21=9+12: PG(2,4)=AG(2,3)+DAG(2,3).

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Dedicated to Prof. Luigia Berardi on the occasion of her seventyth birthday

Abstract. In this paper, we provide a construction of PG(2,4) by a collage of AG(2,3) and its dual DAG(2,3). Moreover, we prove that the construction is unique.

Key words. AG(2,3), DAG(2,3), PG(2,4).

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0. Introduction.

Configuration is one of the oldest combinatorial structure, since it appeared for the first time in 1876 in the second edition of Theodor Reye's book *Geometrie der Lage* [7]. A (v_r,b_k) *configuration* is a pair $C=(P,L)$ where P is a set of v elements, called *points*, and L is a family of b subsets, called *lines*, with k points on each line and r lines through each point. Two different lines intersect each other at most once and two different points are connected by a line at most once. By definition it easy follows that the parameters of a configuration (v_r,b_k) must satisfy $vr=bk$ and $v\geq r(k-1)+1$, see [3]. The (b_k,v_r) configuration $C^d=(L,P)$ is called the *dual configuration* of C. If $v=b$, and, hence, $r=k$, the configuration is said to be *symmetric* and is denoted by v_k. Several configurations are by no means independent. Combinations of configurations may serve the development

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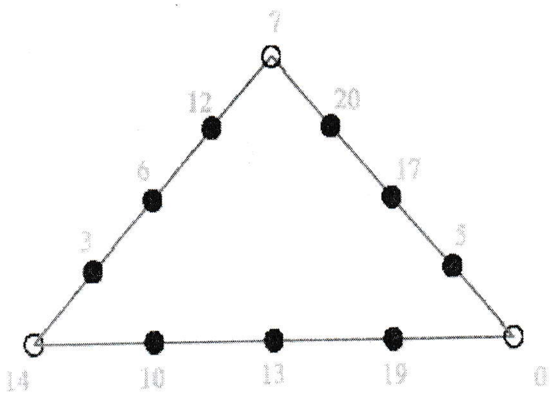
of any of them, and sometimes reveal hidden interrelations, cf. [6]. One of the most remarkable configuration is the 21_5 symmetric configuration, i.e. the projective plane of order 4, PG(2,4), see [1] and [2]. A particularly interesting property of the projective plane of order 4 is that the $(9_4,12_3)$ configuration, i.e. AG(2,3), the affine plane of order 3, are hidden in PG(2,4), cf. [8]. In Section 1 we show that a structure as simple as a vertexless triangle T in PG(2,4) is isomorphic, as point–line geometry, to the affine plane of order 3 and that the complementary set T^c in PG(2,4) is isomorphic, as point–line geometry, to $(12_3,9_4)$ configuration, i.e. the dual of the affine plane of order 3, DAG(2,3). In Section 2 we investigate this connection by providing a construction that combine the two configurations in order to lead to the projective plane of order 4. In Section 3 we prove the uniqueness of the construction which permit to rebuild PG(2,4) by a collage of AG(2,3) and its DAG(2,3).

1. The projective plane of order 4.

In this section we show that PG(2,4), the projective plane of order 4 is the union of an affine plane of order 3 and its dual. For convenience of the reader, we present the projective plane of order 4 by using the Singer [9] difference set {0,3,4,9,11} modulo 21.

0	3	4	9	11		7	10	11	16	18		2	4	14	17	18
1	4	5	10	12		8	11	12	17	19		3	5	15	18	19
2	5	6	11	13		9	12	13	18	20		4	6	16	19	20
3	6	7	12	14		0	10	13	14	19		0	5	7	17	20
4	7	8	13	15		1	11	14	15	20		0	1	6	8	18
5	8	9	14	16		0	2	12	15	16		1	2	7	9	19
6	9	10	15	17		1	3	13	16	17		2	3	8	10	20

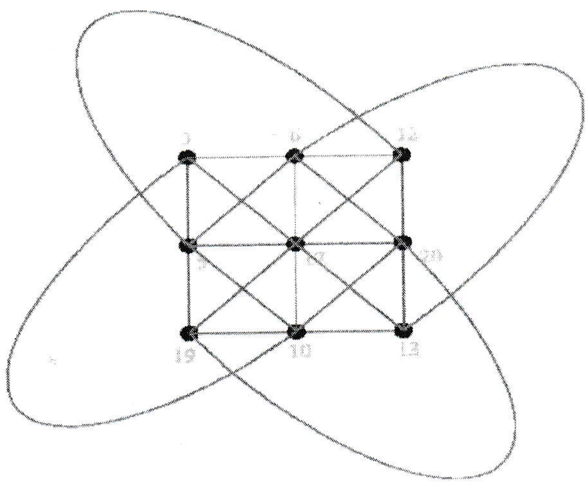
The affine plane of order 3, AG(2,3), can be embedded into PG(2,q) if and only if $q \equiv 0$, 1 (mod 3), as one easily checks by assigning coordinates to the 9 points, cf. [4] and [5]. This embedding is unique up to isomorphism. The three lines in a parallel class of AG(2,3) are concurrent in PG(2,q) if and only if $q \equiv 0$ (mod 3). For $q \equiv 1$ (mod 3), this 9–set can be found as the set of inflections of a non–degenerate cubic. Dualizing we find a dual affine plane of order 3, DAG(2,3), with 12 points and 9 lines embedded in PG(2,q), for $q \equiv 1$ (mod 3).
In PG(2,4), this 9–set can be easily found taking the vertexless triangle of any three non–concurrent lines. For instance, take {0,5,7,17,20}, {0,10,13,14,19} and {3,6,7,12,14}. Consider the vertexless triangle T={5,17,20}$\cup${10,13,19}$\cup${3,6,12}.



A vertexless triangle in PG(2,4).

The 9–set T with its collinear points is isomorphic to AG(2,3), the affine plane of order 3.

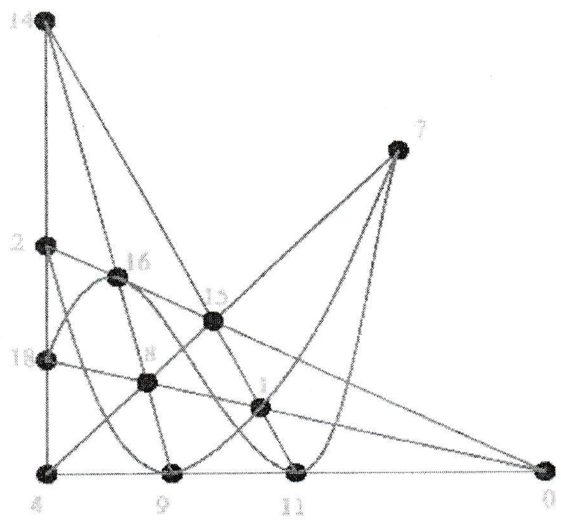
3	6	12		3	5	19		3	10	20		3	13	17
5	17	20		6	10	17		5	6	13		5	10	12
10	13	19		12	13	20		12	17	19		6	19	20



AG(2,3).

The complementary 12–set T^c={0,1,2,4,7,8,9,11,14,15,16,18} with its collinear points is isomorphic to DAG(2,3), the dual of an affine plane of order 3.

0	1	8	18		1	2	7	9		1	11	14	15
0	2	15	16		4	7	8	15		2	4	14	18
0	4	9	11		7	11	16	18		8	9	14	16



DAG(2,3)

2. A construction of PG(2,4) by AG(2,3) and its DAG(2,3).

In this section we construct PG(2,4) by AG(2,3) and its DAG(2,3).
In order to construct DAG(2,3), we label the lines of AG(2,3) by numbers.

3	6	12	1		3	5	19	2
5	17	20	5		6	10	17	6
10	13	19	9		12	13	20	10
3	10	20	3		3	13	17	4
5	6	13	7		5	10	12	8
12	17	19	11		6	19	20	12

By writing the 9 pencil of lines

1	2	3	4		2	5	7	8		1	6	7	12
3	6	8	9		1	8	10	11		4	7	9	10
4	5	6	11		2	9	11	12		3	5	10	12

we obtain DAG(2,3).

Any class of parallelism in AG(2,3) contains three lines. We begin the construction of PG(2,4) by joining to each line of AG(2,3) the two numbers of the other two parallel lines.

3	6	12	5	9		3	5	19	6	10
5	17	20	1	9		6	10	17	2	10
10	13	19	1	5		12	13	20	2	6
3	10	20	7	11		3	13	17	8	12
5	6	13	3	11		5	10	12	4	12
12	17	19	3	7		6	19	20	4	8

The lines of DAG(2,3) are pencil of lines of AG(2,3). We complete PG(2,4) by joining to each line of DAG(2,3) the number of the centre of the pencil.

1	2	3	4	3		2	5	7	8	5		1	6	7	12	6
3	6	8	9	10		1	8	10	11	12		4	7	9	10	13
4	5	6	11	17		2	9	11	12	19		3	5	10	12	20

We get PG(2,4), as one easily checks.

1	3	4	2	3		9	10	3	8	6		11	4	5	17	6
12	4	5	10	12		7	3	12	17	19		3	5	10	6	19
11	5	6	3	13		2	12	13	6	20		4	6	8	19	20
3	6	9	12	5		1	10	13	5	19		1	5	9	17	20
4	9	7	13	10		12	3	5	10	20		1	12	6	7	6
5	7	2	5	8		1	11	12	10	8		12	11	9	2	19
6	2	10	10	17		12	3	13	8	17		11	3	7	10	20

3. The uniqueness of the construction of PG(2,4) by AG(2,3) and DAG(2,3).

In this section we ask us if the construction proposed in Section 2 is the unique construction which permit to rebuilt PG(2,4) by a collage of AG(2,3) and DAG(2,3). Since a line in PG(2,4) contains 5 points we must join two Latin letters to each line of AG(2,3) and one Greek letter to each line of DAG(2,3). Since the lines of DAG(2,3) are pencil of lines of AG(2,3), the two Latin letters are two parallel lines of AG(2,3). Any class of parallelism in AG(2,3) contains three lines. Therefore, to join two parallel lines of AG(2,3) is equivalent to label any line of AG(2,3) with the third lines of the class of parallelism. Moreover, the Greek letter to join to a line of DAG(2,3) cannot belong to a parallel line of any of the lines of the pencil. Thus, it belongs to any of the line of the pencil, i.e. it is the centre of the pencil. This proves the uniqueness of the construction up to isomorphism.

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