The ring design game with fair cost allocation*

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Abstract

In this paper we study the network design game when the underlying network is a ring. In a network design game we have a set of players, each of them aims at connecting nodes in a network by installing links and equally sharing the cost of the installation with other users. The ring design game is the special case in which the potential links of the network form a ring. It is well known that in a ring design game the price of anarchy may be as large as the number of players. Our aim is to show that, despite the worst case, the ring design game always possesses good equilibria. In particular, we prove that the price of stability of the ring design game is at most 3/2, and such bound is tight. Moreover, we observe that the worst Nash equilibrium cannot cost more than 2 times the optimum if the price of stability is strictly larger than 1. We believe that our results might be useful for the analysis of more involved topologies of graphs, e.g., planar graphs.

1 Introduction

In a network design problem the goal is to construct a network which satisfies certain connectivity requirements. Formally we are given a graph $G = (V, E)$, in which each edge $e \in E$ has an associated non-negative cost $c(e)$, and a set of n pairs of nodes $(s_i, t_i)_{i=1\dots n}$. Each edge of the graph corresponds to a potential link of the resulting network, and its cost denotes the effective cost for establishing the corresponding link. Thus the cost of the designed network is given by the total cost for establishing all its links. The requirement for the resulting network is that it must contain a set of links guaranteeing the connectivity between each pair of nodes (s_i, t_i). In other words, the goal is to find a subgraph of G in which each node s_i is connected to the corresponding node t_i. From a combinatorial optimization perspective, the objective is to find the subgraph satisfying the connectivity requirements with minimum total cost, and such problem is known as the minimum Steiner forest problem. In this paper we will look at the network design problem from a decentralized non-cooperative perspective, considering the scenario in which there is not only one but several minimization objectives. More specifically, each connectivity requirement is associated with a player (agent) who greedily choses the path connecting the two endpoints, aiming at minimizing the cost she incurs for such path. In particular, each player is charged for a portion of the cost of each edge she uses, and in this work we

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assume that the cost of each edge is equally shared among its users. Thus the players interact in a non-cooperative game in which the choice of each player affects the choices of the other players. This model was first considered by Anshelevich *et al.* [2] and it is known as network design game.

A state of the game corresponds to a choice of a path by each player, thus guaranteeing all the desired connectivity requirements. The cost of a state is the cost of the corresponding subgraph, i.e., the cost of the edges who appear in at least one path. A state in which each player minimizes his incurred cost, given the choices of the remaining players, is known as *Nash equilibrium*. In this paper we only consider pure Nash Equilibria. One property which makes the network design game particularly interesting also from a theoretical perspective, is that it always admits a Nash equilibrium and it can be obtained as the result of a natural dynamics. More precisely, it is well known that the dynamics of the game in which at each time step there is exactly one player performing an improvement move, i.e., a change of path that strictly decreases the cost incurred by the moving player, converges to a Nash equilibrium within a finite number of steps. This property has been proven by Rosenthal [14] for a more general class of games, called congestion games. He demonstrated the existence of a potential function which at each time step of the dynamics decreases exactly as much as the cost of the moving player. In other words, the Nash equilibrium corresponds to the state which locally minimizes the potential function.

Unfortunately, in a non-cooperative setting, the Nash equilibrium does not necessarily correspond to the state with minimum cost. In particular, this is true for the network design game where a Nash equilibrium, differently from the optimal state, may also contain cycles. To capture the deterioration of the performance due to the presence of different self-interested players, the concepts of *Price of Anarchy* (PoA) [11] and *Price of Stability* (PoS) [2] have been introduced. They are formally defined respectively as the worst-case ratio (PoA) and the best-case ratio (PoS) between the cost of a Nash equilibrium and the cost of the optimal state. Both the measures give the interval in which the performances of the game may range. It is trivial to observe that the PoA in a network design game may be as large as the number of players n, and such bound is tight. On the counterpart, Chekuri *et al.* [5], showed that the cost of any Nash equilibrium reachable by a dynamics starting from an empty state (the state in which no player is making a choice), is $O(\sqrt{n}\log^2 n)$ times the optimal state. Less is known about the PoS. We first notice that the current results in the literature are different depending whether the underlying graph is directed or undirected. Anshelevich *et al.* [2] considered network design games in directed graphs and proved that the price of stability is at most $H_n = 1 + 1/2 + \ldots + 1/n$, and such bound is tight. The technique used to prove such bound is quite easy and generally applicable to other settings. It consists in showing first a worst case ratio of H_n between the potential of the optimal state and the one at any equilibrium, subsequently the result follows by arguing that there exists a Nash equilibrium reachable from the optimal state and thus having a potential lower than the one at the optimal state. Although the upper bound proof carries over to undirected network design games, the lower bound does not. There have been several attempts to give a significant lower bounds for the undirected case, e.g., [9, 8, 3, 1, 6]. The best known lower bound so far of $348/155 \approx 2.245$, has been shown in [3]. Upper bounds lower than H_n have been given only for the special cases of multicast and broadcast games. In the multicast there is a single node to which every player desires to be connected. The broadcast is a special case of the multicast in which every node in the graph requires to be connected to the single source. For broadcast game, Fiat *et al.* [9] proved an upper bound of $O(\log\log n)$, and Lee *et al.* [13] have improved such bound to $O(\log\log\log n)$. Recently Bilò *et al.* [4] proved that the price of stability for broadcast game is $O(1)$, however the constant coming out of such paper is quite large. For multicast game, Li [12] presented an upper bound of $O(\log n/\log\log n)$.

All these upper bounds are not known to be tight. In fact the best known lower bounds for broadcast and multicast games are $12/7 \approx 1.714$ and $42/23 \approx 1.826$ respectively [3].
Finally [10] considered undirected broadcast games and proved upper and lower bounds on the Potential-Optimal Price of Anarchy (resp. Stability) defined as the ratio between the worst (resp. the best) cost of Nash equilibrium with optimal potential (i.e., equilibrium minimizing the value of the potential function) and the minimum social cost. (NEW)

Despite considerable research effort the gaps on the price of stability remain rather large, especially for multicast game. We remark that our study considers the more general case where a player of the game is represented by any pair of nodes of the network. (NEW)

The aim of the current paper is to analyze the network design game when the underlying graph is a ring. We refer to this special case as *ring design game*. Notice that in such a setting the strategy set of each player i is composed by exactly 2 different strategies, i.e., the clockwise and counterclockwise paths connecting s_i and t_i. The ring design game captures the whole spectra of interesting behavior, i.e., PoA remains equal to the number of players. Moreover, the ring is crucial in the sense that it is the first non-trivial topology to analyze in the context of network design and it is the first step in order to cope with more involved topologies, like planar graphs. Hence, we believe that giving a tight bounds here could give some insight for studying more general settings. (NEW)

Our results. Initially, by looking deeper into the structures of Nash equilibria in a ring design game, we observe that, even though in a ring design game in the worst case the price of anarchy can be arbitrarily large, it suffices to guarantee that the optimum is not a Nash equilibrium to show that every Nash equilibrium is always at most 2 times the cost of the social optimum. Subsequently, we show that there always exists a Nash equilibrium of cost at most 3/2 times the cost of an optimal state, thus giving a bound on the PoS. We show that such equilibrium can be reached by an improvement dynamics having as initial state an optimal configuration. Such result also gives some insight on the problem of computing an equilibrium in a ring design game. In fact, it reveals that if the cost of the entire ring is larger than 3/2 times the cost of an optimal state, then the improvement dynamics starting from an optimal state converges quickly, within at most 4 steps, to an equilibrium. We also prove that the bound on the PoS is tight, by showing an instance for which PoS $= 3/2 - \epsilon$. We conclude by showing a more involved instance depending on two parameter ϵ and α, which generalizes the lower bound on the PoS and gives some insight on the relation between the PoA and PoS in a ring design game. In particular, we observe that by suitable choosing a value of ϵ close to 0, when α approaches to 1, both the PoA and PoS tend to 3/2. On the counterpart, when α approaches to 0, the PoA tends to 2 and the PoS gets close to 1. (NEW)

2 Preliminaries

In an undirected network design game, we are given an undirected graph $G = (V, E)$ and edge costs given by a function $c : E \to \mathbb{R}^+$. The edge cost function naturally extends to any subset of edges, that is $c(B) = \sum_{e \in B} c(e)$ for any $B \subseteq E$. We define $c(\emptyset) = 0$. There is a set of n players $[n] = \{1, \ldots, n\}$; each player $i \in [n]$ wishes to establish a connection between two nodes $s_i, t_i \in V$ called the source and destination node of player i, respectively. The set of strategies available to player i consists of all paths connecting nodes s_i and t_i in G. We call a *state* of the game a set of strategies $\sigma \in \Sigma$ (where Σ is the set of all the states of the game), with one strategy per player, i.e., $\sigma = (\sigma_1, \ldots, \sigma_n)$ where σ_i denotes the strategy of player i in σ . Given a state σ, let $n_\sigma(e)$ be the number of players using edge e in σ. Then, the cost of player i in σ is defined as $c_\sigma(i) = \sum_{e \in \sigma_i} \frac{c(e)}{n_\sigma(e)}$. Let $E(\sigma)$ be the set of edges that are used by at least one player in state

σ. The social cost $C(\sigma)$ is simply the total cost of the edges used in state σ which coincides with the sum of the costs of the players, i.e., $C(\sigma) = \sum_{e \in E(\sigma)} c(e) = \sum_{i \in [n]} c_\sigma(i) = c(E(\sigma))$.

Let (σ_{-i}, σ'_i) denote the state obtained from σ by changing the strategy of player i from σ_i to σ'_i. Given a state $\sigma = (\sigma_1, \ldots, \sigma_n)$, an *improving move* of player i in σ is a strategy σ'_i such that $c_{(\sigma_{-i}, \sigma'_i)}(i) < c_\sigma(i)$. A state of the game is a *Nash equilibrium* if and only if no player can perform any improving move. An *improvement dynamics* (shortly *dynamics*) is a sequence of improving moves. A game is said to be *convergent* if, given any initial state σ, any sequence of improving moves leads to a Nash equilibrium. We denote by NE the set of states that are Nash Equilibria. Nash equilibrium can be different from the socially optimal solution. Let $\textsc{Opt}$ be a state of the game minimizing the social cost. We denote by $\textsc{Opt}_i$ the strategy used by player i in $\textsc{Opt}$. Furthermore we write $\overline{\textsc{Opt}} = E \setminus \textsc{Opt}$ and $\overline{\textsc{Opt}}_i = E \setminus \textsc{Opt}_i$. The *price of anarchy* (PoA) of a network design game is defined as the ratio of the maximum social cost among all Nash equilibria over the optimal cost, i.e., $\text{PoA} = \frac{\max_{\sigma \in \text{NE}} C(\sigma)}{C(\textsc{Opt})}$. The *price of stability* (PoS) is defined as the ratio of the minimum social cost among all Nash equilibria over the optimal cost, i.e., $\text{PoS} = \frac{\min_{\sigma \in \text{NE}} C(\sigma)}{C(\textsc{Opt})}$. (NEW)

In this paper we restrict our attention to the network design game played on rings. We refer to such game as *ring design game*. For the sake of clarity, by a ring we mean an undirected graph $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_k\}$, $E = \{e_1, e_2, \ldots, e_k\}$, and $e_i = v_i v_{i+1}$, $i = 1, \ldots, k$ (where $v_{k+1} = v_1$).

It is well known that in a ring design game the PoA may be as large as the number of players n. Consider the graph in Figure 1 of a ring design game consisting of two nodes s and

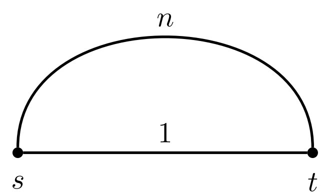

Figure 1: In this example there are n players that want to connect s and t. The optimal solution uses only edge of cost 1, whereas the worst Nash equlibrium uses the edge of cost n. This gives an n lower bound on the PoA.

t with two edges connecting them. The cost of one edge is n, while the cost of the other edge is 1. There are n players who want to connect s to t. The state where all the players use the edge of cost 1 is the optimal state, and it corresponds to a Nash equilibrium as well. The state where all players use the expensive edge, each paying 1, is a Nash equilibrium with a cost of n. Thus this example shows that the PoA of a ring design game may be very large. Notice that it is very unnatural that the players chose the edge of cost n, ending up with the worst equilibrium of cost n. Furthermore we observe that every dynamics starting from any other initial state, beside the worst equilibrium, converges to the best equilibrium. In particular, this holds if we pick as initial state the optimal state. This motivates the study of the quality of an equilibrium resulting from a dynamics starting from an optimal state. It is easy to see that, in a ring design game, any improvement dynamics starting from the optimal state leads to an equilibrium at most 2 times the cost of the optimal state. In fact, either the optimal state is a Nash equilibrium, or there is a player j wishing to switch from his optimal strategy to the alternative path. At the optimum, the cost of player j is at most $C(\textsc{Opt})$, and thus the cost of the alternative path cannot be more than this quantity. Since the alternative path of j contains edges of the ring not belonging to $E(\textsc{Opt})$, it implies that $C(\textsc{Opt})$ is also an upper bound to

$c(E \setminus E(\textsc{Opt}))$. Consequently, the cost of the entire ring, and thus the cost of any state, is at most $2C(\textsc{Opt})$. As we show later, by doing a more careful analysis, we are actually able to prove a tight bound of $3/2 \cdot C(\textsc{Opt})$ on the cost of any equilibrium reachable from the optimum. Obviously, such bound is also a bound on the PoS in a ring design game, as shown in Theorem 1.

3 Upper and lower bounds on the price of stability for ring design game

In this section we provide matching upper and lower bounds on the price of stability of ring design game. We start by upper bounding the price of stability. Our technique to prove the bound on the PoS is different from the previously used ones. Previous techniques used potential function arguments and proved that any equilibrium reached by any dynamics starting by an optimal state has potential value at most $H_n \cdot C(\textsc{Opt})$. Here we also bound the cost of a Nash equilibrium reachable by a dynamics from the optimal state but without using potential function arguments. In particular, the analysis is made by cases on the number of moves, and for each such case we write a linear program that captures the most important inequalities. The most important observation we use is that one needs to consider the cases when at most 4 players move. We prove that for higher number of moves the PoS can only be smaller.

Our notation includes the number m representing the amount of steps in which some fixed dynamics reaches a Nash equilibrium starting from an optimal state $\textsc{Opt}$, the Nash equilibrium N obtained after m steps, as well as players making a move in the dynamics. Let π_j be the player that makes the move at step $j = 1, \ldots, m$ during the dynamics. Note that a player could make a move at many different steps of the dynamics. Let $\sigma^0, \ldots, \sigma^j, \ldots, \sigma^m$ be the states corresponding to the considered dynamics, where $\sigma^0 = \textsc{Opt}$ and $\sigma^m = \mathrm{N}$. Also, let f be a set of players of interest. The set f will be composed by a subset of the players moving in the dynamics. The usage of f will be clear in the proof of Theorem 1. (NEW)

For any $A \subseteq f \neq \emptyset$ the set D_A will denote the set of edges such that each edge in D_A is used by each player in A and no one player from outside of A, formally: (NEW)

$$D_A^f = \{e \in E \mid (\forall i \in A.\ e \in \textsc{Opt}_i) \wedge (\forall i \in f \setminus A.\ e \in \overline{\textsc{Opt}_i}) \wedge (\forall i \notin f.\ e \in \overline{\textsc{Opt}_i})\}.$$

That is an edge e is in D_A^f precisely if in state $\textsc{Opt}$ it is used by all members of A, it is used by no members of $f \setminus A$, and there are no non-members of f who use it. Notice that when $A = \emptyset$, D_A denotes the set of edges used by no player in $\textsc{Opt}$, i.e, $D_\emptyset = \overline{\textsc{Opt}}$.

For any $A \subseteq f \neq \emptyset$ the set R_A will denote the set of edges such that each edge in R_A is used by each player in A and at least one player from outside of f, that is: (NEW)

$$R_A^f = \{e \in E \mid (\forall i \in A.\ e \in \textsc{Opt}_i) \wedge (\forall i \in f \setminus A.\ e \in \overline{\textsc{Opt}_i}) \wedge (\exists i \notin f.\ e \in \textsc{Opt}_i)\},$$

That is an edge e is in R_A^f precisely if in state $\textsc{Opt}$ it is used by all members of A, it is used by no members of $f \setminus A$, and there is at least one non-member of f who use it. Notice that when $A = \emptyset$, R_A denotes the set of edges used in $\textsc{Opt}$ by at least one player not belonging to f and no player belonging to f.

For the sake of simplicity in the sequel we will omit the superscript f when it is clear from the context. In order to make the reader familiar with these definitions, let us consider the instance depicted in Figure 2. As we will see in Theorem 2, the state $\textsc{Opt}$ where player 1 uses edge e_1, player 2 uses e_2, player 3 uses e_3 and player 4 uses the path composed by edges $\{e_1, e_2, e_3\}$ is an optimal state of the instance.

First, suppose that $f = \{1\}$. Then $D_\emptyset = \{e_4\}$, $D_{\{1\}} = \emptyset$, $R_\emptyset = \{e_2, e_3\}$, and $R_{\{1\}} = \{e_1\}$. Let us now consider the setting where $f = \{1, 4\}$. We have: $D_\emptyset = \{e_4\}$, $D_{\{1\}} = \emptyset$, $D_{\{4\}} = \emptyset$, $D_{\{1,4\}} = \{e_1\}$, $R_\emptyset = \{e_2, e_3\}$, $R_{\{1\}} = \emptyset$, $R_{\{4\}} = \emptyset$ and $R_{\{1,4\}} = \emptyset$. Notice that D_A and R_A naturally define a partition of the edges of the ring. Moreover note that for any f we have that:

(NEW)

$$E(\textsc{Opt}) = \Big(\bigcup_{\emptyset \neq A \subseteq f} D_A\Big) \bigcup \Big(\bigcup_{A \subseteq f} R_A\Big) = \Big(\bigcup_{A \subseteq f} D_A \cup R_A\Big) \backslash D_\emptyset. \tag{1}$$

Equation (1) derives from the observation that, for any fixed $\emptyset \neq f \subseteq [n]$, the set of edges $E(\textsc{Opt}) = \bigcup_{i \in [n]} \textsc{Opt}_i$ can be partitioned into two sets: the set of edges used in $\textsc{Opt}$ only by players in f, that is

$$\bigcup_{\emptyset \neq A \subseteq f} D_A = \bigcup_{i \in f} \textsc{Opt}_i$$

and the set of edges used in $\textsc{Opt}$ by at least a player not in f, that is $\bigcup_{A \subseteq f} R_A$.

Moreover let $\lambda > 0$ be such that $c\big(D_\emptyset\big) \le \lambda C(\textsc{Opt})$. Since $E(\textsc{Opt})$ and $D_\emptyset$ is a partition of E and the cost of any equilibrium N can be at most $c(E)$, then: (NEW)

$$PoS \le \frac{C(\mathrm{N})}{C(\textsc{Opt})} \le \frac{C(\textsc{Opt}) + c\big(D_\emptyset\big)}{C(\textsc{Opt})} \le \frac{C(\textsc{Opt}) + \lambda \cdot C(\textsc{Opt})}{C(\textsc{Opt})} \le 1 + \lambda. \tag{2}$$

Now let us write the necessary conditions for the fact that player π_j can move in step j of the dynamics, for any $j = 1, \ldots, m$. Such conditions will be expressed by using the above defined variables D_A and R_A. Unfortunately, we do not know the exact usage of edges in sets R_A. Let us define functions $\mathrm{left}_k, \mathrm{right}_k : \Sigma \to \mathbb{R}$ for any players $k \in f$. Set $d_\sigma(k)$ and $r_\sigma(k)$ to be the collection of subsets A of f such that player k is using (all) edges of D_A and R_A respectively in state σ, i.e., $d_\sigma(k) = \{A \in 2^f \mid k \text{ is using edges of } D_A \text{ in } \sigma\}$, $r_\sigma(k) = \{A \in 2^f \mid k \text{ is using edges of } R_A \text{ in } \sigma\}$. Notice that for every $A, A' \in d_\sigma(k)$, $D_A \cap D_{A'} = \emptyset$, for every $A, A' \in r_\sigma(k)$, $R_A \cap R_{A'} = \emptyset$, and $d_\sigma(k)$ and $r_\sigma(k)$ are disjoint. Thus (NEW)

$$\sigma_k = \Big(\bigcup_{A \in d_\sigma(k)} D_A\Big) \bigcup \Big(\bigcup_{A \in r_\sigma(k)} R_A\Big).$$

It results that the cost of player k in σ is given by (NEW)

$$\begin{aligned} c_\sigma(k) &= \sum_{A \in d_\sigma(k)} \sum_{e \in D_A} \frac{c(e)}{n_\sigma(e)} + \sum_{A \in r_\sigma(k)} \sum_{e \in R_A} \frac{c(e)}{n_\sigma(e)}. \\ &= \sum_{A \in d_\sigma(k)} \frac{c(D_A)}{\hat{n}_\sigma(D_A)} + \sum_{A \in r_\sigma(k)} \sum_{e \in R_A} \frac{c(e)}{n_\sigma(e)}, \end{aligned}$$

where $\hat{n}_\sigma : 2^E \to \mathbb{N}$ as $\hat{n}_\sigma(H) = \#\{i \in f \mid H \subseteq \sigma_i\}$. In fact, notice that, given $A \in d_\sigma(k)$, $\forall e \in D_A$, $n_\sigma(e) = \hat{n}_\sigma(D_A)$.

A lower bound to $c_\sigma(k)$ is

$$\mathrm{left}_\sigma(k) = \sum_{A \in d_\sigma(k)} \frac{c(D_A)}{\hat{n}_\sigma(D_A)},$$

and an upper bound to $c_\sigma(k)$ is (NEW)

$$\mathrm{right}_\sigma(k) = \sum_{A \in d_\sigma(k)} \frac{c(D_A)}{\hat{n}_\sigma(D_A)} + \sum_{A \in r_\sigma(k)} \frac{c(R_A)}{\hat{n}_\sigma(R_A) + 1},$$

in fact, given $A \in r_\sigma(k)$, $\forall e \in R_A$, $n_\sigma(e) \geq \hat{n}_\sigma(R_A)+1$. Summarising, the following inequalities (NEW)
hold for any state $\sigma \in \Sigma$: (NEW)

$$\text{left}_\sigma(k) \leq c_\sigma(k) \leq \text{right}_\sigma(k). \tag{3}$$

The role of functions left_k and right_k is to weaken the inequalities between player's utilities in some neighbour states, so that they become manageable. As we do not know the exact usage of edges in sets R_A, it would be hard to derive the precise bounds. On the lower-hand side we (NEW) do not take into account the cost of edges of R_A while for the upper-hand side we introduce the minimum number of players using edges of R_A in σ, i.e., $\hat{n}_\sigma(R_A)+1$.

Let us start by proving the following very useful lemma.

Lemma 1 *In the ring design game, if in state* OPT *there are at least two players able to perform an improving move (both starting from state* OPT*) then the cost of the whole ring is at most $\frac{3}{2}$ times the cost of an optimal solution, that is $c(E) \leq \frac{3}{2}C(\text{OPT})$.*

Proof: Let an OPT-*neighbour state* be a state $\sigma' = (\text{OPT}_{-i}, x)$, where x is a strategy of player i such that $c_{(\text{OPT}_{-i},x)}(i) < c_{\text{OPT}}(i)$. Fix two OPT-neighbour states σ', σ'', and two players α and β that change strategy in those state respectively (starting by OPT). Then the following inequalities representing necessary condition for player α and β hold:

$$\begin{aligned}\text{left}_{\sigma'}(\alpha) &\leq c_{\sigma'}(\alpha) < c_{\text{OPT}}(\alpha) \leq \text{right}_{\text{OPT}}(\alpha),\\ \text{left}_{\sigma''}(\beta) &\leq c_{\sigma''}(\beta) < c_{\text{OPT}}(\beta) \leq \text{right}_{\text{OPT}}(\beta).\end{aligned}$$

Setting $f = \{\alpha, \beta\}$, we have that

$$\begin{aligned}\frac{c(D_\emptyset)}{1} + \frac{c(D_{\{\beta\}})}{2} &< \frac{c(D_{\{\alpha\}})}{1} + \frac{c(R_{\{\alpha\}})}{2} + \frac{c(D_{\{\alpha,\beta\}})}{2} + \frac{c(R_{\{\alpha,\beta\}})}{3}\\ \frac{c(D_\emptyset)}{1} + \frac{c(D_{\{\alpha\}})}{2} &< \frac{c(D_{\{\beta\}})}{1} + \frac{c(R_{\{\beta\}})}{2} + \frac{c(D_{\{\alpha,\beta\}})}{2} + \frac{c(R_{\{\alpha,\beta\}})}{3}\end{aligned}$$

Summing the above up we get

$$\begin{aligned}\frac{2c(D_\emptyset)}{1} &< \frac{c(D_{\{\alpha,\beta\}})}{1} + \frac{c(D_{\{\alpha\}}) + c(D_{\{\beta\}}) + c(R_{\{\alpha\}}) + c(R_{\{\beta\}})}{2} + \frac{2c(R_{\{\alpha,\beta\}})}{3}\\ &\leq C(\text{OPT}),\end{aligned}$$

where the second inequality follows by (1). Hence, we have $c(D_\emptyset) \leq \frac{1}{2}C(\text{OPT})$, so by (2) PoS is not higher than $\frac{3}{2}$. □

We are now ready to prove our main result.

Theorem 1 *The price of stability for the ring design game is at most $\frac{3}{2}$.*

Proof: The proof is split into five different cases, depending on the amount m of steps in which some fixed dynamics reaches a Nash equilibrium starting from an optimal state OPT. Moreover notice that since in a ring design game the strategy set of each player i is composed by exactly 2 different strategies, i.e., the clockwise and counterclockwise paths connecting s_i and t_i. This implies that $\pi_j \neq \pi_{j+1}$, for any $j = 1, \ldots, m-1$. We remark that in some cases we get the bound by solving a linear program where constraints are naturally defined by using left and right functions, and where objective functions are proper defined in each of the case. The picture will be clear in the following.

Case $m = 0$. The equality $m = 0$ trivializes the instance into an example where OPT is a Nash equilibrium, thus PoS = 1.

Case $m = 1$. In this case the dynamics reaches a Nash Equilibrium N after one step starting from $\textsc{Opt}$. Since player π_1 can perform an improving move starting by state $\textsc{Opt}$, the following inequalities hold:

$$\text{left}_{\mathrm{N}}(\pi_1) \leq c_{\mathrm{N}}(\pi_1) < c_{\textsc{Opt}}(\pi_1) \leq \text{right}_{\textsc{Opt}}(\pi_1).$$

Therefore, by setting $f = \{\pi_1\}$ we have that:

$$\frac{c\big(D_\emptyset\big)}{1} < \frac{c\big(D_{\{\pi_1\}}\big)}{1} + \frac{c\big(R_{\{\pi_1\}}\big)}{2}.$$

The last inequality directly implies that:

$$\begin{aligned} \frac{C(\mathrm{N})}{C(\textsc{Opt})} &= \frac{c\big(D_\emptyset\big) + c\big(R_\emptyset\big) + c\big(R_{\{\pi_1\}}\big)}{c\big(D_{\{\pi_1\}}\big) + c\big(R_\emptyset\big) + c\big(R_{\{\pi_1\}}\big)} \\ &\leq \frac{c\big(D_{\{\pi_1\}}\big) + c\big(R_\emptyset\big) + \frac{3}{2}c\big(R_{\{\pi_1\}}\big)}{c\big(D_{\{\pi_1\}}\big) + c\big(R_\emptyset\big) + c\big(R_{\{\pi_1\}}\big)} \leq \frac{3}{2}. \end{aligned}$$

Case $m = 2$. When $m = 2$, the player π_1 leads the dynamic from $\textsc{Opt}$ to σ^1 and player π_2 leads the dynamics from σ^1 to N. Therefore the following must hold:

$$\begin{aligned} \text{left}_{\sigma^1}(\pi_1) &\leq c_{\sigma^1}(\pi_1) < c_{\textsc{Opt}}(\pi_1) \leq \text{right}_{\textsc{Opt}}(\pi_1), \\ \text{left}_{\mathrm{N}}(\pi_2) &\leq c_{\mathrm{N}}(\pi_2) < c_{\sigma^1}(\pi_2) \leq \text{right}_{\sigma^1}(\pi_2). \end{aligned}$$

Therefore, by setting $f = \{\pi_1, \pi_2\}$ we have that:

$$\begin{aligned} \frac{c\big(D_\emptyset\big)}{1} + \frac{c\big(D_{\{\pi_2\}}\big)}{2} &< \frac{c\big(D_{\{\pi_1\}}\big)}{1} + \frac{c\big(R_{\{\pi_1\}}\big)}{2} + \frac{c\big(D_{\{\pi_1,\pi_2\}}\big)}{2} + \frac{c\big(R_{\{\pi_1,\pi_2\}}\big)}{3} \\ \frac{c\big(D_\emptyset\big)}{2} + \frac{c\big(D_{\{\pi_1\}}\big)}{1} &< \frac{c\big(D_{\{\pi_2\}}\big)}{2} + \frac{c\big(R_{\{\pi_2\}}\big)}{2} + \frac{c\big(D_{\{\pi_1,\pi_2\}}\big)}{1} + \frac{c\big(R_{\{\pi_1,\pi_2\}}\big)}{2} \end{aligned}$$

Without loss of generality we can add the following constraints:

$$\sum_{e \in OPT} c(e) \leq 1, \qquad \forall e \in E.\ c(e) \geq 0.$$

We need to bound the value of $c\big(D_\emptyset\big) - \frac{1}{2}c\big(D_{\{\pi_1,\pi_2\}}\big)$ with respect to the above inequalities. Such a bound can be obtained by forming a linear program from all the above equations including the appropriate objective function. We have solved this linear program on a computer using a standard LP solver. This way we have obtained the following bound:

$$c\big(D_\emptyset\big) - \frac{1}{2}c\big(D_{\{\pi_1,\pi_2\}}\big) \leq \frac{5}{11} < \frac{1}{2}.$$

In the remainder of the proof similar bounds have been obtained in the same way by using a LP solver. Further, the cost of states N and $\textsc{Opt}$ are:

$$\begin{aligned} C(\mathrm{N}) &= c\big(D_\emptyset\big) + c\big(R_\emptyset\big) + c\big(D_{\{\pi_1\}}\big) + c\big(R_{\{\pi_1\}}\big) \\ &+ c\big(D_{\{\pi_2\}}\big) + c\big(R_{\{\pi_2\}}\big) + c\big(R_{\{\pi_1,\pi_2\}}\big), \end{aligned}$$

and

$$
\begin{aligned}
C(\text{OPT}) &= c\big(D_{\{\pi_1,\pi_2\}}\big)+c\big(R_\emptyset\big)+c\big(D_{\{\pi_1\}}\big)+c\big(R_{\{\pi_1\}}\big)\\
&+ c\big(D_{\{\pi_2\}}\big)+c\big(R_{\{\pi_2\}}\big)+c\big(R_{\{\pi_1,\pi_2\}}\big),
\end{aligned}
$$

respectively. Therefore, by using the upper bound on $c\big(D_\emptyset\big)-\frac{1}{2}c\big(D_{\{\pi_1,\pi_2\}}\big)$ we obtain that:

$$
\frac{C(\text{N})}{C(\text{OPT})} \le \frac{16}{11} < \frac{3}{2}.
$$

Case $m=3$**.** Similarly to the previous case, we will construct a suitable linear program. We know that $\pi_1 \neq \pi_2$ and $\pi_2 \neq \pi_3$. If $\pi_1 = \pi_3$ then it means that σ^1 and σ^3 are two OPT-neighbour states and therefore by Lemma 1 we have that $PoS \le \frac{3}{2}$. Therefore we can assume that $\pi_1 \neq \pi_3$. Since $m=3$ then player π_1 leads the dynamic from OPT to σ^1, player π_2 leads the dynamics from σ^1 to σ^2 and finaly player π_3 from σ^2 to N. Therefore the following must hold:

$$
\begin{aligned}
\text{left}_{\sigma^1}(\pi_1) &\le c_{\sigma^1}(\pi_1) < c_{\text{OPT}}(\pi_1) \le \text{right}_{\text{OPT}}(\pi_1),\\
\text{left}_{\sigma^2}(\pi_2) &\le c_{\sigma^2}(\pi_2) < c_{\sigma^1}(\pi_2) \le \text{right}_{\sigma^1}(\pi_2),\\
\text{left}_{\text{N}}(\pi_3) &\le c_{\text{N}}(\pi_3) < c_{\sigma^2}(\pi_3) \le \text{right}_{\sigma^2}(\pi_3).
\end{aligned}
$$

By setting $f=\{\pi_1,\pi_2,\pi_3\}$ we obtain a set of constraints (for the sake of readability and completeness we move them to the Appendix A) that along with $C(\text{OPT}) \le 1$ and maximization target $c\big(D_\emptyset\big)-\frac{1}{2}c\big(D_{\{\pi_1,\pi_2,\pi_3\}}\big)$ constitute a linear program with a solution

$$
c\big(D_\emptyset\big)-\frac{1}{2}c\big(D_{\{\pi_1,\pi_2,\pi_3\}}\big) \le \frac{198}{487} < \frac{1}{2}.
$$

Substituting it into the ratio of the costs of N and OPT we get that:

$$
\frac{C(\text{N})}{C(\text{OPT})} \le \frac{685}{487} < \frac{3}{2}.
$$

Case $m \ge 4$**.** This case is different because we do not know m. However, it is enough to consider the case $m=4$. This is due to the fact that the inequalities obtained by the dynamics of the first 4 players are strong enough to bound the cost of the whole ring. This gives the bound on the cost of any Nash equlibrium the dynamics will converge to, because even if more players move the cost of the final state will be smaller then the cost of the whole ring. We show that if $m=4$ then $c\big(D_\emptyset\big) < \frac{1}{2}\cdot C(\text{OPT})$. Clearly, adding new constraints for $m>4$ cannot increase this bound.

Then let us consider $m=4$. As in the previous case we have that $\pi_1 \neq \pi_2$ and $\pi_2 \neq \pi_3$ and $\pi_1 \neq \pi_3$, anyway we are not able to derive any conclusion about π_4. It follows that we have to consider 3 subcases, i.e., $\pi_4=\pi_1$, $\pi_4=\pi_2$ and $\pi_4 \neq \pi_z$ for $z=1,2,3$. As usually in this proof, we are going to derive sets of constraints that must hold at every step of the dynamics by using functions left and right. In the first two subcases, i.e., $\pi_4=\pi_1$, $\pi_4=\pi_2$, by setting $f=\{\pi_1,\pi_2,\pi_3\}$ the constraints set is composed by the ones corresponding to the case $m=3$ capturing the dynamics by π_1,π_2 and π_3 (constraints in Appendix A), plus a further constraint (Appendix B when $\pi_4=\pi_1$ and Appendix C when $\pi_4=\pi_2$) capturing the fact that players π_1 or π_2 move back at the fourth step of the dynamics respectively. Concerning the subcase where $\pi_4 \neq \pi_z$ for $z=1,2,3$, by setting $f=\{\pi_1,\pi_2,\pi_3,\pi_4\}$ we get the set of constraints included in Appendix D. By summarizing we get three different sets of constraints corresponding to three different linear programs. In each of them it suffices to consider maximization target $c\big(D_\emptyset\big)$

assuming that $C(\text{OPT}) \leq 1$ without loss of generality. In all cases the maximum value of $c\big(D_\emptyset\big)$ turns out to be smaller than $\frac{1}{2}$, specifically for $\pi_4 = \pi_1$, $\pi_4 = \pi_2$ and $\pi_4 \neq \pi_z$ where $z = 1, 2, 3$ the bounds are $\frac{3}{7}$, $\frac{63}{131}$ and $\frac{114}{253}$ respectively. Hence, by (2) for all these cases we know that PoS is bounded by $\frac{3}{2}$. □

It is worth to note that the above theorem give us some insights on the complexity of computing a Nash equilibrium in ring design game. We will further discuss this issue in Section 4. The next corollary directly follows from Theorem 1

Corollary 1 *In a ring design game, if the cost of the entire ring is larger than* $3/2$ *times the cost of an optimal state, then the improvement dynamics starting from an optimal state converges quickly, within at most 3 steps, to a Nash equilibrium.*

We are now ready to show a matching lower bound. Let us start by investigating the structure of Nash equilibria. Notice that, since in any stable configuration each player (given the strategies of other players) selects the least expensive path, then in any Nash equilibrium the following property holds:

Property 1 *If any player chooses a path* w *for connecting nodes* v *and* v'*, then all players crossing nodes* v *and* v' *must share the same path* w*.*

Theorem 2 *Given any* $\epsilon > 0$*, there exists an instance of the ring design game such that the price of stability is at least* $\frac{3}{2} - \epsilon$*.*

Proof: We will construct an instance of the ring design game so that there is a distinct player associated with each edge of the ring that wishes to connect the endpoints of this edge. This construction guarantees that there is a unique Nash equilibrium in which each player uses her corresponding edge. Consider a ring consisting of 4 edges $E = \{e_1, e_2, e_3, e_4\}$ where $c(e_1) = c(e_2) = c(e_3) = 2$ and $c(e_4) = 3 - \epsilon$ as shown in Figure 2. There are four players $\{1, 2, 3, 4\}$; player i is interested in connecting the endpoints of edge e_i for $i = 1, 2, 3, 4$. Observe

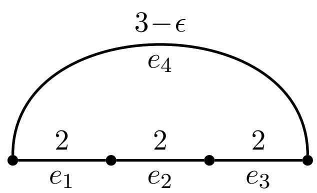

Figure 2: The lower bound example for PoS on the ring. For the detailed description please see proof of Theorem 2.

that in any state of the game the edges used by at least one of the players form a connected spanning subgraph. Therefore the cost of an optimal solution is at least the cost of any minimum spanning tree. It follows that the state of the game where player 1 uses edge e_1, player 2 uses e_2, player 3 uses e_3 and player 4 uses the path composed by edges $\{e_1, e_2, e_3\}$ is an optimal solution of cost 6. Let σ be the state in which each player uses its corresponding edge, i.e., where players $1, 2, 3, 4$ use edges e_1, e_2, e_3, e_4 respectively. It is easy to check that σ is a Nash equilibrium. It remains to show that σ is the only Nash equilibrium. Let us suppose that our instance admits a Nash equilibrium where player 4 uses the path composed by edges $\{e_1, e_2, e_3\}$. By Property 1, players $1, 2, 3$ must use edges e_1, e_2, e_3 respectively. However, in such state player 4 is experiencing a cost of $\frac{2}{2} + \frac{2}{2} + \frac{2}{2} > 3 - \epsilon$, then player 4 can use its corresponding edge e_4 and thus strictly reduces her cost, a contradiction. Let us suppose that our instance admits a Nash equilibrium where player 1 uses the path composed by edges $\{e_2, e_3, e_4\}$. By Property 1, players $2, 3, 4$ must use edges e_2, e_3, e_4 respectively. However, in this state player 1 is experiencing a cost of $\frac{2}{2} + \frac{2}{2} + \frac{3-\epsilon}{2} > 2$, then player 1 can use its corresponding edge e_1 and thus strictly reduces her cost, a contradiction. With similar arguments it is possible to show that in any Nash equilibrium players 2 and 3 have to use their corresponding edges. It follows that in any equilibrium of our instance all players have to use their corresponding edges. This completes the proof. □

The next theorem generalizes the previous Theorem 2, and gives some insight on the relation between the PoA and PoS in a ring design game. In particular, we observe that by choosing a large number of players and setting the value of ϵ to be very small, when α approaches to 1, both the PoA and PoS tend to 3/2. On the counterpart, when α approaches to 0, the PoA tends to 2 and the PoS gets close to 1.

Theorem 3 *Given an integer $n \geq 7$, for every $\epsilon \in [\frac{2}{n-4}, 1)$ and $\alpha \in (0, 1-\epsilon]$, there exists an instance of ring design game with n players, having* $\mathrm{PoA} = 2 - \frac{\alpha}{2(1+\epsilon)} - \frac{2\epsilon}{(1+\epsilon)}$ *and* $\mathrm{PoS} = 1 + \frac{\alpha}{2(1+\epsilon)}$.

Proof: The instance giving the result is depicted in Figure 3. The game is defined as follows. For simplicity we say that a player is associated to an edge, if it is interested in connecting its endpoints. In the game there are $n \geq 7$ players. Player 1 is associated to edge e_1, player 2 is associated to edge e_2, player 3 is associated to edge e_3, player 4 is associated to edge e_4, and the remaining set of $n-4$ players $R = \{5, 6, \ldots, n\}$ are associated to edge e_6.

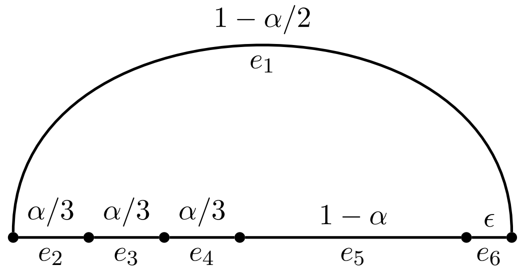

Figure 3: The lower bound example for PoS on the ring. For the detailed description please see proof of Theorem 3

We observe that the optimal state uses the set of edges $E(\textsc{Opt}) = \{e_2, e_3, e_4, e_5, e_6\}$, and it costs $1 + \epsilon$. Subsequently we show that the game admits only two Nash equilibria, namely N_1 and N_2. N_1 is the state in which the players $1, 2, 3, 4$, use their corresponding edge, and each player in R uses their alternative path $\{e_1, e_2, e_3, e_4, e_5\}$. Thus $E(\mathrm{N}_1) = \{e_1, e_2, e_3, e_4, e_5\}$ and $C(\mathrm{N}_1) = 2 - \frac{1}{2}\alpha$. N_2 is the state in which each player uses the associated edge, thus $E(\mathrm{N}_2) = \{e_1, e_2, e_3, e_4, e_6\}$, and $C(\mathrm{N}_2) = 1 + \frac{1}{2}\alpha + \epsilon$. Notice that $C(\mathrm{N}_1) \geq C(\mathrm{N}_2)$. Since $\mathrm{PoA} = C(\mathrm{N}_1)/C(\textsc{Opt})$ and $\mathrm{PoS} = C(\mathrm{N}_2)/C(\textsc{Opt})$, the claim follows.

We first show that N_1 and N_2 are Nash equilibria. In N_1, the cost of player 1 is $1-\alpha/2$, and the alternative path $\{e_2, e_3, e_4, e_5, e_6\}$ would cost $1-\alpha/2+\epsilon/(n-3)$. The cost of player 2 is $\alpha/3$ and the alternative path $\{e_3, e_4, e_5, e_6, e_1\}$ would cost at least $\alpha/3$, since he would share e_3 and e_4 with player 3 and 4 respectively. The same argument holds for players 3 and 4. Finally, the cost of each player in R is $\epsilon/(n-4) \leq 1/3$, and the alternative path $\{e_1, e_2, e_3, e_4, e_5\}$ would cost $3/2 - 3\alpha/4 \geq 3/4$. In N_2, the cost of player 1 is $\frac{1-\alpha/2}{n-3}$, and the alternative path $\{e_2, e_3, e_4, e_5, e_6\}$ would cost at least $\frac{1-\alpha}{n-3}$. The cost of player 2 is $\frac{\alpha/3}{n-3}$, and the alternative path $\{e_3, e_4, e_5, e_6, e_1\}$ would cost at least $\frac{2\alpha/3}{n-2}$, since he would share both e_3 and e_4 with other $n-2$ players (since

$n \geq 7$, it turn out that $\frac{2\alpha/3}{n-2} = \frac{\alpha/3}{n/2-1} \geq \frac{\alpha/3}{n-3}$). The same argument holds also for player 3 and 4. Finally, the cost of each player in R is $\frac{1}{n-3}(1-\alpha/2) + \frac{1}{n-3}\alpha/2 + \frac{1}{n-4}(1-\alpha) \leq \frac{1}{n-3} + \frac{1}{n-4}$, and the alternative path would cost $\epsilon \geq \frac{2}{n-4}$.

It remains to show that no other state, beside N_1 and N_2, is a Nash equilibrium. For this purpose we use Property 1. We first notice that, by Property 1, there may not exist any Nash equilibrium in which two different players in R chose different paths. Therefore, let us focus on states in which all players in R chose the same path $\{e_6\}$ or $\{e_1, e_2, e_3, e_4, e_5\}$. The state in which player 2 chooses the path alternative to e_2 cannot be an equilibrium. In fact, in order to be a Nash equilibrium, by Property 1, it must hold that each other player must use its associated edge. Therefore in such state player 2 would pay more than $\alpha/3$ since, beside the cost incurred because of other edges, he would share e_3 and e_4 with player 3 and 4 respectively. The same argument hold for player 3 and player 4. Finally, by applying Property 1, we notice that the state in which player 1 choses the path $\{e_2, e_3, e_4, e_5, e_6\}$ and players in R chose the path $\{e_1, e_2, e_3, e_4, e_5\}$, cannot be an equilibrium. It results that the only two states left out, are N_1 and N_2. □

4 Conclusions and open problems

In this paper we have succeeded to give the first nontrivial example of network design games where our understanding of PoS is tight. We showed that any dynamics starting from Opt returns an equilibrium of cost at most $(\frac{3}{2}) \cdot C(\textsc{Opt})$ in at most 4 moves, when $c(E) \geq (3/2)C(\textsc{Opt})$. The major open question is whether the dynamics starting from Opt is still quickly convergent to an equilibrium when $c(E) < (3/2)C(\textsc{Opt})$. A positive answer to the previous question would lead to a polynomial algorithm for computing a Nash equilibrium in the ring design game. We remark that in the general design game, (i.e., design game for general graph topology), the problem of computing a Nash equilibrium is PLS-complete [15].

Moreover, we could not manage to find instances with the PoA close to 2 when the PoS is close to 3/2, thus another interesting problem is to investigate the relation between the costs of the best and the worst Nash equilibria. We believe that the insight given by our results can be used for the analysis of more involved graph topologies, e.g., planar graphs. Finally, another investigating issue would be to derive an exact bound on the price of stability for broadcast ring design game. We remark that the broadcast game is a special case of the game in which there is at least one player for every node of the graph requiring to be connected to the single common source. Observe that our lower bound of $\frac{3}{2}$ does not work here. In the following we give the best lower bound on the price of stability for the broadcast ring game. It would be interesting to give an accompanying tight upper bound. Let us consider the Figure 4 where there is one player for each node but the source s. It is well known [3, 9] that in broadcast

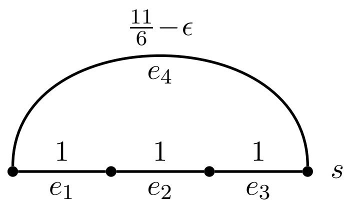

Figure 4: Instance showing a lower bound to the price of stability for broadcast ring design game.

design game the optimum is a minimum cost spanning tree. Moreover, any Nash equilibrium is a spanning tree. Clearly in our instance the optimum state OPT is composed by edges e_1, e_2, e_3. However, the player corresponding to the leftmost vertex of the figure (i.e., the furthest player with respect to the source s in state OPT) is paying $\frac{11}{6}$ and thus OPT is not an equilibrium since this player has an improvement move. It follows that any equilibrium is composed by the edge e_4 and other two edges belonging in $\{e_1, e_2, e_3\}$. Therefore the cost of any equilibrium is $\frac{23}{6} - \epsilon$ while the cost of the optimum is 3. It follows that the price of stability for broadcast ring design game is at least $\frac{23}{18} - \epsilon$ for any $\epsilon > 0$.

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4.1 Appendix A: proof of theorem 1 case m=3

Player π_1 moves:

$$\frac{c(D_\emptyset)}{1}+\frac{c(D_{\{\pi_2\}})}{2}+\frac{c(D_{\{\pi_3\}})}{2}+\frac{c(D_{\{\pi_2,\pi_3\}})}{3}$$
$$\leq$$
$$\frac{c(D_{\{\pi_1\}})}{1}+\frac{c(D_{\{\pi_1,\pi_2\}})}{2}+\frac{c(D_{\{\pi_1,\pi_3\}})}{2}+\frac{c(D_{\{\pi_1,\pi_2,\pi_3\}})}{3}+$$
$$\frac{c(R_{\{\pi_1\}})}{2}+\frac{c(R_{\{\pi_1,\pi_2\}})}{3}+\frac{c(R_{\{\pi_1,\pi_3\}})}{3}+\frac{c(R_{\{\pi_1,\pi_2,\pi_3\}})}{4}.$$

Player π_2 moves:

$$\frac{c(D_\emptyset)}{2}+\frac{c(D_{\{\pi_1\}})}{1}+\frac{c(D_{\{\pi_3\}})}{3}+\frac{c(D_{\{\pi_1,\pi_3\}})}{2}$$
$$\leq$$
$$\frac{c(D_{\{\pi_2\}})}{2}+\frac{c(D_{\{\pi_1,\pi_2\}})}{1}+\frac{c(D_{\{\pi_2,\pi_3\}})}{3}+\frac{c(D_{\{\pi_1,\pi_2,\pi_3\}})}{2}+$$
$$\frac{c(R_{\{\pi_2\}})}{3}+\frac{c(R_{\{\pi_1,\pi_2\}})}{2}+\frac{c(R_{\{\pi_2,\pi_3\}})}{4}+\frac{c(R_{\{\pi_1,\pi_2,\pi_3\}})}{3}.$$

Player π_3 moves:

$$\frac{c(D_\emptyset)}{3}+\frac{c(D_{\{\pi_1\}})}{2}+\frac{c(D_{\{\pi_2\}})}{2}+\frac{c(D_{\{\pi_1,\pi_2\}})}{1}$$
$$\leq$$
$$\frac{c(D_{\{\pi_3\}})}{3}+\frac{c(D_{\{\pi_1,\pi_3\}})}{2}+\frac{c(D_{\{\pi_2,\pi_3\}})}{2}+\frac{c(D_{\{\pi_1,\pi_2,\pi_3\}})}{1}+$$
$$\frac{c(R_{\{\pi_3\}})}{4}+\frac{c(R_{\{\pi_1,\pi_3\}})}{3}+\frac{c(R_{\{\pi_2,\pi_3\}})}{3}+\frac{c(R_{\{\pi_1,\pi_2,\pi_3\}})}{2}.$$

4.2 Appendix B: proof of theorem 1 case $m \geq 4$, and $\pi_4 = \pi_1$

Player π_1 moves back:

$$\frac{c(D_{\{\pi_1\}})}{3}+\frac{c(D_{\{\pi_1,\pi_2\}})}{2}+\frac{c(D_{\{\pi_1,\pi_3\}})}{2}+\frac{c(D_{\{\pi_1,\pi_2,\pi_3\}})}{1}$$
$$\leq$$
$$\frac{c(D_\emptyset)}{3}+\frac{c(D_{\{\pi_2\}})}{2}+\frac{c(D_{\{\pi_3\}})}{2}+\frac{c(D_{\{\pi_2,\pi_3\}})}{1}+$$
$$\frac{c(R_\emptyset)}{4}+\frac{c(R_{\{\pi_2\}})}{3}+\frac{c(R_{\{\pi_3\}})}{3}+\frac{c(R_{\{\pi_2,\pi_3\}})}{2}.$$

4.3 Appendix C: proof of theorem 1 case $m \geq 4$, and $\pi_4 = \pi_2$

Player π_2 moves back:

$$\frac{c(D_{\{\pi_2\}})}{3}+\frac{c(D_{\{\pi_1,\pi_2\}})}{2}+\frac{c(D_{\{\pi_2,\pi_3\}})}{2}+\frac{c(D_{\{\pi_1,\pi_2,\pi_3\}})}{1}$$
$$\leq$$
$$\frac{c(D_\emptyset)}{3}+\frac{c(D_{\{\pi_1\}})}{2}+\frac{c(D_{\{\pi_3\}})}{2}+\frac{c(D_{\{\pi_1,\pi_3\}})}{1}+$$
$$\frac{c(R_\emptyset)}{4}+\frac{c(R_{\{\pi_1\}})}{3}+\frac{c(R_{\{\pi_3\}})}{3}+\frac{c(R_{\{\pi_1,\pi_3\}})}{2}.$$

4.4 Appendix D: proof of theorem 1 case $m \geq 4$, and $\pi_4 \neq \pi_z$ where $z = 1, 2, 3$

Player π_1 moves:

$$\begin{aligned}
&\frac{c\big(D_{\emptyset}\big)}{1} + \frac{c\big(D_{\{\pi_2\}}\big)}{2} + \frac{c\big(D_{\{\pi_3\}}\big)}{2} + \frac{c\big(D_{\{\pi_4\}}\big)}{2} + \\
&\frac{c\big(D_{\{\pi_2,\pi_3\}}\big)}{3} - \frac{c\big(D_{\{\pi_2,\pi_4\}}\big)}{3} - \frac{c\big(D_{\{\pi_3,\pi_4\}}\big)}{3} + \frac{c\big(D_{\{\pi_2,\pi_3,\pi_4\}}\big)}{4} \\
\leq \\
&\frac{c\big(D_{\{\pi_1\}}\big)}{1} + \frac{c\big(D_{\{\pi_1,\pi_2\}}\big)}{2} + \frac{c\big(D_{\{\pi_1,\pi_3\}}\big)}{2} + \frac{c\big(D_{\{\pi_1,\pi_4\}}\big)}{2} + \\
&\frac{c\big(D_{\{\pi_1,\pi_2,\pi_3\}}\big)}{3} + \frac{c\big(D_{\{\pi_1,\pi_2,\pi_4\}}\big)}{3} + \frac{c\big(D_{\{\pi_1,\pi_3,\pi_4\}}\big)}{3} + \frac{c\big(D_{\{\pi_1,\pi_2,\pi_3,\pi_4\}}\big)}{4} + \\
&\frac{c\big(R_{\{\pi_1\}}\big)}{2} + \frac{c\big(R_{\{\pi_1,\pi_2\}}\big)}{3} + \frac{c\big(R_{\{\pi_1,\pi_3\}}\big)}{3} + \frac{c\big(R_{\{\pi_1,\pi_4\}}\big)}{3} + \\
&\frac{c\big(R_{\{\pi_1,\pi_2,\pi_3\}}\big)}{4} + \frac{c\big(R_{\{\pi_1,\pi_2,\pi_4\}}\big)}{4} + \frac{c\big(R_{\{\pi_1,\pi_3,\pi_4\}}\big)}{4} + \frac{c\big(R_{\{\pi_1,\pi_2,\pi_3,\pi_4\}}\big)}{5}.
\end{aligned}$$

Player π_2 moves:

$$\begin{aligned}
&\frac{c\big(D_{\emptyset}\big)}{2} + \frac{c\big(D_{\{\pi_1\}}\big)}{1} + \frac{c\big(D_{\{\pi_3\}}\big)}{3} + \frac{c\big(D_{\{\pi_4\}}\big)}{3} + \\
&\frac{c\big(D_{\{\pi_1,\pi_3\}}\big)}{2} + \frac{c\big(D_{\{\pi_1,\pi_4\}}\big)}{2} + \frac{c\big(D_{\{\pi_3,\pi_4\}}\big)}{4} + \frac{c\big(D_{\{\pi_1,\pi_3,\pi_4\}}\big)}{3} \\
\leq \\
&\frac{c\big(D_{\{\pi_2\}}\big)}{2} + \frac{c\big(D_{\{\pi_1,\pi_2\}}\big)}{1} + \frac{c\big(D_{\{\pi_2,\pi_3\}}\big)}{3} + \frac{c\big(D_{\{\pi_2,\pi_4\}}\big)}{3} + \\
&\frac{c\big(D_{\{\pi_1,\pi_2,\pi_3\}}\big)}{2} + \frac{c\big(D_{\{\pi_1,\pi_2,\pi_4\}}\big)}{2} + \frac{c\big(D_{\{\pi_2,\pi_3,\pi_4\}}\big)}{4} + \frac{c\big(D_{\{\pi_1,\pi_2,\pi_3,\pi_4\}}\big)}{3} + \\
&\frac{c\big(R_{\{\pi_2\}}\big)}{3} + \frac{c\big(R_{\{\pi_1,\pi_2\}}\big)}{2} + \frac{c\big(R_{\{\pi_2,\pi_3\}}\big)}{4} + \frac{c\big(R_{\{\pi_2,\pi_4\}}\big)}{4} + \\
&\frac{c\big(R_{\{\pi_1,\pi_2,\pi_3\}}\big)}{3} + \frac{c\big(R_{\{\pi_1,\pi_2,\pi_4\}}\big)}{3} + \frac{c\big(R_{\{\pi_2,\pi_3,\pi_4\}}\big)}{5} + \frac{c\big(R_{\{\pi_1,\pi_2,\pi_3,\pi_4\}}\big)}{4}.
\end{aligned}$$

Player π_3 moves:

$$\begin{aligned}
&\frac{c\big(D_{\emptyset}\big)}{3} + \frac{c\big(D_{\{\pi_1\}}\big)}{2} + \frac{c\big(D_{\{\pi_2\}}\big)}{2} + \frac{c\big(D_{\{\pi_4\}}\big)}{4} + \\
&\frac{c\big(D_{\{\pi_1,\pi_2\}}\big)}{1} + \frac{c\big(D_{\{\pi_1,\pi_4\}}\big)}{3} + \frac{c\big(D_{\{\pi_2,\pi_4\}}\big)}{3} + \frac{c\big(D_{\{\pi_1,\pi_2,\pi_4\}}\big)}{2} \\
\leq \\
&\frac{c\big(D_{\{\pi_3\}}\big)}{3} + \frac{c\big(D_{\{\pi_1,\pi_3\}}\big)}{2} + \frac{c\big(D_{\{\pi_2,\pi_3\}}\big)}{2} + \frac{c\big(D_{\{\pi_3,\pi_4\}}\big)}{4} + \\
&\frac{c\big(D_{\{\pi_1,\pi_2,\pi_3\}}\big)}{1} + \frac{c\big(D_{\{\pi_1,\pi_3,\pi_4\}}\big)}{3} + \frac{c\big(D_{\{\pi_2,\pi_3,\pi_4\}}\big)}{3} + \frac{c\big(D_{\{\pi_1,\pi_2,\pi_3,\pi_4\}}\big)}{2} + \\
&\frac{c\big(R_{\{\pi_3\}}\big)}{4} + \frac{c\big(R_{\{\pi_1,\pi_3\}}\big)}{3} + \frac{c\big(R_{\{\pi_2,\pi_3\}}\big)}{3} + \frac{c\big(R_{\{\pi_3,\pi_4\}}\big)}{5} + \\
&\frac{c\big(R_{\{\pi_1,\pi_2,\pi_3\}}\big)}{2} + \frac{c\big(R_{\{\pi_1,\pi_3,\pi_4\}}\big)}{4} + \frac{c\big(R_{\{\pi_2,\pi_3,\pi_4\}}\big)}{4} + \frac{c\big(R_{\{\pi_1,\pi_2,\pi_3,\pi_4\}}\big)}{3}.
\end{aligned}$$

Player π_4 moves:

$$\begin{aligned}
&\frac{c\big(D_{\emptyset}\big)}{4} + \frac{c\big(D_{\{\pi_1\}}\big)}{3} + \frac{c\big(D_{\{\pi_2\}}\big)}{3} + \frac{c\big(D_{\{\pi_3\}}\big)}{3} + \\
&\frac{c\big(D_{\{\pi_1,\pi_2\}}\big)}{2} + \frac{c\big(D_{\{\pi_1,\pi_3\}}\big)}{2} + \frac{c\big(D_{\{\pi_2,\pi_3\}}\big)}{2} + \frac{c\big(D_{\{\pi_1,\pi_2,\pi_3\}}\big)}{1} \\
\leq\; & \\
&\frac{c\big(D_{\{\pi_4\}}\big)}{4} + \frac{c\big(D_{\{\pi_1,\pi_4\}}\big)}{3} + \frac{c\big(D_{\{\pi_2,\pi_4\}}\big)}{3} + \frac{c\big(D_{\{\pi_3,\pi_4\}}\big)}{3} + \\
&\frac{c\big(D_{\{\pi_1,\pi_2,\pi_4\}}\big)}{2} + \frac{c\big(D_{\{\pi_1,\pi_3,\pi_4\}}\big)}{2} + \frac{c\big(D_{\{\pi_2,\pi_3,\pi_4\}}\big)}{2} + \frac{c\big(D_{\{\pi_1,\pi_2,\pi_3,\pi_4\}}\big)}{1} + \\
&\frac{c\big(R_{\{\pi_4\}}\big)}{5} + \frac{c\big(R_{\{\pi_1,\pi_4\}}\big)}{4} + \frac{c\big(R_{\{\pi_2,\pi_4\}}\big)}{4} + \frac{c\big(R_{\{\pi_3,\pi_4\}}\big)}{4} + \\
&\frac{c\big(R_{\{\pi_1,\pi_2,\pi_4\}}\big)}{3} + \frac{c\big(R_{\{\pi_1,\pi_3,\pi_4\}}\big)}{3} + \frac{c\big(R_{\{\pi_2,\pi_3,\pi_4\}}\big)}{3} + \frac{c\big(R_{\{\pi_1,\pi_2,\pi_3,\pi_4\}}\big)}{2}.
\end{aligned}$$