = \left\{\hat{f}^{(r)}\right\} \tag{34}$$

from which the condensed dynamic stiffness matrix $\left[\hat{Z}^{(r)}\right]$ can be identified as:

$$\left[\hat{Z}^{(r)}\right] = \left[Z^{(r)}\right]_{MM} - \left[Z^{(r)}\right]_{MS} (\left[Z^{(r)}\right]_{SS})^{-1} \left[Z^{(r)}\right]_{SM} \tag{35}$$

The submatrices $[Z^{(r)}]_{MM}$, $[Z^{(r)}]_{MS}$, $[Z^{(r)}]_{SM}$ and $[Z^{(r)}]_{SS}$ can be easily built by using the Boolean matrices $[P_M^{(r)}]$ and $[P_S^{(r)}]$, as:

$$\begin{aligned} \left[Z^{(r)}\right]_{MM} &= \left[P_M^{(r)}\right]\left[Z^{(r)}\right]\left[P_M^{(r)}\right]^T, \quad \left[Z^{(r)}\right]_{MS} = \left[P_M^{(r)}\right]\left[Z^{(r)}\right]\left[P_S^{(r)}\right]^T \\ \left[Z^{(r)}\right]_{SM} &= \left[P_S^{(r)}\right]\left[Z^{(r)}\right]\left[P_M^{(r)}\right]^T, \quad \left[Z^{(r)}\right]_{SS} = \left[P_S^{(r)}\right]\left[Z^{(r)}\right]\left[P_S^{(r)}\right]^T \end{aligned} \tag{36}$$

Therefore, the condensed dynamic stiffness matrix $[\hat{Z}^{(r)}]$ can be expressed as:

$$\begin{aligned} \left[\hat{Z}^{(r)}\right] = &\left[P_M^{(r)}\right]\left[Z^{(r)}\right]\left[P_M^{(r)}\right]^T - \left[P_M^{(r)}\right]\left[Z^{(r)}\right]\left[P_S^{(r)}\right]^T \times \\ &\times \left(\left[P_S^{(r)}\right]\left[Z^{(r)}\right]\left[P_S^{(r)}\right]^T\right)^{-1}\left[P_S^{(r)}\right]\left[Z^{(r)}\right]\left[P_M^{(r)}\right]^T \end{aligned} \tag{37}$$

The condensed dynamic stiffness matrix must be computed at all the frequencies of interest. It can be noticed that the matrix to be inverted, $[Z^{(r)}]_{SS} = [P_S^{(r)}][Z^{(r)}][P_S^{(r)}]^T$, is the dynamic stiffness matrix of a system where all master DoFs of the substructure (r) are grounded. That is, the matrix to be inverted becomes singular at the resonant frequencies of substructure (r) with master DoFs grounded. At those frequencies, the determinant of the condensed dynamic stiffness matrix $[\hat{Z}^{(r)}]$ tends to infinity.

B Nullspace of $[\hat{H}^{AB}]$ and $[\hat{H}^{B}]$

At each of the resonant frequencies of substructure B with master DoFs grounded, $[Z^B]_{SS} = [Z^B]_{uu}$ is singular (see Section 3.1.1). Then an eigenvector $\left\{\tilde{u}^B\right\}^*$ exists such that:

$$\left[Z^B\right]_{SS}\left\{\tilde{u}^B\right\}^* = 0 \tag{38}$$

For the residual subsystem B, Eq. (33) can be written:

$$\begin{bmatrix} [Z^{B}]_{MM} & [Z^{B}]_{MS} \\ [Z^{B}]_{SM} & [Z^{B}]_{SS} \end{bmatrix} \begin{Bmatrix} \{\hat{u}^{B}\} \\ \{\tilde{u}^{B}\} \end{Bmatrix} = \begin{Bmatrix} \left\{\hat{f}^{B}\right\} \\ \{0\} \end{Bmatrix} \tag{39}$$

Therefore, by substituting $\left\{\tilde{u}^B\right\}^*$ in the second block of the previous equation, it is, in view of Eq. (38):

$$\left[Z^B\right]_{SM}\left\{\hat{u}^B\right\}^* = 0 \qquad \Rightarrow \qquad \left\{\hat{u}^B\right\}^* = 0 \tag{40}$$

On the other side, using the FRF matrix, one can write:

$$\left[\hat{H}^B\right]\left\{\hat{f}^B\right\}^* = \left\{\hat{u}^B\right\}^* = 0 \tag{41}$$

where $\{\hat{f}^B\}^*$ is the vector of forces acting on master DoFs when the displacements on slave DoFs are given by the eigenvectors $\{\tilde{u}^B\}^*$. The vector $\{\hat{f}^B\}^*$ that spans the one-dimensional nullspace of $[\hat{H}^B]$, can be computed from the first block of Eq. (39) as:

$$\left\{\hat{f}^B\right\}^* = \left[Z^B\right]_{MS}\left\{\tilde{u}^B\right\}^* = \begin{bmatrix}\left[Z^B\right]_{cu}\\ \left[Z^B\right]_{iu}\end{bmatrix}\left\{\tilde{u}^B\right\}_u^* \tag{42}$$

Similarly, the vector $\{\hat{f}^{AB}\}^*$ that spans the one-dimensional nullspace of $[\hat{H}^{AB}]$, is:

$$\left\{\hat{f}^{AB}\right\}^* = \left[Z^{AB}\right]_{MS}\left\{\tilde{u}^{AB}\right\}^* = \begin{bmatrix}\left[Z^B\right]_{cu} & \left[Z^A\right]_{ca}\\ \left[Z^B\right]_{iu} & [0]_{ia}\end{bmatrix}\begin{Bmatrix}\left\{\tilde{u}^{AB}\right\}_u^*\\ \left\{\tilde{u}^{AB}\right\}_a^*\end{Bmatrix} \tag{43}$$

Each pair of eigenvectors (of the coupled system AB and of the residual system B), corresponding to the same resonant frequency, spans the same one-dimensional subspace, i.e. for the coupled system AB, the sub-vector corresponding to internal DoFs of subsystem B is proportional to the eigenvector of the residual system B and the components corresponding to internal DoFs of subsystem A are zero. This can be expressed as:

$$\left\{\tilde{u}^{AB}\right\}^* = \begin{Bmatrix}\left\{\tilde{u}^{AB}\right\}_u^*\\ \left\{\tilde{u}^{AB}\right\}_a^*\end{Bmatrix} = \begin{Bmatrix}\alpha\left\{\tilde{u}^B\right\}^*\\ 0\end{Bmatrix} \tag{44}$$

Therefore:

$$\left\{\hat{f}^{AB}\right\}^* = \begin{bmatrix}\left[Z^B\right]_{cu} & \left[Z^A\right]_{ca}\\ \left[Z^B\right]_{iu} & [0]_{ia}\end{bmatrix}\begin{Bmatrix}\alpha\left\{\tilde{u}^B\right\}^*\\ 0\end{Bmatrix} = \begin{bmatrix}\left[Z^B\right]_{cu}\\ \left[Z^B\right]_{iu}\end{bmatrix}\alpha\left\{\tilde{u}^B\right\}^* \quad \Rightarrow \quad \left\{\hat{f}^{AB}\right\}^* = \alpha\left\{\hat{f}^B\right\}^* \tag{45}$$

in view of Eq. (42).