

**SUITABLE WEAK SOLUTIONS TO THE 3D  
NAVIER-STOKES EQUATIONS ARE CONSTRUCTED  
WITH THE VOIGT APPROXIMATION**

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ABSTRACT. In this paper we consider the Navier-Stokes equations supplemented with either the Dirichlet or vorticity-based Navier boundary conditions. We prove that weak solutions obtained as limits of solutions to the Navier-Stokes-Voigt model satisfy the local energy inequality. Moreover, in the periodic setting we prove that if the parameters are chosen in an appropriate way, then we can construct suitable weak solutions through a Fourier-Galerkin finite-dimensional approximation in the space variables.

1. INTRODUCTION

We prove that weak solutions to the 3D Navier-Stokes Equations (1.1) (from now on NSE) obtained as limits of solutions to the Navier-Stokes-Voigt model (1.6) (from now on NSV) are *suitable weak solutions*.

To set the problem, we recall that the initial boundary value problem for the incompressible NSE with unit viscosity, zero external force, in a smooth bounded domain  $\Omega \subset \mathbb{R}^3$  is

$$u_t - \Delta u + (u \cdot \nabla) u + \nabla p = 0 \quad \text{in } (0, T) \times \Omega, \quad (1.1)$$

$$\operatorname{div} u = 0 \quad \text{in } (0, T) \times \Omega, \quad (1.2)$$

$$u = 0 \quad \text{on } (0, T) \times \Gamma, \quad (1.3)$$

$$u(x, 0) = u_0(x) \quad \text{in } \Omega, \quad (1.4)$$

where  $u : (0, T) \times \Omega \rightarrow \mathbb{R}^3$  is the velocity vector field and  $p : (0, T) \times \Omega \rightarrow \mathbb{R}$  is the scalar pressure, and  $u_0(x)$  is a divergence-free vector initial datum. We are writing the problem with vanishing Dirichlet boundary conditions, but we will treat also a Navier-type boundary condition.

It is well-known that important issues as global regularity and uniqueness of weak solutions for the 3D NSE are still open and very far to be understood, see Galdi [19] and Constantin and Foias [16]. A keystone regularity result for weak solutions is the partial regularity theorem of Caffarelli, Kohn, and Nirenberg [13] which asserts that the set of interior (possible) singularities has vanishing one-dimensional parabolic Hausdorff measure. Concerning partial regularity results up to the boundary with Dirichlet boundary conditions, see [28]. In the case of Navier boundary condition (1.10) we could not find a specific reference, however we suspect and conjecture that similar

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results of partial regularity can be obtained also in this case with minor changes.

In any case, the partial regularity theorem holds for a particular subclass of Leray weak solutions, called starting from [13] “*suitable weak solutions*.” Beside technical regularity properties, the most important additional requirement of suitable weak solutions (see Scheffer [36] and see also Sec. 2 below for precise definitions), is the following inequality, often called in literature *local or generalized energy inequality*:

$$\partial_t \left( \frac{1}{2} |u|^2 \right) + \nabla \cdot \left( \left( \frac{1}{2} |u|^2 + p \right) u \right) - \Delta \left( \frac{1}{2} |u|^2 \right) + |\nabla u|^2 \leq 0 \quad (1.5)$$

in  $\mathcal{D}'((0, T) \times \Omega)$ .

Since at present results of uniqueness for weak solutions are not known, we cannot exclude that each method used to construct weak solutions can produce its own class of solutions and these solutions could not satisfy the local energy inequality. For this particular issue see also the recent review in Robinson, Rodrigo, and Sadowski [35]. It is then a relevant question to check whether weak solutions obtained by different methods are suitable or not, especially those constructed with methods which are well established by physical or computational motivations. Together with the construction given in [13] by retarded mollifiers, they also recall that in [36] existence of suitable solution (even if the name did not exist yet) has been obtained for the Cauchy problem without external force. The technical improvements to obtain partial regularity with external forces in the natural  $L^2(\Omega)$  space –or even  $H^{-1/2}(\Omega)$ – arrived only recently with the work of Kukavica [27] (in fact in [13] the force needed to be in  $L^{\frac{5}{3}+\delta}(\Omega)$ ). Here we do not consider external forces and we are only treating the problem of showing if the solutions constructed by certain approximations satisfy (1.5).

In the development of the concept of local energy inequality we recall –in earlier times– the two companion papers by Beirão da Veiga [2, 3] dealing with the hyper-viscosity and a general approximation theorem. Recent results in an exterior domain with the Yosida approximation are also those by Farwig, Kozono, and Sohr [18]. Finally, we mention also that the existence of suitable weak solution has been proved by using some artificial compressibility method, see [11, 17].

Beside the pioneering work in [13], the interest for the notion of suitable solutions has been recently renewed in the context of Large Eddy Simulation (LES) and turbulence models. The interest for suitable solutions comes also from the fact that the local energy inequality seems a natural request (representing a sort of entropy) for any reasonable approximation of the NSE. It is especially in the field of turbulent models and LES that Guermond *et al.* [22, 24, 25] –making a parallel with the notion of entropy solutions– suggested that LES models should select “*physically relevant*” solutions of the Navier-Stokes equations, that is those satisfying the local energy inequality.

In the periodic setting it is known that most of the models belonging to the  $\alpha$ -family produce suitable weak solutions, in the limit as  $\alpha \rightarrow 0^+$ ; recall e.g. the results on Leray- $\alpha$  approximation [23], see also the last Sec. 5. The LES models are aimed at producing approximation of the average velocity

and dimensionally  $\alpha$  is a length, connected with the smallest resolved scale. Most of the LES models are designed for the space-periodic setting, to simulate homogeneous turbulence, and their introduction is generally considered *challenging* in presence of boundaries, see [7, 15]. Among these methods the NSV (also very close to the simplified Bardina model) seems one of the most promising, since it does not require extra boundary conditions. The initial-boundary value problem for the NSV system with Dirichlet data reads as follows: given  $\alpha > 0$  solve

$$u_t^\alpha - \alpha^2 \Delta u_t^\alpha - \Delta u^\alpha + (u^\alpha \cdot \nabla) u^\alpha + \nabla p^\alpha = 0 \quad \text{in } (0, T) \times \Omega, \quad (1.6)$$

$$\nabla \cdot u^\alpha = 0 \quad \text{in } (0, T) \times \Omega, \quad (1.7)$$

$$u^\alpha = 0 \quad \text{on } (0, T) \times \Gamma, \quad (1.8)$$

$$u^\alpha(0, x) = u_0^\alpha(x) \quad \text{in } \Omega. \quad (1.9)$$

The NSV method has been introduced by Oskolkov [32, 33] to model visco-elastic fluids, but the interest in the theory of turbulence and LES models came with the work of Titi *et al.* [14, 34] and also with the simplified Bardina model by Layton and Lewandowski [31]. For these reasons here we consider the NSV model which is one of the few well-posed also in the case of a domain with boundaries. We observe that the problem of convergence towards a suitable weak solutions is not hard in the space-periodic setting, see the last section. On the other hand, for the boundary value problem the result is –as far as we know–not solved yet and as the reader will see it requires some technical care to be tackled. In particular, we will focus on the problems arising from the presence boundaries and first we consider the Dirichlet case, which is the most common. Then, we will consider another popular (especially in turbulence problems [7, 15]) set of boundary condition: the slip at the wall (Navier boundary conditions), which are particularly interesting when studying wall effects for non-homogeneous turbulence. These boundary conditions, which replace (1.8), have been recently studied for various applied and theoretical reasons in [5, 6, 9, 10, 44] and they read as follows

$$\begin{aligned} u \cdot n &= 0 & \text{on } (0, T) \times \Gamma, \\ \omega \times n &= 0 & \text{on } (0, T) \times \Gamma, \end{aligned} \quad (1.10)$$

where  $\omega := \text{curl } u$  and  $n$  is the unit outward normal vector on the boundary  $\Gamma$ . The same conditions can be also used to complement the NSV with the variables  $u^\alpha$ ,  $\omega^\alpha = \text{curl } u^\alpha$ , hence substituting (1.8) by

$$\begin{aligned} u^\alpha \cdot n &= 0 & \text{on } (0, T) \times \Gamma, \\ \omega^\alpha \times n &= 0 & \text{on } (0, T) \times \Gamma. \end{aligned} \quad (1.11)$$

We will recall the notion of solution for both problems and we will show the differences to handle the two set of boundary conditions.

Then, we prove the main results of the paper which are the following two theorems.

**Theorem 1.1.** *Let  $\{u_0^\alpha\}_\alpha \subset H_{0,\sigma}^1(\Omega) \cap H^2(\Omega)$  be a sequence of initial data converging strongly in  $H_{0,\sigma}^1(\Omega)$  to  $u_0$ . Let  $(u^\alpha, p^\alpha)$  be the corresponding unique weak solution of the NSV system (1.6)-(1.9) with Dirichlet boundary*

conditions (1.8). Then, there exists  $(u, p)$  such that –up to a sub-sequence still labeled as  $\{(u^\alpha, p^\alpha)\}_\alpha$ – it holds as  $\alpha \rightarrow 0^+$

- 1)  $u^\alpha \rightarrow u$  strongly in  $L^2((0, T) \times \Omega)$ ,
- 2)  $\nabla u^\alpha \rightharpoonup \nabla u$  weakly in  $L^2((0, T) \times \Omega)$ ,
- 3)  $p^\alpha \rightharpoonup p$  weakly in  $H^{-r}(0, T; H^{\frac{3}{10}}(\Omega))$  for all  $r > \frac{2}{5}$ ,
- 4) The couple  $(u, p)$  is a weak solution of the NSE (1.1) and it satisfies the local energy inequality (1.5).

*Remark 1.2.* It is also possible to show that the solution can be slightly changed to be suitable in the usual sense, see Corollary 4.2.

**Theorem 1.3.** Let  $\{u_0^\alpha\}_\alpha \subset L^2_\sigma(\Omega) \cap H^2(\Omega)$  be a sequence of initial data satisfying (1.11) and converging strongly in  $L^2_\sigma(\Omega) \cap H^1(\Omega)$  to  $u_0$ . Let  $(u^\alpha, p^\alpha)$  be the corresponding unique weak solution of the NSV system (1.6)-(1.9) with Navier-type boundary conditions (1.8). Then, there exists  $(u, p)$  such that –up to a sub-sequence still labeled as  $\{(u^\alpha, p^\alpha)\}_\alpha$ – it holds as  $\alpha \rightarrow 0^+$

- 1)  $u^\alpha \rightarrow u$  strongly in  $L^2((0, T) \times \Omega)$ ,
- 2)  $\nabla u^\alpha \rightharpoonup \nabla u$  weakly in  $L^2((0, T) \times \Omega)$ ,
- 3)  $p^\alpha \rightharpoonup p$  weakly in  $L^{\frac{5}{3}}((0, T) \times \Omega)$ ,
- 4) The couple  $(u, p)$  is a suitable weak solution of the NSE (1.1).

Another important open problem is whether Faedo-Galerkin approximation methods produce solutions which are suitable or not. As far as we know, there are only partial results of Guermond [20, 21], concerning approximations made by a special (but very large) class of Finite-Element spaces for the velocity and pressure. Anyway, the Fourier-Galerkin method (obtained by Fourier series expansion in the space-periodic case) is a case still not covered by the theory. In the last section we will consider the space-periodic case, that is  $\Omega = \mathbb{T} := (\mathbb{R}/2\pi\mathbb{Z})^3$  (the three-dimensional torus), and we require that all variables have vanishing mean value on  $\Omega$ . We will recall some results making connections with the Fourier-Galerkin methods and in the final section we also give the following result, inspired by Biryuk, Craig, and Ibrahim [12].

**Theorem 1.4.** Let in the space-periodic setting  $\{(u^{\alpha_n, n}, p^{\alpha_n, n})\}_{n \in \mathbb{N}}$  be the Fourier-Galerkin approximation for the system NSV (1.6) up to the wave-number  $|k| = n$ , and corresponding to the regularization parameter  $\alpha = \alpha_n$ . If  $\frac{1}{\alpha_n} = o(n^6)$  then there exists  $(u, p)$  such that, as  $n \rightarrow +\infty$

- 1)  $u^{\alpha_n, n} \rightarrow u$  strongly in  $L^2((0, T) \times \Omega)$ ,
- 2)  $\nabla u^{\alpha_n, n} \rightharpoonup \nabla u$  weakly in  $L^2((0, T) \times \Omega)$ ,
- 3)  $p^{\alpha_n, n} \rightharpoonup p$  weakly in  $L^{\frac{5}{3}}((0, T) \times \Omega)$ ,
- 4)  $(u, p)$  is a suitable weak solution of the space-periodic NSE.

**Plan of the paper.** In Section 2 we are going to fix the notation that we use in the paper and we recall the main definition and tools used. In Section 3 there are the a priori estimates we will use in Section 4 to prove Theorem 1.1-1.3. Finally, in Section 5 we prove Theorem 1.4.

## 2. NOTATION AND PRELIMINARIES

We start by recalling the functional spaces we will use. Let  $\Omega \subset \mathbb{R}^3$  be a smooth and bounded open set. The space of compactly supported smooth functions on  $\Omega$  will be denoted by  $\mathcal{D}(\Omega)$ . We will denote with  $(L^p(\Omega), \|\cdot\|_p)$  the standard Lebesgue space and to simplify the notation we will denote by  $\|\cdot\|$  the  $L^2(\Omega)$ -norm and the corresponding scalar product by  $(\cdot, \cdot)$ . The Sobolev space of functions with  $k$ -distributional derivatives in  $L^p(\Omega)$  is denoted by  $W^{k,p}(\Omega)$  and their norm with  $\|\cdot\|_{W^{k,p}}$ . As customary we define  $H^k(\Omega) := W^{k,2}(\Omega)$ . We will also use the Bochner spaces  $L^p(0, T; W^{k,q}(\Omega))$ ,  $L^p(0, T; H^s(\Omega))$  and  $H^r(0, T; H^s(\Omega))$ , for the precise definitions see for instance [16, 43]. We use the subscript "σ" to denote the subspace of solenoidal vector fields, obtained by using the Leray projection operator  $P$  over tangential and divergence-free vector fields, see [19, 38]. In particular, it is standard to introduce the following spaces

$$\begin{aligned} L_\sigma^2(\Omega) &= \{u \in L^2(\Omega) : \nabla \cdot u = 0, u \cdot n = 0 \text{ on } \partial\Omega\}, \\ H_{0,\sigma}^1(\Omega) &= \{u \in H_0^1(\Omega) : \nabla \cdot u = 0, u = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

Then, the Stokes operator  $(A, D(A))$  is defined as follows:

$$\begin{aligned} A : D(A) &\rightarrow L_\sigma^2(\Omega), \\ D(A) &= H_{0,\sigma}^1(\Omega) \cap H^2(\Omega), \\ Au &= -P\Delta u, \end{aligned} \tag{2.1}$$

where  $-\Delta : H^2(\Omega) \cap H_0^1(\Omega) \rightarrow L^2(\Omega)$  denotes the Laplace operator with homogeneous Dirichlet boundary condition and  $P$  denotes the projection in  $L^2(\Omega)$  onto  $L_\sigma^2(\Omega)$ . We choose the open set  $\Omega \subset \mathbb{R}^3$  smooth enough such that the following inequality holds true

$$\|u\|_{H^2} \leq C \|Au\| \quad \text{for some } C \geq 0.$$

We recall some technical results on the Navier-Stokes equations we will need during the proof of the main theorems.

**2.1. Initial Value boundary problem for the NSE.** In this subsection we define the suitable weak solutions of NSE with boundary condition (1.3) or (1.10) and initial datum  $u_0 \in L_\sigma^2(\Omega)$ . With Dirichlet conditions the notion of Leray-Hopf weak solution for the NSE is well-known, see for instance [19]. The notion of weak solution to the NSV can be found in [14, 29].

We recall now the definition of suitable weak solution.

**Definition 2.1.** *Let  $(u, p)$  be such that  $u \in L^2(0, T; H_{0,\sigma}^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ , and  $p \in L^{\frac{5}{3}}((0, T) \times \Omega)$ , and  $u$  is a Leray-Hopf weak solution to the Navier-Stokes equation (1.1). The pair  $(u, p)$  is said a suitable weak solutions if the local energy balance (1.5) holds in the distributional sense.*

As we said in the introduction one of the main differences between Leray-Hopf weak solutions and suitable weak solutions is the inequality (1.5). Indeed, by using the results of [39] in the case of boundary conditions (1.3) and the Poisson equation associated to the pressure in the case of boundary conditions (1.10), see also Lemma 3.10, it is always possible to associate to  $u$  a scalar pressure  $p$  with the regularity stated in Definition 2.1. However, it not

know how to prove inequality (1.5) for general Leray-Hopf weak solutions. Indeed (1.5) is an entropy inequality and formally is derived by multiplying the first equation of (1.1) by  $u\phi$ , with  $\phi$  a positive, compactly supported test function, and by integrating by parts in space and time. The regularity of a weak solution is not enough to directly justify all the calculation leading to (1.5).

The technicalities often rely in showing certain regularity of the pressure field. This can be done rather easily for the problem without boundaries, since the pressure satisfies the Poisson equation

$$-\Delta p^\alpha = \partial_i \partial_j (u_i^\alpha u_j^\alpha). \quad (2.2)$$

In the case of periodic boundary condition or in the whole space, from (2.2) it is easy to get the necessary regularity to treat the pressure term in (1.5), once good estimates are known on the velocity.

In a bounded domain with Dirichlet boundary conditions the situation becomes more subtle. One possible approach would be that of using the semigroup theory, in the spirit of the estimates by Sohr and von Wahl [39] and Solonnikov [40, 41], but this seems difficult because the approximate system NSV doesn't fit directly in the type of equations used in that theory, being a pseudo-parabolic system. Hence, in this paper we use a different approach consisting of getting a pressure estimate in negative fractional Sobolev spaces which is weaker than the classical  $L^p((0, T); L^q(\omega))$  mixed estimates available for the NSE. This method has been successfully used by Guermond [21] to prove that some special Galerkin methods yield suitable weak solutions. In particular, the difficulties arising from the boundary condition can be circumvented by using appropriate fractional powers of the Stokes operator. It is for this reason that we need a very detailed treatment of the Stokes operator.

**2.2. Stokes operator and negative fractional Sobolev space.** We focus now on some special properties of the Stokes operator  $-P\Delta$ . As said in the introduction in the case of Dirichlet boundary conditions (1.3) we will get uniform bound for the pressure in negative fractional Sobolev space and we will make an extensively use of the Stokes operator and its powers. In this subsection we recall the main definitions and the main properties we will use in the sequel, following very closely [21]. Let  $\mathcal{H}$  be an Hilbert space and  $L^q(\mathbb{R}; \mathcal{H})$  with  $q \geq 1$  be the associated Bochner space. For  $\psi \in L^1(\mathbb{R}; \mathcal{H})$  and  $\xi \in \mathbb{R}$  we define the Fourier transform (with respect to the time variable)  $\widehat{\psi}(\xi)$  as follows

$$\widehat{\psi}(\xi) := \int_{\mathbb{R}} \psi(t) e^{-2i\pi\xi t} dt.$$

The previous definition can be extended to the space of tempered distribution  $\mathcal{S}'(\mathbb{R}; \mathcal{H})$ . As usual we define  $\mathcal{H}^\gamma(\mathbb{R}; \mathcal{H})$  the space of tempered distributions  $v \in \mathcal{S}'(\mathbb{R}; \mathcal{H})$  such that

$$\int_{\mathbb{R}} (1 + |\xi|)^{2\gamma} \|\widehat{v}(\xi)\|_{\mathcal{H}}^2 d\xi < +\infty.$$

The space  $H^\gamma(0, T; \mathcal{H})$  is defined by those distributions that can be extended to  $\mathcal{S}'(\mathbb{R}; \mathcal{H})$  and whose extension is in  $H^\gamma(0, T; \mathcal{H})$ . The norm in  $H^\gamma(0, T; \mathcal{H})$



is the quotient norm, namely,

$$\|v\|_{H^\gamma(0,T;\mathcal{H})} = \inf_{v=u \text{ a.e. on } (0,T)} \|v\|_{H^\gamma(\mathbb{R};\mathcal{H})}.$$

As Hilbert spaces  $\mathcal{H}$  we will take the classical Sobolev spaces  $H^s(\Omega)$ , allowing also for fractional values of  $s$ . Since we are considering functions in a bounded domain with zero boundary value, we need to be rather precise about the definition of the function space, when  $s \notin \mathbb{N}$ .

The space  $H^s(\Omega)$  is defined via the real method of interpolation as follows

$$H^s(\Omega) = \begin{cases} [L^2(\Omega), H^1(\Omega)]_s & \text{for } s \in (0, 1), \\ [H^1(\Omega), H^2(\Omega)]_s & \text{for } s \in (1, 2). \end{cases}$$

Next, to deal with zero traces, we introduce the space  $H_0^s(\Omega)$  which is the closure of  $\mathcal{D}(\Omega)$  in  $H^s(\Omega)$  and for any  $s \in (0, 1)$ . The space  $\widetilde{H}_0^s(\Omega)$  is defined as follows

$$\widetilde{H}_0^s(\Omega) = \begin{cases} [L^2(\Omega), H_0^1(\Omega)]_s & \text{for } s \in [0, 1], \\ H^s(\Omega) \cap H_0^1(\Omega) & \text{for } s \in (1, 2]. \end{cases}$$

Finally, for  $s > 0$  we denote with  $\widetilde{H}_0^{-s}(\Omega)$  the dual space of  $\widetilde{H}_0^s(\Omega)$  and by  $H^{-s}(\Omega)$  the space defined via the norm coming from the duality

$$\|u\|_{H^{-s}} = \sup_{0 \neq w \in \mathcal{D}(\Omega)} \frac{(u, w)}{\|w\|_{H^s}}.$$

It is well-known that the following spaces coincide with equivalent norms:

$$\begin{aligned} H^s(\Omega) &\cong H_0^s(\Omega) & s \in [0, 1/2], \\ H^s(\Omega) &\cong \widetilde{H}_0^s(\Omega) & s \in [0, 1/2), \\ H^{-s}(\Omega) &\cong \widetilde{H}_0^{-s}(\Omega) & s \in [0, 1/2) \cup (1/2, 3/2). \end{aligned}$$

Since we will not consider the critical case  $s = \frac{1}{2}$  we will always denote these spaces with  $H^s(\Omega)$ . From the definition (2.1) it follows that the operator  $A$  is positive and self-adjoint. By using the spectral theorem we can define its fractional powers, specifically we can define  $A^s$ , for any  $s \in \mathbb{R}$ , on its domain which we denote with  $D(A^s)$ . We have that the quantity  $(u, A^s u)$  is a norm on the subspace  $D(A^{\frac{s}{2}})$ . The following equivalences of norms will be frequently used in the sequel: There exists  $c_1, c_2 > 0$  such that

$$\begin{aligned} c_1 \|u\|_{\widetilde{H}^s} &\leq (u, A^s u)^{\frac{1}{2}} \text{ for any } u \in D(A^{\frac{s}{2}}), \quad s \in (-1/2, 2], \\ (u, A^s u)^{\frac{1}{2}} &\leq c_2 \|u\|_{\widetilde{H}^s} \text{ for any } u \in D(A^{\frac{s}{2}}), \quad s \in [-2, 2]. \end{aligned} \tag{2.3}$$

For the proof of the inequalities (2.3) see [26].

**2.3. The Navier-Stokes-Voigt model.** In this subsection we recall the main result regarding the approximating system (1.6).

**Theorem 2.2.** *Let  $\alpha > 0$  be fixed and  $u_0^\alpha \in H_{0,\sigma}^1(\Omega)$ . Then, there exists a unique weak solution  $u^\alpha \in L^\infty(0, T; H_{0,\sigma}^1(\Omega))$  and  $p^\alpha \in L^2(0, T; L^2(\Omega))$  (with norms depending on  $\alpha > 0$ ) of the initial value boundary problem (1.6).*

*In addition, if  $u_0^\alpha \in H^2(\Omega) \cap H_{0,\sigma}^1(\Omega)$ . Then,  $u^\alpha \in L^\infty(0, T; H^2(\Omega) \cap H_{0,\sigma}^1(\Omega))$  and  $p^\alpha \in L^2(0, T; H^1(\Omega))$ .*

We do not prove this theorem. The existence and uniqueness parts can be found in [14] and the regularity can be proved by standard energy estimates. For the sequel are very relevant the estimates in terms of  $\alpha > 0$  for several norms, see Lemmas 3.1- 3.2. Especially the first one is crucial also for the existence of weak solutions. In particular, uniform estimates on the Galerkin-approximate system allow to pass to the limit with standard compactness results [14]. More delicate is the question of space regularity. Since the NSV system is pseudo-parabolic there is not an increase of regularity as for the parabolic equations, but the solutions keep the regularity of the initial datum, as in the hyperbolic case [29, 30]. An explicit example of this is given in [8]. This motivates the request that the initial datum belongs to  $H^2(\Omega)$ , since otherwise calculations of the next sections would be formal and not justified.

*Remark 2.3.* With a procedure of approximation the condition on the initial datum could be slightly relaxed.

**2.4. On slip boundary conditions.** First we recall some definitions and technical facts when dealing with the NSE and NSV with the Navier-type boundary conditions (1.10)-(1.11), respectively.

**Definition 2.4.** *We say that  $u \in L^\infty(0, T; L_\sigma^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ , is a (Leray-Hopf) weak solution of the NSE (1.1) with boundary conditions (1.10) if the two following hold true:*

$$\begin{aligned} \int_0^T \int_\Omega (-u\phi_t + \nabla u \nabla \phi - (u \cdot \nabla) \phi u) \, dx dt + \int_0^T \int_\Gamma u \cdot (\nabla n)^T \cdot \phi \, dS dt \\ = \int_\Omega u_0 \phi(0) \, dx, \end{aligned}$$

for all vector-fields  $\phi \in C_0^\infty([0, T] \times \overline{\Omega})$  such that  $\nabla \cdot \phi = 0$  in  $[0, T] \times \Omega$ , and  $\phi \cdot n = 0$  on  $[0, T] \times \Gamma$ . Moreover, the following energy estimate

$$\frac{1}{2} \|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2 \, ds + \int_0^t \int_\Gamma u \cdot (\nabla n)^T \cdot u \, dS ds \leq \frac{1}{2} \|u_0\|^2, \quad (2.4)$$

is satisfied for all  $t \in [0, T]$ .

With this definition we have the following result.

**Theorem 2.5.** *Let be given any positive  $T > 0$  and  $u_0 \in L_\sigma^2(\Omega)$ , then there exists at least a weak solution  $u$  of the Navier-Stokes equations (1.1) on  $[0, T]$ .*

The proof of global existence of weak solution in the sense of the Definition 2.4 can be found for instance in [44, § 6]. We observe that an equivalent formulation can be given. To this end we recall the following formulas for integration by parts (see [4] for the proof).

**Lemma 2.6.** *Let  $u$  and  $\phi$  be two smooth enough vector fields, tangential to the boundary  $\Gamma$ . Then it follows*

$$-\int_\Omega \Delta u \phi \, dx = \int_\Omega \nabla u \nabla \phi \, dx - \int_\Gamma (\omega \times n) \phi \, dS + \int_\Gamma u \cdot (\nabla n)^T \cdot \phi \, dS,$$



where  $\omega = \operatorname{curl} u$ . Moreover, if  $\nabla \cdot u = 0$ , then  $-\Delta u = \operatorname{curl} \operatorname{curl} u$ , and

$$\int_{\Omega} \operatorname{curl} \omega \phi \, dx = - \int_{\Omega} \Delta u \phi \, dx = \int_{\Omega} \omega (\operatorname{curl} \phi) \, dx + \int_{\Gamma} (\omega \times n) \phi \, dS.$$

With the above formulas the weak formulation can be written as follows

$$\int_0^T \int_{\Omega} (-u \phi_t + \omega \operatorname{curl} \phi - (u \cdot \nabla) \phi u) \, dx d\tau = \int_{\Omega} u_0 \phi(0) \, dx,$$

for all vector-fields  $\phi \in C_0^\infty([0, T[ \times \bar{\Omega})$  such that  $\nabla \cdot \phi = 0$  in  $[0, T[ \times \Omega$ , and  $\phi \cdot n = 0$  on  $[0, T[ \times \Gamma$ . Moreover, the following energy estimate holds true

$$\frac{1}{2} \|u(t)\|^2 + \int_0^t \|\omega(s)\|^2 \, ds \leq \frac{1}{2} \|u_0\|^2 \quad \forall t \in [0, T]. \quad (2.5)$$

Next, when we consider the NSV system with Navier conditions and we have the following definition.

**Definition 2.7.** *We say that  $u \in L^\infty(0, T; L_\sigma^2(\Omega) \cap H^1(\Omega))$ , weak solution of the NSV (1.6) with boundary conditions (1.11) if the two following condition hold:*

$$\begin{aligned} \int_0^T \int_{\Omega} (-u^\alpha \phi_t - \alpha^2 \omega^\alpha \operatorname{curl} \phi_t + \nabla u^\alpha \nabla \phi - (u^\alpha \cdot \nabla) \phi u^\alpha) \, dx dt \\ = \int_{\Omega} u_0^\alpha \phi(0) + \alpha^2 \nabla u_0^\alpha \nabla \phi(0) \, dx, \end{aligned}$$

for all vector-fields  $\phi \in C_0^\infty([0, T[ \times \bar{\Omega})$  such that  $\nabla \cdot \phi = 0$  in  $[0, T[ \times \Omega$ , and  $\phi \cdot n = 0$  on  $[0, T[ \times \Gamma$ . Moreover, the following energy estimate

$$\frac{1}{2} \|u^\alpha(t)\|^2 + \frac{\alpha^2}{2} \|\omega^\alpha(t)\|^2 + \int_0^t \|\omega^\alpha(s)\|^2 \, ds = \frac{1}{2} \|u_0^\alpha\|^2 + \frac{\alpha^2}{2} \|\omega_0^\alpha\|^2, \quad (2.6)$$

is satisfied for all  $t \in [0, T]$ .

With this definition one can easily prove the following result (for which we did not find any reference)

**Theorem 2.8.** *Let be given any  $T > 0$ ,  $\alpha > 0$  and  $u_0^\alpha \in L_\sigma^2(\Omega) \cap H^1(\Omega)$ , then there exists a unique weak solution  $u^\alpha$  of the NSV (1.6) with Navier boundary conditions (1.11) on  $[0, T]$ . Moreover, if  $u_0^\alpha \in L_\sigma^2(\Omega) \cap H^2(\Omega)$  and  $\omega_0^\alpha \times n = 0$  at  $\partial\Omega$ , then the unique solution belongs also to  $L^\infty(0, T; H^2(\Omega) \cap L_\sigma^2(\Omega))$ .*

The proof of this result goes through the a priori estimates obtained testing with  $u^\alpha$  and  $-P\Delta u^\alpha$ , see especially those obtained in Lemmas 3.7-3.8.

### 3. A PRIORI ESTIMATES INDEPENDENT OF $\alpha$ FOR SOLUTIONS OF THE NAVIER-STOKES-VOIGT MODEL

In this section we are going to prove the main  $\alpha$ -independent *a priori* estimates needed in the proof of Theorems 1.1-1.3.

**3.1. The case of vanishing Dirichlet Boundary Conditions.** The first estimate we prove is the standard energy-type estimate one obtained by multiplying equations (1.6) by  $u^\alpha$  and by integrating by parts over  $\Omega$ . We have the following result.

**Lemma 3.1.** *Let  $u^\alpha$  be a weak solution of the NSV (1.6)-(1.9) with initial datum  $u_0^\alpha \in H_{0,\sigma}^1(\Omega)$ . Then, for any  $t \in (0, T)$*

$$\|u^\alpha(t)\|^2 + \alpha^2 \|\nabla u^\alpha(t)\|^2 + 2 \int_0^t \|\nabla u^\alpha(s)\|^2 ds = \|u_0^\alpha\|^2 + \alpha^2 \|\nabla u_0^\alpha\|^2. \quad (3.1)$$

Then we prove a simple weighted (in  $\alpha$ ) estimate for  $u_t^\alpha$  which will be useful to pass to the limit to get the local energy inequality (1.5).

**Lemma 3.2.** *Let  $u^\alpha$  be a weak solution of the NSV (1.6)-(1.9) with initial datum  $u_0^\alpha \in H_{0,\sigma}^1(\Omega)$ . Then, there exists  $c > 0$  independent of  $\alpha$  such that for any  $t \in (0, T)$*

$$\alpha^3 \|\nabla u^\alpha\|^2 + \alpha^3 \int_0^t \|u_t^\alpha(s)\|^2 ds + \alpha^5 \int_0^t \|\nabla u_t^\alpha(s)\|^2 ds \leq c.$$

*Proof.* Let us multiply the momentum equation in (1.6) by  $\alpha^3 u_t^\alpha$ . By integrating by parts over  $(0, T) \times \Omega$  we get

$$\begin{aligned} \frac{\alpha^3}{2} \|\nabla u^\alpha(t)\|^2 + \int_0^t \alpha^3 \|u_t^\alpha(s)\|^2 + \alpha^5 \|\nabla u_t^\alpha(t)\|^2 ds \\ \leq \alpha^3 \int_0^t \int_\Omega |u^\alpha| |\nabla u^\alpha| |u_t^\alpha| dx ds + \frac{\alpha^3}{2} \|\nabla u_0^\alpha\|^2. \end{aligned}$$

We estimate the right-hand side by using Hölder and the standard Gagliardo-Nirenberg type interpolation inequality (of  $L^4(\Omega)$  with  $L^2(\Omega)$  and  $H_0^1(\Omega)$ ) as follows

$$\begin{aligned} \alpha^3 \int_0^t \int_\Omega |u^\alpha| |\nabla u^\alpha| |u_t^\alpha| dx ds &\leq \alpha^3 \int_0^t \|u^\alpha\|_4 \|\nabla u^\alpha\| \|u_t^\alpha\|_4 ds \\ &\leq c \alpha^3 \int_0^t \|u^\alpha\|^{\frac{1}{4}} \|\nabla u^\alpha\|^{\frac{7}{4}} \|\nabla u_t^\alpha\|^{\frac{3}{4}} \|u_t^\alpha\|^{\frac{1}{4}} ds \\ &\leq c \alpha^3 (\|u_0^\alpha\|^2 + \alpha^2 \|\nabla u_0^\alpha\|^2)^{\frac{1}{8}} \int_0^t \|\nabla u^\alpha\|^{\frac{7}{4}} \|\nabla u_t^\alpha\|^{\frac{3}{4}} \|u_t^\alpha\|^{\frac{1}{4}} ds, \end{aligned}$$

where in the second line we use, by Lemma 3.1, the fact the  $u^\alpha$  is bounded in  $L^\infty(0, T; L^2(\Omega))$  independently of  $\alpha$ . Now, we use Young inequality with  $p_1 = \frac{8}{3}$ ,  $p_2 = 8$  and  $p_3 = 2$  and we get

$$\begin{aligned} \alpha^3 \int_0^t \int_\Omega |u^\alpha| |\nabla u^\alpha| |u_t^\alpha| dx ds \\ \leq c (\|u_0^\alpha\|^2 + \alpha^2 \|\nabla u_0^\alpha\|^2)^{\frac{1}{4}} \int_0^t \alpha^{\frac{3}{2}} \|\nabla u^\alpha\|^{\frac{3}{2}} \|\nabla u^\alpha\|^2 ds + \frac{\alpha^3}{2} \int_0^t \|u_t\|^2 ds \\ + \frac{\alpha^5}{2} \int_0^t \|\nabla u_t\|^2 ds. \end{aligned}$$

Then, by using (3.1) we have that  $\alpha^{\frac{3}{2}} \|\nabla u^\alpha(s)\|_{\frac{3}{2}} \leq (\|u_0^\alpha\|^2 + \alpha^2 \|\nabla u_0^\alpha\|^2)^{\frac{3}{4}}$ , and consequently we get

$$\begin{aligned} \frac{\alpha^3}{2} \|\nabla u^\alpha(t)\|^2 + \int_0^T \alpha^3 \|u_t^\alpha(s)\|^2 ds + \alpha^5 \int_0^T \|\nabla u_t^\alpha(s)\|^2 ds \\ \leq \frac{\alpha^3}{2} \|\nabla u_0^\alpha\|^2 + c(\|u_0^\alpha\|^2 + \alpha^2 \|\nabla u_0^\alpha\|^2) \int_0^T \|\nabla u^\alpha(s)\|^2 ds \\ \leq \frac{\alpha^3}{2} \|\nabla u_0^\alpha\|^2 + c(\|u_0^\alpha\|^2 + \alpha^2 \|\nabla u_0^\alpha\|^2)^2 \leq c, \end{aligned}$$

if  $0 < \alpha < 1$ . We can suppose that  $\alpha$  is always smaller than one, since we are interested in the behavior as  $\alpha \rightarrow 0^+$ . In particular, for our purposes the most relevant estimate is that there exists a constant  $c$  independent of  $\alpha > 0$  such that

$$\alpha^3 \int_0^T \|u_t^\alpha(s)\|^2 ds \leq c. \quad (3.2)$$

□

**3.1.1. Estimates in fractional Sobolev space.** This section is devoted to the proof of the estimate for the pressure in the case of Dirichlet boundary conditions. We follow the same line of [21]. Let be  $p, q, \bar{r}$  and  $s$  real positive numbers such that the following relations hold:

$$\frac{2}{p} + \frac{3}{q} = 4, \quad p \in [1, 2], \quad q \in [1, \frac{3}{2}], \quad \frac{s}{3} := \frac{1}{q} - \frac{1}{2}, \quad \bar{r} := \frac{1}{p} - \frac{1}{2}. \quad (3.3)$$

If  $p, q$  satisfy (3.3) we have by Sobolev embedding that

$$L^p(0, T; L^q(\Omega)) \subset H^{-r}(0, T; H^{-s}(\Omega)).$$

The first lemma we recall is an estimate for the nonlinear term in negative-fractional Sobolev spaces. See [21, Lemma 3.4].

**Lemma 3.3.** *Let  $u^\alpha$  be a weak solution to NSV, then for any  $s \in [\frac{1}{2}, \frac{3}{2}]$ , there exists a constant  $c > 0$ , independent of  $\alpha$  such that*

$$\|(u^\alpha \cdot \nabla) u^\alpha\|_{H^{-r}(0, T; H^{-s})} \leq c. \quad (3.4)$$

*Proof.* By Sobolev embedding and interpolation inequality we have that if  $u^\alpha \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ , then by duality

$$\|(u^\alpha \cdot \nabla) u^\alpha\|_{L^p(0, T; H^{-s})} \leq c, \quad (3.5)$$

for any  $p \in [1, 2]$  and  $s \in [\frac{1}{2}, \frac{3}{2}]$  such that  $\frac{2}{p} + s = \frac{7}{2}$ . In particular, in the above lemma is exactly the same which is valid for Leray-Hopf weak solutions to the NSE (1.1).

Then, we extend  $(u^\alpha \cdot \nabla) u^\alpha$  to 0 out of  $(0, T)$  and we take the Fourier transform with respect to the time variable. By using Hölder inequality and Hausdorff-Young inequality we get (3.4). □

Then, we use this information on the convective term, when considered as a right-hand side, to infer further properties of  $(u^\alpha, p^\alpha)$ . This is more or less the same approach as in [21] and it is based on an extension of classical results on fractional derivatives. The relevant point is that one can use Hilbert-space techniques at the price of working with negative norms. The

following Lemma is a refined estimate of the velocity in fractional Sobolev spaces.

**Lemma 3.4.** *For any  $\chi \in [\frac{1}{4}, \frac{1}{2})$  and  $\tau < \bar{\tau} = \frac{2}{5}(1 + \chi)$  there exists  $c > 0$ , independent of  $\alpha$ , such that*

$$\|u^\alpha\|_{H^\tau(0,T;H^{-\chi})} \leq c. \quad (3.6)$$

*Proof.* We write the system (1.6) in the following way

$$\begin{aligned} u_t^\alpha - \alpha^2 \Delta u_t^\alpha - \Delta u^\alpha + \nabla p^\alpha &= -(u^\alpha \cdot \nabla) u^\alpha & \text{in } (0, T) \times \Omega, \\ \nabla \cdot u^\alpha &= 0 & \text{in } (0, T) \times \Omega. \end{aligned} \quad (3.7)$$

By applying  $P$  to the equations (3.7) we get

$$u_t^\alpha + \alpha^2 A u_t^\alpha + A u^\alpha = -P((u^\alpha \cdot \nabla) u^\alpha) \quad \text{in } (0, T). \quad (3.9)$$

where  $A$  is the Stokes operator. Since we are going to use Fourier transform with respect to time we need to extend all the functions from  $[0, T]$  to  $\mathbb{R}$ . We extend  $u^\alpha$  by  $(t+1)u_0^\alpha$  on  $[-1, 0]$  and by 0 on  $[T+1, \infty)$ . We denote this extension by  $\bar{u}^\alpha$ . Let  $\varphi \in C^\infty(\mathbb{R})$  be such that  $\text{supp}(\varphi) \subset (-1, T+1)$  and  $\varphi \equiv 1$  on  $[0, T]$ , we denote with a slight abuse of notation

$$u^\alpha = \varphi \bar{u}^\alpha.$$

Next, we define the following function

$$f^\alpha = \begin{cases} (1+t)\varphi'(t)(I + \alpha^2 A)u_0^\alpha + \varphi(t)(I + \alpha^2 A)u_0^\alpha \\ \quad - \varphi(t)(1+t)A u_0^\alpha & t \in (-1, 0), \\ -\varphi(t)P((u^\alpha \cdot \nabla) u^\alpha) + \varphi'(t)(I + \alpha^2 A)u^\alpha & t \notin (-1, 0). \end{cases}$$

It follows that  $u^\alpha$  e  $f^\alpha$  are well defined on  $(-\infty, +\infty)$ . Then, (3.9) becomes

$$u_t^\alpha + \alpha^2 A u_t^\alpha + A u^\alpha = f^\alpha \quad \text{in } t \in \mathbb{R}. \quad (3.10)$$

By using (3.4) with  $s = \frac{3}{2}$  we get that for any  $r > 0$  there exists  $c > 0$ , independent of  $\alpha$ , such that

$$\|f^\alpha\|_{H^{-r}(0,T;H^{-\frac{3}{2}})} \leq c. \quad (3.11)$$

Next, we take the Fourier transform of (3.10) with respect to  $t$  we get the following (abstract) equation in  $L^2(\Omega)$

$$2\pi i \xi (\widehat{u}^\alpha + \alpha^2 A \widehat{u}^\alpha) + A \widehat{u}^\alpha = P \widehat{f}^\alpha, \quad (3.12)$$

and it is at this point that we require the initial datum in  $H^2(\Omega)$  in order that  $A \widehat{u}^\alpha$  is well defined in  $L^2(\Omega)$ . Let  $\chi$  as in the statement and let us take the  $L^2(\Omega)$ -scalar product of the equations (3.12) and  $A^{-\chi} \widehat{u}^\alpha$ . We then obtain

$$2\pi i \xi [(\widehat{u}^\alpha, A^{-\chi} \widehat{u}^\alpha) + \alpha^2 (A \widehat{u}^\alpha, A^{-\chi} \widehat{u}^\alpha)] + (A \widehat{u}^\alpha, A^{-\chi} \widehat{u}^\alpha) = (P \widehat{f}^\alpha, A^{-\chi} \widehat{u}^\alpha).$$

Then, since  $A$  is self-adjoint and positive,  $(A \widehat{u}^\alpha, A^{-\chi} \widehat{u}^\alpha)$  is real and non-negative. By taking the imaginary part of both sides we get

$$|\xi| |(\widehat{u}^\alpha, A^{-\chi} \widehat{u}^\alpha)| \leq c |(P \widehat{f}^\alpha, A^{-\chi} \widehat{u}^\alpha)|.$$

Since  $\chi < \frac{1}{2}$  we can use the norm equivalence (2.3) and

$$\|\widehat{u}^\alpha\|_{H^{-\chi}}^2 \leq c (\widehat{u}^\alpha, A^{-\chi} \widehat{u}^\alpha).$$

Concerning the right-hand side of (3.12) by using again (2.3) we have

$$\begin{aligned} \|A^{-\chi}\widehat{u}^\alpha\|_{H^{\frac{3}{2}}}^2 &\leq (A^{-\chi}\widehat{u}^\alpha, A^{\frac{3}{2}}A^{-\chi}\widehat{u}^\alpha) \\ &\leq (\widehat{u}^\alpha, A^{\frac{3}{2}-2\chi}\widehat{u}^\alpha) \leq c\|\widehat{u}^\alpha\|_{H^{\frac{3}{2}-2\chi}}^2. \end{aligned}$$

Then, we get

$$|\xi|\|\widehat{u}^\alpha\|_{H^{-\chi}}^2 \leq C\|\widehat{f}^\alpha\|_{H^{-\frac{3}{2}}}\|\widehat{u}^\alpha\|_{H^{\frac{3}{2}-2\chi}}. \quad (3.13)$$

Note that for all  $\chi \in [\frac{1}{4}, \frac{1}{2})$  we have that

$$-\chi < \frac{3}{2} - 2\chi < 1,$$

hence we can interpolate as follows  $H^{\frac{3}{2}-2\chi}(\Omega)$  between  $H^{-\chi}(\Omega)$  and  $H^1(\Omega)$

$$\|\widehat{u}^\alpha\|_{H^{\frac{3}{2}-2\chi}} \leq \|\widehat{u}^\alpha\|_{H^{-\chi}}^\gamma \|\widehat{u}^\alpha\|_{H^1}^{1-\gamma}, \quad (3.14)$$

with  $\gamma = \frac{4\chi-1}{2+2\chi}$ . Inserting (3.14) in (3.13) we have

$$|\xi|\|\widehat{u}^\alpha\|_{H^{-\chi}}^{2-\gamma} \leq c\|\widehat{f}^\alpha\|_{H^{-\frac{3}{2}}}\|\widehat{u}^\alpha\|_{H^1}^{1-\gamma}, \quad (3.15)$$

Since  $L^2(\Omega) \subset H^{-\chi}(\Omega)$  for any  $\chi > 0$ , for  $\gamma \in (0, 1)$  we have the following inequality

$$\|\widehat{u}^\alpha\|_{H^{-\chi}}^{2-\gamma} \leq \|\widehat{u}^\alpha\|_{H^1}^{2-\gamma}. \quad (3.16)$$

By summing up (3.15) and (3.16) we get

$$(1 + |\xi|)\|\widehat{u}^\alpha(\xi)\|_{H^{-\chi}}^{2-\gamma} \leq \|\widehat{f}^\alpha(\xi)\|_{H^{-\frac{3}{2}}}\|\widehat{u}^\alpha(\xi)\|_{H^1}^{1-\gamma} + \|\widehat{u}^\alpha(\xi)\|_{H^1}^{2-\gamma}. \quad (3.17)$$

Let  $r > 0$  and set  $\nu = \frac{2r}{2-\gamma}$ . By dividing both sides by  $(1 + |\xi|)^\nu$  we have

$$\begin{aligned} (1 + |\xi|)^{\frac{2}{2-\gamma}-\nu}\|\widehat{u}^\alpha(\xi)\|_{H^{-\chi}}^2 \\ \leq c(1 + |\xi|)^{-\nu}\|\widehat{f}^\alpha(\xi)\|_{H^{-s}}^{\frac{2}{2-\gamma}}\|\widehat{u}^\alpha(\xi)\|_{H^1}^{\frac{2(1-\gamma)}{2-\gamma}} + \|\widehat{u}^\alpha(\xi)\|_{H^1}^2. \end{aligned} \quad (3.18)$$

By integrating (3.18) with respect to  $\xi \in \mathbb{R}$  and by using Hölder inequality we get

$$\begin{aligned} \int_{\mathbb{R}} (1 + |\xi|)^{\frac{2(1-r)}{2-\gamma}}\|\widehat{u}^\alpha(\xi)\|_{H^{-\chi}}^2 d\xi \\ \leq c\|\widehat{f}^\alpha\|_{H^{-r}(0,T;H^{-\frac{3}{2}})}^{\frac{1}{2-\gamma}}\|\widehat{u}^\alpha\|_{L^2(0,T;H^1)}^{\frac{(1-\gamma)}{2-\gamma}} + \|\widehat{u}^\alpha\|_{L^2(0,T;H^1)}^2. \end{aligned}$$

Note that since  $r > 0$  we have

$$\tau = \frac{1-r}{2-\gamma} < \frac{1}{2-\gamma} = \frac{2}{5}(1+\chi) = \bar{\tau}.$$

By using that  $u^\alpha \in L^2(0, T; H^1(\Omega))$ , which becomes by Plancherel theorem  $\|\widehat{u}^\alpha(\xi)\|_{H^1} \in L^2(\mathbb{R})$ , and (3.11) finally we get (3.6).  $\square$

Once we have an estimate on  $u^\alpha$  in fractional spaces, we can derive a corresponding estimate for  $\Delta u^\alpha$ , by the properties of the Stokes operator.

**Lemma 3.5.** *For all  $s \in [\frac{1}{2}, \frac{3}{2})$  and  $r > \bar{r}$ , there exists  $c$  independent of  $\alpha$  such that*

$$\|\Delta u^\alpha\|_{H^{-r}(0,T;H^{-s})} \leq c. \quad (3.19)$$

*Proof.* First we have that

$$\|\Delta u^\alpha\|_{H^{-s}}^2 \leq c \|Au^\alpha\|_{H^{-s}}^2. \quad (3.20)$$

Note that since  $\frac{1}{2} < 2 - s < \frac{3}{2}$  we could use (2.3). We multiply (3.12) by  $A^{1-s}\widehat{u}^\alpha$  and we get, taking now only the real part,

$$\|A\widehat{u}^\alpha(\xi)\|_{H^{-s}}^2 \leq c \|\widehat{f}^\alpha(\xi)\|_{H^{-s}} \|A\widehat{u}^\alpha(\xi)\|_{H^{-s}}.$$

By simplifying the square of  $\|A\widehat{u}^\alpha\|_{H^{-s}}$ , using (3.20) and integrating in time we get (3.19).  $\square$

Finally we come back to the equations without the Leray projection over divergence-free vector fields. We prove by comparison an estimate for the pressure, which will be used to prove Theorem 1.1.

**Lemma 3.6.** *For any  $r > \frac{2}{5}$  there exists  $c > 0$ , independent of  $\alpha$ , such that*

$$\|p^\alpha\|_{H^{-r}(0,T;H^{\frac{3}{10}})} \leq c. \quad (3.21)$$

*Proof.* We come back to the (1.6) and let us start to estimate the term with the Laplacian of  $u_t^\alpha$ .

$$\begin{aligned} \|\alpha^2 \Delta u_t^\alpha\|_{H^{-r}(0,T;H^{-\frac{7}{10}})} &\leq \|\alpha^2 Au_t^\alpha\|_{H^{-r}(0,T;H^{-\frac{7}{10}})} \\ &\leq \|u_t^\alpha\|_{H^{-r}(0,T;H^{-\frac{7}{10}})} + \|Au^\alpha\|_{H^{-r}(0,T;H^{-\frac{7}{10}})} \\ &\quad + \|f^\alpha\|_{H^{-r}(0,T;H^{-\frac{7}{10}})} \\ &\leq c + \|u^\alpha\|_{H^{1-r}(0,T;H^{-\chi})}, \end{aligned} \quad (3.22)$$

where  $\chi < \frac{1}{2}$  and we have used (2.3), (3.4) and (3.19). Then, we have

$$\begin{aligned} \|p^\alpha\|_{H^{-r}(0,T;H^{\frac{3}{10}})} &\leq \|\nabla p\|_{H^{-r}(0,T;H^{-\frac{7}{10}})} \\ &\leq \|u_t^\alpha\|_{H^{-r}(0,T;H^{-\frac{7}{10}})} + \|\Delta u^\alpha\|_{H^{-r}(0,T;H^{-\frac{7}{10}})} \\ &\quad + \|f^\alpha\|_{H^{-r}(0,T;H^{-\frac{7}{10}})} + \|\alpha^2 \Delta u_t^\alpha\|_{H^{-r}(0,T;H^{-\frac{7}{10}})} \\ &\leq c + 2\|u^\alpha\|_{H^{1-r}(0,T;H^{-\chi})}. \end{aligned} \quad (3.23)$$

In order to estimate  $u^\alpha$  in (3.23) we are going to use Lemma 3.4. We have to find  $\chi \in [\frac{1}{4}, \frac{1}{2})$  such that

$$1 - r < \bar{\tau} = \frac{2}{5}(1 + \chi),$$

namely,  $\frac{3}{2} - \frac{5}{2}r < \chi$ . This is actually always possible because

$$\frac{3}{2} - \frac{5}{2}r < \frac{1}{2},$$

then we can always find  $\chi \in [\frac{1}{4}, \frac{1}{2})$  such that

$$\frac{3}{5} - \frac{5}{2}r < \chi < \frac{1}{2}.$$

With this choice of  $\chi$  by Lemma 3.4 we get (3.21).  $\square$

**3.2. Navier boundary conditions.** In this section we prove the estimates needed to prove Theorem 1.3 that concerns the problem with the Navier boundary conditions (1.10). We report here the following estimates, which are counterpart of those obtained in the Dirichlet case.

**Lemma 3.7.** *Let  $u^\alpha$  be a solution of the NSV with Navier condition, then in addition to the estimate (2.6) we have also that*

$$\|u^\alpha(t)\|^2 + \alpha^2 \|\nabla u^\alpha(t)\|^2 + \int_0^t \|\nabla u^\alpha(s)\|^2 ds \leq c(\|u_0^\alpha\|^2 + \alpha^2 \|\nabla u_0^\alpha\|^2), \quad (3.24)$$

where the constant  $c$  depends only on  $\Omega$ .

*Proof.* The proof follows easily by observing that Lemma 2.6 implies that

$$\|\omega^\alpha\|^2 = \|\nabla u^\alpha\|^2 + \int_\Gamma u^\alpha \cdot \nabla n \cdot u^\alpha dS,$$

hence by trace theorems

$$\|\nabla u^\alpha\|^2 \leq \|\omega^\alpha\|^2 + c \int_\Gamma |u^\alpha|^2 dS \leq \|\omega^\alpha\|^2 + \frac{1}{2} \|\nabla u^\alpha\|^2 + c \|u^\alpha\|^2,$$

Then substituting and by using the estimate for  $u^\alpha \in L^\infty(0, T; L^2(\Omega))$  coming from the definition of weak solution we have the thesis.  $\square$

We then prove a simple weighted (in  $\alpha$ ) estimate for  $u_t^\alpha$  we will use to pass to the limit in the local energy inequality.

**Lemma 3.8.** *Let  $u^\alpha$  be a solution of (1.6). Then, there exists  $c > 0$  independent of  $\alpha$  such that for any  $t \in (0, T)$*

$$\alpha^3 \|\nabla u^\alpha\|^2 + \alpha^3 \int_0^t \|u_t^\alpha(s)\|^2 ds + \alpha^5 \int_0^t \|\nabla u_t^\alpha(s)\|^2 ds \leq c.$$

*Proof.* The proof is very similar to that of Lemma 3.2. We start by multiplying the momentum equation in (1.6) by  $\alpha^3 u_t^\alpha$  and by integrating by parts we get

$$\begin{aligned} \frac{\alpha^3}{2} \|\omega^\alpha(t)\|^2 + \int_0^t \alpha^3 \|u_t^\alpha(s)\|^2 + \alpha^5 \|\omega_t^\alpha(t)\|^2 ds \\ \leq \alpha^3 \int_0^t \int_\Omega |u^\alpha| |\nabla u^\alpha| |u_t^\alpha| dx ds + \frac{\alpha^3}{2} \|\omega_0^\alpha\|^2. \end{aligned}$$

We estimate the right-hand side by using Hölder inequality, the standard convex interpolation, and the Sobolev embedding  $H^1(\Omega) \subset L^6(\Omega)$  we get

$$\|u^\alpha\|_4 \leq c \|u^\alpha\|_{\frac{1}{4}}^{\frac{1}{4}} \|u^\alpha\|_6^{\frac{3}{4}} \leq c(\|u^\alpha\| + \|u^\alpha\|_{\frac{1}{4}}^{\frac{1}{4}} \|u^\alpha\|_{\frac{3}{4}}^{\frac{3}{4}}).$$

By using also Lemma 2.6, the basic energy type inequality, and the previous Lemma 3.7 we obtain the thesis.

Again, for our purposes, the most relevant estimate is the bound (independent of  $\alpha$ )  $\alpha^{3/2} u_t \in L^2(0, T; L^2(\Omega))$ .  $\square$

The main difference with respect to the Dirichlet case is the treatment of the pressure, which is now much simpler. In particular, the use of the Navier-type conditions allow us to infer the following lemma, see [5, 10].



**Lemma 3.9.** *Let  $v$  be a smooth vector field satisfying  $(\operatorname{curl} v) \times n = 0$  on  $\Gamma$ . Then,  $\zeta = \operatorname{curl} \operatorname{curl} v$  is a vector field tangential to the boundary, i.e.,  $\zeta \cdot n = 0$ . In particular in our case, since  $\nabla \cdot u^\alpha = 0$ , then we have  $\operatorname{curl} \operatorname{curl} u^\alpha = -\Delta u^\alpha$  in  $\Omega$ . Moreover since  $u^\alpha \cdot n = u_t^\alpha \cdot n = 0$  on  $\Gamma$ , we finally get that*

$$\Delta u^\alpha \cdot n = \Delta u_t^\alpha \cdot n = 0 \quad \text{on } \Gamma.$$

For a detailed proof see [6, Lemma 7.4]. In that reference many other results on the Navier conditions are reviewed.

Let  $\mathcal{U} \subset \mathbb{R}^3$  be a neighbourhood of  $\Gamma$  and  $n : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a smooth function with compact support in  $\mathcal{U}$  and such that  $n|_\Gamma$  is the normal vector to  $\Gamma$ .

**Lemma 3.10.** *Let  $(u^\alpha, p^\alpha)$  be a smooth solution to the NSV system (1.6) with boundary condition (1.11). Then  $p^\alpha$  satisfy the following Neumann problem*

$$\begin{cases} -\Delta p^\alpha &= \partial_i \partial_j (u_i^\alpha u_j^\alpha) & \text{in } \Omega, \\ \frac{\partial p^\alpha}{\partial n} &= u_i^\alpha \nabla_j n_i u_i^\alpha & \text{on } \partial\Omega. \end{cases} \quad (3.25)$$

Consequently, there exists  $c > 0$ , independent of  $\alpha$ , such that the following estimate holds true for all  $t \in (0, T)$

$$\int_0^t \|p^\alpha(s)\|_{\frac{5}{3}}^{\frac{5}{3}} ds \leq c. \quad (3.26)$$

*Proof.* By taking the divergence of the momentum equation we get

$$-\Delta p^\alpha = \operatorname{div}(u^\alpha \cdot \nabla) u^\alpha,$$

and by classical interpolation inequality we have  $(u^\alpha \cdot \nabla) u^\alpha$  is uniformly bounded in  $L^{\frac{5}{3}}(0, T; L^{\frac{15}{14}}(\Omega))$  with respect to  $\alpha$ . This holds true because we are using only that  $u^\alpha \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$  to obtain the estimate. (The regularity inherited by Leray-Hopf weak solutions)

By multiplying the momentum equation (restricted to  $\Gamma$ ) by  $n$  and by using the fact that  $u^\alpha \cdot n = 0$  on  $(0, T) \times \Gamma$  we get

$$\frac{\partial p^\alpha}{\partial n} = \left( \Delta u^\alpha \cdot n + \alpha^2 \Delta u_t^\alpha - u_t - (u^\alpha \cdot \nabla) u^\alpha \right) \cdot n = u_i^\alpha u_j^\alpha \partial_j n_i,$$

where we have used Lemma 3.9 and that on  $(0, T) \times \Gamma$  the equality  $u \cdot \nabla u \cdot n = -u \cdot \nabla n \cdot u$  holds true, see [10]. Then,  $(u^\alpha, p^\alpha)$  satisfies (3.25). By using a trace theorem and the fact that  $u_i^\alpha u_j^\alpha$  is in  $L^{\frac{5}{3}}(0, T; W^{1, \frac{15}{14}}(\Omega))$  we have that  $u_i^\alpha u_j^\alpha \partial_j n_i \in L^{\frac{5}{3}}(0, T; W^{1-\frac{14}{15}}(\Gamma))$  uniformly in  $\alpha$ . Then, by classical  $L^p$  estimates for the scalar Neumann problem, see [1, 42] we get

$$\|\nabla p^\alpha\|_{L^{\frac{5}{3}}(0, T; L^{\frac{15}{14}})} \leq c,$$

with  $c > 0$  independent of  $\alpha$ . By using a Sobolev embedding inequality, and since  $p^\alpha$  is with zero mean value, we finally get (3.26).  $\square$

## 4. LOCAL ENERGY INEQUALITY AND THE PROOFS OF THEOREMS 1.1-1.3

In this section we prove the convergence of  $\{(u^\alpha, p^\alpha)\}_\alpha$  to a suitable weak solutions of the NSE when  $\alpha \rightarrow 0$ . Note that, the passage to the limit in the weak formulation of the NSV to show that the limit satisfy the NSE is standard. Specifically, either in the case of boundary condition (1.3) or in the case (1.10) it is possible to prove that there exists  $u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$  a Leray-Hopf weak solution such that, up to sub-sequences,

$$\begin{aligned} u^\alpha &\rightarrow u \text{ strongly in } L^2((0, T) \times \Omega), \\ \nabla u^\alpha &\rightharpoonup \nabla u \text{ weakly in } L^2((0, T) \times \Omega). \end{aligned} \quad (4.1)$$

Then, we need only to prove the the local energy inequality holds true. At this point, due to the fact that in the definition of local energy inequality there test functions which are with compact support, the role of boundary conditions is limited. We wish to mention that in case of regularity results up to the boundary slightly different notions are used, see the work of Seregin *et al.* summarized in [37].

*Proof of Theorem 1.1.* We start by multiplying equations (1.6) by  $u^\alpha \phi$  for some  $0 \leq \phi \in C_c^\infty((0, T) \times \Omega)$ , and after some integration by parts over  $(0, t) \times \Omega$  we get

$$\begin{aligned} \int_0^T (|\nabla u^\alpha|^2, \phi) dt &= \int_0^T \left( \frac{|u^\alpha|^2}{2}, \phi_t + \Delta \phi \right) + (u^\alpha \frac{|u^\alpha|^2}{2}, \nabla \phi) dt \\ &\quad + \int_0^T \alpha^2 (\Delta u_t^\alpha, u^\alpha \phi) + (u^\alpha p^\alpha, \nabla \phi) dt. \end{aligned} \quad (4.2)$$

We estimate the terms from the right-hand side of (4.2): By weak lower semicontinuity of the  $L^2$ -norm, the fact that  $u^\alpha \rightharpoonup u$  weakly in  $L^2(0, T; H^1)$ , and  $\phi \geq 0$  we have that

$$\int_0^T (|\nabla u|^2, \phi) dt \leq \liminf_{\alpha \rightarrow 0} \int_0^T (|\nabla u^\alpha|^2, \phi) dt.$$

Then, since  $u^\alpha \rightarrow u$  strongly in  $L^2(0, T; L^2(\Omega))$ , we get

$$\int_0^T \left( \frac{|u^\alpha|^2}{2}, \phi_t + \Delta \phi \right) dt \rightarrow \int_0^T \left( \frac{|u|^2}{2}, \phi_t + \Delta \phi \right) dt \quad \text{as } \alpha \rightarrow 0. \quad (4.3)$$

Next, by interpolation we also have that  $u^\alpha \rightarrow u$  strongly in  $L^2(0, T; L^3(\Omega))$  and that  $u^\alpha$  is bounded in  $L^4(0, T; L^3(\Omega))$ , so it follows that

$$\int_0^T (u^\alpha \frac{|u^\alpha|^2}{2}, \nabla \phi) dt \rightarrow \int_0^T (u \frac{|u|^2}{2}, \nabla \phi) dt \quad \text{as } \alpha \rightarrow 0.$$

of (4.2). To estimate the third term, we observe that since  $\phi$  is with space-time compact support in  $(0, T) \times \Omega$  we can freely integrate by parts without appearance of boundary terms. With a first integration by parts with respect

to the time variable and then with respect to the space variables we get

$$\begin{aligned}
\int_0^T \int_{\Omega} \Delta u_t^\alpha u^\alpha \phi \, dx dt &= - \int_0^T \int_{\Omega} \Delta u^\alpha u_t^\alpha \phi + \Delta u^\alpha u^\alpha \phi_t \, dx dt \\
&= \int_0^T \int_{\Omega} \nabla u^\alpha \nabla u_t^\alpha \phi + \nabla u^\alpha \nabla \phi u_t^\alpha + |\nabla u^\alpha|^2 \phi_t + \nabla u^\alpha u^\alpha \nabla \phi_t \, dx dt \\
&= \int_0^T \int_{\Omega} \partial_t \frac{|\nabla u^\alpha|^2}{2} \phi + \nabla u^\alpha \nabla \phi u_t^\alpha + |\nabla u^\alpha|^2 \phi_t + \nabla \frac{|u^\alpha|^2}{2} \nabla \phi_t \, dx dt.
\end{aligned}$$

Hence, with further integration by parts of the first and last term

$$\begin{aligned}
\int_0^T \int_{\Omega} \Delta u_t^\alpha u^\alpha \phi \, dx dt &= \\
&= \int_0^T \int_{\Omega} -\frac{|\nabla u^\alpha|^2}{2} \phi_t + \nabla u^\alpha \nabla \phi u_t^\alpha + |\nabla u^\alpha|^2 \phi_t - \frac{|u^\alpha|^2}{2} \Delta \phi_t \, dx dt \\
&= \int_0^T \int_{\Omega} \frac{|\nabla u^\alpha|^2}{2} \phi_t + \nabla u^\alpha \nabla \phi u_t^\alpha + |\nabla u^\alpha|^2 \phi_t - \frac{|u^\alpha|^2}{2} \Delta \phi_t \, dx dt.
\end{aligned}$$

Consequently, we can prove the following estimate

$$\begin{aligned}
\alpha^2 \left| \int_0^T \int_{\Omega} \Delta u_t^\alpha u^\alpha \phi \, dx dt \right| & \\
&\leq \frac{\alpha^2}{2} \int_0^T \int_{\Omega} |\nabla u^\alpha|^2 |\phi_t| \, dx dt + \frac{\alpha^2}{2} \int_0^T \int_{\Omega} |u^\alpha|^2 |\Delta \phi_t| \, dx dt \\
&\quad + \alpha^2 \left| \int_0^T \int_{\Omega} u_t^\alpha \nabla u^\alpha \nabla \phi \, dx dt \right|.
\end{aligned}$$

By using the fact that  $u^\alpha$  and  $\nabla u^\alpha$  are uniformly bounded in  $L^2((0, T) \times \Omega)$  and that  $\phi$  is  $C_c^\infty((0, T) \times \Omega)$  we have that the first two integrals from the right-hand side vanish when  $\alpha \rightarrow 0$ . Concerning the last term we argue in the following way

$$\begin{aligned}
\alpha^2 \left| \int_0^T \int_{\Omega} u_t^\alpha \nabla u^\alpha \nabla \phi \, dx dt \right| &\leq C \alpha^2 \int_0^T \|u_t^\alpha\| \|\nabla u^\alpha\| \, dt \\
&\leq C \alpha^{\frac{1}{2}} \int_0^T \alpha^{\frac{3}{2}} \|u_t^\alpha\| \|\nabla u^\alpha\| \, dt \\
&\leq C \alpha^{\frac{1}{2}} \left( \alpha^3 \int_0^T \|u_t^\alpha\|^2 \, dt \right)^{\frac{1}{2}} \left( \int_0^T \|\nabla u^\alpha\|^2 \, dt \right)^{\frac{1}{2}} \\
&\leq C \alpha^{\frac{1}{2}},
\end{aligned}$$

where we have used Lemma 3.2. By letting  $\alpha$  go to 0 all these integrals vanish.

*Remark 4.1.* All arguments used up to now are true in the case of both boundary conditions (1.3) and (1.10).

Now, we estimate the last two term from the right-hand side of (4.2).

$$\int_0^T (u^\alpha p^\alpha, \nabla \phi) \, dt. \tag{4.4}$$

In the case of the Dirichlet boundary condition (1.3) by using Lemma 3.6 we have that

$$p^\alpha \rightharpoonup p \quad \text{in} \quad H^{-r}(0, T; H^{\frac{3}{10}}(\Omega)).$$

for any  $r > \frac{2}{5}$ . Since  $\frac{1}{4} < \frac{3}{10}$  we have that  $H^{-\frac{1}{4}}(\Omega) \subset H^{-\frac{3}{10}}(\Omega)$  with compact embedding. Moreover, with  $\chi = \frac{1}{2}$  we have that  $\bar{\tau} = \frac{2}{5}(1 + \chi) = \frac{1}{2}$ . Then, we can find  $r$  and  $\tau$  such that

$$\frac{2}{5} < r < \tau < \frac{1}{5}.$$

With this choice of parameters we have that

$$H^r(0, T; H^{-\frac{1}{4}}(\Omega)) \subset H^r(0, T; H^{-\frac{3}{10}}(\Omega)).$$

Then, by using Lemma 3.4 and classical compactness argument (a variant of Aubin-Lions lemma, see [21]) we get that

$$u^\alpha \rightarrow u \quad \text{in} \quad H^r(0, T; H^{-\frac{3}{10}}(\Omega)).$$

Then, it follows that

$$\int_0^T (p^\alpha u^\alpha, \nabla \phi) dt \rightarrow \int_0^T (p u, \nabla \phi) dt \quad \text{as } \alpha \rightarrow 0$$

and this proves that the local energy inequality (1.5) holds true.  $\square$

In the Theorem 1.1 the local energy inequality is satisfied, but the pressure has not the usual regularity. We show now how to slightly change the pressure to obtain a genuine suitable solution with pressure in  $L^{\frac{5}{3}}((0, T) \times \Omega)$ .

**Corollary 4.2.** *It is possible to associate to  $p$  another scalar  $\tilde{p} \in L^{\frac{5}{3}}((0, T) \times \Omega)$  such that the couple  $(u, \tilde{p})$  is a suitable weak solution.*

*Proof.* We start by recalling that In Sohr and von Wahl [39] (see also [13]) it is proved that the pressure associated to Leray-Hopf weak solution belongs to the space  $L^{\frac{5}{3}}((0, T) \times \Omega)$ . This is obtained by considering the linear Stokes problem

$$v_t - \Delta v + \nabla q = -(u \cdot \nabla) u \quad \text{in } (0, T) \times \Omega, \quad (4.5)$$

$$\operatorname{div} v = 0 \quad \text{in } (0, T) \times \Omega, \quad (4.6)$$

$$v = 0 \quad \text{on } (0, T) \times \Gamma, \quad (4.7)$$

$$v(x, 0) = u_0(x) \quad \text{in } \Omega, \quad (4.8)$$

and by employing classical  $L^p$ -estimates, together with the uniqueness for the linear problem. We observe that in Theorem 1.1  $p \in H^{-r}(0, T; H^{\frac{3}{10}}(\Omega))$ , while considering (4.5) with right-hand side  $-(u \cdot \nabla) u$  we get that

$$q \in L^{\frac{5}{3}}(0, T) \times \Omega).$$

Then it follows that  $p$  and  $q$  are almost the same, in particular we have

$$\int_0^T \int_\Omega (\nabla p - \nabla q) \phi dx dt = 0,$$

implying that

$$q(t, x) = p(t, x) + \mathcal{G}(t)$$

for some function  $\mathcal{G}(t)$  depending only on the time variable. It follows that the couple  $(u, q)$  is again a weak solution to the NSE with the same initial datum as  $(u, p)$ . Next, we consider the approximate NSV problems

$$\begin{aligned} u_t^\alpha - \alpha^2 \Delta u_t^\alpha - \Delta u^\alpha + (u^\alpha \cdot \nabla) u^\alpha + \nabla(p^\alpha + \mathcal{G}) &= 0 && \text{in } (0, T) \times \Omega, \\ \nabla \cdot u^\alpha &= 0 && \text{in } (0, T) \times \Omega, \\ u^\alpha &= 0 && \text{on } (0, T) \times \Gamma, \\ u^\alpha(0, x) &= u_0^\alpha(x) && \text{in } \Omega, \end{aligned}$$

for which we can prove exactly the same estimates as before. Since the function  $\mathcal{G}(t)$  has (at least) the same time-regularity as  $p$ , we can prove with the same arguments as before that Now, we estimate the last two term from the right-hand side of (4.2).

$$\int_0^T (u^\alpha(p^\alpha + \mathcal{G}(t)), \nabla \phi) dt \rightarrow \int_0^T (u(p + \mathcal{G}(t)), \nabla \phi) dt, \quad (4.9)$$

hence proving that  $(u, p + \mathcal{G}(t))$  satisfy the local energy inequality.  $\square$

In the case of Navier boundary conditions the proof of the local energy inequality is very similar to the previous case, only the convergence of the term with the pressure requires a different treatment

*Proof of Theorem 1.3.* In the case of boundary conditions (1.10) the passage to the limit  $\alpha \rightarrow 0$  is very standard. The proof is the same as in Theorem 1.1 up to the treatment of the pressure. In this case we have that by using Lemma 3.10

$$p^\alpha \rightharpoonup p \text{ weakly in } L^{\frac{5}{3}}((0, T) \times \Omega),$$

while by standard interpolation argument

$$u^\alpha \rightarrow u \text{ strongly in } L^{\frac{5}{2}}((0, T) \times \Omega).$$

Then, it straightforward to pass to the limit in the term (4.4).  $\square$

## 5. FOURIER-GALERKIN APPROXIMATIONS AND SUITABLE WEAK SOLUTIONS

In this section we consider the NSE (1.1) and NSV (1.6) equations in the space-periodic setting and we address the problem of construction of suitable solutions by means of the Fourier-Galerkin method. The standard (Fourier) Galerkin method to approximate system in  $\mathbb{T} := \mathbb{R}^3 / (2\pi\mathbb{Z})^3$  can be implemented in the following way. Let  $P$  denote the Leray projector of  $L_0^2(\mathbb{T})$  (the subspace of  $L^2(\mathbb{T})$  with zero mean value) onto divergence free vector fields denoted by  $L_\sigma^2(\mathbb{T})$ , which explicitly reads in the orthogonal Hilbert basis of complex exponentials as follows:

$$P : g(x) = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} g_k e^{ik \cdot x} \mapsto Pg(x) = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \left[ g_k - \frac{(g_k \cdot k)k}{|k|^2} \right] e^{ik \cdot x}.$$

It is well-known that the Leray projector is continuous also as an operator  $H^s(\mathbb{T}) \cap L_0^2(\mathbb{T}) \mapsto H_\sigma^s(\mathbb{T}) := H^s(\mathbb{T}) \cap L_s^2(\mathbb{T})$  for all positive  $s$ . For any  $n \in \mathbb{N}$ ,

we denote by  $P_n$  the projector of  $L^2(\mathbb{T})$  on the finite-dimensional sub-space  $V_n := P_n(L^2_\sigma(\mathbb{T}))$  given by the following formula

$$P_n : g(x) = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} g_k e^{ik \cdot x} \mapsto P_n g(x) = \sum_{0 < |k| \leq n} \left[ g_k - \frac{(g_k \cdot k)k}{|k|^2} \right] e^{ik \cdot x}.$$

Then the Galerkin approximation for the NSE (1.1) is the following Cauchy problem for systems of ordinary differential equations in the unknowns  $c_k^n(t)$ , with  $|k| \leq n$

$$u_t^n - \Delta u^n + P_n((u_n \cdot \nabla) u_n) = 0 \quad \text{in } (0, T) \times \mathbb{T}, \quad (5.1)$$

$$u^n(0, x) = P_n u_0(x) \quad \text{in } \mathbb{T}, \quad (5.2)$$

where

$$u^n(t, x) = \sum_{0 < |k| \leq n} c_k^n(t) e^{ik \cdot x}, \quad \text{with } k \cdot c_k^n = 0.$$

It is important to point to that in the standard Fourier-Galerkin method there is an explicit formula for the Leray projector and even if the pressure  $p^n$  disappears, it can explicitly computed.

One main unsolved question is to prove (or disprove) that  $\{(u^n, p^n)\}_n$  converges as  $n \rightarrow +\infty$  to a suitable weak solution. This special setting, with an approximation which is spectral and consequently *non local* seems to require tools completely different from those successfully used in [20] to handle finite element approximations. Some conditional results linking the hyper-dissipative NSE and the problem of Fourier-Galerkin approximation are treated in [12]. In that reference there is also an interesting link between the *global energy equality* and the local energy inequality (1.5).

By following the same spirit, we consider then the approximation by a Fourier-Galerkin-NSV system and show that, under a link between the coefficient  $\alpha$  and the order of approximation  $n \in \mathbb{N}$  one can show the local energy inequality. We use the symbol  $u^{\alpha, n}$  to denote the approximate Fourier-Galerkin solution to the NSV, in the unknowns  $d_k^n(t)$  with modes up to  $|k| \leq n$

$$\begin{aligned} u_t^{\alpha, n} - \alpha^2 \Delta u_t^{\alpha, n} - \Delta u^{\alpha, n} + P_n((u_n \cdot \nabla) u_n) &= 0 & \text{in } (0, T) \times \mathbb{T}, \\ u^{\alpha, n}(0, x) &= P_n u_0(x) & \text{in } \mathbb{T}, \end{aligned} \quad (5.3)$$

where

$$u^{\alpha, n}(t, x) = \sum_{0 < |k| \leq n} d_k^n(t) e^{ik \cdot x}, \quad \text{with } k \cdot d_k^n = 0.$$

The Galerkin approximation for the NSV (1.6) is the following Cauchy problem for systems of ordinary differential equations

The estimates which may give to the local energy inequality are obtained by testing the equations(5.3) by  $u^{\alpha, n} \phi$ , where  $\phi$  which is a non-negative, space-periodic, and with compact support in  $(0, T)$ . In general  $u^{\alpha, n} \phi \notin V_n$ , hence we cannot use directly this approach. We have two possible choices to project  $u^{\alpha, n}$ , or to rewrite the equations (5.3) in such a way to have the pressure and a formulation which avoids the projection over  $V_n$ . It is natural, since we are in the periodic case, to define  $p^{\alpha, n}$  as solution of the

Poisson problem

$$-\Delta p^{\alpha,n} = \sum_{i,j=1}^3 \partial_i \partial_j (u_i^{\alpha,n} u_j^{\alpha,n}), \quad (5.4)$$

endowed with periodic conditions and normalized with vanishing mean value. After having defined the operator  $Q_n := P - P_n$  we get the following equations for  $u^{\alpha,n}$

$$\begin{aligned} u_t^{\alpha,n} - \alpha_n^2 \Delta u_t^{\alpha,n} - \Delta u^{\alpha,n} + P((u^{\alpha,n} \cdot \nabla) u^{\alpha,n}) \\ - Q_n((u^{\alpha,n} \cdot \nabla) u^{\alpha,n}) = 0 \quad \text{in } (0, T) \times \mathbb{T}, \end{aligned}$$

which can be rewritten in the following way

$$\begin{aligned} u_t^{\alpha,n} - \alpha^2 \Delta u_t^{\alpha,n} - \Delta u^{\alpha,n} + (u^{\alpha,n} \cdot \nabla) u^{\alpha,n} \\ - Q_n((u^{\alpha,n} \cdot \nabla) u^{\alpha,n}) + \nabla p^{\alpha,n} = 0 \quad \text{in } (0, T) \times \mathbb{T}. \end{aligned} \quad (5.5)$$

We can now freely test (5.5) with  $u^{\alpha,n} \phi$ , but at the price of being able to obtain good estimates on  $Q_n((u^{\alpha,n} \cdot \nabla) u^{\alpha,n})$ . It is at this step that in the finite element setting that a special choice of the function spaces can be used to prove the local energy inequality. In the Fourier-Galerkin setting it is not known whether this methodology works or not, since the available estimates are not strong enough to handle the remainder term involving  $Q_n((u^{\alpha,n} \cdot \nabla) u^{\alpha,n})$ .

**5.1. A priori estimates.** in this section we prove the main weighted estimates needed to prove the local energy inequality under certain assumptions on the parameter  $\alpha$  of the Voigt regularization. We begin with the standard a priori estimate on the solution  $(u^{\alpha,n}, p^{\alpha,n})$ .

**Lemma 5.1.** *Let  $u_0 \in H_\sigma^1(\mathbb{T})$ . Then, for all  $\alpha > 0$  and  $n \in \mathbb{N}$ , the unique solution of (5.3) satisfies for all  $t > 0$  the following equality*

$$\|u^{\alpha,n}(t)\|^2 + \alpha^2 \|\nabla u^{\alpha,n}(t)\|^2 + 2 \int_0^t \|\nabla u^{\alpha,n}(s)\|^2 ds = \|P_n u_0\|^2 + \alpha^2 \|\nabla P_n u_0\|^2.$$

This lemma is just the standard energy type equality, which is –by the way– satisfied also in the limit  $n \rightarrow +\infty$  by weak solutions of the NSV.

The following Lemma gives a weighted a priori estimate for second order space derivatives and will be used to pass to the limit to get the generalized energy inequality.

**Lemma 5.2.** *Let be given  $u_0 \in H_\sigma^2(\mathbb{T})$  and let  $u^{\alpha,n}$  be the unique solution of (5.3) with initial datum  $P_n u_0$ . Then, there exists  $c > 0$ , independent of  $\alpha > 0$  and of  $n \in \mathbb{N}$ , such that for all  $t \in (0, T)$*

$$\alpha^6 \int_0^t \|\Delta u^{\alpha,n}(s)\|^2 ds \leq c.$$



*Proof.* We multiply by  $-\Delta u^{\alpha,n}$  the equations (5.3) and after an integration by parts we get (recall that due to the choice of the basis  $\Delta u^{\alpha,n} \in V_n$ )

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\nabla u^{\alpha,n}\|^2 + \alpha^2 \|\Delta u^{\alpha,n}\|^2) + \|\Delta u^{\alpha,n}\|^2 \\ &= \int_{\Omega} P_n((u^{\alpha,n} \cdot \nabla) u^{\alpha,n}) \cdot \Delta u^{\alpha,n} \, dx \\ &= \int_{\Omega} (u^{\alpha,n} \cdot \nabla) u^{\alpha,n} \cdot P_n(\Delta u^{\alpha,n}) \, dx \\ &= \int_{\Omega} (u^{\alpha,n} \cdot \nabla) u^{\alpha,n} \cdot \Delta u^{\alpha,n} \, dx. \end{aligned}$$

Integrating in time over  $(0, t)$  and since the initial datum belongs to  $H^2(\mathbb{T})$  we get

$$\begin{aligned} & \|u^{\alpha,n}\|^2 + \alpha^2 \|\Delta u^{\alpha,n}\|^2 + 2 \int_0^t \|\Delta u^{\alpha,n}(s)\|^2 \, ds \leq \|u_0\|^2 + \alpha^2 \|\Delta u_0\|^2 \\ &+ 2 \int_0^t \int_{\Omega} |u^{\alpha,n}| |\nabla u^{\alpha,n}| |\Delta u^{\alpha,n}| \, dx \, ds \\ &\leq \|u_0\|^2 + \alpha^2 \|\Delta u_0\|^2 \\ &+ c \int_0^t \|u^{\alpha,n}(s)\|^{\frac{1}{4}} \|\nabla u^{\alpha,n}(s)\| \|\Delta u^{\alpha,n}(s)\|^{\frac{7}{4}} \, ds \\ &\leq \|u_0\|^2 + \alpha^2 \|\Delta u_0\|^2 + c \sup_{0 < t < t} \|\nabla u^{\alpha,n}(t)\|^6 \int_0^t \|\nabla u^{\alpha,n}(s)\|^2 \, ds, \end{aligned}$$

where we have used the usual Gagliardo-Nirenberg, Hölder, and Young inequalities and Lemma 5.1. By multiplying the above inequality on both side by  $\alpha^6$  and using again Lemma 5.1 we get that, for all  $0 < \alpha \leq 1$  and for all  $t \in (0, T)$ ,

$$\alpha^6 \|\nabla u^{\alpha,n}(t)\|^2 + \alpha^8 \|\Delta u^{\alpha,n}(t)\|^2 + \alpha^6 \int_0^t \|\Delta u^{\alpha,n}(s)\|^2 \, ds \leq c,$$

with a constant  $c$  independent of  $\alpha$  and of  $n$ , thus ending the proof.  $\square$

Moreover, we have also the analogue of Lemma 3.2, we omit the proof since it is essentially the same.

**Lemma 5.3.** *Let  $u_0 \in H_{\sigma}^1(\mathbb{T})$  and let  $u^{\alpha,n}$  be the corresponding solution of (5.3). Then, there exists  $c > 0$ , independent of  $\alpha > 0$  and of  $n \in \mathbb{N}$ , such that for all  $t \in (0, T)$*

$$\alpha^3 \int_0^t \|\partial_t u^{\alpha,n}(s)\|^2 \, dt \leq c.$$

From the elliptic equations associated to  $p^{\alpha,n}$  we can easily prove the following estimate

**Lemma 5.4.** *Let  $u_0 \in H_{\sigma}^1(\mathbb{T})$  and let  $(u^{\alpha,n}, p^{\alpha,n})$  be a solution of (5.3), where the pressure is defined through (5.4). Then, there exists  $c > 0$ , independent of  $\alpha > 0$  and of  $n \in \mathbb{N}$ , such that for all  $t \in (0, T)$  such that*

$$\int_0^t \|p^{\alpha,n}(s)\|_{\frac{5}{3}}^{\frac{5}{3}} \, ds \leq c.$$

We observe that from the previous estimates it follows that in the space-periodic case we can prove that the unique weak solution  $(u^\alpha, p^\alpha)$  of the space-periodic NSV (1.6) obtained as  $\lim_{n \rightarrow +\infty} (u^{\alpha, n}, p^{\alpha, n})$  satisfies the same estimates. Hence, we can easily infer the following result, which we recalled in the introduction.

**Theorem 5.5.** *Let in the space-periodic case  $(u^\alpha, p^\alpha)$  be a weak solution to the NSV equation (1.6). Then, as  $\alpha \rightarrow 0$  the couple  $(u^\alpha, p^\alpha)$  converges (up to sub-sequences) to a suitable weak solution to the space-periodic NSE (1.1).*

*Proof.* The proof is apart some simplifications the same as in the case of Navier conditions of Theorem 1.3. In some sense the use of Navier-conditions allows to use, even in a slightly more complicated way, the same tools of reconstructing the pressure via the solution of a Poisson equation.  $\square$

Concerning the Galerkin approximation, the challenging point is studying the limit obtained in the other way around: first the limit as  $\alpha \rightarrow 0$  and then that as  $n \rightarrow \infty$ . This is still an open problem. A possible way to partially handle this problem is to link  $\alpha$  and  $n$  as stated in Theorem 1.4. To this end we first recall the following lemma, which is proved as one of the steps in [12, Lemma 4.4].

**Lemma 5.6.** *Let us define*

$$g(t) := \|Q_n(\phi(t) u^{\alpha, n}(t))\|_{L^\infty(\mathbb{T})},$$

where  $\phi \in C^\infty((0, T) \times \mathbb{T})$  is space-periodic and with support contained in  $(0, T)$ . Then, there exists a constant  $c$  depending only on  $\phi$  such that if  $u^{\alpha, n}(t, x) = \sum_{0 < |k| \leq n} d_k^n(t) e^{ik \cdot x}$ , then

$$g(t)^2 \leq c \left( n^2 \sum_{|k| \geq \frac{n}{2}} |d_k^n(t)|_2^2 + \frac{1}{n} \sum_{k \in \mathbb{Z}^3} |d_k^n(t)|_2^2 \right).$$

This lemma will be used to estimate the integral

$$\int_0^T \int_{\mathbb{T}} Q_n((u^{\alpha, n} \cdot \nabla) u^{\alpha, n}) u^{\alpha, n} \phi \, dx dt,$$

which comes out when testing (5.5) by  $u^{\alpha, n} \phi$ .

**5.2. Proof of the Theorem 1.4.** First, we note that the convergence  $\lim_{n \rightarrow +\infty} u^{\alpha, n} \rightarrow u$  towards a Leray-Hopf weak solutions is standard. In particular, we get from the basic a priori estimate and Lemma 5.4 that there exists  $u \in L^\infty(0, T; L^2(\mathbb{T})) \cap L^2(0, T; H^1(\mathbb{T}))$  such that the following convergences hold true

$$\begin{aligned} u^{\alpha, n} &\rightarrow u \text{ strongly in } L^2(0, T; L^2(\mathbb{T})), \\ u^{\alpha, n} &\rightharpoonup u \text{ weakly in } L^2(0, T; H^1(\mathbb{T})), \\ p^{\alpha, n} &\rightharpoonup p \text{ weakly in } L^{\frac{5}{3}}(0, T; L^{\frac{5}{3}}(\mathbb{T})). \end{aligned} \tag{5.6}$$

As in the proof of Theorem 1.1 we need to prove that the local energy inequality is satisfied. We multiply the equations (5.5) by  $u^{\alpha, n} \phi$  with  $\phi$  non-negative positive and belonging to  $C_c^\infty((0, T) \times \mathbb{T})$ . After standard

integrations by parts and by using the fact that  $Q_n$  is a projector on the orthogonal of  $V_n$  we get

$$\begin{aligned} & \int_0^T (|\nabla u^{\alpha_n, n}|^2, \phi) dt \\ &= \int_0^T \frac{|u^{\alpha_n, n}|^2}{2} (\phi_t + \Delta \phi) + (u^{\alpha_n, n} \frac{|u^{\alpha_n, n}|^2}{2}, \nabla \phi) + (u^{\alpha_n, n} p^{\alpha_n, n}, \nabla \phi) dt \\ & \quad + \alpha_n^2 \int_0^T (\Delta u_t^{\alpha_n, n}, u^{\alpha_n, n} \phi) dt - \int_0^T ((u^{\alpha_n, n} \cdot \nabla) u^{\alpha_n, n}, Q_n(u^{\alpha_n, n} \phi)) dt. \end{aligned}$$

By using the convergences from Eq. (5.6), Lemmas 5.3-5.4, and the results of the previous section, it is possible to show how pass to the limit as  $n \rightarrow +\infty$  in all the terms of the above equality, except the last one. To successfully handle this we have to assume a particular behavior for the sequence  $\{\alpha_n\}_n$ . Indeed, by using Hölder inequality we have that

$$\begin{aligned} & \left| \int_0^T ((u^{\alpha_n, n} \cdot \nabla) u^{\alpha_n, n}, Q_n(u^{\alpha_n, n} \phi)) dt \right| \\ & \leq \int_0^T \|u^{\alpha_n, n}(t)\| \|\nabla u^{\alpha_n, n}(t)\| \|Q_n(u^{\alpha_n, n} \phi)(t)\|_{L^\infty} dt \\ & \leq \|u^{\alpha_n, n}(t)\|_{L^\infty(0, T; L^2)} \|\nabla u^{\alpha_n, n}(t)\|_{L^2(0, T; L^2)} \|g(t)\|_{L^2(0, T)} \\ & \leq c \left( \int_0^T n^2 \sum_{|k| \geq \frac{n}{2}} |u_k^{\alpha_n, n}(t)|^2 + \frac{1}{n} \sum_{k \in \mathbb{Z}^3} |u_n^k(t)|^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

where we have used the fact that, by the basic energy estimate of Lemma 5.1, both  $\|u^{\alpha_n, n}\|_{L^\infty(0, T; L^2)}$  and  $\|\nabla u^{\alpha_n, n}\|_{L^2(0, T; L^2)}$  are uniformly bounded in  $n \in \mathbb{N}$ . Moreover, the second term in the right-hand side of the above inequality converges to 0 as  $n \rightarrow \infty$  because the sum is bounded, again by using the standard a priori estimate.

Then, we have only to show that the term

$$\int_0^T n^2 \sum_{|k| \geq \frac{n}{2}} |u_k^{\alpha_n, n}(t)|^2 dt,$$

converges to zero, as  $n \rightarrow +\infty$ .

In particular, by using the a-priori estimates from Lemma 5.2 we have that

$$\begin{aligned} \int_0^T n^2 \sum_{|k| \geq \frac{n}{2}} |u_k^{\alpha_n, n}(t)|^2 &= \frac{n^2 \alpha_n^6}{n^2 \alpha_n^6} \int_0^T \sum_{|k| \geq \frac{n}{2}} n^2 |u_k^{\alpha_n, n}(t)|^2 dt \\ &\leq 4 \frac{\alpha_n^6}{n^2 \alpha_n^6} \int_0^T \sum_{|k| \geq \frac{n}{2}} |k|^4 |u_k^{\alpha_n, n}(t)|^2 dt \\ &\leq \frac{4}{n^2 \alpha_n^6} \alpha_n^6 \int_0^T \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} |k|^4 |u_k^{\alpha_n, n}(t)|^2 dt \\ &\leq \frac{c}{n^2 \alpha_n^6} \alpha_n^6 \int_0^T \|\Delta u^{\alpha_n, n}(t)\|^2 dt \leq \frac{c}{n^2 \alpha_n^6}. \end{aligned}$$

Then, for any positive sequence  $\{\alpha_n\}_n$  such that

$$\lim_{n \rightarrow +\infty} \alpha_n = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \alpha_n^6 n^2 = 0$$

we get that

$$\int_0^T g^2(t) dt \rightarrow +\infty,$$

then the generalized energy inequality is proved.

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