Totally Asymmetric Limit for Models of Heat Conduction

Leonardo De Carlo & Davide Gabrielli

Journal of Statistical Physics
1

ISSN 0022-4715
Volume 168
Number 3

J Stat Phys (2017) 168:508-534
DOI 10.1007/s10955-017-1823-x

Springer

Your article is protected by copyright and all rights are held exclusively by Springer Science+Business Media, LLC. This e-offprint is for personal use only and shall not be self-archived in electronic repositories. If you wish to self-archive your article, please use the accepted manuscript version for posting on your own website. You may further deposit the accepted manuscript version in any repository, provided it is only made publicly available 12 months after official publication or later and provided acknowledgement is given to the original source of publication and a link is inserted to the published article on Springer's website. The link must be accompanied by the following text: "The final publication is available at link.springer.com".

Author's personal copy

J Stat Phys (2017) 168:508–534
DOI 10.1007/s10955-017-1823-x

Totally Asymmetric Limit for Models of Heat Conduction

Leonardo De Carlo[1] · **Davide Gabrielli**[2]

Received: 21 December 2016 / Accepted: 31 May 2017 / Published online: 8 June 2017
© Springer Science+Business Media, LLC 2017

Abstract We consider one dimensional weakly asymmetric boundary driven models of heat conduction. In the cases of a constant diffusion coefficient and of a quadratic mobility we compute the quasi-potential that is a non local functional obtained by the solution of a variational problem. This is done using the dynamic variational approach of the macroscopic fluctuation theory (Bertini et al. in Rev Mod Phys 87:593, 2015). The case of a concave mobility corresponds essentially to the exclusion model that has been discussed in Bertini et al. (J Stat Mech L11001, 2010; Pure Appl Math 64(5):649–696, 2011; Commun Math Phys 289(1):311–334, 2009) and Enaud and Derrida (J Stat Phys 114:537–562, 2004). We consider here the convex case that includes for example the Kipnis-Marchioro-Presutti (KMP) model and its dual (KMPd) (Kipnis et al. in J Stat Phys 27:6574, 1982). This extends to the weakly asymmetric regime the computations in Bertini et al. (J Stat Phys 121(5/6):843–885, 2005). We consider then, both microscopically and macroscopically, the limit of large externalfields. Microscopically we discuss some possible totally asymmetric limits of the KMP model. In one case the totally asymmetric dynamics has a product invariant measure. Another possible limit dynamics has instead a non trivial invariant measure for which we give a duality representation. Macroscopically we show that the quasi-potentials of KMP and KMPd, which are non local for any value of the external field, become local in the limit. Moreover the dependence on one of the external reservoirs disappears. For models having strictly positive quadratic mobilities we obtain instead in the limit a non local functional having a structure similar to the one of the boundary driven asymmetric exclusion process.

Keywords Non equilibrium statistical mechanics · Large deviations · Stochastic lattice gases

✉ Davide Gabrielli
gabriell@univaq.it

Leonardo De Carlo
neoleodeo@gmail.com

1 GSSI, Viale Francesco Crispi 7, 67100 Coppito, L'Aquila, Italy

2 DISIM, Università dell'Aquila, Via Vetoio, 67100 Coppito, L'Aquila, Italy

Author's personal copy

Mathematics Subject Classification 60F10 · 60K35 · 82C05

1 Introduction

Understanding the structure of stationary non equilibrium states is a major issue in non equilibrium statistical mechanics. In recent years some one dimensional boundary driven stochastic lattice models have been exactly solved. A stationary non equilibrium state can be described microscopically exhibiting the invariant measure of the model or macroscopically describing the structure of the fluctuations. This second approach is usually based on large deviations theory and the corresponding rate functional has some thermodynamic interpretations [13,18]. Apart some special cases, the rate functional is non local or correspondingly the invariant measure has long range correlations.

Stochastic lattice gases for which the hydrodynamic scaling limit has been derived are essentially of two types. Driven diffusive models that have a non trivial diffusive scaling limit typically given by a non linear diffusive equation [30,34] and asymmetric models whose natural scaling is the Eulerian one and have as a scaling limit a first order hyperbolic conservation law [2,30,34]. There are examples of solvable one dimensional stationary non equilibrium states in both classes of models.

In the case of diffusive systems, the first computation of large deviations rate functionals was obtained for the exclusion model starting from an exact representation of the invariant measure [18–20]. The same result was then obtained with the dynamic variational approach of the macroscopic fluctuation theory [5,6,13]. This macroscopic approach was then generalized to a wider class of models characterized by a constant diffusion matrix and a quadratic mobility [7,13].

For the weakly asymmetric exclusion the computation of the rate functional has been done in [23] starting from an exact representation of the invariant measure and then in [9–11] using the macroscopic fluctuation theory. A weakly asymmetric model is a microscopic model having a behavior whose asymmetry, encoded by an external field E, is suitably going to zero when the underlying lattice size is going to zero.

For asymmetric models we have the following. The large deviations rate functional for the invariant measure of the boundary driven asymmetric exclusion process has been computed using an exact representation [21,22]. The corresponding rate functional is not local. The dynamic large deviations for asymmetric models is less understood. The rate functional for the exclusion process with periodic boundary conditions is discussed in [36]. The case with boundary sources has been discussed in [16]. Once the dynamic rate functional is identified then it is possible to define and in some cases to compute the corresponding quasi-potential [1,13]. In particular in [1] the functionals in [21,22] and the quasi-potentials for other conservation laws with convex mobilities have been computed.

Other examples of exact computations of large deviations rate functionals for stationary non equilibrium states are for example [3] for a two component diffusive system and [27] for the totally asymmetric exclusion process with particles of different classes on a ring.

A different way of obtaining the functionals in [21,22] for the asymmetric exclusion process is to compute the functional for the weakly asymmetric case and then to consider the limit for large values of the field E. This has been done in [9–11]. With this approach the existence of Lagrangian phase transitions for finite but large external fields [10,11] has also been proved. In this paper we use exactly this approach generalizing it to the case of models having constant diffusion matrix and convex quadratic mobility.

Author's personal copy

We consider also the problem from a microscopic perspective. We can define several totally asymmetric dynamics having the same weakly asymmetric diffusive scaling limit. We discuss two possible boundary driven totally asymmetric versions of the KMP model. In one case the invariant measure is given by a product measure of exponentials. For the other dynamics the invariant measure has not a simple structure and we give a representation in terms of convex combinations of products of Gamma distributions. This is done using a duality representation of the process. The same could be done also for the dual model of the KMP (that we shortly call KMPd).

Macroscopically we start computing the large deviations rate functional for the empirical measure when the particles are distributed according to the invariant measure of weakly asymmetric models of heat conduction. Instead of discussing the general case we concentrate on three prototypal models corresponding respectively to quadratic and convex mobilities having two coinciding roots, two distinct roots and no roots. The first case corresponds microscopically to the KMP model. The second case corresponds microscopically to the dual of the KMP model (KMPd). We study the third case only macroscopically, indeed it is difficult to build up a corresponding explicit microscopic model (see the comments after (7.7) for the technical reasons). We call KMPx this unspecified model. In this case the density can assume also negative values. For all these models, in the one dimensional case, we can compute the quasi-potential using the dynamic variational approach of the macroscopic fluctuation theory [13]. The corresponding rate functionals are generically not local and depends on the value of the constant external field E generating the asymmetry.

We study the asymptotic behavior when $E \to \pm\infty$ of the quasi-potentials. In the case of the exclusion model these limits allow to recover the large deviations rate functionals of the totally asymmetric exclusion process [9–11]. In the case of KMP and KMPd we have that the functionals become local in the limit. Moreover the dependence on one of the boundary sources disappears. More precisely the limit functionals correspond to the large deviations rate functionals for empirical measures when the variables are distributed according to product measures. We recover in this way the large deviations rate functionals for one of the two versions of the totally asymmetric KMP dynamics. The model KMPx has instead a very different behavior. The limiting functional is again non local and has a structure very similar to the one of the exclusion. In this case the possibility of Lagrangian phase transitions appears for large values of the field acting in the same direction with respect to the external reservoirs.

1.1 Main Results

For the readers convenience we summarize the main results of the paper. At microscopic level we have the following. We select two possible totally asymmetric boundary driven models of heat conduction. For one of them we show that the invariant measure is product, for the other one we obtain a representation of the invariant measure as a mixture of products of gamma distributions. This is done using a kind of duality based on convex analytic arguments. At macroscopic level we have the following. We compute the quasi-potential for any diffusive, one dimensional, weakly asymmetric, boundary driven model having a constant diffusion coefficient and a quadratic mobility. This is generically a non-local functional. We then compute the limit of the quasi-potentials for large values of the external field. We consider only the cases of convex mobilities. The totally asymmetric limit is a local functional except in the case of single site variables that can assume all the possible real values. In this case the limiting quasi-potential has a non local structure similar to that of the exclusion process. This result suggests the natural conjecture that a large class of totally asymmetric models of heat conduction should have a local rate functional for the invariant measure.

Author's personal copy

1.2 Organization of the Paper

The structure of the paper is the following In Sect. 2 we describe microscopically the symmetric versions of the models of heat conduction that we are going to study, describe the weakly asymmetric versions, study the invariant measures and describe the instantaneous current that is the basic object to understand the connection between the microscopic and the macroscopic description. In Sect. 3 we introduce two possible totally asymmetric versions of the KMP dynamics and study the corresponding invariant measures in the boundary driven case. For one model the invariant measure is of product type while for the other one this is not the case and we give a duality representation using convex analytic arguments. In Sect. 4 we give a short overview of the scaling limit for particle systems that is the main bridge between the microscopic and macroscopic descriptions. We discuss the transport coefficients and introduce macroscopically the three prototypal models we are going to study. In Sect. 5 we study the stationary solutions of the hydrodynamic equations and the corresponding associated currents. In particular we discuss the asymptotic behaviors for large fields E. In Sect. 6 we outline the structure of dynamic large deviations for our class of models and show the relation between the quasi-potential and the large deviations for the invariant measure. In Sect. 7 we compute the quasi-potential for boundary driven one dimensional weakly asymmetric models. In Sect. 8 we compute the limit of the quasi-potentials for $E \to \pm\infty$.

2 Models of Heat Conduction

We consider generalized stochastic lattice gases having random energies/masses associated to the vertices of a lattice. More precisely let $\Lambda \subseteq \mathbb{R}^d$ be a bounded domain and let $\Lambda_N := \frac{1}{N}\mathbb{Z}^d \cap \Lambda$ be its discretization with a lattice of mesh $\frac{1}{N}$. We denote by $\xi = \{\xi(x)\}_{x\in\Lambda_N}$ the energy/mass configuration of the system. The value $\xi(x) \in \mathbb{R}$ is the energy/mass associated to the site x. When $\xi(x) \in \mathbb{R}^+$, we use the interpretation as a configuration of mass. We will mainly consider the one dimensional case for which Λ is an interval and Λ_N is a linear lattice. We call $x \in \Lambda_N$ an internal vertex if all its nearest neighbors $y \in \frac{1}{N}\mathbb{Z}^d$ belong also to Λ_N. A vertex $x \in \Lambda_N$ that is not internal is instead a boundary vertex. We denote by $\partial\Lambda_N$ the set of boundary vertices. The stochastic evolution is encoded in the generator that is of the type

$$\mathcal{L}_N f = \sum_{x\sim y} L_{x,y} f + \sum_{x\in\partial\Lambda_N} L_x f. \tag{2.1}$$

The first sum in (2.1) is a sum over unordered nearest neighbor sites of Λ_N, i.e. each pair is considered once. The first term in (2.1) is the bulk contribution to the stochastic evolution while the second term is the boundary part of the dynamics that models the interaction of the system with external reservoirs.

Let $\varepsilon^x = \{\varepsilon^x(y)\}_{y\in\Lambda_N}$ be the configuration of mass with all the sites different from x empty and having unitary mass at site x. This means that $\varepsilon^x(y) = \delta_{x,y}$ where δ is the Kronecker symbol. The bulk contribution to the stochastic dynamics is given by

$$L_{x,y} f(\xi) = \int_{\mathbb{R}} \Gamma^{\xi}_{x,y}(dj)\big[f\left(\xi - j\left(\varepsilon^x - \varepsilon^y\right)\right) - f(\xi)\big]. \tag{2.2}$$

This means that, after random exponential times, across each bond of the system there is a random flow of mass. The parameters of the exponential times and the distribution of the flow are determined by the positive measure $\Gamma^{\xi}_{x,y}(dj)$. The new configuration $\xi - j\,(\varepsilon^x - \varepsilon^y)$

Author's personal copy

is the starting configuration minus the divergence of a current on the lattice different from zero on the single edge (x, y) where it assumes the value j. If for any $A \subseteq \mathbb{R}$ we have $\Gamma^{\xi}_{x,y}(A) = \Gamma^{\xi}_{y,x}(-A)$, then (2.2) is symmetric in x, y so that we can consider without ambiguity a sum over unordered pairs in (2.1). We will use the same notation both for a measure and the corresponding density. If we consider $\Gamma^{\xi}_{x,y}(dj)$ uniform in $[-\xi(y), \xi(x)]$ we obtain

$$L_{x,y} f(\xi) := \int_{-\xi(y)}^{\xi(x)} \frac{dj}{\xi(x)+\xi(y)} \big[f(\xi - j\left(\varepsilon^x - \varepsilon^y\right)) - f(\xi) \big], \tag{2.3}$$

that corresponds to the KMP dynamics [29] for which the variables ξ remain always positive if they are positive at the beginning. Another choice is the discrete uniform distribution on the integer points in $[-\xi(y), \xi(x)]$. This means that ξ is a configuration of mass assuming only integer values and

$$\Gamma^{\xi}_{x,y}(dj) = \frac{1}{\xi(x)+\xi(y)+1} \sum_{i \in [-\xi(y),\xi(x)] \cap \mathbb{Z}} \delta_i(dj) \tag{2.4}$$

where $\delta_i(dj)$ is the delta measure concentrated at i and the sum is over the integer values belonging to the interval. If the initial configuration is such that the values of the variables ξ are all integers then this fact is preserved by the dynamics and we obtain a model that can be interpreted as a model of evolving particles. This is exactly the dual model of KMP [29]. We call KMPd the stochastic dynamics associated to the choice (2.4), where the last letter means *dual*.

The boundary part of the generator can be defined in several ways. Let us fix a possible definition for KMP that is good for symmetric and weakly asymmetric models. The system is in contact with external reservoirs with chemical potential $\lambda(x) < 0$, $x \in \partial\Lambda_N$. The effect of the interaction with the source is that at rate one the value of the variable $\xi(x)$ is substituted by a random value exponentially distributed with parameter $-\lambda(x)$

$$L_x f(\xi) = \int_{-\xi(x)}^{+\infty} |\lambda(x)| e^{\lambda(x)(\xi(x)+j)} \left[f(\xi + \varepsilon^x j) - f(\xi) \right] dj. \tag{2.5}$$

For KMPd the boundary dynamics can be fixed similarly to (2.5). In this case it is natural to substitute the exponential distribution by a geometric one.

Other possible solvable models with exchange of mass between neighboring sites are possible (see for example [28]) and some of them will have the same macroscopic behavior of the models here discussed. For example a natural choice is obtained fixing $\Gamma^{\xi}_{x,y}$ as the law of a Gaussian random variable with mean $\frac{(\xi(x)-\xi(y))}{2}$ and variance a parameter γ^2. In this case the interpretation in terms of mass is missing since the variables can assume every real value.

Suppose that on the lattice we have a discrete vector field $\mathbb{F}$. This is a collection of numbers $\mathbb{F}(x, y)$ for any ordered pair of nearest neighbors lattice points satisfying the antisymmetry relationships $\mathbb{F}(x, y) = -\mathbb{F}(y, x)$. If the vector field is time dependent these number are time dependent. The motion of the mass is influenced by the presence of the field and we have a perturbed measure $\Gamma^{\mathbb{F}}$ in the bulk dynamics

$$L^{\mathbb{F}}_{x,y} f(\xi) := \int \Gamma^{\xi,\mathbb{F}}_{x,y}(dj) \left[f\left(\xi - j\left(\varepsilon^x - \varepsilon^y\right)\right) - f(\xi) \right]. \tag{2.6}$$

The natural choice of the measure $\Gamma^{\mathbb{F}}$ is

$$\Gamma^{\xi,\mathbb{F}}_{x,y}(dj) = \Gamma^{\xi}_{x,y}(dj) e^{\frac{\mathbb{F}(x,y)}{2} j}. \tag{2.7}$$

Author's personal copy

The factor $\frac{1}{2}$ in the exponent appears just for convenience of notation in the following. A perturbation of this type is for example the one used in [7] to compute dynamic large deviations for the KMP model and corresponds therefore to the choice

$$\Gamma^{\xi,\mathbb{F}}_{x,y}(dj) = \frac{e^{\frac{\mathbb{F}(x,y)}{2}j}}{\xi(x)+\xi(y)}\chi_{[-\xi(y),\xi(x)]}(j)\,dj. \tag{2.8}$$

An equivalent formula obtained applying (2.7) holds also for KMPd. By the symmetry of the measure Γ and the antisymmetry of the discrete vector field $\mathbb{F}$ we have that $\Gamma^{\xi,\mathbb{F}}_{x,y}(j) = \Gamma^{\xi,\mathbb{F}}_{y,x}(-j)$ and we can define the generator considering sums over unordered bonds.

A special case is when the parameter N is large and the discrete vector field is obtained as a discretization of a smooth vector field on Λ. In this case since the mesh of the lattice is $\frac{1}{N}$ the values of the discrete vector field are $O\left(\frac{1}{N}\right)$. For this reason we say in this case that we have a *weakly asymmetric model*. A natural discretization is as follows.

Let $F(x) = (F_1(x), \ldots, F_d(x))$ be a smooth vector field. We associate to F a discrete vector field $\mathbb{F}$ on the lattice that corresponds to a discretized version of the continuous vector field defined by

$$\mathbb{F}(x,y) = \int_{(x,y)} F(z)\cdot dz. \tag{2.9}$$

In (2.9) (x, y) is an oriented edge of the lattice, the integral is a line integral that corresponds to the work done by the vector field F when a particle moves from x to y. The value of $\mathbb{F}(y, x)$, by antisymmetry, corresponds to minus the value in (2.9).

2.1 Stationarity

Considering KMP models like in (2.8), for any $x \sim y$ the detailed balance condition

$$\mu_N^{\lambda(\cdot)}(\xi)\Gamma^{\xi,\mathbb{F}}_{x,y}(j) = \mu_N^{\lambda(\cdot)}\left(\xi - j\left(\varepsilon^x - \varepsilon^y\right)\right)\Gamma^{\xi-j(\varepsilon^x-\varepsilon^y),\mathbb{F}}_{x,y}(-j) \tag{2.10}$$

is satisfied provided that $\mathbb{F}(x, y) = \lambda(y) - \lambda(x)$ and

$$\mu_N^{\lambda(\cdot)}(\xi) = \prod_z |\lambda(z)|e^{\lambda(z)\xi(z)} \tag{2.11}$$

is the density of an inhomogeneous product of exponentials.

For a boundary dynamics like in (2.5) we have for a boundary site x the detailed balance relationship

$$\mu_N^{\lambda(\cdot)}(\xi)\Gamma^{\xi}_{x}(j) = \mu_N^{\lambda(\cdot)}(\xi + j\varepsilon^x)\Gamma^{\xi+j\varepsilon^x}_{x}(-j) \tag{2.12}$$

when $\mu_N^{\lambda(\cdot)}$ is like (2.11) and the value $\lambda(x)$ coincides with the parameter λ in (2.5). In (2.12) we called

$$\Gamma^{\xi}_{x}(dj) = |\lambda|e^{\lambda(\xi(x)+j)}\chi_{[-\xi(x),+\infty)}(j)\,dj.$$

By the above computations we obtain, in agreement with the results in [8], that a KMP model is reversible if the external field is of gradient type $\mathbb{F}(x, y) = \psi(y) - \psi(x)$ for a function ψ such that $\psi(x) = \lambda(x)$ for $x \in \partial\Lambda_N$, where $\lambda(x)$ is the parameter of the external source at x. In this case the invariant measure is product like in (2.11) with $\lambda(\cdot)$ replaced by $\psi(\cdot)$. When the model is not reversible the invariant measure is not known. A similar result can be obtained also for KMPd. The Gaussian model briefly discussed above (2.6) satisfies the detailed balance condition with respect to a product of Gaussian distributions having the same

Author's personal copy

arbitrary mean value and variance equal to $2\gamma^2$ where γ^2 is the variance of the stochastic current across an edge.

A special situation is when the system is in contact with sources having all the same chemical potential and there is not an external field. In this case we have that the KMP and the KMPd models are equilibrium models reversible with respect to homogeneous product measures. Given a reference measure μ on $\mathbb{R}$, we denote by μ^λ the probability measure obtained inserting a chemical potential term of the form

$$\mu^\lambda(dx) = \frac{\mu(dx)e^{\lambda x}}{Z(\lambda)}, \tag{2.13}$$

where $Z(\lambda)$ is the normalization constant. The corresponding average density $\rho[\lambda] := \int \mu_\lambda(dx)x = (\log Z(\lambda))'$ is increasing in λ and we call $\lambda[\rho]$ the inverse function. In the case of the KMP model it is natural to fix $\mu(dx) = dx$ and restrict to negative values of λ. In this case $Z(\lambda) = -\lambda^{-1} = \rho[\lambda]$ and $\lambda[\rho] = -\rho^{-1}$. For the KMPd model we fix $\mu(dx) = \sum_{k=0}^{+\infty} \delta_k(dx)$ and again we restrict to negative values of λ. In this case we have $Z(\lambda) = (1-e^\lambda)^{-1}$, $\rho[\lambda] = \frac{e^\lambda}{1-e^\lambda}$ and $\lambda[\rho] = \log \frac{\rho}{1+\rho}$.

2.2 Instantaneous Current

The instantaneous current for the bulk dynamics is defined as

$$j_\xi(x, y) := \int \Gamma^\xi_{x,y}(dj)j. \tag{2.14}$$

We call symmetric model those for which $j_\xi(x, y) = 0$ when $\xi(x) = \xi(y)$. The importance of this definition is in the following fact. Let ξ_t be the random configuration of particles at time t. Let also $\mathcal{J}_t(x, y)$ be the net total amount of mass that has flown from x to y in the time window $[0, t]$. Then by a simple argument [34] we have that

$$\mathcal{J}_t(x, y) - \int_0^t j_{\xi_s}(x, y)\, ds, \tag{2.15}$$

is a martingale. For example the instantaneous current across the edge (x, y) for the KMP process is given by

$$\int_{-\xi(y)}^{\xi(x)} \frac{jdj}{\xi(x)+\xi(y)} = \frac{1}{2}\left(\xi(x) - \xi(y)\right). \tag{2.16}$$

This computation shows that the KMP model is of gradient type. In general a model of stochastic particles on a lattice is called of gradient type [30,34] if the instantaneous current can be written as

$$j_\xi(x, y) = \tau_x h(\xi) - \tau_y h(\xi), \tag{2.17}$$

where h is a local function and τ_z is the shift operator by the vector z. Formula (2.16) shows for example that for KMP formula (2.17) holds with $h(\xi) = \frac{\xi(0)}{2}$. Also KMPd is gradient with respect to the same function h.

By a direct computation we get that the instantaneous current for the weakly asymmetric KMP model in the case of a constant external field E in the direction from x to y is given by

$$j^E_\xi(x, y) = \frac{1}{2}\big(\xi(x) - \xi(y)\big) + \frac{E}{6}\big[\xi(x)^2 + \xi(y)^2 - \xi(x)\xi(y)\big] + o(E). \tag{2.18}$$

Author's personal copy

For the KMPd model we get instead

$$j_\xi^E(x,y)=\frac{1}{2}\big(\xi(x)-\xi(y)\big)+\frac{E}{12}\big[2\xi(x)^2+2\xi(y)^2-2\xi(x)\xi(y)+\xi(x)+\xi(y)\big]+o(E). \tag{2.19}$$

The computations in this section will be important in the discussion of the transport coefficients in Sect. 4.2.

3 Asymmetric Models

We consider now some possible one dimensional totally asymmetric models for which the mass can move only in one preferred direction. If on a bond (x,y) the asymmetry is from x to y then we have that the measure $\Gamma^\xi_{x,y}$ has a support contained on the interval $[0,\xi(x)]$. We consider only the KMP case ($\xi(x)\in\mathbb{R}^+$) and assume that the density $\Gamma^\xi_{x,y}$ depends only on $\xi(x)$ and not on $\xi(y)$. We discuss indeed two different cases. One has a product invariant measure while for the other one we discuss a duality representation of the invariant measure using a convex analytic approach. Both models can be imagined as limits for large values of an external constant field of weakly asymmetric models having an expansion like (2.18). A natural question is if they have the same large deviations for the invariant measure.

The macroscopic domain is $\Lambda=(0,1)$ and the asymmetry is in the positive direction. For simplicity of notation we consider the models defined on the lattice $\{1,2,\ldots,N\}$ instead of Λ_N. Since the computations in this section are only microscopic the lattice size is not relevant.

3.1 Totally Asymmetric KMP Model Version 1

In this section we discuss a model with distribution of the current flowing across a bond in the bulk given by $\Gamma^\xi_{x,x+1}=\chi_{[0,\xi(x)]}(j)\,dj$. We fix the interaction with the boundary left source like in [8]

$$L_1f(\xi)=\int_0^{+\infty}e^{\lambda j}\left[f(\xi+j\varepsilon^1)-f(\xi)\right]dj. \tag{3.1}$$

At the right boundary site N with rate $\xi(N)$ the amount of mass present is transformed into $\xi'(N)$ that is uniformly distributed on $[0,\xi(N)]$. Consider the product measure (recall $\lambda<0$)

$$\mu_N^\lambda(d\xi)=\prod_{x\in\Lambda_N}|\lambda|e^{\lambda\xi(x)}d\xi(x). \tag{3.2}$$

With a change of variables we get

$$\mathbb{E}_{\mu_N^\lambda}\left[L_{x,x+1}f\right]=\int_{(\mathbb{R}^+)^N}\mu_N^\lambda(d\xi)f(\xi)\left[\xi(x+1)-\xi(x)\right]. \tag{3.3}$$

Still with a changes of variables at the boundaries we get

$$\begin{cases}\mathbb{E}_{\mu_N^\lambda}\left[L_1f\right]=\int_{(\mathbb{R}^+)^N}\mu_N^\lambda(d\xi)f(\xi)\left[\xi(1)+\lambda^{-1}\right],\\ \mathbb{E}_{\mu_N^\lambda}\left[L_Nf\right]=-\int_{(\mathbb{R}^+)^N}\mu_N^\lambda(d\xi)f(\xi)\left[\lambda^{-1}+\xi(N)\right].\end{cases} \tag{3.4}$$

Summing up (3.3) and (3.4) we obtain that (3.2) is invariant for the dynamics.

Author's personal copy

3.2 Totally Asymmetric KMP Version 2

In this section we discuss a second possible totally asymmetric limit dynamics. This is the model that is obtained considering a constant external field $\mathbb{F}$ in (2.8) and taking the limit $\mathbb{F} \to +\infty$, suitably normalizing the rates. The dynamics in the bulk is defined by a distribution of the current flowing across a bond given by $\Gamma^{\xi}_{x,x+1} = \delta_{\xi(x)}$. This means that at rate one all the mass present on a site jumps to the nearest neighbor site on the right. On the torus this dynamics is not irreducible since eventually all the mass will concentrate on a single lattice site moving randomly like an asymmetric random walk. The boundary driven case has not this problem and the dynamics is irreducible.

Given two probability measures μ and ν on $\mathbb{R}$ we denote by $\mu * \nu = \nu * \mu$ their convolution. Let us also define the family of Gamma measures $\{\gamma_n\}_{n\geq 0}$ of parameter $|\lambda|$ as follows. We set $\gamma_0 := \delta_0$, then we define γ_1 as the absolute continuous probability measure on $\mathbb{R}^+$ having density $|\lambda| e^{\lambda x}$. Finally we define $\gamma_n := \gamma_1^{*n}$ where the right hand side symbol means a n-times convolution of γ_1. Note that $\gamma_j * \gamma_i = \gamma_{i+j}$. We fix the dynamics at the boundaries by

$$\begin{cases} L_1 f(\xi) = \int_0^{+\infty} \gamma_1(j) \left[f(\xi + j\varepsilon^1) - f(\xi) \right] dj \\ L_N f(\xi) = \left[f\left(\xi - \xi(N)\varepsilon^N\right) - f(\xi) \right]. \end{cases} \tag{3.5}$$

The invariant measure for this second version of the totally asymmetric KMP model is not of product type and it seems not to have a simple expression. We give a representation of this measure as a convex combination of products of Gamma distributions. This is done developing a kind of duality between this process and a totally asymmetric version of KMPd.

Consider a product measure ν_N having marginals $\nu^{(x)}$, i.e.

$$\nu_N(d\xi) := \prod_{x=1}^{N} \nu^{(x)}(d\xi(x)) =: \otimes_{x=1}^{N} \nu^{(x)}. \tag{3.6}$$

For a measure of this type we have for any k

$$\begin{aligned} &\int_{(\mathbb{R}^+)^n} \nu_N(d\xi) \int_0^{+\infty} \gamma_k(j) f(\xi + j\varepsilon^1)\, dj \\ &\quad = \int_{(\mathbb{R}^+)^n} (\nu^{(1)} * \gamma_k)(d\xi(1)) \nu^{(2)}(d\xi(2)) \dots \nu^{(N)}(d\xi(N))\, f(\xi). \end{aligned} \tag{3.7}$$

Likewise we have

$$\begin{aligned} &\int_{(\mathbb{R}^+)^n} \nu_N(d\xi) f\left(\xi + \xi(x)(\varepsilon^{x+1} - \varepsilon^x)\right) \\ &\quad = \int_{(\mathbb{R}^+)^n} \nu^{(1)}(d\xi(1)) \dots \gamma_0(d\xi(x)) \left(\nu^{(x)} * \nu^{(x+1)})(d\xi(x+1)\right) \dots \nu^{(N)}(d\xi(N))\, f(\xi) \end{aligned}$$

The right hand side in the above formula is the expected value of the function f with respect to a product measure having x-marginal equal to γ_0, $(x+1)-$marginal equal to $\nu^{(x)} * \nu^{(x+1)}$ and all the remaining marginal equal to the one of ν_N. Finally we have also that

$$\int_{(\mathbb{R}^+)^n} \nu_N(d\xi) f\left(\xi - \xi(N)\varepsilon^N\right) = \int_{(\mathbb{R}^+)^n} \nu^{(1)}(d\xi(1)) \dots \gamma_0(d\xi(N))\, f(\xi). \tag{3.8}$$

Author's personal copy

If we call $\mathcal{L}_{N,a}$ the generator of this asymmetric model, for a measure ν_N like in (3.6) we obtained

$$\begin{aligned}\nu_N\mathcal{L}_{N,a} = &\left\{\left[\left(\nu^{(1)} * \gamma_1\right)\otimes \nu^{(2)}\otimes\cdots\otimes\nu^{(N)}\right]-\left[\nu^{(1)}\otimes\cdots\otimes\nu^{(N)}\right]\right\}\\ &+\sum_x\left\{\left[\nu^{(1)}\otimes\ldots\gamma_0\otimes\left(\nu^{(x)} * \nu^{(x+1)}\right)\otimes\cdots\otimes\nu^{(N)}\right]-\left[\nu^{(1)}\otimes\cdots\otimes\nu^{(N)}\right]\right\}\\ &+\left\{\left[\nu^{(1)}\otimes\cdots\otimes\nu^{(N-1)}\otimes\gamma_0\right]-\left[\nu^{(1)}\otimes\cdots\otimes\nu^{(N)}\right]\right\}. \end{aligned}\tag{3.9}$$

We can now show that there is a solution of the Kolmogorov equation $\partial_t\nu_N(t) = \nu_N(t)\mathcal{L}_{N,a}$ that can be written in the form

$$\nu_N(t) = \sum_{\eta\in\mathbb{N}^N} c_t(\eta)\gamma_{\eta(1)}\otimes\gamma_{\eta(2)}\otimes\cdots\otimes\gamma_{\eta(N)}. \tag{3.10}$$

In the formula (3.10) $\eta = (\eta(1),\ldots\eta(N)) \in \mathbb{N}^N$ can be interpreted as a configuration of particles on the lattice and $c_t(\eta) \geq 0$ for any fixed t is a suitable probability measure on $\mathbb{N}^N$ to be determined. Formula (3.10) says that we are searching for a solution that can be written as a mixture of products of Gamma measures for any time. Defining

$$\gamma^\eta := \gamma_{\eta(1)}\otimes\gamma_{\eta(2)}\otimes\cdots\otimes\gamma_{\eta(N)}$$

we write compactly (3.10) as $\sum_\eta c_t(\eta)\gamma^\eta$. For a measure of the type (3.10) we have

$$\nu_N(t)\mathcal{L}_{N,a} = \sum_\eta c_t(\eta)\left(\gamma^\eta\mathcal{L}_{N,a}\right), \tag{3.11}$$

and we can now use formula (3.9). Reorganizing the terms, the right hand side of (3.11) becomes

$$\begin{aligned}\sum_\eta\gamma^\eta\Bigg\{&\left[c_t(\eta-\varepsilon^1)\chi(\eta(1)>0)-c_t(\eta)\right]\\ &+\left[\chi(\eta(N)=0)\sum_{k=0}^{+\infty}c_t(\eta+k\varepsilon^N)-c_t(\eta)\right]\\ &+\sum_{x=1}^{N-1}\sum_{k=0}^{\eta(x+1)}\left[\chi(\eta(x)=0)c_t(\eta+k(\varepsilon^x-\varepsilon^{x+1}))-c_t(\eta)\right]\Bigg\}.\end{aligned}\tag{3.12}$$

where χ denotes the indicator function. Using (3.12) we can write formula (3.11) compactly as

$$\sum_\eta\gamma^\eta\partial_t c_t(\eta) = \nu_N(t)\mathcal{L}_{N,a} = \sum_\eta\gamma^\eta\left(c_t(\eta)\mathcal{L}^d_{N,a}\right) \tag{3.13}$$

where $\mathcal{L}^d_{N,a}$ is a Markov generator of a stochastic dynamics on the variables η. We interpret (3.13) as a duality relationship between the two stochastic dynamics $\mathcal{L}_{N,a}$ and $\mathcal{L}^d_{N,a}$. The upper index d is the shorthand of *dual*. The variables η represent configurations of particles on the lattice and $\eta(x)$ that is always an integer number is the number of particles at site x. By formula (3.12) the stochastic dynamics associated to $\mathcal{L}^d_{N,a}$ can be described as follows. In the bulk the dynamics has a distribution of current given by $\Gamma^\eta_{x,x+1} = \delta_{\eta(x)}$. At the left boundary one particle is created with rate 1 while all the particle at the right boundary are erased at rate 1. This is a totally asymmetric version of the model KMPd. To have the above interpretation we remark that we interpret $c_t(\eta)$ as a probability measure on configurations

Author's personal copy

of particles and in (3.12) we have, as written in (3.13), the action of $\mathcal{L}^d_{N,a}$ on measures. In particular in (3.12) appear values $c_t(\eta')$ for configurations η' that can be transformed into η with an elementary move of the dynamics $\mathcal{L}^d_{N,a}$. Formula (3.9) has a different structure (similar to the action of the generator on functions) because we used the special form of the measures.

We proved that the model with generator $\mathcal{L}_{N,a}$ starting at time zero with a distribution of the type $\sum_\eta c_0(\eta)\gamma^\eta(d\xi)$ will have a distribution of energies at time t that is $\sum_\eta c_t(\eta)\gamma^\eta(d\xi)$ where $c_t(\eta)$ is the distribution of particles at time t for the model with generator $\mathcal{L}^d_{N,a}$ starting at time 0 with the distribution of particles given by $c_0(\eta)$. In particular, considering the limit for $t\to+\infty$, this relationship between the two processes will hold also for the corresponding invariant measures for which we get

$$\mu_N(d\xi)=\sum_\eta \mu_{N,d}(\eta)\gamma^\eta(d\xi). \tag{3.14}$$

In (3.14) μ_N is the invariant measure for the process $\mathcal{L}_{N,a}$ while $\mu_{N,d}$ is the invariant measure for the process $\mathcal{L}^d_{N,a}$.

It is not clear if there is a usual duality relationship between the two processes that we are intertwining. It is interesting to analyze this duality within the general approach to duality in [17]. We conjecture that the large deviations rate functional for the empirical measure when particles are distributed according to the invariant measure of the original model is the same of the corresponding one associated to a product of exponentials. A direct microscopic computation of this rate functional would be very interesting.

4 Scaling Limits

4.1 Scaling Limit

The KMP model is gradient and the hydrodynamic behavior is relatively well understood even if a complete rigorous proof of the scaling limit and the corresponding large deviations estimates is still missing (see [7, Remark 3.3] for a discussion). We review here some general facts and we refer to [7,30,34] for more details. We consider the one dimensional case with $\Lambda=(0,1)$. In the symmetric case the model is diffusive and the natural scaling of the system is obtained considering a lattice of mesh $\frac{1}{N}$ and rescaling time by a factor N^2. This is done simply multiplying by N^2 the rates of jump (for notational convenience we will multiply by a factor $2N^2$). The observable that describe macroscopically the evolution of the mass of the system is the empirical measure. This is a positive measure on Λ associated to any fixed microscopic configuration ξ. It is defined as a convex combination of delta measures as

$$\pi_N(\xi):=\frac{1}{N}\sum_{x\in\Lambda_N}\xi(x)\delta_x. \tag{4.1}$$

Integrating a continuous function $f:\Lambda\to\mathbb{R}$ with respect to $\pi_N(\xi)$ we get

$$\int_\Lambda f\,d\pi_N(\xi)=\frac{1}{N}\sum_{x\in\Lambda_N} f(x)\xi(x). \tag{4.2}$$

In the hydrodynamic scaling limit the empirical measure, that for any finite N is atomic and random, becomes deterministic and absolutely continuous. For suitable initial conditions ξ_0 that are associated to a given density profile $\gamma(x)dx$ in the sense that

Author's personal copy

$$\lim_{N\to+\infty}\int_\Lambda f\,d\pi_N(\xi_0)=\int_\Lambda f(x)\gamma(x)dx \tag{4.3}$$

we have that $\pi_N(\xi_t)$ is associated to the density profile $\rho(x,t)dx$ where ρ is the solution to the heat equation with initial condition γ. The boundary conditions are fixed by the interactions with the external sources [24]. We have then that ρ is the solution of the Cauchy problem

$$\begin{cases}\partial_t\rho=\Delta\rho,\\ \rho(x,0)=\gamma(x),\end{cases} \tag{4.4}$$

with Dirichelet boundary condition $\rho(0,t)=\rho_-$, $\rho(1,t)=\rho_+$. Without loss of generality we will always consider the case $\rho_-\le\rho_+$. Consider a weakly asymmetric dynamics having the bulk part of the generator obtained as a sum of possibly time dependent contributions like (2.6) multiplied by N^2. In particular we consider a model with rates determined by (2.8) or more generally having an expansion like (2.18) where the external field E is however substituted by a space and time dependent vector field $\mathbb{F}$ obtained by a discretization of a smooth vector field on Λ like (2.9). The hydrodynamic behavior of this model is similar to the symmetric one and the external field appears macroscopically with a new term

$$\partial_t\rho(x,t)=\Delta\rho(x,t)-\nabla\cdot\left(\rho^2(x,t)F(x,t)\right), \tag{4.5}$$

The specific form of Eqs. (4.4) and (4.5) for KMP, will be explained in the following section.

4.2 Transport Coefficients

The general form of the hydrodynamic equation associated to weakly asymmetric diffusive stochastic particle systems in any dimension is

$$\partial_t\rho=\nabla\cdot\left(D(\rho)\nabla\rho-\sigma(\rho)F\right). \tag{4.6}$$

The symmetric and positive definite matrix D is called the diffusion matrix while the symmetric and positive definite matrix σ is called the mobility. For all the models that we are discussing the diffusion matrix coincides with the identity matrix while the mobility is a multiple of the identity matrix $\sigma(\rho)\mathbb{I}$ (we are calling σ both the matrix and the scalar value on the diagonal). In the following we will always consider just the one dimensional case. The transport coefficients can be explicitly computed for gradient models for which at equilibrium the invariant measure is known.

It is convenient to write the hydrodynamic equation (4.6) as a conservation law $\partial_t\rho+\nabla\cdot J_F(\rho)=0$ where

$$J_F(\rho):=-\nabla\rho+\sigma(\rho)F \tag{4.7}$$

is the typical current observed in correspondence to the density profile ρ.

We briefly illustrate just the idea of the proof of the hydrodynamic limit for gradient reversible models that allows to identify and compute exactly the transport coefficients [30, 34]. This argument is the bridge between the microscopic and the macroscopic description of a system and allows to identify the hydrodynamic equations associated to the microscopic models.

Let us consider weakly asymmetric models subject to an external field obtained by the discretization (2.9) of a smooth vector field F. We consider before the general d dimensional case but then we restrict again to dimension one. We consider the time window $[0,t]$ and we speed up the process by a N^2 factor. This is obtained multiplying by N^2 the transition rates. The scalar product of the flow of mass in this time window with a smooth test vector field H in any dimension is given by

Author's personal copy

$$\frac{1}{N^d}\sum_{x\sim y}\mathcal{J}_t(x,y)\mathbb{H}(x,y). \tag{4.8}$$

In formula (4.8) the sum is over unordered nearest neighbor sites. By the antisymmetry of the two vector fields there is no ambiguity in this definition. The factor N^{-d} is due to the fact that the scaling limit normalizes the mass by this factor. For simplicity we consider the one dimensional KMP model with periodic boundary conditions and with a macroscopic constant external field F. Microscopically this corresponds to consider $E=\frac{F}{N}$ in (2.18). The value $\frac{F}{N}$ is indeed obtained by formula (2.9) when F is constant (recall that the lattice that has size $1/N$). Using (2.15) and (2.18) (accelerated by N^2) and the gradient condition we can write (4.8) up to an infinitesimal term as

$$N^{-1}\int_0^t\sum_x \tau_x h(\xi_s)\nabla\cdot H(x)\,ds+FN^{-1}\int_0^t\sum_x\tau_x g(\xi_s)H(x)\,ds, \tag{4.9}$$

where $h(\xi)=\frac{\xi(0)}{2}$ and $g(\xi)=\frac{1}{6}\big(\xi(0)^2+\xi(1/N)^2-\xi(0)\xi(1/N)\big)$ is the function that, suitably shifted, multiplies E in the right hand side of (2.18). In the case of KMPd we have instead to use formula (2.19).

At this point the main issue in proving hydrodynamic behavior is the prove the validity of a local equilibrium property. Given a local function k we call $C(\rho):=\mathbb{E}_{\mu_N^{\lambda[\rho]}}(k(\xi))$, where μ_N^{λ} is the product of measures (2.13) associated to the chemical potential λ and we recall that $\lambda[\rho]$ is the chemical potential associated to the density ρ (see the discussion after (2.13)). The local equilibrium property is explicitly stated through a replacement lemma that says that for a smooth test function ψ we have

$$\frac{1}{N}\sum_x\int_0^t\tau_x k(\xi_s)\psi(x)\;ds\simeq\frac{1}{N}\sum_x\int_0^t C\left(\frac{\int_{B_x}d\pi_N(\xi_s)}{|B_x|}\right)\psi(x)\;ds \tag{4.10}$$

where B_x is a microscopically large but macroscopically small volume around the point $x\in\Lambda_N$. This allows to write, up to infinitesimal corrections, Eq. (4.9) in terms only of the empirical measure.

Applying the replacement Lemma we have that with high probability when N is diverging (4.9) converges to $\int_0^t ds\int_0^1 J_F(\rho)\cdot H\,dx$ with

$$J_F(\rho)=-D(\rho)\nabla\rho+\sigma(\rho)F, \tag{4.11}$$

and

$$\sigma(\rho)=\mathbb{E}_{\mu_N^{\lambda[\rho]}}[g(\eta)],\qquad D(\rho)=\frac{d\left(\mathbb{E}_{\mu_N^{\lambda[\rho]}}[h(\eta)]\right)}{d\rho}. \tag{4.12}$$

This is the typical current associated to a density profile ρ in presence of an external field F. Recall that for notational convenience we are multiplying the rates by a factor of 2. Observe that the hydrodynamic behavior is determined by the first two orders in the expansions (2.18), (2.19). In particular any perturbed model having the same first two orders terms will have the same hydrodynamic behavior.

Formula (4.12) for our prototype models gives $D(\rho)$ as the identity and the first two lines of

$$\sigma(\rho)=\begin{cases}\rho^2 & \text{KMP}\\ \rho(\rho+1) & \text{KMPd}\\ \rho^2+1 & \text{KMPx.}\end{cases} \tag{4.13}$$

Author's personal copy

We recall that we are studying the KMPx model only macroscopically and (4.13) together with the unitary diffusion coefficient define macroscopically this model.

A general identity holding for diffusive systems is the Einstein relation between the transport coefficients [13,34]

$$D(\rho) = \sigma(\rho) f''(\rho). \tag{4.14}$$

In (4.14) f is the free energy density that will be introduced and discussed in Sect. 7.1. We have that $f'(\rho) = \lambda[\rho]$ where we recall $\lambda[\cdot]$ is the chemical potential as a function of the density introduced in Sect. 2.1.

5 Stationary Solutions and Currents

The stationary solution $\bar{\rho}_E$ of the hydrodynamic equation (4.6) with a constant external field E, in one dimension with boundary conditions $\rho_\pm$ is obtained as the solution of

$$\begin{cases} \Delta\rho - E\nabla\sigma(\rho) = 0, \\ \rho(0) = \rho_-, \quad \rho(1) = \rho_+. \end{cases} \tag{5.1}$$

Recalling the typical current (4.7), Eq. (5.1) can be written as $\nabla \cdot J_E(\rho) = 0$. In one dimension this implies that the typical current in the stationary state is spatially constant. We are interested in the asymptotic behavior of the stationary solution of the hydrodynamic equation in the limit of a large external field. The asymptotic behavior of the solution can be obtained either by a direct computation or using the general theory [33, Chap. 15]. According to this the limiting value is the stationary solution of the conservation law obtained removing the second order derivative term and with Bardos Leroux Nédélec boundary conditions. This should be also the stationary solution of the hydrodynamic equation for the asymmetric models discussed in Sect. 3 [2,31]. For our aims a weaker result is enough. We use the fact that the unique solution of (5.1) is monotone and the asymptotic behavior of the current for large fields.

Equation (5.1) can be integrated obtaining

$$\nabla\rho - E\sigma(\rho) = -J_E \tag{5.2}$$

where J_E is the integration constant that coincides with $J_E(\bar{\rho}_E)$ the typical current in the stationary state.

The monotonicity of $\bar{\rho}_E$ follows by the fact that if there is a non constant solution $\tilde{\rho}$ of the equation in (5.1) such that $\nabla\tilde{\rho}(y) = 0$ for some $y \in [0, 1]$ then we have two different solutions to the Cauchy problem determined by the conditions $\nabla\rho(y) = 0$ and $\rho(y) = \tilde{\rho}(y)$. One is $\tilde{\rho}$ itself and the other one is the constant one.

Since the solution ρ in (5.2) is monotone the constant J_E is determined integrating (5.2) and using the boundary conditions for ρ

$$\int_{\rho_-}^{\rho_+} \frac{d\rho}{E\sigma(\rho) - J_E} = 1. \tag{5.3}$$

The left hand side of (5.3) is monotone on J_E that can be uniquely fixed for any choice of $\rho_\pm$ and E. Once J_E has been fixed $\bar{\rho}_E$ is uniquely obtained by a direct integration of (5.2). We distinguish the stationary states according to the sign of the stationary current. For any choice of $\rho_\pm$ there exists an external field E^* for which the typical value of the current in the stationary state vanishes. This field is obtained selecting $J_{E^*} = J_{E^*}(\bar{\rho}_{E^*}) = 0$ in (5.3) and using (4.14)

Author's personal copy

$$E^* = \lambda[\rho_+] - \lambda[\rho_-]. \tag{5.4}$$

As the intuition suggests if we have a field $E > E^*$ then $J_E(\bar{\rho}_E) > 0$ while $J_E(\bar{\rho}_E) < 0$ for a field $E < E^*$.

It is convenient to introduce the variable $\alpha = \frac{1}{E}$ and the function $\mathcal{E}(\alpha) := \alpha J_{\frac{1}{\alpha}}$. Condition (5.3) becomes

$$\alpha \int_{\rho_-}^{\rho_+} \frac{d\rho}{\sigma(\rho) - \mathcal{E}(\alpha)} = 1. \tag{5.5}$$

For any $\alpha \neq 0$ the value $\mathcal{E}(\alpha)$ cannot belong to the interval $\{\sigma(\rho), \rho \in [\rho_-, \rho_+]\}$ because otherwise the integral on the left hand side of (5.5) is divergent. Moreover when $\alpha < 0$ then we need to have $\mathcal{E}(\alpha) \geq \sigma(\rho)$ for any ρ while if $\alpha > 0$ we get $\mathcal{E}(\alpha) \leq \sigma(\rho)$ for any ρ. This follows by the fact that otherwise the sign of the integral in (5.5) is not positive. When $|\alpha| \to 0$ the value of $\mathcal{E}(\alpha)$ cannot stay far from the interval $\{\sigma(\rho), \rho \in [\rho_-, \rho_+]\}$ since otherwise the equality (5.5) cannot be satisfied. Since depending on the sign of α we have that $\mathcal{E}(\alpha)$ is always above or below the interval we deduce that

$$\begin{cases} \lim_{\alpha \uparrow 0} \mathcal{E}(\alpha) = \lim_{E \to -\infty} J_E/E = \max_{\rho \in [\rho_-, \rho_+]} \sigma(\rho), \\ \lim_{\alpha \downarrow 0} \mathcal{E}(\alpha) = \lim_{E \to +\infty} J_E/E = \min_{\rho \in [\rho_-, \rho_+]} \sigma(\rho). \end{cases} \tag{5.6}$$

This is the main result of this section. The values of these limits will be used in the computations of the limits of the quasi-potentials.

6 Dynamic Large Deviations and Quasi-potential

The hydrodynamics gives a space time law of large numbers and the corresponding fluctuations can be described by a large deviations principle [7]. Given a space and time dependent density profile $\rho(x,t)dx$ the probability that the empirical measure will be in a suitable neighborhood of it is exponentially unlikely with a corresponding rate functional that we call dynamic large deviations rate functional. This is the main ingredient for the dynamic variational study of stationary non equilibrium states of the macroscopic fluctuation theory [13]. The dynamic rate functional can be described as follows. Consider the class of perturbations obtained adding an external field that is the gradient of a potential assuming the value zero at the boundaries. This means $F(x,t) = \nabla H(x,t)$ with $H(0,t) = H(1,t) = 0$. Given a space time dependent density profile $\rho(x,t)dx$ we compute the potential H solving the equation

$$\begin{cases} \partial_t \rho(x,t) = \Delta\rho(x,t) - \nabla \cdot (\sigma(\rho(x,t))\nabla H(x,t)) \\ H(0,t) = H(1,t) = 0. \end{cases} \tag{6.1}$$

The dynamic rate function for a symmetric model in the time window $[0,T]$ is then obtained by [12,13,30]

$$I_{[0,T]}(\rho) = \frac{1}{4}\int_0^T dt \int_\Lambda dx\, \sigma(\rho)\, (\nabla H)^2\,, \tag{6.2}$$

if the density profile satisfies the boundary conditions $\rho(x,t) = \rho_-$ and $\rho(x,t) = \rho_+$, while instead is identically equal to $+\infty$ if the boundary conditions are violated.

If the original process for which we want to compute large deviations is not the symmetric one but is already a weakly asymmetric one with for example a constant external field then the dynamic rate functional has still the form (6.2) but the potential H has to be computed using the equation

Author's personal copy

$$\begin{cases} \partial_t \rho(x,t) = \Delta\rho(x,t) - \nabla\cdot(\sigma(\rho(x,t))(E + \nabla H(x,t))) \\ H(0,t) = H(1,t) = 0. \end{cases} \tag{6.3}$$

The quasi-potential W_E [13,26] associated to a dynamic rate functional like (6.2), for a model having a constant external field E, is defined through the following variational problem

$$W_E(\rho) := \inf_{T>0} \inf_{\hat\rho\in\mathcal{A}_{\rho,T}} I_{[-T,0]}\left(\hat\rho\right). \tag{6.4}$$

In the above equation $\rho = \rho(x)dx$ is a space dependent density, the lower index E denotes the external field and $\hat\rho$ is a space and time dependent density profile belonging to the set of space time profiles

$$\mathcal{A}_{\rho,T} := \left\{\hat\rho \,:\, \hat\rho(x,-T) = \bar\rho_E,\ \hat\rho(x,0) = \rho(x)\right\}. \tag{6.5}$$

The problem we are interested in, is the computation of the large deviation rate functional for the empirical measure when the mass is distributed according to the invariant measure $\mu_{N,E}$. In general the computation of the invariant measure in the non reversible case is a difficult problem. We give a description of the invariant measure at a large deviations scale. This asymptotic is described by the associate rate functional by

$$\mathbb{P}_{\mu_{N,E}}(\pi_N(\eta) \sim \rho(x)dx) \simeq e^{-NV_E(\rho)}. \tag{6.6}$$

Again we denote by a lower index the dependence on the external vector field.

Under general conditions [15,25,26] the large deviations rate functional for the invariant measure V_E and the quasi-potential W_E coincide $V_E(\rho) = W_E(\rho)$. We can then compute the large deviations asymptotic of the invariant measure solving the dynamic variational problem (6.4) without entering into the details of the invariant measure [13].

7 Quasi-potential for Weakly Asymmetric Models

In [7] it was shown that for all the boundary driven one dimensional symmetric models having constant diffusion and quadratic mobility it is possible to compute the corresponding non local quasi-potential. We show that this is possible for the same class of models also in the case of a weak constant asymmetry. The corresponding quasi-potential is still non local and has a structure similar to the one of weakly asymmetric exclusion [9,11,23].

7.1 The Reversible Case

In the case of reversible models (see Sect. 2.1) the computation of the quasi-potential is direct and we do not need to solve the variational problem (6.4). This is due to the fact that the invariant measure is product and the corresponding large deviations rate functional can be computed directly as the Legendre transform of a scaled cumulant generating function [32,35]. Let us show this in a more general framework. Consider a family of probability measures μ^λ on $\mathbb{R}$ depending on the real parameter λ of the form (2.13). We call

$$P_\lambda(\phi) = \log \int_{\mathbb{R}} \mu^\lambda(dx)e^{\phi x} \tag{7.1}$$

its cumulant generating function. Using the expression (2.13) we have that

$$P_\lambda(\phi) = P(\phi+\lambda) - P(\lambda) \tag{7.2}$$

Author's personal copy

where $P(\cdot) = \log Z(\cdot)$. We call

$$f_\lambda(\alpha) := \sup_\phi \{\alpha\phi - P_\lambda(\phi)\} \tag{7.3}$$

the Legendre transform of P_λ. Using (7.2) we obtain $f_\lambda(\rho) = f(\rho) + P(\lambda) - \lambda\rho$ where $f(\cdot)$ is the Legendre transform of P and, in the case of models with equilibrium product measures, it is called the free energy density. Recall that $\rho[\lambda]$ and $\lambda[\rho]$ are the monotone functions determining respectively the density as a function of the chemical potential and the chemical potential as a function of the density. We have that $\lambda[\rho] = f'(\rho)$, $P'(\lambda) = \rho[\lambda]$. By the Legendre duality we obtain

$$f_\lambda(\rho) = f(\rho) - f\big(\rho[\lambda]\big) - f'\big(\rho[\lambda]\big)\big(\rho - \rho[\lambda]\big). \tag{7.4}$$

The free energy density f satisfies the Einstein relation (4.14) and we obtain

$$f(\rho) = \begin{cases} -\log\rho & \text{KMP}, \\ \rho\log\rho - (1+\rho)\log(1+\rho) & \text{KMPd}, \\ \rho\arctan\rho - \frac{1}{2}\log(1+\rho^2) & \text{KMPx}. \end{cases} \tag{7.5}$$

Consider a slowly varying product measure $\mu_N^{\lambda(\cdot)} = \prod_{x\in\Lambda_N} \mu^{\lambda(x)}(d\xi(x))$ where $\lambda(\cdot)$ is a continuous function on Λ. Let $g : \Lambda \to \mathbb{R}$ be a continuous test function. We can compute

$$\mathcal{P}(g) = \lim_{N\to+\infty} \frac{1}{N^d} \log \int_{\mathbb{R}^{\Lambda_N}} \prod_{x\in\Lambda_N} \mu^{\lambda(x)}(d\xi(x)) e^{N^d \int_\Lambda g d\pi_N(\xi)} = \int_\Lambda P_{\lambda(x)}(g(x))dx. \tag{7.6}$$

A general theorem [32,35] implies that the large deviations rate functional $\mathcal{V}_\lambda$ for $\pi_N(\xi)$ when the configuration is distributed according to the slowly varying product measure with marginals $\mu^{\lambda(x)}$ is obtained as the Legendre transform of (7.6)

$$\mathcal{V}_\lambda(\rho) = \sup_g \left\{ \int_\Lambda \rho(x)g(x)dx - \mathcal{P}(g) \right\} = \int_\Lambda f_{\lambda(x)}(\rho(x))dx. \tag{7.7}$$

In the case of $\lambda(\cdot) = \lambda$ constant we obtain the rate functionals for the empirical measure associated to product of independent random variables having distribution μ^λ. This means that the free energy densities in (7.5) encode the distribution of the microscopic system at equilibrium. The distribution μ^λ can be obtained by $f(\rho)$ with a Legendre transform and a double sided inverse Laplace transform. For the first two expressions in (7.5) we obtain respectively exponentials and geometric distributions. For the third expression in (7.5) we could not find a closed analytic expression after these two operations and this is the main reason why we have not an explicit microscopic model for the KMPx dynamics.

Consider a model having transition rates (2.7) where $\mathbb{F}$ is the discretization of $F = \nabla\psi$ with ψ is a smooth function on the domain Λ such that $\psi|_{\partial\Lambda_N} = \lambda$. By the general result on non homogeneous reversible models in Sect. 2.1 we have that the invariant measure is of product type slowly varying and indeed coincides with $\mu_N^{\psi(\cdot)}$. The quasi-potential can be computed using the approach described above and we have $V_{\nabla\psi}(\rho) = \mathcal{V}_\psi(\rho)$.

7.2 A Hamilton–Jacobi Equation

From the general theory [13] associated to the variational problem (6.4) there is an infinite dimensional Hamilton–Jacobi equation that in this specific case is written as

$$\int_\Lambda \left[\nabla\frac{\delta V(\rho)}{\delta\rho} \cdot \sigma(\rho)\nabla\frac{\delta V(\rho)}{\delta\rho} + \frac{\delta V(\rho)}{\delta\rho}\nabla\cdot\Big(\nabla\rho - E\sigma(\rho)\Big)\right] dx = 0. \tag{7.8}$$

Author's personal copy

In the above formula $\frac{\delta V(\rho)}{\delta \rho}$ denotes the functional derivative. The rate functional V_E that coincides with the quasi-potential W_E is a solution to (7.8). We show that it is possible to find the relevant solutions of this equation for all the weakly asymmetric one dimensional models discussed here. The cases with zero external field unitary diffusion matrix and quadratic mobility were discussed in [7]. The symmetric and weakly asymmetric cases with unitary diffusion matrix and quadratic concave mobility correspond essentially to exclusion models and have been already discussed in [9, 11, 23]. Here we complete the class of solvable models discussing the cases of unitary diffusion, quadratic and convex mobilities and in presence of a constant external field E. In this way we complete the picture obtaining the expression of the quasi-potential for any boundary driven one dimensional system with constant diffusion, quadratic mobility and in presence of any constant external field. A similar computation for a model of oscillators having the same dynamic rate functional as the KMP model has been done in [4]. In [14] the KMP and its totally asymmetric limit are considered.

We show in this section how to find solutions of the Hamilton–Jacobi equation (7.8). Later on we discuss more precisely the relevant variational problems corresponding to the different values of the external field. This second step is not discussed in full detail since a complete analysis requires a long discussion. The variational problems are however very similar to the corresponding ones for the exclusion process and we refer to [9–11] for the details. Following the approach of [5–7, 13] we search for a solution of (7.8) of the form

$$\frac{\delta V(\rho(x))}{\delta \rho} = f'(\rho(x)) - f'(\phi(x)), \tag{7.9}$$

where ϕ has to be determined by the equation (7.8). By the general theory [13] ϕ has to satisfy the boundary conditions $\phi(0) = \rho_-$, $\phi(1) = \rho_+$. We write the generic quadratic mobility as $\sigma(\rho) = c_2\rho^2 + c_1\rho + c_0$ for suitable constants c_i. We consider only the cases with $c_2 \neq 0$. The cases $c_2 = 0$ correspond to special models (zero range, Ginzburg–Landau) that have a local quasi-potential and can be studied directly. We Insert (7.9) into (7.8) and use the quadratic expression of the mobility with some manipulations like in [5–7, 13]. After one integration by parts whose boundary terms disappear since ρ and ϕ satisfy the same boundary conditions we obtain

$$\int_\Lambda \left[\nabla\big(f'(\phi) - f'(\rho)\big)\sigma(\rho)\nabla f'(\phi)\right] dx + E\int_\Lambda \big(f'(\phi) - f'(\rho)\big)\,\nabla\sigma(\rho)\,dx = 0. \tag{7.10}$$

The first term in (7.10) can be developed as follows. First we compute the derivatives and add and subtract suitable terms. Using the Einstein relation (4.14) and the fact that the diffusion coefficient is 1 we get

$$\int_\Lambda \left[\nabla(\phi - \rho)\,\frac{\nabla\phi}{\sigma(\phi)} + (\sigma(\rho) - \sigma(\phi))\left(\frac{\nabla\phi}{\sigma(\phi)}\right)^2\right] dx. \tag{7.11}$$

Then we integrate by parts the first term in (7.11) and use the identity

$$\sigma(\rho) - \sigma(\phi) = (\rho - \phi)(c_2(\rho + \phi) + c_1) \tag{7.12}$$

obtaining

$$\int_\Lambda \left[\frac{\rho - \phi}{\sigma(\phi)}\Delta\phi + c_2\left(\frac{\rho - \phi}{\sigma(\phi)}\right)^2 (\nabla\phi)^2\right] dx. \tag{7.13}$$

Author's personal copy

For the second term in (7.10) we integrate by parts and use again (7.12) obtaining

$$E\int_\Lambda\left[\frac{(\phi-\rho)\nabla\phi}{\sigma(\phi)}\big(c_2(\rho+\phi)+c_1\big)\right]dx. \tag{7.14}$$

Putting together (7.13) and (7.14) we obtain that the Hamilton–Jacobi equation can be written as

$$\int_\Lambda\frac{(\rho-\phi)}{\sigma^2(\phi)}\Big[\Delta\phi\sigma(\phi)+c_2(\nabla\phi)^2(\rho-\phi)-E\sigma(\phi)\nabla\phi\left(c_2(\rho+\phi)+c_1\right)\Big]dx=0. \tag{7.15}$$

A possible way of solving the above equation is to impose that the term inside squared parenthesis is zero. Let us introduce the following functional on ρ,ϕ

$$\mathcal{G}_E(\rho,\phi):=\int_\Lambda\Big[f(\rho)-f(\phi)-f'(\phi)(\rho-\phi)\Big]dx+\mathcal{R}(\phi), \tag{7.16}$$

where

$$\mathcal{R}(\phi)=\int_\Lambda\frac{1}{c_2E\sigma(\phi)}\Big[(\nabla\phi-E\sigma(\phi))\log|\nabla\phi-E\sigma(\phi)|-\nabla\phi\log|\nabla\phi|\Big]. \tag{7.17}$$

The case $E=0$ in [7] can be recovered as a limit. Observe that the first term in (7.16) corresponds to an equilibrium rate functional with typical density profile ϕ (see (7.7)) and we have to add the new term (7.17). We have for the functional (7.16) that $\frac{\delta\mathcal{G}_E}{\delta\rho}=f'(\rho)-f'(\phi)$ while instead with a long but straightforward computation we have

$$\frac{\delta\mathcal{G}_E}{\delta\phi}=\frac{\phi\nabla\phi}{\sigma(\phi)(\nabla\phi-E\sigma(\phi))}-\frac{\rho}{\sigma(\phi)}-\frac{\Delta\phi}{c_2\nabla\phi(\nabla\phi-E\sigma(\phi))}+\frac{E(c_2\phi+c_1)}{c_2(\nabla\phi-E\sigma(\phi))}. \tag{7.18}$$

If the right hand side of (7.18) is zero then also the term inside the squared parenthesis in (7.15) is zero. Let us call ϕ^ρ a critical point satisfying $\frac{\delta\mathcal{G}_E(\rho,\phi^\rho)}{\delta\phi}=0$. We obtain that the functional of ρ defined by $\mathcal{G}_E(\rho,\phi^\rho)$ solves the Hamilton–Jacobi equation (7.8). Since in general the critical points are not unique we discuss more in detail the specific identification of the relevant ϕ^ρ that gives the quasi-potential for our models. We distinguish the 2 cases $E<E^*$ and $E>E^*$. When $E=E^*$ there is no current in the stationary state $J_{E^*}(\bar\rho_{E^*})=0$. This is the condition of macroscopic reversibility [13] that corresponds microscopically to the inhomogeneous reversible product measure discussed at the end of Sect. 2.1. The quasi-potential V_{E^*} is local and can be computed both microscopically like in Sect. 7.1 and macroscopically using (7.8). We obtain that $V_{E^*}(\rho)=\mathcal{V}_{\lambda[\bar\rho_{E^*}]}(\rho)$ and $\lambda\big[\bar\rho_{E^*}(x)\big]$ linearly interpolates $\lambda[\rho_-]$ and $\lambda[\rho_+]$ when $x\in[0,1]$.

7.3 The Case $E<E^*$

For some computations it is more convenient to use the variable $\psi=\lambda[\phi]$. Instead of discussing the general case we consider the three prototype models. Recall that in the case of the KMP model this change of variables corresponds to $\psi=-\frac{1}{\phi}$. We discuss first the KMP case showing then how to modify the computations to cover also the other cases. We consider the functional (7.16) with a fixed determination of the signs of the moduli in (7.17). This is enough to identify the correct solution. We write the functional in terms of the variable ψ. The choice of the sign of the two logarithmic terms has to be + since otherwise the function ψ cannot satisfy the boundary conditions. Consequently we have to restrict the domain of definition of $\mathcal{G}_E$. We add also a suitable constant to fix the normalization. In terms of ψ the functional becomes

Author's personal copy

$$
\begin{aligned}
\mathcal{G}_E(\rho,\psi) &= \int_\Lambda \left[\left(\frac{\nabla\psi}{E}-1\right)\log\left(\nabla\psi - E\right) - \frac{\nabla\psi}{E}\log\left(\nabla\psi\right)\right]dx \\
&\quad + \int_\Lambda \left[-\rho\psi + \log\left(-\frac{\psi}{\rho}\right) - 1\right]dx + K_E
\end{aligned}
\tag{7.19}
$$

where the constant K_E is

$$
K_E = \log\left(-J_E\right) + \frac{1}{E}\int_{\rho_-}^{\rho_+} \frac{d\rho}{\sigma(\rho)} \log\left(1 - \frac{\sigma(\rho)E}{J_E}\right). \tag{7.20}
$$

The functions ψ that we are considering belong to

$$
\mathcal{F}_E := \left\{\psi \in C^1(\Lambda)\ :\ \nabla\psi \geq \max\{E, 0\},\ \psi(0) = -\frac{1}{\rho_-}, \psi(1) = -\frac{1}{\rho_+}\right\}. \tag{7.21}
$$

For a $\psi \in \mathcal{F}_E$ the functional (7.19) is well defined. Formula (7.19) is not well defined in the special case $E = 0$. This case corresponds to the symmetric KMP process and has been already discussed in [7]. It is possible to obtain the corresponding functional for this special case as a limit of (7.19) when $E \to 0$. The constant K_E has been fixed in such a way that $\inf_{\rho,\psi\in\mathcal{F}_E} \mathcal{G}_E(\rho,\psi) = 0$. We discuss later in the general framework this point.

By a direct computation it is possible to check that for fixed ρ the functional $\mathcal{G}_E(\rho,\cdot)$ is neither concave nor convex. Its critical points are determined by the Euler-Lagrange equation

$$
\frac{\Delta\psi}{\nabla\psi(E - \nabla\psi)} + \frac{1}{\psi} = \rho. \tag{7.22}
$$

We define the functional

$$
S_E(\rho) = \inf_{\psi\in\mathcal{F}_E} \mathcal{G}_E(\rho,\psi). \tag{7.23}
$$

We can identify the quasi-potential $W_E = V_E$ with the infimum (7.23), i.e. $V_E = W_E = S_E$. This is based on the interpretation of $\mathcal{G}_E$ as the pre-potential in a Hamiltonian framework obtained interpreting (6.2) as a Lagrangian action [10,11,13]. The pre-potential is defined on the unstable manifold for the Hamiltonian flow relative to a suitable equilibrium point associated to the stationary solution $\bar{\rho}_E$. The value $\mathcal{G}_E(\rho,\psi)$ coincides with the value of the pre-potential when the pair (ρ,ψ) belongs to the unstable manifold. Since the unstable manifold can be characterized by the stationary condition (7.22) we can then consider simply the infimum in (7.23) since all the critical points are belonging to the unstable manifold. In this case ($E < E^*$) there is not uniqueness in the minimizer in (6.4) and equivalently in (7.23). In correspondence the unstable manifold is not a graph and there is the possibility to have Lagrangian phase transitions. We are not going to discuss the details of these arguments and we refer to [10,11] for the analogous computation in the case of the exclusion.

For the KMPd model we use again the change of variables $\psi = \lambda[\phi] = \log\frac{\phi}{\phi+1}$ and the corresponding functional $\mathcal{G}_E$ has a form very similar to (7.19). In particular we give a general form that works for all the three models that we are considering and that depends on the free energy density and the transport coefficients. In particular (7.19) can be obtained as a special case. The general form is constituted by three terms. The first one depends only on $\nabla\psi$ and coincides with the first term in (7.19). The second one can be written in general as

$$
\int_\Lambda \left[f(\rho) - f\left(\rho[\psi]\right) - \psi\big(\rho - \rho[\psi]\big) - \log\sigma\big(\rho[\psi]\big)\right] dx. \tag{7.24}
$$

Author's personal copy

The third term is the constant (7.20). When ρ is fixed the general form of the Euler Lagrange equation for $\mathcal{G}_E(\rho,\cdot)$ can be written in the general form

$$\frac{\Delta\psi}{\nabla\psi(E-\nabla\psi)}+\rho[\psi]-\sigma'\big(\rho[\psi]\big)=\rho. \tag{7.25}$$

These general expressions work also for the KMPx model. The change of variable is again $\psi=\lambda[\phi]=\arctan\phi$.

The functional space for the ψ is like (7.21) with just a difference in the boundary values. In particular for KMPd we have $\psi(0)=\log\frac{\rho_-}{1+\rho_-}$ and $\psi(1)=\log\frac{\rho_+}{1+\rho_+}$ while for KMPx we have $\psi(0)=\arctan\rho_-$ and $\psi(1)=\arctan\rho_+$.

To compute the global infimum of $\mathcal{G}_E$, that is relevant for the determination of the normalizing constant, it is convenient to minimize before in ρ keeping fixed ψ. The stationary condition that corresponds to a minimum is $\lambda[\rho]=\psi$ and in correspondence the term (7.24) reduces to $-\int_\Lambda\log\sigma\big(\rho[\psi]\big)\,dx$. We minimize now over ψ and obtain the stationary condition

$$\frac{\Delta\psi}{\nabla\psi(E-\nabla\psi)}=\sigma'\big(\rho[\psi]\big). \tag{7.26}$$

Using the change of variable $\rho=\rho[\psi]$ Eq. (7.26) becomes the stationary equation in (5.1) that has a unique solution. We obtain then $\inf_{\rho,\psi\in\mathcal{F}_E}\mathcal{G}_E(\rho,\psi)=\mathcal{G}_E\left(\bar\rho_E,\lambda\big[\bar\rho_E\big]\right)$ and imposing that this value is zero we get the general formula (7.20).

7.4 The Case $E>E^*$

Also in this case we use the variable $\psi=\lambda[\phi]$. We consider the functional (7.16) in terms of this new variable. In this case the sign of the modulus in the second logarithmic term in (7.17) is still $+$ while the first one has to be fixed as $-$. This is because the values of the field are different and this is the choice that allows to satisfy the boundary conditions for ψ. Consequently we have to restrict the functions considered. In the case of the KMP model we have

$$\begin{aligned}\mathcal{G}_E(\rho,\psi)=&\int_\Lambda\left[\left(\frac{\nabla\psi}{E}-1\right)\log(E-\nabla\psi)-\frac{\nabla\psi}{E}\log\nabla\psi\right]dx\\&+\int_\Lambda\left[-\rho\psi+\log\left(-\frac{\psi}{\rho}\right)-1\right]dx+K_E,\end{aligned} \tag{7.27}$$

where the constant K_E is

$$K_E=\log(J_E)+\frac{1}{E}\int_{\rho_-}^{\rho_+}\frac{d\rho}{\sigma(\rho)}\log\left(\frac{\sigma(\rho)E}{J_E}-1\right). \tag{7.28}$$

The function ψ belongs to the set

$$\mathcal{F}_E:=\left\{\psi\in C^1(\Lambda)\,:\,0\le\nabla\psi\le E,\ \psi(0)=-\frac{1}{\rho_-},\psi(1)=-\frac{1}{\rho_+}\right\}. \tag{7.29}$$

We have that the function $\left(\frac{\alpha}{E}-1\right)\log(E-\alpha)-\frac{\alpha}{E}\log\alpha$ is concave when $\alpha\in[0,E]$ and also the function $\log(-\alpha)$ defined on the negative real line. The concavity of the functions implies the concavity of the functional $\mathcal{G}_E(\rho,\cdot)$ for any fixed ρ. Like in [9] there exists in $\mathcal{F}_E$ a unique critical point of $\mathcal{G}_E(\rho,\cdot)$ that is then a maximum. This is obtained as the unique solution in $\mathcal{F}_E$ to the Euler-Lagrange equation

$$\frac{\Delta\psi}{\nabla\psi(\nabla\psi-E)}+\frac{1}{\psi}=\rho. \tag{7.30}$$

Author's personal copy

We define the functional

$$S_E(\rho) = \sup_{\psi \in \mathcal{F}_E} \mathcal{G}_E(\rho, \psi) = \mathcal{G}_E(\rho, \psi^\rho) \tag{7.31}$$

where ψ^ρ is the maximizer solving (7.30). Again we have $S_E = V_E = W_E$. In this case the uniqueness of the solution of (7.30) is related to the fact that there is a unique critical point for the variational problem related to the quasi-potential (6.4). The minimizer for the computation of the quasi-potential is related to the time reversal of the hydrodynamic of the adjoint process [9,13]. Also in this case $\mathcal{G}_E$ can be interpreted as the pre-potential in a Hamiltonian framework but in this case for any ρ there is a unique (ρ, ψ) belonging to the unstable manifold. We have then not to minimize over the different points of the unstable manifold. It turns out that we have instead to maximize because the unstable manifold is exactly characterized by the stationary condition (7.30) and the critical point corresponds to a maximum by concavity [9].

Also in this case we can obtain similar expressions of $\mathcal{G}_E$ for the models KMPd and KMPx and write a general expression for it. Like in the previous case we have that $\mathcal{G}_E$ is composed by the sum of three terms. The first one depending only on $\nabla\psi$ coincides with the first term in (7.27). The second term has a general form that coincides with (7.24). The additive constant has the form (7.28). Also for these models the functional $\mathcal{G}_E(\rho, \cdot)$ is concave.

8 Totally Asymmetric Limit

In this section we study the asymptotic limit of the quasi-potential when the external field is large. This is done, like in [9–11], studying first the limit of $\mathcal{G}_E$ and then solving a corresponding variational problem. The limiting domains $\mathcal{F}_\pm := \lim_{E\to\pm\infty} \mathcal{F}_E$ for the functions ψ are be in both cases

$$\mathcal{F}_- = \mathcal{F}_+ = \left\{\psi \in C^1(\Lambda) \,:\, \nabla\psi \geq 0, \psi(0) = \lambda[\rho_-], \psi(1) = \lambda[\rho_+], \right\}. \tag{8.1}$$

This is due to the fact that for large values of the field the constraints on the derivative are irrelevant.

8.1 The Case $E \to -\infty$

We study first the limit when $E \to -\infty$ of the auxiliary functional $\mathcal{G}_E$. Since we are interested in the limiting value we can assume that $E < 0$ and it is convenient to add and subtract the term $\log(-E)$ in (7.19). We add this factor to the first term of (7.19) that becomes $\int_\Lambda s\left(-\frac{\nabla\psi}{E}\right) dx$ with $s(\alpha) := \alpha \log\alpha - (1+\alpha)\log(1+\alpha)$. Since $\lim_{\alpha\downarrow 0} s(\alpha) = 0$ this term is converging to zero in the limit of large and negative field. We subtract $\log(-E)$ to K_E obtaining

$$\log\frac{J_E}{E} + \frac{1}{E}\int_{\rho_-}^{\rho_+} \frac{d\rho}{\sigma(\rho)} \log\left(1 - \frac{\sigma(\rho)E}{J_E}\right). \tag{8.2}$$

The asymptotic behavior of (8.2) can be easily understood since several terms depend just on the ratio $\frac{J_E}{E}$ whose behavior in the limit for large fields is given by (5.6). In particular in this case the second term in (8.2) converges to zero while the first one converges to $2\log\rho_+$.

Author's personal copy

The second term in (7.19) does not depend on E and remains identical in the limit. We obtained then that $\mathcal{G}_- := \lim_{E\to-\infty}\mathcal{G}_E$ is given by

$$\mathcal{G}_-(\rho,\psi) = \int_0^1 \left[-\rho\psi + \log\left(-\frac{\psi}{\rho}\right) - 1\right] dx + 2\log\rho_+. \tag{8.3}$$

The domain for the ψ is (8.1). We can obtain (see [9,11]) $V_-(\rho) := \lim_{E\to-\infty} V_E(\rho)$ as

$$V_-(\rho) = S_-(\rho) := \inf_{\psi\in\mathcal{F}_-}\mathcal{G}_-(\rho,\psi). \tag{8.4}$$

Since the function $\psi \to -\rho\psi + \log(-\frac{\psi}{\rho})$ is decreasing when $\psi \in [\frac{-1}{\rho_-}, \frac{-1}{\rho_+}]$, the minimum of the integrand in (8.3) over the ψ for a fixed ρ is obtained for $\psi = -\frac{1}{\rho_+}$. This means that we can construct a minimizing sequence in $\mathcal{F}_-$ approximating a function that takes the value $-\frac{1}{\rho_-}$ at 0 and then immediately jumps to the value $-\frac{1}{\rho_+}$. We then obtain

$$V_-(\rho) = \inf_{\psi\in\mathcal{F}_-}\mathcal{G}(\rho,\psi) = \int_\Lambda \left[\frac{\rho}{\rho_+} - \log\frac{\rho}{\rho_+} - 1\right] dx \tag{8.5}$$

that is the large deviations rate function for masses distributed according to a product of exponentials distributions of parameter $\frac{1}{\rho_+}$. This means that the right hand side of (8.5) can be obtained by (7.7) using a product of exponentials with constant parameter. This is the large deviations rate functional for the invariant measure of the version 1 of the totally asymmetric KMP dynamics discussed in Sect. 3.1 if we invert the direction of the asymmetry there.

For the KMPd model the asymptotic behavior is very similar and with the same computations we obtain that $V_- = \lim_{E\to-\infty} V_E$ is given by

$$V_-(\rho) = S_-(\rho) := \inf_{\psi\in\mathcal{F}_-}\mathcal{G}_-(\rho,\psi) = \int_\Lambda \left[\rho\log\frac{\rho}{\rho_+} - (1+\rho)\log\frac{1+\rho}{1+\rho_+}\right] dx \tag{8.6}$$

that is the large deviation rate functional for a product measure with marginals given by geometric distributions of parameter $\frac{1}{1+\rho_+}$.

The limiting behavior of the KMPx model is instead quite different and exhibits a behavior similar to the exclusion process. Recalling (5.6), for the KMPx model the limiting value of $\frac{J_E}{E}$ when $E \to -\infty$ is given by $\max\{1+\rho_-^2, 1+\rho_+^2\}$. Let us call

$$\bar\rho := \begin{cases} \rho_- & \text{if } |\rho_-| \ge |\rho_+|, \\ \rho_+ & \text{if } |\rho_-| < |\rho_+|. \end{cases} \tag{8.7}$$

The limiting functional $\mathcal{G}_-$ in this case becomes

$$\mathcal{G}_-(\rho,\psi) = \int_0^1 \left[\rho\arctan\rho - \frac{\log(1+\rho^2)}{2} - \rho\psi - \frac{\log(1+(\tan\psi)^2)}{2}\right] dx + \log(1+\bar\rho^2), \tag{8.8}$$

and the function ψ has to belong to (8.1). Since the function $\psi \to -\log(1+(\tan\psi)^2)$ is concave the functional $\mathcal{G}_-(\rho,\cdot)$ is also concave. To compute $V_-(\rho) = S_-(\rho)$ we need to minimize $\mathcal{G}_-(\rho,\cdot)$ over the convex set (8.1). The infimum is then realized on the extremal points of a suitable closure of that convex set. The extremal elements are the function of the form

$$\psi^y(x) = \arctan(\rho_-)\chi_{[0,y)}(x) + \arctan(\rho_+)\chi_{[y,1]}(x), \qquad y \in (0,1]. \tag{8.9}$$

Author's personal copy

This means piecewise constant functions jumping from $\arctan(\rho_-)$ to $\arctan(\rho_+)$ at the single point y. Let us define the functional

$$\begin{aligned}\tilde{\mathcal{G}}_-(\rho, y) := \mathcal{G}_-(\rho, \psi^y) = &\int_0^1 \left[\rho \arctan \rho - \frac{1}{2}\log(1+\rho^2)\right] dx \\ &- \arctan(\rho_-)\int_0^y \rho(x)\,dx - \arctan(\rho_+)\int_y^1 \rho(x)\,dx \\ &- \frac{y}{2}\log(1+\rho_-^2) - \frac{1-y}{2}\log(1+\rho_+^2) + \log(1+\bar{\rho}^2).\end{aligned} \tag{8.10}$$

This is the functional $\mathcal{G}_-$ computed in correspondence of the extremal elements. The infimum of $\mathcal{G}_-(\rho, \cdot)$ coincides with the infimum of the function of one real variable $\tilde{\mathcal{G}}_-(\rho, \cdot)$ on the interval $[0, 1]$. Consequently we have

$$V_-(\rho) = S_-(\rho) = \inf_{y\in(0,1]} \tilde{\mathcal{G}}_-(\rho, y).$$

This is a minimum problem for a function of one real variable whose critical points are obtained equating to zero the derivative

$$\rho(y)\left[\arctan \rho_+ - \arctan \rho_-\right] = \frac{1}{2}\log\frac{1+\rho_-^2}{1+\rho_+^2}. \tag{8.11}$$

The second derivative establishing if a critical point is a local minimum or a local maximum is given by $\nabla\rho(y)\left[\arctan \rho_+ - \arctan \rho_-\right]$. To find the global minimum we have to consider also the values of the functions on the two extrema of the interval. Like for the exclusion process it is possible to have that the minimum is obtained in more than one single point y and this phenomenon is related to the existence of dynamic phase transitions [10] for finite and large enough negative fields.

8.2 The Case $E \to +\infty$

We study first the limit when $E \to +\infty$ of the auxiliary functionals $\mathcal{G}_E$ [9,11]. We start with the KMP model. As before it is convenient to add and subtract the term $\log E$. We add this factor to the first term of (7.27) that becomes $\int_\Lambda \tilde{s}\left(\frac{\psi_x}{E}\right) dx$ with $\tilde{s}(\alpha) = -\alpha\log\alpha - (1-\alpha)\log(1-\alpha)$. Since $\lim_{\alpha\downarrow 0}\tilde{s}(\alpha) = 0$ this term is converging to zero in the limit of large and positive field. The factor that we subtract is inserted in the additive constant K_E and we obtain

$$\log\frac{J_E}{E} + \frac{1}{E}\int_{\rho_-}^{\rho_+}\frac{d\rho}{\sigma(\rho)}\log\left(\frac{\sigma(\rho)E}{J_E} - 1\right). \tag{8.12}$$

The asymptotic behavior of (8.12) can be easily understood since several terms depend just on the ratio $\frac{J_E}{E}$ whose behavior in the limit for large fields is given by (5.6). In particular in the KMP case the second term in (8.12) converges to zero while the first one converges to $2\log\rho_-$.

The second term in (7.27) does not depend on E and remains identical in the limit. We deduce that $\lim_{E\to+\infty}\mathcal{G}_E = \mathcal{G}_+$ defined as

$$\mathcal{G}_+(\rho, \psi) = \int_0^1\left[-\rho\psi + \log\left(-\frac{\psi}{\rho}\right) - 1\right] dx + 2\log\rho_-, \tag{8.13}$$

with $\psi \in \mathcal{F}_+$.

Author's personal copy

Since the function $\psi \to -\rho\psi + \log\left(-\frac{\psi}{\rho}\right)$ is decreasing when $\psi \in \left[-\frac{1}{\rho_-}, -\frac{1}{\rho_+}\right]$ we can compute the supremum of (8.13) over the ψ for a fixed ρ obtaining a minimizing sequence converging to a function that assumes the value $-\frac{1}{\rho_-}$ on all the interval except the point 1 where it assumes the value $-\frac{1}{\rho_+}$. We then obtain

$$V_+(\rho) = S_+(\rho) := \sup_{\psi\in\mathcal{F}_+} \mathcal{G}_+(\rho, \psi) = \int_0^1 \left[\frac{\rho}{\rho_-} - \log\frac{\rho}{\rho_-} - 1\right] dx \tag{8.14}$$

that is the large deviation rate function for masses distributed according to a product of exponentials of parameter $\frac{1}{\rho_-}$. This is exactly the large deviations rate functional for the invariant measure of the version 1 of the totally asymmetric KMP dynamics discussed in Sect. 3.1.

Again The limiting behavior for the KMPd model is very similar and we obtain

$$V_+(\rho) = S_+(\rho) := \sup_{\psi\in\mathcal{F}_+} \mathcal{G}_+(\rho, \psi) = \int_0^1 \left[\rho\log\frac{\rho}{\rho_-} - (1+\rho)\log\frac{1+\rho}{1+\rho_-}\right] \tag{8.15}$$

that is the large deviation rate functional for a product measure with marginals given by geometric distributions of parameter $\frac{1}{1+\rho_-}$.

For the KMPx model the limiting value of $\frac{J_E}{E}$ when $E \to +\infty$ has three different possible values. Let us define

$$\bar\rho := \begin{cases} \rho_- & \text{if } 0 \le \rho_- \le \rho_+, \\ 0 & \text{if } \rho_- \le 0 \le \rho_+, \\ \rho_+ & \text{if } \rho_- \le \rho_+ \le 0. \end{cases} \tag{8.16}$$

With this definition we have $\frac{J_E}{E} \to \sigma(\bar\rho)$. The limiting functional $\mathcal{G}_+$ in this case becomes

$$\mathcal{G}_+(\rho, \psi) = \int_0^1 \left[\rho\arctan\rho - \frac{\log(1+\rho^2)}{2} - \rho\psi - \frac{\log(1+(\tan\psi)^2)}{2}\right] dx + \log(1+(\bar\rho)^2). \tag{8.17}$$

and the function ψ belongs to (8.1) We have $V_+(\rho) = S_+(\rho) = \sup_{\psi\in\mathcal{F}_+} \mathcal{G}_+(\rho, \psi)$ and the maximizer ψ_M^ρ such that $V_+(\rho) = \mathcal{G}_+\left(\rho, \psi_M^\rho\right)$ can be described as follows. Let $H(x) = -\int_0^x \rho(y)\,dy$ and $G = co(H)$ its convex envelope. We define

$$\phi_M^\rho(x) = \begin{cases} \rho_- & \text{if } \nabla G(x) \le \rho_- \\ \rho_+ & \text{if } \nabla G(x) \ge \rho_+ \\ \nabla G(x) & \text{otherwise.} \end{cases} \tag{8.18}$$

We have that the maximizer is $\psi_M^\rho = \arctan\phi_M^\rho$. To prove this statement it is convenient to re-write (8.17) again in terms of the variable ϕ related to ψ by $\phi = \tan\psi$. We need to prove that ϕ_M^ρ is the minimizer of

$$\inf_{\phi\in\tilde{\mathcal{F}}_+} \int_0^1 \left[\rho\arctan\phi + \frac{1}{2}\log\left(1+\phi^2\right)\right] dx = \inf_{\phi\in\tilde{\mathcal{F}}_+} \mathcal{B}(\rho, \phi), \tag{8.19}$$

where the last equality defines the functional $\mathcal{B}$ and

$$\tilde{\mathcal{F}}_+ := \left\{\phi \in C^1(\Lambda) : \nabla\phi \ge 0, \phi(0) = \rho_-, \phi(1) = \rho_+\right\}. \tag{8.20}$$

Author's personal copy

To identify the minimizer ϕ_M^ρ in (8.19) we can adapt the argument in [22]. Since $H(0) = G(0)$ and $H(1) = G(1)$ with an integration by parts we obtain

$$\int_0^1 (\rho + \nabla G) \arctan(\phi)\, dx = \int_0^1 (H - G)\, \nabla \arctan(\phi)\, dx \geq 0. \tag{8.21}$$

The last inequality follows by the fact that $\nabla\phi \geq 0$, the function arctan is increasing and by definition $G \leq H$. We have that the second term of (8.21) is the integral of the product of two non negative terms and consequently it is non negative. Inequality (8.21) can be written as $\mathcal{B}(\rho, \phi) \geq \mathcal{B}(-\nabla G, \phi)$.

The derivative with respect to ϕ of the function $\alpha \arctan\phi + \frac{1}{2}\log\left(1 + \phi^2\right)$ is negative for $\phi < -\alpha$ and positive for $\phi > -\alpha$. Since the function $\phi(x)$ takes values only on the interval $[\rho_-, \rho_+]$ we have that

$$\inf_{\phi\in[\rho_-,\rho_+]} \left\{\alpha \arctan\phi + \frac{1}{2}\log\left(1 + \phi^2\right)\right\}$$

is obtained at ρ_- when $\alpha \geq -\rho_-$, it is obtained at ρ_+ when $\alpha \leq -\rho_+$ and it is obtained at $-\alpha$ when $\alpha \in [-\rho_+, -\rho_-]$. This fact implies that we have the inequality $\mathcal{B}(-\nabla G, \phi) \geq \mathcal{B}(-\nabla G, \phi_M^\rho)$. The last fact that remains to prove is that the equality $\mathcal{B}(-\nabla G, \phi_M^\rho) = \mathcal{B}(\rho, \phi_M^\rho)$ holds. This follows by the following argument. We split the interval $[0, 1]$ on intervals where either $-\nabla G = \rho$ or ∇G is constant. On an interval where $-\nabla G = \rho$ we have obviously the same contribution. On an interval $[a, b]$ where ∇G is constant correspondingly also ϕ_M^ρ is constant. Moreover the constant value of ∇G coincides with $\frac{\int_a^b \rho(y)\, dy}{a-b}$. Since ϕ_M^ρ is constant on the interval the contribution coming from intervals of this type still coincide and we get the equality. Summarizing we have the following chain

$$\mathcal{B}(\rho, \phi) \geq \mathcal{B}(-\nabla G, \phi) \geq \mathcal{B}\left(-\nabla G, \phi_M^\rho\right) = \mathcal{B}\left(\rho, \phi_M^\rho\right) \tag{8.22}$$

that implies that ϕ_M^ρ is the minimizer in (8.19). Since $\psi = \arctan\phi$ we obtain that the maximizer to compute $V_+(\rho) = S_+(\rho)$ is given by $\psi_M^\rho = \arctan\phi_M^\rho$.

Acknowledgements D.G. acknowledges L. Bertini, A. De Sole, G. Jona Lasinio and C. Landim for a long collaboration on the general subject. We especially thank C. Bernardin and L. Bertini for very useful discussions on this specific problem and for informing us of their related computations [4,14]. We thank the referees for the very careful reading of the paper.

References

1. Bahadoran, C.: A quasi-potential for conservation laws with boundary conditions. arXiv:1010.3624 (2010)
2. Bahadoran, C.: Hydrodynamics and hydrostatics for a class of asymmetric particle systems with open boundaries. Commun. Math. Phys. **310**(1), 124 (2012)
3. Bernardin, C.: Stationary nonequilibrium properties for a heat conduction model. Phys. Rev. E (3) **3**(78), 021134 (2008)
4. Bernardin, C.: Private communication
5. Bertini, L., De Sole, A., Gabrielli, D., Jona-Lasinio, G., Landim, C.: Fluctuations in stationary nonequilibrium states of irreversible processes. Phys. Rev. Lett. **87**(4), 040601 (2001)
6. Bertini, L., De Sole, A., Gabrielli, D., Jona-Lasinio, G., Landim, C.: Macroscopic fluctuation theory for stationary non-equilibrium states. J. Stat. Phys. **107**(3–4), 635–675 (2002)
7. Bertini, L., Gabrielli, D., Lebowitz, J.L.: Large deviations for a stochastic model of heat conduction. J. Stat. Phys. **121**(5/6), 843–885 (2005)
8. Bertini, L., De Sole, A., Gabrielli, D., Jona-Lasinio, G., Landim, C.: Stochastic interacting particle systems out of equilibrium. J. Stat. Mech. **7**, P07014 (2007)

Author's personal copy

9. Bertini, L., Gabrielli, D., Landim, C.: Strong asymmetric limit of the quasi-potential of the boundary driven weakly asymmetric exclusion process. Commun. Math. Phys. **289**(1), 311–334 (2009)
10. Bertini, L., De Sole, A., Gabrielli, D., Jona-Lasinio, G., Landim, C.: Lagrangian phase transitions in nonequilibrium thermodynamic systems. J. Stat. Mech. L11001 (2010)
11. Bertini, L., De Sole, A., Gabrielli, D., Jona-Lasinio, G., Landim, C.: Action functional and quasi-potential for the Burgers equation in a bounded interval. Commun. Pure Appl. Math. **64**(5), 649–696 (2011)
12. Bertini, L., Faggionato, A., Gabrielli, D.: Large deviation principles for nongradient weakly asymmetric stochastic lattice gases. Ann. Appl. Probab. **23**(1), 165 (2013)
13. Bertini, L., De Sole, A., Gabrielli, D., Jona-Lasinio, G., Landim, C.: Macroscopic fluctuation theory. Rev. Mod. Phys. **87**, 593 (2015)
14. Bertini, L.: Private communication
15. Bodineau, T., Giacomin, G.: From dynamic to static large deviations in boundary driven exclusion particles systems Stoch. Proc. Appl. **110**, 6781 (2004)
16. Bodineau, T., Derrida, B.: Current large deviations for asymmetric exclusion processes with open boundaries. J. Stat. Phys. **123**(2), 277–300 (2006)
17. Carinci, G., Giardiná, C., Redig, F., Sasamoto, T.: Asymmetric stochastic transport models with $\mathcal{U}_q(\mathfrak{su}(1,1))$ symmetry. J. Stat. Phys. **163**(2), 239–279 (2016)
18. Derrida, B.: Non-equilibrium steady states: fluctuations and large deviations of the density and of the current. J. Stat. Mech. **7**, P07023 (2007)
19. Derrida, B., Lebowitz, J.L., Speer, E.R.: Free energy functional for nonequilibrium systems: an exactly soluble case. Phys. Rev. Lett. **87**, 150601 (2001)
20. Derrida, B., Lebowitz, J.L., Speer, E.R.: Large deviation of the density drofile in the steady state of the symmetric simple exclusion process. J. Stat. Phys. **107**, 599–634 (2002a)
21. Derrida, B., Lebowitz, J.L., Speer, E.R.: Exact free energy functional for a driven diffusive open stationary nonequilibrium system. Phys. Rev. Lett. **89**, 030601 (2002b)
22. Derrida, B., Lebowitz, J.L., Speer, E.R.: Exact large deviation functional of a stationary open driven diffusive system: the asymmetric exclusion process. J. Stat. Phys. **110**, 775810 (2003)
23. Enaud, C., Derrida, B.: Large deviation functional of the weakly asymmetric exclusion process. J. Stat. Phys. **114**, 537–562 (2004)
24. Eyink, G.L., Lebowitz, J.L., Spohn, H.: Hydrodynamics and fluctuations outside of local equilibrium: driven diffusive systems. J. Stat. Phys. **83**(3–4), 385–472 (1996)
25. Farfan, J.: Stationary large deviations of boundary driven exclusion processes. arXiv:0908.1798 (2009)
26. Freidlin, M.I., Wentzell, A.D.: Random Perturbations of Dynamical Systems. Grundlehren der Mathematischen Wissenschaften, vol. 260, 3rd edn. Springer, Heidelberg (2012)
27. Gabrielli, D.: From combinatorics to large deviations for the invariant measures of some multiclass particle systems. Markov Process. Relat. Fields **14**(3), 365–402 (2008)
28. Van Ginkel, B., Redig, F., Sau, F.: Duality and stationary distributions of the immediate exchange model and its generalizations. J. Stat. Phys. **163**(1), 92–112 (2016)
29. Kipnis, C., Marchioro, C., Presutti, E.: Heat flow in an exactly solvable model. J. Stat. Phys. **27**, 6574 (1982)
30. Kipnis, C., Landim, C.: Scaling Limits of Interacting Particle Systems. Springer, New York (1999)
31. Popkov, V., Schütz, G.M.: Steady-state selection in driven diffusive systems with open boundaries. Europhys. Lett. **48**(3), 257 (1999)
32. Rassoul-Agha, F., Seppäläinen, T.: A Course on Large Deviations with an Introduction to Gibbs Measures. Graduate Studies in Mathematics, vol. 162. American Mathematical Society, Providence (2015)
33. Serre, D.: Systems of Conservation Laws. Geometric Structures, Oscillations, and Initialboundary Value Problems, vol. 2. Cambridge University Press, Cambridge (2000)
34. Spohn, H.: Large Scale Dynamics of Interacting Particles. Springer, New York (1991)
35. Touchette, H.: The large deviation approach to statistical mechanics. Phys. Rep. **478**(1–3), 169 (2009)
36. Varadhan, S.R.S.: Large Deviations for the Asymmetric Simple Exclusion Process. Stochastic Analysis on Large Scale Interacting Systems. Advanced Studies in Pure Mathematics, vol. 39. Mathematical Society of Japan, Tokyo (2004)