

# NON-UNIQUENESS AND PRESCRIBED ENERGY FOR THE CONTINUITY EQUATION

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ABSTRACT. In this note we provide new non-uniqueness examples for the continuity equation by constructing infinitely many weak solutions with prescribed energy.

## 1. INTRODUCTION

In this paper we consider the *continuity equation* for a bounded scalar function  $u: \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  with a bounded divergence-free vector field  $\mathbf{b}: \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ :

$$\partial_t u + \operatorname{div}(u\mathbf{b}) = 0, \quad (1)$$

$$\operatorname{div} \mathbf{b} = 0. \quad (2)$$

This equation appears in various problems of mathematical physics, in particular fluid mechanics and kinetic theory. In the smooth setting (and assuming suitable integrability) the *energy*

$$\mathcal{E}(t) := \int_{\mathbb{R}^d} u^2(t, x) dx$$

of the solution  $u$  is conserved:

$$\frac{d}{dt} \mathcal{E}(t) = 0. \quad (3)$$

Indeed, since  $\mathbf{b}$  is divergence-free, by multiplying (1) with  $u$ , using the chain rule and integrating over  $\mathbb{R}^d$  one immediately obtains (3).

In many applications one has to study (1) in a nonsmooth setting. Roughly speaking, since (1) is linear, the conservation of energy (3) implies uniqueness of weak solutions to the corresponding initial-value problem for (1). In fact, conservation of energy is a consequence of the so-called *renormalization property*, which was proved by [DL89] for any vector field  $\mathbf{b}$  with Sobolev regularity and later extended by Ambrosio [Amb] to the case when  $b$  has bounded variation. We refer to [DL08, AC14] for a detailed review of recent results in this direction.

On the other hand, when the regularity of the vector field  $\mathbf{b}$  is too low, the conservation of energy (3) fails in general. In a nonsmooth setting several counterexamples to the uniqueness, and therefore to the conservation of energy, are known, see [Aiz78, CLR03, Dep03, ABC14, ABC13]. A similar phenomenon occurs in the context of the Euler equations. For example, in the papers [Sch93, Shn97, DLS09] weak solutions of the Euler equations were constructed with compact support in space time.

In particular the example in [Dep03] gives a bounded vector field  $\mathbf{b}$  and a bounded scalar field  $u$  which satisfy (1) and (2) such that

$$\mathcal{E}(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 & \text{for } t > 0. \end{cases} \quad (4)$$

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In this paper, for any given nonnegative bounded function  $E: \mathbb{R} \rightarrow \mathbb{R}$  which is continuous on an open interval and zero outside we construct infinitely many pairs  $(\mathbf{b}, u)$  satisfying (1) and (2), such that  $\mathcal{E}(t) = E(t)$  for a.e.  $t$ . Thus, in contrast with (4), we provide more general profiles for the energy. Our results are also connected to the chain rule problem for the divergence operator, see [ADLM07, BG14, CGSW].

We construct such pairs  $(\mathbf{b}, u)$  using the method of convex integration, and our techniques are similar to the ones used in [DLS09, Szé12]. The latter reference contains an appendix giving a general framework for convex integration, but for the problem at hand we need to consider a nonlinear constraint that depends on the points in the domain (as was the case e.g. in [DLS10], albeit in a different functional setting). For this reason we adapt the framework from [Szé12] to this more general situation (see §2). We then apply this abstract framework to the specific situation of the continuity equation (see §3).

Finally, let us mention [CFG11, Shv11, BLFNL], where results were obtained by convex integration, respectively, that yield as a byproduct counterexamples to the energy conservation for continuity equations. However, in these works the energy profile is always piecewise constant.

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## 2. DIFFERENTIAL INCLUSIONS WITH NON-CONSTANT NONLINEAR CONSTRAINT

We start with the so-called Tartar framework (cf. e.g. [DLS09]). Consider a system of  $m$  linear partial differential equations

$$\sum_{i=1}^D A_i \partial_i z = 0 \quad (5)$$

in an open set  $\mathcal{D} \subset \mathbb{R}^D$ , where  $A_i$  are constant  $m \times n$  matrices and  $z: \mathcal{D} \rightarrow \mathbb{R}^n$ . Consider a nonlinear constraint

$$z(y) \in K_y \quad (6)$$

for a.e.  $y$  in  $\mathcal{D}$ , where  $K_y \subset \mathbb{R}^n$  is a compact set for any  $y \in \mathcal{D}$ .

For any  $y \in \mathcal{D}$  let  $U_y := \text{int conv } K_y$ , where with  $\text{conv}$  we denote the convex hull of the set  $K_y$  and with  $\text{int}$  we denote its interior. Let  $\mathcal{U} \subset \mathcal{D}$  be a bounded open set.

**Definition 1** (Subsolutions). *We say that  $z \in L^2(\mathcal{D})$  is a subsolution of (5), (6) if  $z$  is a weak solution of (5) in  $\mathcal{D}$ ,  $z$  is continuous on  $\mathcal{U}$ , (6) holds for a.e.  $y \in \mathcal{D} \setminus \mathcal{U}$  and*

$$z(y) \in U_y \quad (7)$$

for any  $y \in \mathcal{U}$ .

**Definition 2** (Localized plane waves/wave cone). *A set  $\Lambda \subset \mathbb{R}^n$  is called wave cone if there exists a constant  $C > 0$  such that for any  $\bar{z} \in \Lambda$  there exists a sequence  $w_k \in C_0^\infty(B_1(0); \mathbb{R}^n)$  solving (5) in  $\mathbb{R}^D$  such that*

- $\text{dist}(w_k(x), [-\bar{z}, \bar{z}]) \rightarrow 0$  for all  $x \in B_1(0)$  uniformly as  $k \rightarrow \infty$ ,
- $w_k \rightarrow 0$  in  $L^2$  as  $k \rightarrow \infty$ ,
- $\int |w_k|^2 dy > C|\bar{z}|^2$  for all  $k \in \mathbb{N}$ .

In the above definition we denoted the segment with endpoints  $x \in \mathbb{R}^n$  and  $y \in \mathbb{R}^n$  with  $[x, y] := \text{conv}\{x, y\}$ . The functions  $w_k$  are called *localized plane waves*. We make the following assumptions:

**Assumption 1.** *There exists a wave cone  $\Lambda$  dense in  $\mathbb{R}^n$ .*

Let  $\mathcal{K}$  denote the set of all compact subsets of  $\mathbb{R}^n$ , endowed with the Hausdorff metric  $d_{\mathcal{H}}$ . It is well-known that  $\mathcal{K}$  is a complete metric space.

**Assumption 2** (Continuity of the nonlinear constraint). *The map  $f: \mathcal{U} \ni y \mapsto K_y \in \mathcal{K}$  is continuous and bounded in the Hausdorff metric.*

Our main abstract result is the following:

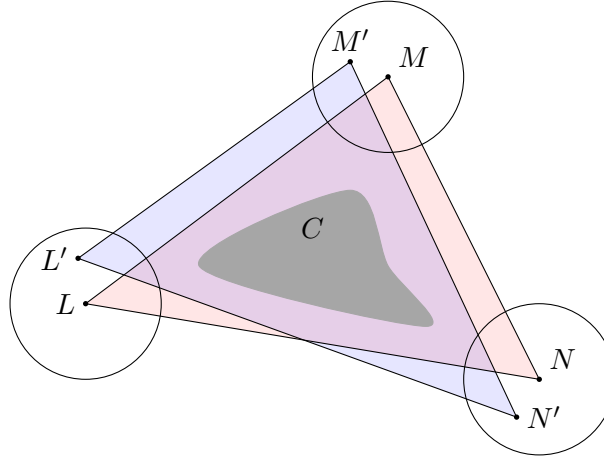
**Theorem 1.** *Suppose that Assumptions 1 and 2 hold. Suppose that  $z_0$  is a subsolution of (5), (6). Then there exist infinitely many weak solutions  $z \in L^2(\mathcal{D})$  of (5) which agree with  $z_0$  a.e. on  $\mathcal{D} \setminus \mathcal{U}$  and satisfy (6) for a.e.  $y \in \mathcal{D}$ .*

**2.1. Geometric preliminaries.** The next lemma shows that compact subsets of the interior of the convex hull of a compact set  $K$  are stable with respect to sufficiently small perturbations of  $K$  in the Hausdorff metric.

**Lemma 1.** *Let  $K \subset \mathbb{R}^n$  be a compact set. Then for any compact set  $C \subset \text{int conv } K$  there exists  $\varepsilon > 0$  such that for any compact set  $K' \subset \mathbb{R}^n$  with  $d_{\mathcal{H}}(K, K') < \varepsilon$  we have*

$$C \subset \text{int conv } K'.$$

FIGURE 1. An illustration of Lemma 1 in the case when  $K = \{L, M, N\}$  and  $K' = \{L', M', N'\}$ .



*Proof.* Since  $\text{int conv } K$  is open, for any point  $x \in C$  there exists a simplex  $S_x$  with vertices  $\{v_i\}_{i=1..n+1} \subset \text{conv } K$  such that  $x$  belongs to the inner open simplex

$$I_x := \left\{ \sum_{i=1}^{n+1} \lambda_i v_i \mid \lambda_i \in \left( \frac{1}{2(n+1)}, \frac{2}{n+1} \right), \sum_{i=1}^{n+1} \lambda_i = 1, i = 1..n+1 \right\}.$$

Since  $C$  is a compact set and the inner simplices  $\{I_x\}_{x \in C}$  cover  $C$  we can extract a finite subcover  $\{I_{x_k}\}_{k=1..m}$  of  $C$ . Let us fix  $k \in 1..m$  and consider the simplex  $S := S_{x_k}$  with vertices  $\{v_i\}_{i=1..n+1} \subset \text{conv } K$ . Let  $I := I_{x_k}$  denote the corresponding inner simplex.

If  $\varepsilon < \text{dist}(\partial I, \partial S)$  then for any points  $v'_i \in B_\varepsilon(v_i)$ ,  $i = 1..n+1$  one has

$$I \subset \text{conv}\{v'_1, v'_2, \dots, v'_{n+1}\}. \quad (8)$$

Observe that for any  $\varepsilon > 0$  and  $i = 1..n+1$  the ball  $B_\varepsilon(v_i)$  contains a point  $v'_i \in \text{conv } K'$ . Indeed, by Caratheodory's theorem  $v_i = \sum_{j=1}^{n+1} \mu_j z_j$  for some  $z_j \in K$  and  $\mu_j \in [0, 1]$  with  $\sum_{j=1}^{n+1} \mu_j = 1$ . Since  $d_{\mathcal{H}}(K, K') < \varepsilon$  there exist points  $z'_j \in K'$  such that  $z'_j \in B_\varepsilon(z_j)$ , where  $j = 1..n+1$ . Let

$$v'_i := \sum_{j=1}^{n+1} \mu_j z'_j,$$

then  $|v_i - v'_i| \leq \sum_{j=1}^{n+1} \mu_j |z_j - z'_j| < \varepsilon$ . Hence by (8) we have  $I \subset \text{conv}\{v'_1, v'_2, \dots, v'_{n+1}\}$  provided that  $\varepsilon$  is small enough. But  $v'_i \in \text{conv } K'$ , hence  $I \subset \text{conv } K'$ . Since  $I$  is open we can also write  $I \subset \text{int conv } K'$ .

Since we have finitely many simplices, we can choose  $\varepsilon > 0$  in such a way that the inclusion  $I_{x_k} \subset \text{int conv } K'$  holds for any  $k = 1..m$  (provided that  $d_{\mathcal{H}}(K, K') < \varepsilon$ ). Then

$$C \subset \cup_{k=1..m} I_{x_k} \subset \text{int conv } K'. \quad \square$$

We will also need the following elementary lemma:

**Lemma 2.** *Suppose that  $z \in C(\mathcal{U}; \mathbb{R}^n)$  where  $\mathcal{U} \subset \mathbb{R}^D$  is an open set. Suppose that for any  $y \in \mathcal{U}$  we have a compact set  $K_y \subset \mathbb{R}^n$  and the function  $y \mapsto K_y$  is continuous in the Hausdorff metric. Then the function  $F: y \mapsto \text{dist}(z(y), K_y)$  is continuous on  $\mathcal{U}$ .*

*Proof.* One can prove directly that the function  $(z, K) \mapsto \text{dist}(z, K)$  is continuous on  $\mathbb{R}^n \times \mathcal{K}$ . The function  $y \mapsto (z(y), K_y)$  is continuous in view of the assumptions. Hence the function  $F$  is continuous as a composition of continuous functions.  $\square$

In general the distance from a point  $z$  to a compact set  $K$  does not control from below the distance from  $z$  to the boundary of  $\text{conv } K$ . However the following lemma shows that there exists a segment inside  $\text{int conv } K$  with midpoint  $z$  and length controlled from below by  $\text{dist}(z, K)$ :

**Lemma 3** (Geometric lemma). *Let  $K \subset \mathbb{R}^n$  be a compact set. For any  $z \in \text{int conv } K$  there exists  $\bar{z} \in \mathbb{R}^n$  such that*

- $[z - \bar{z}, z + \bar{z}] \subset \text{int conv } K$
- $|\bar{z}| \geq \frac{1}{2n} \text{dist}(z, K)$

(This is exactly Lemma 5.3 from [DLS12].)

**2.2. Convex integration.** The following lemma is the main building block of the convex integration scheme:

**Lemma 4** (Perturbation lemma). *Suppose that Assumptions 1 and 2 hold and  $z$  is a subsolution of (5) and (6) such that*

$$\int_{\mathcal{U}} \text{dist}^2(z(y), K_y) dy = \varepsilon > 0.$$

*Then there exists  $\delta = \delta(\varepsilon) > 0$  and a sequence  $\{z_k\}_{k \in \mathbb{N}}$  of subsolutions of (5) and (6) such that*

- $z_k = z$  on  $\mathcal{D} \setminus \mathcal{U}$  for any  $k \in \mathbb{N}$
- $\int_{\mathcal{U}} |z - z_k|^2 dy \geq \delta$  for any  $k \in \mathbb{N}$
- $z_k \rightharpoonup z$  in  $L^2(\mathcal{U})$  as  $k \rightarrow \infty$ .

*Proof. Step 1.* Let  $y \in \mathcal{U}$ . Since  $z(y) \in U_y$  we can apply Lemma 3 to obtain  $\bar{z}_*(y)$  such that

$$\begin{aligned} [z(y) - \bar{z}_*(y), z(y) + \bar{z}_*(y)] &\subset U_y, \\ |\bar{z}_*(y)| &\geq \frac{1}{2n} \text{dist}(z(y), K_y), \end{aligned}$$

Since  $\Lambda$  is dense in  $\mathbb{R}^n$  and  $U_y$  is open we can find  $\bar{z}(y) \in \Lambda$  such that

$$[z(y) - \bar{z}(y), z(y) + \bar{z}(y)] \subset U_y, \quad (9)$$

$$|\bar{z}(y)| \geq \frac{1}{4n} \text{dist}(z(y), K_y). \quad (10)$$

Due to (9) there exists  $\rho(y) > 0$

$$[z(y) - \bar{z}(y), z(y) + \bar{z}(y)] + \overline{B_{2\rho(y)}(0)} \subset U_y.$$

Hence using Assumption 2, Lemma 1 and the continuity of  $z$  we can find  $R(y) > 0$  such that

$$[z(x) - \bar{z}(y), z(x) + \bar{z}(y)] + \overline{B_{\rho(y)}(0)} \subset U_x \quad (11)$$

for all  $x \in B_{R(y)}(y) \subset \mathcal{U}$ . Moreover, in view of Lemma 2 we can choose  $R(y)$  in such a way that

$$\text{dist}(z(x), K_x) \leq 2 \text{dist}(z(y), K_y) \quad (12)$$

for all  $x \in B_{R(y)}(y)$ .

Using Assumption 1 for any fixed  $y \in \mathcal{U}$  we can construct a sequence  $\{w_{y,k}\}_{k \in \mathbb{N}} \subset C_0^\infty(B_1(0))$  such that

- $w_{y,k}(x) \in [-\bar{z}(y), \bar{z}(y)] + B_{\rho(y)}(0)$  for all  $x \in B_1(0)$  and  $k \in \mathbb{N}$ ,
- $w_{y,k} \rightharpoonup 0$  in  $L^2$  as  $k \rightarrow \infty$ ,
- $\int |w_{y,k}|^2 dx > C|\bar{z}(y)|^2$  for all  $k \in \mathbb{N}$ .

*Step 2.* Let  $\varepsilon := \int_{\mathcal{U}} \text{dist}^2(z(y), K_y) dy$ . The balls  $\{B_r(y) \mid y \in \mathcal{U}, r \in (0, R(y))\}$  cover  $\mathcal{U}$ , so using Vitali's covering theorem (see e.g. [Bog07], Theorem 5.5.2) and the absolute continuity of the Lebesgue integral we can find finitely many points  $\{y_i\}_{i=1..N} \subset \mathcal{U}$  and radii  $r_i \in (0, R(y_i))$  such that

$$\sum_{i=1}^N \int_{B_i} \text{dist}^2(z(y), K_y) dy > \frac{1}{2} \varepsilon, \quad (13)$$

where the balls  $B_i := B_{r_i}(y_i)$  are pairwise disjoint.

For each  $i = 1..N$  let us introduce the scaled and translated perturbations  $w_{i,k}(x) := w_{y_i,k}(\frac{x-y_i}{r_i})$ . These functions belong to  $C_0^\infty(B_i)$  and satisfy

- (i)  $w_{i,k}(x) \in [-\bar{z}(y_i), \bar{z}(y_i)] + B_{\rho(y_i)}(0)$  for all  $x \in B_i$ ,  $k \in \mathbb{N}$ ,  $i = 1..N$ ;
- (ii)  $w_{i,k} \rightharpoonup 0$  in  $L^2$  as  $k \rightarrow \infty$  (for each fixed  $i = 1..N$ );
- (iii)  $\int |w_{i,k}|^2 dx > C|\bar{z}(y_i)|^2 \mathcal{L}^D(B_i)$  for all  $k \in \mathbb{N}$ .

In view of (i) and (11) we have  $z(x) + w_{i,k}(x) \in U_x$  for all  $x \in \mathcal{U}$  and  $i = 1..N$ , hence  $z + w_{i,k} \in X_0$ . Since the balls  $B_i$  are pairwise disjoint the function

$$z_k := z + \sum_{i=1}^N w_{i,k}$$

also belongs to  $X_0$ .

Using successively (iii), (10), (12) and (13) we obtain:

$$\begin{aligned}
\int_{\mathcal{U}} |z - z_k|^2 dy &= \sum_{i=1}^N \int_{B_i} |w_{i,k}(y)|^2 dy \stackrel{(iii)}{>} C \sum_{i=1}^N |\bar{z}(y_i)|^2 \mathcal{L}^D(B_i) \\
&\stackrel{(10)}{\geq} \frac{C}{16n^2} \sum_{i=1}^N \text{dist}^2(z(y_i), K_{y_i}) \mathcal{L}^D(B_i) = \frac{C}{16n^2} \sum_{i=1}^N \int_{B_i} \text{dist}^2(z(y_i), K_{y_i}) dx \\
&\stackrel{(12)}{>} \frac{C}{32n^2} \sum_{i=1}^N \int_{B_i} \text{dist}^2(z(x), K_x) dx \stackrel{(13)}{>} \frac{C}{64n^2} \varepsilon.
\end{aligned}$$

It remains to observe that since  $N$  is finite and the points  $y_i$  are fixed we have  $z_k \rightarrow z$  in  $L^2$  as  $k \rightarrow \infty$ .  $\square$

**2.3. Proof of Theorem 1.** We are now ready to prove our main abstract theorem.

*Proof of Theorem 1.* Let  $X_0$  denote a set of all subsolutions of (5) and (6) which agree with  $z_0$  on  $\mathcal{D} \setminus \mathcal{U}$ . Let  $X$  be the closure of  $X_0$  in the weak topology of  $L^2(\mathcal{U})$ , endowed with the corresponding induced weak topology. Clearly any  $z \in X$  solves (5) and satisfies (2) a.e. on  $\mathcal{D} \setminus \mathcal{U}$ .

For any  $z \in X$  let us define

$$I(z) := \int_{\mathcal{U}} |z(y)|^2 dy.$$

This functional is a Baire-1 function on  $X$ . Indeed, for any  $j \in \mathbb{N}$  let

$$I_j(z) := \int_{\{y \in \mathcal{U} \mid \text{dist}(y, \partial U) > 1/j\}} |(\omega_{1/j} * z)(y)|^2 dy$$

where for any  $\varepsilon > 0$  we denote by  $\omega_\varepsilon(\cdot) = \varepsilon^{-D} \omega(\cdot/\varepsilon)$  the standard convolution kernel. Then for any  $j \in \mathbb{N}$  the functional  $I_j$  is continuous on  $X$ , and for any  $z \in X$  we have  $I_j(z) \rightarrow I(z)$  as  $j \rightarrow \infty$ .

In view of Assumption 2  $X$  is a *bounded* subset of  $L^2(\mathcal{U})$ . Since the weak topology is metrizable on the norm-bounded subsets of  $L^2(\mathcal{U})$ , we can consider  $X$  as a complete metric space with some metric  $d_X$ .

Then by Baire category theorem (see also Theorem 7.3 from [Oxt80]) the set

$$Y := \{z \in X \mid I \text{ is continuous at } z\}$$

is residual in  $X$  (and hence is infinite). We claim that  $z \in Y$  implies  $J(z) = 0$ , where

$$J(z) := \int_{\mathcal{U}} \text{dist}^2(z(y), K_y) dy.$$

Indeed, suppose that  $J(z) = \varepsilon > 0$  for some  $z \in Y$ . Let  $z_j \in X_0$  be a sequence such that  $z_j \rightarrow z$  in  $L^2(\mathcal{U})$  as  $j \rightarrow \infty$ . Since  $I$  is continuous at  $z$  this implies that  $I(z_j) \rightarrow I(z)$  and consequently  $z_j \rightarrow z$  in  $L^2(\mathcal{U})$  as  $j \rightarrow \infty$ .

Then in view of Assumption 2 we have  $J(z_j) \rightarrow J(z)$  as  $j \rightarrow \infty$  and hence without loss of generality we can assume that  $J(z_j) > \varepsilon/2$  for all  $j \in \mathbb{N}$ .

Applying Lemma 4 to  $z_j$  for each  $j \in \mathbb{N}$  we can find  $\tilde{z}_j \in X_0$  such that  $d_X(\tilde{z}_j, z_j) < 2^{-j}$  and  $\int_{\mathcal{U}} |\tilde{z}_j - z_j|^2 dy \geq \delta > 0$ , where  $\delta = \delta(\varepsilon)$  is independent of  $j$ .

Since  $d_X(\tilde{z}_j, z) \leq d_X(\tilde{z}_j, z_j) + d_X(z_j, z) \rightarrow 0$  as  $j \rightarrow \infty$  we also have  $\tilde{z}_j \rightarrow z$  in  $L^2$ . Since  $z$  is a point of continuity of  $I$  we also have  $z_j \rightarrow z$  in  $L^2(\mathcal{U})$  as  $j \rightarrow \infty$ . But then  $\tilde{z}_j - z_j \rightarrow 0$  in  $L^2(\mathcal{U})$ , which contradicts the construction of  $\tilde{z}_j$ .  $\square$

## 3. APPLICATION TO THE CONTINUITY EQUATION

In this section we apply our abstract framework to the case of the continuity equation.

**Theorem 2.** *Suppose that  $d \geq 2$ . Let  $E: \mathbb{R} \rightarrow \mathbb{R}$  be a non-negative bounded function which is continuous on some bounded open interval  $I \subset \mathbb{R}$  and vanishes on  $\mathbb{R} \setminus I$ . Then there exist infinitely many bounded, compactly supported  $u: \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  and  $\mathbf{b}: \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  which satisfy (1) and (2) in sense of distributions and such that*

$$\int_{\mathbb{R}^2} u^2(t, x) dx = E(t) \quad \text{for a.e. } t \in I.$$

*Remark 1.* It is well-known that a representative of  $u$  can be chosen such that the map  $t \mapsto u(t, \cdot)$  is continuous with values in  $L^2$  equipped with the weak topology. Then the question arises whether the assertion in the theorem holds even for *every*, and not just almost every, time  $t$ . We expect this to be true: indeed this should follow by methods similar to those of [DLS10]. We will however not pursue this question further in this article.

*Remark 2.* When  $d = 2$  and  $f$  is a characteristic function of an interval, the statement of Theorem 2, essentially, follows from the example constructed in [Dep03]. This particular case of Theorem 2 was also proved in [Gus11] using the convex integration method.

*Remark 3.* A similar problem can be addressed for more general equation of the form  $\operatorname{div}(u\mathbf{B}) = 0$  instead of (1). For this equation the problem is stated as follows: given a distribution  $g$  is it possible to construct compactly supported bounded functions  $u: \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\mathbf{B}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that

$$\operatorname{div}(u\mathbf{B}) = 0, \quad \operatorname{div} \mathbf{B} = 0, \quad \operatorname{div}(u^2\mathbf{B}) = g ?$$

This is related to the so-called *chain rule problem* for the divergence operator [ADLM07]. When  $n = 2$  such a construction is not possible for  $g \neq 0$  in view of [BG14], while for  $n \geq 3$  it is possible and is obtained using convex integration and rank-2 laminates in [CGSW].

Let us put the continuity equation in the framework of the previous section. Fix a bounded open set  $\Omega \subset \mathbb{R}^d$ . Let  $\mathcal{U} := I \times \Omega$  and

$$F(t, x) := \frac{E(t)}{\mathcal{L}^d(\Omega)} \mathbf{1}_\Omega(x),$$

where  $\mathbf{1}_\Omega$  denotes the characteristic function of  $\Omega$ .

We consider equations (1) and (2) as a linear system

$$\partial_t u + \operatorname{div}_x \mathbf{m} = 0, \tag{14}$$

$$\operatorname{div}_x \mathbf{b} = 0 \tag{15}$$

in  $\mathcal{D} := \mathbb{R} \times \mathbb{R}^d$  with  $u: \mathcal{D} \rightarrow \mathbb{R}$ ,  $\mathbf{m}: \mathcal{D} \rightarrow \mathbb{R}^d$  and  $\mathbf{b}: \mathcal{D} \rightarrow \mathbb{R}^d$  such that  $z := (u, \mathbf{m}, \mathbf{b})$  satisfies the constraint

$$z(y) \in K_y := \begin{cases} \{(u, \mathbf{m}, \mathbf{b}) \mid \mathbf{m} = u\mathbf{b}, \quad |\mathbf{b}| = 1, \quad u^2 = F(y)\} & \text{if } y \in \mathcal{U} \\ 0 & \text{if } y \in \mathcal{D} \setminus \mathcal{U} \end{cases} \tag{16}$$

for a.e.  $y = (x, t) \in \mathcal{D}$ .

Suppose that  $z = (u, \mathbf{m}, \mathbf{b}) \in L^\infty(\mathcal{D})$  satisfies (14) and (15) in sense of distributions and moreover (16) holds a.e. in  $\mathcal{D}$ . Then the couple  $(u, \mathbf{b})$  satisfies the assertion of Theorem 2.

Let us check the assumption of Theorem 2.

**Lemma 5.** *Suppose that  $A, B \subset \mathbb{R}^n$  are compact sets and  $r > 0$  is such that*

- *for any  $z \in A$  there exists  $z' \in B \cap B_r(z)$*

- for any  $z \in B$  there exists  $z' \in A \cap B_r(z)$

Then  $d_{\mathcal{H}}(A, B) < r$ .

*Proof.* Suppose that  $d_{\mathcal{H}}(A, B) \geq r$ . Then without loss of generality we can assume that there exists  $z \in A$  such that for any  $z' \in B$  we have  $z \notin B_r(z')$ . But then the ball  $B_r(z)$  cannot contain any point of  $B$ , which leads to a contradiction.  $\square$

**Lemma 6.** *If  $F: \mathcal{U} \rightarrow \mathbb{R}$  is continuous, bounded and non-negative then the map  $y \mapsto K_y$  is continuous and bounded (w.r.t.  $d_{\mathcal{H}}$ ) on  $\mathcal{U}$ .*

*Proof.* Let  $f(y) := \sqrt{F(y)}$ . Let us fix  $y \in \mathcal{U}$ . For any  $\varepsilon > 0$  let  $\delta > 0$  be such that  $|f(y) - f(y')| < \varepsilon$  for any  $y' \in B_{\delta}(y) \subset \mathcal{U}$ . Let us prove that  $d_{\mathcal{H}}(K_y, K_{y'}) < 2\varepsilon$  for all  $y' \in B_{\delta}(y)$ .

For any  $z \in K_y$  there exist  $\sigma \in \{\pm 1\}$  and  $\mathbf{b} \in \mathbb{R}^d$  with  $|\mathbf{b}| = 1$  such that  $z = (\sigma f(y), \sigma f(y)\mathbf{b}, \mathbf{b})$ . Then  $z' := (\sigma f(y'), \sigma f(y')\mathbf{b}, \mathbf{b})$  belongs to  $K_{y'}$  and  $|z - z'| \leq 2|f(y) - f(y')|$ . Hence there exists  $z' \in K_{y'} \cap B_{2\varepsilon}(z)$ .

Similarly, for any  $z' \in K_{y'}$  there exist  $\sigma \in \{\pm 1\}$  and  $\mathbf{b} \in \mathbb{R}^d$  with  $|\mathbf{b}| = 1$  such that  $z' = (\sigma f(y'), \sigma f(y')\mathbf{b}, \mathbf{b})$ . Then  $z := (\sigma f(y), \sigma f(y)\mathbf{b}, \mathbf{b})$  belongs to  $K_y$  and  $|z - z'| \leq 2|f(y) - f(y')|$ . Hence there exists  $z \in K_y \cap B_{2\varepsilon}(z')$ .

Therefore by Lemma 5 we have  $d_{\mathcal{H}}(K_y, K_{y'}) < 2\varepsilon$ .  $\square$

**Lemma 7.** *Assumption 1 holds for the system (14)–(16).*

*Proof.* Let  $\phi: \mathcal{D} \rightarrow \mathbb{R}$  be a non-negative smooth function such that  $0 \leq \phi \leq 1$  on  $\mathcal{D}$ ,  $\phi = 0$  on  $\mathcal{D} \setminus B_{1/2}(0)$  and  $\phi = 1$  on  $B_{1/2}(0)$ .

*Part 1.* Suppose that  $d > 2$ . Let us show that Assumption 1 holds with  $\Lambda = \mathbb{R}^{2d+1}$ . Fix  $\bar{u} \in \mathbb{R}$ ,  $\bar{\mathbf{m}} \in \mathbb{R}^d$  and  $\bar{\mathbf{b}} \in \mathbb{R}^d$  and let  $\bar{z} = (\bar{u}, \bar{\mathbf{m}}, \bar{\mathbf{b}})$ . Since  $d > 2$  there exists a unit vector  $\mathbf{n} \in \mathbb{R}^d$  such that  $\mathbf{n} \cdot \bar{\mathbf{m}} = \mathbf{n} \cdot \bar{\mathbf{b}} = 0$ . Denote  $\hat{\mathbf{n}} = (0, \mathbf{n})$ ,  $\bar{\mathbf{a}} = (\bar{u}, \bar{\mathbf{m}})$ . For any  $k \in \mathbb{N}$  define  $\bar{\mathbf{a}}_k: \mathcal{D} \rightarrow \mathbb{R}^{d+1}$  by

$$\bar{\mathbf{a}}_k(y) := \bar{\mathbf{a}}(\hat{\mathbf{n}} \cdot \nabla_y(\phi \Pi_k)) - \hat{\mathbf{n}}(\bar{\mathbf{a}} \cdot \nabla_y(\phi \Pi_k))$$

where  $y = (t, x)$  and

$$\Pi_k(y) := \frac{\sin(k\hat{\mathbf{n}} \cdot y)}{k}.$$

Observe that

$$\operatorname{div}_y \bar{\mathbf{a}}_k = (\bar{\mathbf{a}} \cdot \nabla_y)(\hat{\mathbf{n}} \cdot \nabla_y)(\phi \Pi_k) - (\hat{\mathbf{n}} \cdot \nabla_y)(\bar{\mathbf{a}} \cdot \nabla_y)(\phi \Pi_k) = 0.$$

Let  $(u_k, \mathbf{m}_k)$  denote the components of  $\bar{\mathbf{a}}_k$ , then by the equation above we have  $\partial_t u_k + \operatorname{div}_x \mathbf{m}_k = 0$ .

Similarly let

$$\mathbf{b}_k(t, x) := \bar{\mathbf{b}}(\mathbf{n} \cdot \nabla_x(\phi \Pi_k)) - \mathbf{n}(\bar{\mathbf{b}} \cdot \nabla_x(\phi \Pi_k))$$

Then arguing as above  $\operatorname{div} \mathbf{b}_k = 0$ .

Now we introduce  $w_k := (u_k, \mathbf{m}_k, \mathbf{b}_k)$ . Then

$$w_k(y) = \bar{z}\phi \cos(k\hat{\mathbf{n}} \cdot y) + f\Pi_k$$

where  $f$  does not depend on  $k$  and vanishes on  $B_{1/2}(0)$ .

On the other hand,

$$\begin{aligned} \int_{\mathcal{D}} |w_k|^2 dy &\geq \int_{B_{1/2}(0)} |w_k|^2 dy = \int_{B_{1/2}(0)} |\bar{z}|^2 \cos^2(k\hat{\mathbf{n}} \cdot y) dy = \\ &= \int_{B_{1/2}(0)} |\bar{z}|^2 \frac{1 + \cos(2k\hat{\mathbf{n}} \cdot y)}{2} dy \geq \frac{|\bar{z}|^2}{4} |B_{1/2}(0)| \quad (17) \end{aligned}$$

provided that  $k$  is sufficiently large.



*Part 2.* Suppose that  $d = 2$  and fix  $\bar{z} = (\bar{u}, \bar{\mathbf{m}}, \bar{\mathbf{b}}) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$  with  $\bar{u} \neq 0$ . Let us look for a localized plane wave in the following form:

$$w_k = (\mathbf{a}_k, \mathbf{b}_k)$$

with

$$\begin{aligned}\mathbf{a}_k(y) &= \nabla_y \times \left( \phi \mathbf{A} \frac{\sin(k\mathbf{n} \cdot y)}{k} \right) \\ \mathbf{b}_k(t, x) &= \nabla_x^\perp \left( \phi \frac{\sin(k\mathbf{n} \cdot (t, x))}{k} \right)\end{aligned}$$

where  $\mathbf{n} = (n_t, \mathbf{n}_x) \in \mathbb{R} \times \mathbb{R}^2$  and  $\mathbf{A} \in \mathbb{R}^3$  are to be chosen and  $k \in \mathbb{N}$ . Then, by construction

$$\operatorname{div}_y \mathbf{a}_k = 0, \quad \operatorname{div}_x \mathbf{b}_k = 0.$$

Then, we get

$$w_k = \hat{z} \phi \cos(k\mathbf{n} \cdot y) + f \frac{\sin(k\mathbf{n} \cdot y)}{k}$$

where  $\hat{z} = (\mathbf{A} \times \mathbf{n}, \mathbf{n}_x^\perp)$  and  $f$  does not depend on  $k$  and vanishes on  $B_{1/2}(0)$ .

In order to have  $\hat{z} = \bar{z}$  the vectors  $\mathbf{A}$  and  $\mathbf{n}$  must satisfy

$$\begin{aligned}\mathbf{A} \times \mathbf{n} &= (\bar{u}, \bar{\mathbf{m}}), \\ \mathbf{n}_x^\perp &= \bar{\mathbf{b}}.\end{aligned}$$

From the second equation we immediately obtain that  $\mathbf{n}_x = -\bar{\mathbf{b}}^\perp$ . Since  $\bar{u} \neq 0$  there exists  $n_t$  such that  $\mathbf{n} \perp (\bar{u}, \bar{\mathbf{m}})$ . Then, we can always find  $\mathbf{A}$  such that the first equation is satisfied. It remains to observe that the estimate (17) holds also in the considered case. We thus have verified Assumption 1 for  $\Lambda = \mathbb{R}^5 \setminus \{\bar{u} = 0\}$ .  $\square$

*Proof of Theorem 2.* By symmetry of  $K_y$  for any  $y \in \mathcal{U}$  we have  $0 \in \operatorname{int} \operatorname{conv} K_y$ . On the other hand  $K_y = \{0\}$  for any  $y \in \mathcal{D} \setminus \mathcal{U}$ . Therefore  $u \equiv 0$ ,  $\mathbf{m} \equiv 0$  and  $\mathbf{b} \equiv 0$  is a subsolution of (14)–(16). Then the result follows from Lemma 2, Lemma 7 and Theorem 1.  $\square$

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