Topologically sensitive dynamical systems

Alessandro Fedeli [1]

Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università dell'Aquila
67100 L'Aquila, Italy

Abstract. In this paper we introduce a notion of sensitivity for topological dynamical systems and show some of its basic features and relation to dynamical properties such as transitivity and weak mixing. Finally, we will restrict our attention to the related class of weakly positively expansive dynamical systems.

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1. Introduction and preliminaries

Sensitive dependence on initial conditions is a metric-dependant property which gives some informations on the unpredictability of a dynamical system and it is one of the most relevant concept in chaotic dynamics (see, e.g.,[3], [19]). In this paper we start the investigation of a topological version of sensitive dependence on initial conditions and we will relate this property with the notion of weak positive expansiveness introduced by Richeson and Wiseman [26].

Let $\mathbb{N}$ and $\mathbb{N}_0$ be the sets of positive integers and nonnegative integers, respectively. A continuous self-map f on a metric space (X, d) is said to have *sensitive dependence on initial conditions* (f is *sensitive* for short) if there is some $\varepsilon > 0$ (called *sensitivity constant*) such that, for any $x \in X$ and any

[1] *E-mail address* : afedeli@univaq.it

open neighbourhood V of x in (X, d), there exist $y \in V$ and some $\kappa \in \mathbb{N}_0$ for which $d(f^\kappa(x), f^\kappa(y)) \geq \varepsilon$ (see, e.g., [13]).

By a *dynamical system* we mean a pair (X, f) where f is a continuous self-map on a (nonempty) topological space X (called phase space). We say that the dynamical system (X, f) (or simply the map f) is :

(i) (*topologically*) *transitive* if for every pair U and V of nonempty open subsets of X there is some $\kappa \in \mathbb{N}$ such that $f^\kappa(U) \cap V \neq \emptyset$;

(ii) *periodically dense* if the set of periodic points of f is dense in X.

Let (X, f) be a dynamical system, where X is a metric space. Two of the most popular definitions of chaos in which sensitivity plays an important role are the following: (X, f) is said to be *chaotic in the sense of*

(i) *Devaney* whenever f is transitive, periodically dense and sensitive [10].

(ii) *Auslander-Yorke* whenever f is transitive and sensitive [6].

Let us recall that, for instance, every transitive self-map on an interval is Devaney chaotic [29], while every hypercyclic operator on a separable Fréchet space is Auslander-Yorke chaotic ([14], see also [17, Prop. 2.30]). A discussion of the various notions of chaos can be found in [5],[8] and [27].

We refer the reader to [1],[17],[21] and [30] for more informations on topological dynamics and to [31] for undefined topological notions.

2. Topological sensitivity

Our topological version of sensitivity is based upon the following special covers.

Definition 2.1. Let (X, f) be a dynamical system and let $\mathcal{U}$ be an open cover of X. $\mathcal{U}$ is called *sensitivity cover* (*s-cover* for short) for (X, f) if for every nonempty open subset G of X there exist $x, y \in G$ and $n \in \mathbb{N}_0$ such that $(x, y) \notin \bigcup\{f^{-n}(U) \times f^{-n}(U) : U \in \mathcal{U}\}$.

Definition 2.2. A dynamical system (X, f) (or simply the map f) is called *topologically sensitive* if (X, f) has an s-cover.

Note that we put no restriction on the cardinality of the s-cover and observe also that the phase space of a topologically sensitive dynamical system cannot have isolated points. If (X, d) is a metric space, τ_d will denote the topology on X generated by the metric d and $B(p, \varepsilon)$ the open ball with center p and radius ε.

The basic links between sensitive dependence on initial conditions and topological sensitivity are summarized in the following

Theorem 2.3. *Let f be a continuous self-map on a metric space (X, d) and let us consider the following conditions:*

(i) $f : (X, d) \to (X, d)$ is sensitive;

(ii) There exists some $\varepsilon > 0$ such that $\{B(p, \varepsilon) : p \in X\}$ is an s-cover for (X, f) (with respect to τ_d);

(iii) $f : (X, \tau_d) \to (X, \tau_d)$ is topologically sensitive.

Then (i) $\iff$ (ii) $\implies$ (iii). If, in addition, (X, τ_d) is compact, then (iii) is equivalent to (i) (and (ii)).

Proof. *(i)* $\iff$ *(ii)*. Let δ be a sensitivity constant for (X, f) and set $\varepsilon = \frac{\delta}{2}$. Then $\mathcal{U} = \{B(p, \varepsilon) : p \in X\}$ is an s-cover for (X, f). In fact let G be a nonempty open subset of X and let $x \in G$, then there is some $y \in G$ and $n \in \mathbb{N}_0$ such that $d(f^n(x), f^n(y)) \geq \delta$. So $\{f^n(x), f^n(y)\} \not\subset B(p, \varepsilon)$ for every $p \in X$ (otherwise $d(f^n(x), f^n(y)) < 2\varepsilon = \delta$) and this is equivalent to say that $(x, y) \notin \bigcup\{f^{-n}(B(p, \varepsilon)) \times f^{-n}(B(p, \varepsilon)) : p \in X\}$. Therefore $\mathcal{U}$ is an s-cover for (X, f).

Now let us suppose that $\{B(p, \varepsilon) : p \in X\}$ is an s-cover for (X, f), for some $\varepsilon > 0$. Now let $x \in X$ and let U be an open neighbourhood of x. Then there are $y, z \in U$ and a nonnegative integer n such that $(y, z) \notin \bigcup\{f^{-n}(B(p, \varepsilon)) \times f^{-n}(B(p, \varepsilon)) : p \in X\}$, so $d(f^n(y), f^n(z)) \geq \varepsilon$ (otherwise $\{f^n(y), f^n(z)\} \subset B(f^n(y), \varepsilon)$). Hence $d(f^n(x), f^n(y)) \geq \frac{\varepsilon}{2}$ or $d(f^n(x), f^n(z)) \geq \frac{\varepsilon}{2}$ and $\frac{\varepsilon}{2}$ is a sensitivity constant for f.

(ii) $\implies$ *(iii)* is clear.

(iii) $\implies$ *(i)*. Let $\mathcal{U}$ be an s-cover for (X, f) and let ε be a Lebesgue number of $\mathcal{U}$ (i.e., ε is a positive number such that every subset of X with diameter less than or equal to ε is included in a member of $\mathcal{U}$). Then $\frac{\varepsilon}{4}$ is a sensitivity constant for f. In fact let $x \in X$ and let V be an open neighbourhood of x, then there are $y, z \in V$ and $n \in \mathbb{N}_0$ such that $(y, z) \notin \bigcup\{f^{-n}(U) \times f^{-n}(U) : U \in \mathcal{U}\}$. So $d(f^n(y), f^n(z)) \geq \frac{\varepsilon}{2}$ (otherwise $\{f^n(y), f^n(z)\} \subset B(f^n(z), \frac{\varepsilon}{2}) \subset U$ for some $U \in \mathcal{U}$). Hence $d(f^n(x), f^n(y)) \geq \frac{\varepsilon}{4}$ or $d(f^n(x), f^n(z)) \geq \frac{\varepsilon}{4}$.

Remarks. (i) The compactness condition in Theorem 2.3 cannot be omitted. In fact let f be the self-map on $\mathbb{R}$ given by $f(x) = x + 1$ and let us consider the following metric on $\mathbb{R}$: $\rho(x, y) = |e^x - e^y|$. This metric is equivalent to the usual metric d, i.e., $\tau_d = \tau_\rho$. Since the map $f : (\mathbb{R}, \rho) \to (\mathbb{R}, \rho)$ is sensitive, it follows that $f : (\mathbb{R}, \tau_d) \to (\mathbb{R}, \tau_d)$ is topologically sensitive, nonetheless $f : (\mathbb{R}, d) \to (\mathbb{R}, d)$ is not sensitive.

(ii) It is worth noting that our definition of topological sensitivity is a natural extension of the notion of sensitivity in the uniform setting. Let (X, f) be a dynamical system and let V be a neighbourhood of the diagonal $\Delta = \{(x, x) : x \in X\}$ in $X \times X$. We say that f is V-sensitive if for every nonempty open subset G of X there exist $x, y \in G$ and $n \in \mathbb{N}_0$ such that $(f^n(x), f^n(y)) \notin V$, i.e., $G \times G \not\subset \bigcap\{(f \times f)^{-m}(V) : m \in \mathbb{N}_0\}$.

A (uniformly continuous) self-map f on a uniform space $(X, \mathcal{V})$ is called sensitive if $f : (X, \tau_\mathcal{V}) \to (X, \tau_\mathcal{V})$ is V-sensitive for some entourage $V \in \mathcal{V}$ (where $\tau_\mathcal{V}$ is the topology on X induced by the uniformity $\mathcal{V}$).

Now let $\mathcal{U}$ be an open cover of a space X and let $V = \bigcup\{U \times U : U \in \mathcal{U}\}$. Observe that $\mathcal{U}$ is an s-cover for a dynamical system (X, f) if and only if $f : X \to X$ is V-sensitive.

Sensitivity is a dynamical property in the realm of compact metric spaces. Moreover a continuous self-map f on a compact metric space is sensitive if and only if f^n is sensitive for each $n \in \mathbb{N}$.

For topologically sensitive maps we have the following general result.

Theorem 2.4. *Let $f : X \to X$ and $g : Y \to Y$ be continuous maps and let $n \in \mathbb{N}$.*

(i) If f and g are conjugate and g is topologically sensitive, then f is topologically sensitive.

(ii) f is topologically sensitive if and only if f^n is topologically sensitive.

Proof. (i) Let $\varphi : X \to Y$ be a homeomorphism such that $\varphi \circ f = g \circ \varphi$ and let $\mathcal{V}$ be an s-cover for (Y, g). Then $\mathcal{U} = \{\varphi^{-1}(V) : V \in \mathcal{V}\}$ is an s-cover for (X, f). In fact, let G be a nonempty open subset of X, then $\varphi(G)$ is a nonempty open subset of Y, so there are $y_1, y_2 \in \varphi(G)$ and $n \in \mathbb{N}_0$ such that $(y_1, y_2) \notin \bigcup\{g^{-n}(V) \times g^{-n}(V) : V \in \mathcal{V}\}$. Let $x_1, x_2 \in G$ be such that $y_1 = \varphi(x_1)$ and $y_2 = \varphi(x_2)$, then $(x_1, x_2) \notin \bigcup\{f^{-n}(U) \times f^{-n}(U) : U \in \mathcal{U}\}$ as required.

(ii) If $\mathcal{U}$ is an s-cover for (X, f^n), then $\mathcal{U}$ is also an s-cover for (X, f).

Now let us assume that f is topologically sensitive and let us take an s-cover $\mathcal{U}$ for (X, f). For every $\kappa \in \{0, ..., n-1\}$ let $\mathcal{U}_\kappa = \{f^{-\kappa}(U) : U \in \mathcal{U}\}$. Call $\mathcal{V}$ the join of the covers $\mathcal{U}_0$,..., $\mathcal{U}_\kappa$, i.e., let $\mathcal{V}$ be the open cover of X consisting of all (nonempty) sets of the form $U_0 \cap f^{-1}(U_1) \cap ... \cap f^{-(n-1)}(U_{n-1})$ with $U_\kappa \in \mathcal{U}$ for $\kappa = 0, ..., n-1$. We claim that $\mathcal{V}$ is an s-cover for (X, f^n). So let G be a nonempty open subset of X, then there are $x, y \in G$ and $m \in \mathbb{N}_0$ such that $(x, y) \notin \bigcup\{f^{-m}(U) \times f^{-m}(U) : U \in \mathcal{U}\}$. Let $q \in \mathbb{N}_0$ and $0 \leq p < n$ be such that $m = qn + p$. Now $\{(f^n)^q(x), (f^n)^q(y)\} = \{f^{m-p}(x), f^{m-p}(y)\} \not\subset V$ for every $V \in \mathcal{V}$ (if $\{f^{m-p}(x), f^{m-p}(y)\} \subset V$ for some $V = U_0 \cap f^{-1}(U_1) \cap ... \cap f^{-(n-1)}(U_{n-1}) \in \mathcal{V}$, then $\{f^m(x), f^m(y)\} \subset U_p \in \mathcal{U}$, a contradiction). So $\mathcal{V}$ is an s-cover for (X, f^n) and f^n is topologically sensitive.

It is well-known that sensitivity is a redundant condition in Devaney chaos if X is infinite ([7],[28]), that is, a transitive periodically dense self-map on an infinite metric space is sensitive.

For the general case we need to consider the following property: a space X is called *Urysohn* if for every pair of distinct points x and y there are two open sets U and V in X such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$. It is worth noting that the class of Urysohn spaces lies strictly between the class of T_3-spaces and the class of Hausdorff spaces (see, e.g., [31, Pb. 14F.]). Observe also that there are Hausdorff spaces such that $\overline{U} \cap \overline{V} \neq \emptyset$ for every pair of nonempty open subsets U and V of X (see, e.g., [15]).

Moreover let $f : X \to X$ and recall that $x \in X$ is said to be:

1) *eventually periodic* whenever there exists $m \in \mathbb{N}_0$ such that $f^m(x)$ is periodic;

2) *almost periodic* whenever for every open neighbourhood U of x there is some $r \in \mathbb{N}_0$ satisfying the following condition: for every $n \in \mathbb{N}_0$ there is some nonnegative integer $\kappa \leq r$ such that $f^{n+\kappa}(x) \in U$.

Theorem 2.5. *Let $f : X \to X$ be a transitive map with a dense set of almost periodic points, where X is an infinite Urysohn space. If f has an eventually periodic point, then (X, f) has an s-cover with two elements. Therefore f is topologically sensitive.*

Proof. Let p be an eventually periodic point of f and let us take some $x \in X \setminus O(f, p)$ where $O(f, p) = \{f^n(x) : n \in \mathbb{N}_0\}$ is the orbit of p (observe that $O(f, p)$ is finite). Since X is a Urysohn space, there exist two open subsets U and V of X such that $x \in U$, $O(f, p) \subset V$, and $\overline{U} \cap \overline{V} = \emptyset$.

We claim that the open cover $\mathcal{H} = \{X \setminus \overline{U}, X \setminus \overline{V}\}$ of X is an s-cover for (X, f). So let us take a nonempty open subset G of X. By hypothesis there is an almost periodic point q such that $q \in G \cap f^{-l}(U)$ for some $l \in \mathbb{N}$. Since $f^{-l}(U)$ is an open neighbourhood of q, there is a nonnegative integer r such that for every $n \in \mathbb{N}_0$ $f^{n+\kappa}(q) \in f^{-l}(U)$ for some $\kappa \leq r$. Now set $W = \bigcap\{f^{-j}(V) : l \leq j \leq l + r\}$ and observe that W is a nonempty open subset of X (note that $p \in W$). So by transitivity of f, we can find $z \in G$ and $s \in \mathbb{N}$ such that $f^s(z) \in W$. Now let us take some $\kappa \leq r$ such that $f^{s+\kappa}(q) \in f^{-l}(U)$. Then $f^{l+s+\kappa}(q) \in U$ and $f^{l+s+\kappa}(z) = f^{l+\kappa}(f^s(z)) \in f^{l+\kappa}(W) \subset V$. So $q, z \in G$, $f^{l+s+\kappa}(q) \notin X \setminus \overline{U}$ and $f^{l+s+\kappa}(z) \notin X \setminus \overline{V}$. Set $n = l + s + \kappa$, then $(q, z) \notin \bigcup\{f^{-n}(H) \times f^{-n}(H) : H \in \mathcal{H}\}$. Therefore $\mathcal{H}$ is an s-cover for (X, f) and f is topologically sensitive.

In particular we have

Corollary 2.6. *Every transitive and periodically dense self-map on an infinite Urysohn space is topologically sensitive.*

Theorem 2.5 should be compared with the following result: let $f : X \to X$ be a transitive map with a dense set of almost periodic points, where X is a compact metric space. Then f is either minimal (i.e., every orbit is dense) or sensitive ([2], [13]).

A dynamical system (X, f) (or simply the map f) is called *weakly mixing* if the system $(X \times X, f \times f)$ is transitive, where $f \times f$ is the self-map on $X \times X$ given by $(f \times f)(x, y) = (f(x), f(y))$ for $x, y \in X$.

Clearly f is weakly mixing if and only if for any four nonempty open subsets G, U, H and V of X there is some $n \in \mathbb{N}$ such that $f^n(G) \cap U \neq \emptyset$ and $f^n(H) \cap V \neq \emptyset$.

Every weakly mixing self-map on a metric space with at least two points is sensitive (see, e.g., [30]). The counterpart for topological dynamical systems is the following

Proposition 2.7. *Let X be a topological space with two nonempty open subsets U and V such that $\overline{U} \cap \overline{V} = \emptyset$ and let $f : X \to X$ be a weakly mixing map. Then (X, f) has an s-cover with two elements, so f is topologically sensitive.*

Proof. Let us consider the open cover $\mathcal{W} = \{X \setminus \overline{U}, X \setminus \overline{V}\}$ of X. We claim that $\mathcal{W}$ is an s-cover for (X, f). Let G be a nonempty open subset of X. Since f is weakly mixing, there is some $n \in \mathbb{N}$ such that $G \cap f^{-n}(U) \neq \emptyset$

and $G \cap f^{-n}(V) \neq \emptyset$. So let us take $x, y \in G$ such that $f^n(x) \in U$ and $f^n(y) \in V$. Since $f^n(x) \notin X \setminus \overline{U}$ and $f^n(y) \notin X \setminus \overline{V}$, it follows that $\mathcal{W}$ is an s-cover for (X, f) and f is topologically sensitive.

Corollary 2.8. *Let f be a weakly mixing self-map on a space X which is disconnected or a Urysohn space with at least two points. Then f is topologically sensitive.*

As observed by the referee, Theorem 2.5 and Proposition 2.7 raise the question of whether there exists a sensitive homeomorphism on a compact metric space without s-covers with two elements.

Remarks. (i) A transitive and periodically dense (or weakly mixing) self-map on a T_1-space (without isolated points) need not be topologically sensitive. In fact, let X be an infinite cofinite space, then every surjective continuous self-map on X is weakly mixing (every pair of nonempty open subsets of X intersect). So, for instance, if f is the identity, then f is a weakly mixing (and periodically dense) map. On the other hand f is not topologically sensitive. Since X is compact, it is enough to see that there are no finite s-covers. So let $\mathcal{U} = \{U_1, ...U_\kappa\}$ be an open cover of X, with $U_i \neq \emptyset$ for every i. Then $F = \bigcup\{X \setminus U_i : i = 1, ...\kappa\}$ is finite and for every x, y in the open set $X \setminus F$ we have $\{f^n(x), f^n(y)\} = \{x, y\} \subset U_i$ for every $n \in \mathbb{N}_0$ and $i = 1, ..., \kappa$. So $\mathcal{U}$ is not an s-cover for (X, f).

(ii) Let $X = ([0, 1], \tau)$, where τ is the cocountable topology on $[0, 1]$, and let $T : X \to X$ be the tent map, i.e., $T(x) = 1 - |2x - 1|$ for every $x \in [0, 1]$. Clearly T is continuous, weakly mixing (as above, every pair of nonempty open sets has nonempty intersection), moreover T is not periodically dense (the set of periodic points of T is contained in $\mathbb{Q}$ and $\mathbb{Q}$ is closed in $([0, 1], \tau)$).

T is not topologically sensitive. Since X is a Lindelöf space, it is enough to check that there are no countable s-covers. So let us take a countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}_0\}$ of X with $U_n \neq \emptyset$ for every n. Then $F = \bigcup\{[0, 1] \setminus U_n : n \in \mathbb{N}_0\}$ is countable and $G = [0, 1] \setminus \bigcup\{T^{-n}(F) : n \in \mathbb{N}_0\}$ is a nonempty open subset of X. If $x, y \in G$, then $T^m(x), T^m(y) \in [0, 1] \setminus F = \bigcap\{U_n : n \in \mathbb{N}_0\}$ for every $m \in \mathbb{N}_0$. So $\{T^m(x), T^m(y)\} \subset U_n$ for every $m, n \in \mathbb{N}_0$. Therefore $\mathcal{U}$ is not an s-cover for (X, f).

However, it is unclear if there exists a transitive self-map on a noncompact metric space which is not topologically sensitive.

We have already observed that the phase space X of a topologically sensitive dynamical system (X, f) cannot have isolated points, more generally X cannot have almost P-points. A point p of a space X is called *almost P-point* if every G_δ-set containing p has nonempty interior (a *G_δ-set* is a countable intersection of open subsets). If every point of a space X is an almost P-point, then X is called almost P-space (see, e.g., [23]). It is worth noting that $\beta X \setminus X$ is an almost P-space (where βX is the Stone-Čech compactification of X) for every space X which is locally compact and realcompact (a space is called realcompact if it can be embedded as a closed subset of a product of copies of the real line endowed with the usual topology) [12]. Thus, for example, $\beta\mathbb{N} \setminus \mathbb{N}$, $\beta\mathbb{R} \setminus \mathbb{R}$ and $\beta[0, +\infty) \setminus [0, +\infty)$ are almost P-spaces.

Proposition 2.9. *A space X with an almost P-point does not support a topologically sensitive map.*

Proof. Let f be a continuous self-map on X and let $\mathcal{U}$ be an open cover of X. Let $\mathcal{U}_n = \{f^{-n}(U) : U \in \mathcal{U}\}$ for every $n \in \mathbb{N}_0$. Let x be an almost P-point of X and let us take $U_n \in \mathcal{U}_n$ such that $x \in U_n$ for every $n \in \mathbb{N}_0$. Then $\bigcap\{U_n : n \in \mathbb{N}_0\}$ is a G_δ-set containing x. Set $G = \text{Int}(\bigcap\{U_n : n \in \mathbb{N}_0\})$ and observe that G is a nonempty open subset of X such that for every $n \geq 0$ there is some $U \in \mathcal{U}$ with $f^n(G) \subset U$. So for every $x, y \in G$ and every $n \in \mathbb{N}_0$ there exists some $U \in \mathcal{U}$ such that $(x, y) \in f^{-n}(U) \times f^{-n}(U)$. Therefore $\mathcal{U}$ is not a s-cover for (X, f) and f is not topologically sensitive.

Remark. It is worth pointing out that an infinite Hausdorff space with an almost P-point does not support a transitive map. In fact, let X be a Hausdorff space with an almost P-point x and let f be a transitive self-map on X, then x is a periodic point and $X = O(f, x)$. Let us suppose that $f^n(x) \neq x$ for every positive integer n and let us take two disjoint open sets G_n and H_n such that $x \in G_n$, $f^n(x) \in H_n$ for every $n \in \mathbb{N}$. Since every f^n is continuous, there is an open neighbourhood V_n of x such that $V_n \subset G_n$ and $f^n(V_n) \subset H_n$ for every n. Therefore $V = \text{Int}\bigcap\{V_n : n \in \mathbb{N}\}$ is a nonempty open subset of X such that $f^n(V) \cap V = \emptyset$ for every $n \in \mathbb{N}$, a contradiction. So x is a periodic point. Moreover if there is some $y \in X \setminus O(f, x)$, then there are disjoint open subsets G and H of X such that $O(f, x) \subset G$ and $y \in H$. Let us take an open neighbourhood W_n of x such that $f^n(W_n) \subset G$ for each $n \in \mathbb{N}$. Then $W = \text{Int}\bigcap\{W_n : n \in \mathbb{N}\}$ is a nonempty open subset of X such that $f^n(W) \cap H = \emptyset$ for every $n \in \mathbb{N}$, a contradiction. Hence $X = O(x, f)$.

By Proposition 2.9, it follows that $\beta\mathbb{N} \setminus \mathbb{N}$, $\beta\mathbb{R} \setminus \mathbb{R}$ and $\beta[0, +\infty) \setminus [0, +\infty)$ do not admit topologically sensitive maps. On the other hand we have

Examples 2.10. $\beta[0,+\infty)$ and $\beta\mathbb{R}$ are (nonmetrizable) compact Hausdorff spaces which support a topologically sensitive map.

Let (X, f) and (Y, g) be two dynamical systems. We say that (Y, g) is quasiconjugate to (X, f) if there is a continuous map $\phi : X \to Y$ with dense range such that $\phi \circ f = g \circ \phi$. A property $\mathcal{P}$ for dynamical systems is preserved under quasiconjugacy if the following holds: if (X, f) has $\mathcal{P}$ then every dynamical system (Y, g) that is quasiconjugate to (X, f) also has property $\mathcal{P}$. For every continuous map $f : X \to X$, where X is a Tychonoff space, there exists a continuous map $\beta f : \beta X \to \beta X$ (called Stone extension of f) such that $\beta f \circ e = e \circ f$, where $e : X \to \beta X$ is a dense embedding (e is the evaluation map induced by the collection of all bounded continuous real-valued functions on X). So $(\beta X, \beta f)$ is quasiconjugate to the dynamical system (X, f). Now let us consider the map $f : [0,+\infty) \to [0,+\infty)$ defined by $f(2^n x) = 2^{n+2}T(x-1)$ where $x \in [1, 2]$, $n \in \mathbb{Z}$, $f(0) = 0$ and T is the tent map. This map is bitransitive, i.e., f^2 is transitive [28], therefore f is transitive and periodically dense. Since transitivity and periodic density are preserved under quasiconjugacy (see, e.g., [17, Ch. 1]), it follows that the Stone extension $\beta f : \beta[0,+\infty) \to \beta[0,+\infty)$ of f is transitive and periodically dense, so, by Corollary 2.6, βf is topologically sensitive. In a similar vein, the Stone extension of any (Devaney) chaotic self-map on $\mathbb{R}$ (see, e.g., [28]) is topologically sensitive.

Remarks. (i) Since every point of $\mathbb{N}$ is isolated in $\beta\mathbb{N}$, it follows that $\beta\mathbb{N}$ does not support a topologically sensitive map.

(ii) Let $T : [0, 1] \to [0, 1]$ be the tent map (where $[0, 1]$ is endowed with the usual topology). The restriction $g : [0, 1] \cap \mathbb{Q} \to [0, 1] \cap \mathbb{Q}$ of the tent map is transitive and periodically dense , therefore the Stone extension of g $\beta g : \beta([0, 1] \cap \mathbb{Q}) \to \beta([0, 1] \cap \mathbb{Q})$ is topologically sensitive.

(iii) If f is a continuous self-map on a Tychonoff space X and βf is topologically sensitive, then f is topologically sensitive. More generally, if g is a topologically sensitive self-map on a paracompact space X and D is a dense invariant subset of X, then $f = g|D$ is topologically sensitive. In fact, let $\mathcal{U} = \{U_s : s \in \mathcal{S}\}$ be an s-cover for (X, g). Since X is paracompact, it follows that there is a locally finite open cover $\mathcal{V} = \{V_s : s \in \mathcal{S}\}$ of X such that $\overline{V}_s \subset U_s$ for every s (see, e.g., [11, Remark 5.1.7]). Set $U = \bigcup\{U_s \times U_s : s \in \mathcal{S}\}$, $V = \bigcup\{V_s \times V_s : s \in \mathcal{S}\}$ and observe that $\overline{V} = \bigcup\{\overline{V_s} \times \overline{V_s} : s \in \mathcal{S}\}$ (recall that $\mathcal{V}$ is locally finite) and $\overline{V} \subset U$. Let $W_s = V_s \cap D$ for every s and $W = \bigcup\{W_s \times W_s : s \in \mathcal{S}\}$. Now let us take a nonempty open subset

G of D and an open set H of X such that $H \cap D = G$. Since $\mathcal{U}$ is an s-cover for g, it follows that $H \times H \not\subset \bigcap\{(g \times g)^{-n}(U) : n \in \mathbb{N}_0\}$. So $(H \times H) \setminus \bigcap\{(g \times g)^{-n}(\overline{V}) : n \in \mathbb{N}_0\}$ is a nonempty open subset of $X \times X$. Since $D \times D$ is dense in $X \times X$, it follows that $[(H \times H) \setminus \bigcap\{(g \times g)^{-n}(\overline{V}) : n \in \mathbb{N}_0\}] \cap (D \times D) \neq \emptyset$. So $G \times G \not\subset \bigcap\{(f \times f)^{-n}(W) : n \in \mathbb{N}_0\}$ and $\mathcal{W} = \{W_s : s \in \mathcal{S}\}$ is an s-cover for (D, f). Hence (D, f) is topologically sensitive.

(iv) If f is a continuous self-map on a normal space X and (X, f) has a finite s-cover, then $(\beta X, \beta f)$ has a finite s-cover. In fact, let $\mathcal{U} = \{U_1, ..., U_\kappa\}$ be an s-cover for (X, f) and let us apply the Ex operator to the members of $\mathcal{U}$, namely let us consider the following open subsets of βX: Ex $U_i = \beta X \setminus \overline{X \setminus U_i}$ for every i. Observe that Ex U_i is the largest open subset of βX whose intersection with X is U_i.

Since X is normal, Ex $(\bigcup\{U_i : i = 1, ...\kappa\}) = \bigcup\{\text{Ex } U_i : i = 1, ...\kappa\}$ (see for instance [11, Lemma 7.1.13]). Therefore $\mathcal{V} = \{\text{Ex } U_i : I = 1, ..., \kappa\}$ is an open cover of βX. Now let G be a nonempty open subset of βX, since $\mathcal{U}$ is an s-cover for (X, f), it follows that there are $x, y \in G \cap X$ and $n \in \mathbb{N}_0$ such that $\{f^n(x), f^n(y)\} \not\subset U_i$ for every i. So $\{(\beta f)^n(x), (\beta f)^n(y)\} \not\subset \text{Ex } U_i$ for every i and $\mathcal{V}$ is an s-cover for $(\beta X, \beta f)$.

Now let us consider a dynamical system (X, f), where X is a metric space with metric d. (X, f) is *equicontinuous* at the point x (or x is an *equicontinuity point*) if for every $\varepsilon > 0$ there is an open neighbourhood U of x such that $f^n(U) \subset B(f^n(x), \varepsilon)$ for every $n \in \mathbb{N}_0$. Let $Eq(X, f)$ be the set of all equicontinuity points in X and observe that a sensitive system cannot have equicontinuity points, i.e., $\text{Eq}(X, f) = \emptyset$. The converse is not true in general (see, e.g., [21, Ex. 2.28] and [30, Ex. 8 p. 330]), nonetheless, a well-known result of Akin, Auslander and Berg states that a transitive self-map on a compact metric space is either sensitive or has equicontinuity points [2]. It is worth noting that a topologically sensitive self-map f on a metrizable space (X, τ_d) can have equicontinuity points (with respect to the metric d). In fact let $X = (0, \frac{1}{2}]$, d the usual metric, ρ the metric given by $\rho(x, y) = |\frac{1}{x} - \frac{1}{y}|$ and let f be the self-map on X defined by $f(x) = x^2$ for $x, y \in X$. Since d and ρ are equivalent and $f : (X, \rho) \to (X, \rho)$ is sensitive (even positively expansive, see the next section for the definition), it follows that $f : (X, \tau_d) \to (X, \tau_d)$ is topologically sensitive, on the other hand every point of X is an equicontinuity point with respect to the usual metric d, see, e.g., [30, Ex. 7 p. 329-330].

Remark. Let f be a continuous self-map on a topological space X and let V be a neighbourhood of the diagonal in $X \times X$. Let $\mathrm{E}_f(V)$ be the union of the open sets U of X such that $U \times U \subset \bigcap\{(f \times f)^{-n}(V) : n \in \mathbb{N}_0\}$ and note that $\mathrm{E}_f(V)$ is open and inversely invariant, i.e., $f^{-1}(\mathrm{E}_f(V)) \subset \mathrm{E}_f(V)$. Observe that f is topologically sensitive if and only if $\mathrm{E}_f(V) = \emptyset$ for some neighbourhood V of the diagonal. Let us set $\mathrm{E}(X, f) = \bigcap\{\mathrm{E}_f(V) : V$ is a neighbourhood of $\Delta\}$ and let $\mathrm{Tr}(X, f)$ be the set of points with dense orbit. If f is not topologically sensitive, then $\mathrm{Tr}(X, f) \subset \mathrm{E}(X, f)$. In fact, let $x \in \mathrm{Tr}(X, f)$, then for every neighbourhood V of the diagonal there is some $n \geq 0$ such that $f^n(x) \in \mathrm{E}_f(V)$. Since $\mathrm{E}_f(V)$ is inversely invariant, it follows that $x \in \mathrm{E}_f(V)$ for every neighbourhood V of Δ. Therefore $x \in \mathrm{E}(X, f)$. From this it may be concluded that a system (X, f) with a dense orbit is either topologically sensitive or $\mathrm{E}(X, f) \neq \emptyset$.

On the other hand, as observed by the referee, $\mathrm{E}(X, f) \subset \mathrm{Tr}(X, f)$ whenever f is a transitive self-map on a paracompact space X. In fact, let $x \in \mathrm{E}(X, f)$ and suppose that $y \in X \setminus \overline{O(f, X)}$. Let V be a symmetric open neighbourhood of Δ such that $(V \circ V)[y] \subset X \setminus \overline{O(f, X)}$ and let us take an open neighbourhood U of x such that $U \times U \subset \bigcap\{(f \times f)^{-n}(V) : n \in \mathbb{N}_0\}$. Since f is transitive, there are some $z \in U$ and $m > 0$ such that $f^m(z) \in V[y]$. Since $(x, z) \in U \times U$, it follows that $(f^m(x), f^m(z)) \in V$. Now $(f^m(z), y) \in V$, so $(f^m(x), y) \in V \circ V$ and $f^m(x) \in (V \circ V)[y]$, a contradiction. Therefore the orbit of x is dense.

Now let us show a possible way to fill the gap between sensitivity and topological sensitivity. Let us call a dynamical system $((X, \tau), f)$ (or simply the map f) *subsensitive* if there are a metric d on X such that $\tau_d \subset \tau$ and a positive constant ε satisfying the following condition: for every nonempty open subset G of (X, τ) there are $x, y \in G$ and $n \in \mathbb{N}_0$ such that $d(f^n(x), f^n(y)) \geq \varepsilon$ (spaces admitting a weaker metrizable topology are called *submetrizable*, see, e.g., [16] and [25]).

Remarks. (i) If $f : (X, d) \to (X, d)$ is sensitive , then the map $f : (X, \tau_d) \to (X, \tau_d)$ is subsensitive. Observe also that every subsensitive map $f : (X, \tau) \to (X, \tau)$ is topologically sensitive. In fact, let d be a metric on X and $\varepsilon > 0$ witnessing the subsensitivity of f. Then $\mathcal{U} = \{B(p, \frac{\varepsilon}{2}) : p \in X\}$ is an s-cover for $((X, \tau), f)$.

(ii) Let d be the usual metric on $\mathbb{R}$ and let $f : (\mathbb{R}, d) \to (\mathbb{R}, d)$ be the map given by $f(x) = x + 1$ for every $x \in \mathbb{R}$. Then f is not sensitive, nonetheless $f : (\mathbb{R}, \tau_d) \to (\mathbb{R}, \tau_d)$ is subsensitive. In fact let ρ be the metric on $\mathbb{R}$ given by $\rho(x, y) = |e^x - e^y|$ for every $x, y \in \mathbb{R}$, then $\tau_d = \tau_\rho$ and $f : (X, \rho) \to (X, \rho)$ is sensitive.

(iii) Since a compact Hausdorff space does not admit a strictly weaker Hausdorff topology, it follows that nonmetrizable compact Hausdorff spaces do not support subsensitive maps.

If $f : (X, \tau) \to (X, \tau)$ is subsensitive, then there is a metric d on X such that $\tau_d \subset \tau$ and $f : (X, d) \to (X, d)$ is a (not necessarily continuous) map for which is satisfied the sensitivity condition. On the other hand if $f : (X, d) \to (X, d)$ is sensitive and τ is a (metrizable) topology such that $\tau \subset \tau_d$, then $f : (X, \tau) \to (X, \tau)$ need not be topologically sensitive, as the following example shows.

Example 2.11. Let $f : (\mathbb{R}, d) \to (\mathbb{R}, d)$ be the sensitive map given by $f(x) = 2x$ for every $x \in \mathbb{R}$, where d is the usual metric. Let us consider the topology τ on $\mathbb{R}$ in which the basic neighbourhoods of each point other than the origin are the usual open interval centered at x, while the basic neighbourhoods of 0 are the sets of the form $(-\varepsilon, \varepsilon) \cup (-\infty, -n) \cup (n, +\infty)$ for all $\varepsilon > 0$ and $n \in \mathbb{N}$. Observe that τ is a separable metrizable topology on $\mathbb{R}$ weaker than the usual topology τ_d. $(\mathbb{R}, \tau)$ is called *looped line* (see, e.g., [31, Pb. 4D.]). Now the continuous map $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$ is not topologically sensitive. In fact, let us consider an open cover $\mathcal{U}$ of $(\mathbb{R}, \tau)$. We claim that $\mathcal{U}$ is not an s-cover for $((\mathbb{R}, \tau), f)$. Let us take some $U \in \mathcal{U}$ such that $0 \in U$, we may assume, without loss of generality, that $U = (-\varepsilon, \varepsilon) \cup (-\infty, -m) \cup (m, +\infty)$ for some $m \in \mathbb{N}$ and $\varepsilon > 0$. . Now $(m, +\infty)$ is a nonempty open subset of $(\mathbb{R}, \tau)$ such that $\{f^n(x), f^n(y)\} \subset U$ for every $x, y \in (m, +\infty)$ and every $n \in \mathbb{N}_0$. Therefore $\mathcal{U}$ is not an s-cover and f is not topologically sensitive.

We end this section by showing that topological sensitivity and subsensitivity are equivalent for a well-known class of submetrizable spaces.

We recall that:

(i) A space X has a *G_δ-diagonal* if the diagonal Δ is a G_δ-set in $X \times X$.

(ii) A *zero-set* of a space X is a set of the form $f^{-1}(0)$ for some continuous map $f : X \to [0, 1]$.

(iii) A subset of a normal space is a zero-set if and only if it is a closed G_δ-set.

Theorem 2.12. *Let f be a topologically sensitive self-map on a paracompact space X with a G_δ-diagonal. Then f is subsensitive.*

Proof. Let $\{V_n : n \in \mathbb{N}\}$ be a countable family of open subsets of $X \times \beta X$ such that $\Delta = \bigcap\{V_n \cap (X \times X) : n \in \mathbb{N}\}$ (we may assume, without loss of generality, that X is a subspace of βX). Since Δ is a closed set in $X \times \beta X$ and $X \times \beta X$ is normal (see, e.g., [31, Theorem 21.1]), we may define inductively a sequence $\{W_n : n \in \mathbb{N}\}$ of open subsets of $X \times \beta X$ such that $\Delta \subset W_1 \subset \overline{W}_1 \subset V_1$ and $\Delta \subset W_n \subset \overline{W}_n \subset V_n \cap W_{n-1}$ for $n \geq 2$. So $Z = \bigcap\{W_n : n \in \mathbb{N}\} = \bigcap\{\overline{W}_n : n \in \mathbb{N}\}$ is a closed G_δ-set of $X \times \beta X$ such that $\Delta = Z \cap (X \times X)$. So, again by normality of $X \times \beta X$, it follows that Z is a zero-set of $X \times \beta X$, i.e., there is a continuous map $F : X \times \beta X \to [0, 1]$ such that $Z = F^{-1}(0)$. Now let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an s-cover for (X, f) and let us take, for every $\lambda \in \Lambda$, an open set Y_λ of βX such that $U_\lambda = Y_\lambda \cap X$. Set $K = (X \times \beta X) \setminus \bigcup\{U_\lambda \times Y_\lambda : \lambda \in \Lambda\}$ and observe that Δ and K are disjoint closed subsets of $X \times \beta X$. So, by Urysohn's lemma, there is a continuous map $G : X \times \beta X \to [0, 1]$ such that $G(\Delta) \subset \{0\}$ and $G(K) \subset \{1\}$. Let $H : X \times \beta X \to [0, 1]$ be the continuous map defined by $H(x, y) = \max\{F(x, y), G(x, y)\}$ and observe that $\Delta = H^{-1}(0) \cap (X \times X)$. Now let $d(x, y) = \sup\{|H(x, z) - H(y, z)| : z \in \beta X\}$ for every $x, y \in X$ and observe that d is a metric on X: it is enough to note that $d(x, y) = 0$ yields $H(x, z) - H(y, z) = 0$ for every $z \in \beta X$, in particular $H(x, y) - H(y, y) = H(x, y) = 0$, so $(x, y) \in H^{-1}(0) \cap (X \times X) = \Delta$, i.e., $x = y$.

Now let us verify that the topology τ_d on X generated by the metric d is weaker than the original topology τ on X. It suffices to show that for every $x \in X$ and $\varepsilon > 0$, there is some open set U of X such that $x \in U \subset B(x, \varepsilon)$. Let $0 < \delta < \frac{\varepsilon}{2}$, since H is continuous in (x, z) for every $z \in \beta X$, there is a basic open neighbourhood $A_z \times B_z$ of (x, z) in $X \times \beta X$ such that $H(A_z \times B_z) \subset (H(x, z) - \delta, H(x, z) + \delta)$. Set $\mathcal{V} = \{A_z \times B_z : z \in \beta X\}$ and observe that $\mathcal{V}$ is an open family in $X \times \beta X$ which covers the compact set $\{x\} \times \beta X$. Thus there are $A_1 \times B_1, ..., A_n \times B_n \in \mathcal{V}$ such that $\{x\} \times \beta X \subset \bigcup\{A_i \times B_i : i = 1, ..., n\}$. Now let $U = \bigcap\{A_i : i = 1, ..., n\}$ and observe that U is an open neighbourhood of x in X such that $U \subset B(x, \varepsilon)$. In fact let $p \in U$, then for every $z \in \beta X$ there is some $j \in \{1, ..., n\}$ such that $(p, z) \in A_j \times B_j$. Since $H(x, z), H(p, z) \in H(A_j \times B_j)$ it follows that $|H(x, z) - H(p, z)| < 2\delta$ for every $z \in \beta X$. So $d(x, p) < \varepsilon$, i.e., $p \in B(x, \varepsilon)$.

It remains to show that f is subsensitive. Let G be a nonempty open subset of X, since $\mathcal{U}$ is an s-cover for (X, f) there are $x, y \in G$ and $n \in \mathbb{N}_0$ such that $(x, y) \notin \bigcup\{f^{-n}(U_\lambda) \times f^{-n}(U_\lambda) : \lambda \in \Lambda\}$. Therefore $(f^n(x), f^n(y)) \notin U_\lambda \times U_\lambda$ for every $\lambda \in \Lambda$, so $(f^n(x), f^n(y)) \in K = (X \times \beta X) \setminus \bigcup\{U_\lambda \times$

$Y_\lambda : \lambda \in \Lambda\}$. So $d(f^n(x), f^n(y)) \geq |H(f^n(x), f^n(y)) - H(f^n(y), f^n(y))| = H(f^n(x), f^n(y)) = 1$, and the proof is complete.

3. Weakly positively expansive maps

Let f be a continuous self-map on a metric space X, with metric d. The system (X, f) (or simply f) is called *positively expansive* if there is a real number $\varepsilon > 0$ such that for every pair of distinct points x and y there is some $n \in \mathbb{N}_0$ such that $d(f^n(x), f^n(y) \geq \varepsilon$.

Observe that every positively expansive self-map on a metric space without isolated points is sensitive.

Now let f be a continuous self-map on a topological space X. An *expansivity neighbourhood* for f is a closed neighbourhood C of the diagonal Δ in X^2 such that for every pair of distinct points x and y of X there is some $n \in \mathbb{N}_0$ such that $(f^n(x), f^n(y)) \notin C$. Note that a closed neighbourhood C of Δ is an expansivity neighbourhood for f if and only if $\Delta = \bigcap\{(f \times f)^{-n}(C)) : n \geq 0\}$. A system (X, f) (or simply the map f) is called *weakly positively expansive* if there is an expansivity neighbourhood for f [26]. If $f : (X, d) \to (X, d)$ is positively expansive, then $f : (X, \tau_d) \to (X, \tau_d)$ is weakly positively expansive (if ε is an expansive constant, then $C = \{(x, y) \in X^2 : d(x, y) \leq \varepsilon\}$ is an expansivity neighbourhood for f). If, in addition, (X, τ_d) is compact, the conditions above are equivalent [26].

To investigate the relationship between topological sensitivity and weak positive expansiveness we need, as in the definition of topological sensitivity, a particular type of covers.

Definition 3.1. Let (X, f) be a dynamical system and let $\mathcal{U}$ be an open cover of X. $\mathcal{U}$ is called *e-cover* for (X, f) if for every $x, y \in X$, with $x \neq y$, there is some $n \in \mathbb{N}_0$ such that $(x, y) \notin \bigcup\{f^{-n}(U) \times f^{-n}(U) : U \in \mathcal{U}\}$.

Remark. Let f be a continuous self-map on a space X and let $\mathcal{U}$ be an open cover of X. It is easy to see that $\mathcal{U}$ is an e-cover for (X, f) if and only if $|\bigcap\{f^{-n}(U_n) : n \in \mathbb{N}_0\}| \leq 1$ for every sequence $\{U_n : n \in \mathbb{N}_0\}$ of members of $\mathcal{U}$. Therefore the notion of finite e-cover coincides with the well-known concept of one-sided (weak) generator, so a continuous self-map f on a compact metric space X is positively expansive if and only if (X, f) has a finite e-cover (see, e.g., [20]).

The next result will clarify, in a rather general setting, the connections between e-covers and weak positive expansiveness.

Theorem 3.2. *Let (X, f) be a dynamical system.*

(i) If f is weakly positively expansive, then there is an e-cover for (X, f).
(ii) If (X, f) has an e-cover and X^2 is normal, then f is weakly positively expansive.

Proof. (i) Let C be an expansivity neighbourhood of Δ. For each $x \in X$ let us take an open neighbourhood U_x of x such that $U_x \times U_x \subset C$. Then $\mathcal{U} = \{U_x : x \in X\}$ is an e-cover for (X, f). In fact, let $\{U_n : n \geq 0\}$ be a sequence of members of $\mathcal{U}$ and $x, y \in \bigcap\{f^{-n}(U_n) : n \geq 0\}$. Then $((f^n(x), f^n(y)) \in U_n \times U_n \subset C$ for every $n \in \mathbb{N}_0$, so $x = y$ and we have $|\bigcap\{f^{-n}(U_n) : n \geq 0\}| \leq 1$. Therefore, by the previous remark, $\mathcal{U}$ is an e-cover for (X, f).

(ii) Now let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an e-cover for (X, f) and set $U = \bigcup\{U_\alpha \times U_\alpha : \alpha \in A\}$. Then U is an open set of X^2 containing the diagonal Δ. Since X^2 is a normal space (and Δ is closed) there is some open subset V of X^2 such that $\Delta \subset V \subset \overline{V} \subset U$. The closed neighbourhood $\overline{V}$ of Δ is an expansivity neighbourhood for f. In fact for every pair x, y of distinct points of X there is some $n \geq 0$ such that $(x, y) \notin \bigcup\{f^{-n}(U_\alpha) \times f^{-n}(U_\alpha) : \alpha \in A\}$, i.e., $(f^n(x), f^n(y)) \notin U$. So $(f^n(x), f^n(y)) \notin \overline{V}$ and f is weakly positively expansive.

Corollary 3.3. *Every weakly positively expansive self-map on a space without isolated point is topologically sensitive.*

Proof. By Theorem 3.2.(i) there is an e-cover $\mathcal{U}$ for (X, f). Since every nonempty open subset of X has at least two points, it follows that $\mathcal{U}$ is an s-cover for (X, f). So f is topologically sensitive.

Recall that a space X has a *regular G_δ-diagonal* if there is a countable family $\{U_n : n \geq 0\}$ of open subsets of X^2 such that $\Delta = \bigcap\{U_n : n \geq 0\} = \bigcap\{\overline{U}_n : n \geq 0\}$ (see, e.g., [32]). Note that every space X which has a regular G_δ-diagonal must be a Urysohn space: if x and y are two distinct points of X, then $(x, y) \notin \Delta = \bigcap\{U_n : n \geq 0\} = \bigcap\{\overline{U}_n : n \geq 0\}$. So there is some n such that $(x, y) \notin \overline{U}_n$. Take a basic open neighbourhood $G \times H$ of (x, y) in X^2 disjoint from the open set U_n. Then $\overline{G \times H} \cap U_n = \emptyset$, so $\overline{G} \times \overline{H} \cap \Delta = \emptyset$.

Therefore G and H are open subsets of X such that $x \in G$, $y \in H$ and $\overline{G} \cap \overline{H} = \emptyset$.

Now let f be a continuous self-map on a space X. It is straightforward to see that the weak positive expansiveness of f is equivalent to the existence of an open neighbourhood U of Δ in X^2 satisfying the following conditions:

$$\Delta = \bigcap\{(f \times f)^{-n}(U) : n \in \mathbb{N}_0\} = \bigcap\{(f \times f)^{-n}(\overline{U}) : n \in \mathbb{N}_0\}.$$

So the phase space of a weakly positively expansive dynamical system (X, f) has a regular G_δ-diagonal: since $\overline{(f \times f)^{-n}(U)} \subset (f \times f)^{-n}(\overline{U})$, it follows that $\Delta = \bigcap\{U_n : n \in \mathbb{N}_0\} = \bigcap\{\overline{U}_n : n \in \mathbb{N}_0\}$, where U_n is the open set $(f \times f)^{-n}(U)$ of X^2 for every $n \in \mathbb{N}_0$.

In the next remarks we will see how this observation can be used to show that certain spaces do not admit a weakly positively expansive map.

Remarks. (i) Since every separable space with a regular G_δ-diagonal can have at most the cardinality of the continuum $\mathfrak{c}$ (a more general statement can be found in [9]), it follows that every separable space of cardinality greater than $\mathfrak{c}$ does not support a weakly positively expansive map. So, for instance, $\beta\mathbb{R}$ and $\beta[0, +\infty)$ do not admit weakly positively expansive maps (cf. Examples 2.10).

(ii) A space X is called pseudocompact if every continuous real-valued function defined on X is bounded. Since every pseudocompact Tychonoff space with a regular G_δ-diagonal is (compact and) metrizable ([24], see also [4]), it follows that a nonmetrizable pseudocompact Tychonoff space does not admit a weakly positively expansive map. In particular a compact Hausdorff space supporting a weakly positively expansive map must be metrizable.

(iii) It is well-known that the unit interval I (with the usual topology) does not admit a positively expansive map (this is true, more generally, for every compact connected topological manifold with boundary, see [18]).

A topological space X is called locally euclidean (of dimension n) if every point of X has a neighbourhood which is homeomorphic to $\mathbb{R}^n$ (endowed with the usual topology). Since every locally connected and locally compact T_2-space with a regular G_δ-diagonal is metrizable [16, Th. 2.15], it follows that every nonmetrizable locally euclidean T_2-space does not support a weakly positively expansive map (the long line is an example of a connected locally euclidean T_2-space which is not metrizable, see, e.g., [22, Pb. 4-6]).

It is worth noting, despite the remarks above, that there are weakly positively expansive dynamical systems whose phase space is a (Hausdorff) nonmetrizable space.

Example 3.4. Let τ be the usual topology on $\mathbb{R}$ and let us define a topology σ on $\mathbb{R}$ by declaring open all sets of the form $U \setminus A$, where $U \in \tau$ and A is countable. $X = (\mathbb{R}, \sigma)$ is a Hausdorff space which is not regular (so, a fortiori, X is not metrizable). Let $f : X \to X$ be the map defined by $f(x) = 2x$ for every $x \in \mathbb{R}$. Clearly f is continuous. Moreover (X, f) is weakly positively expansive, in fact $C = \{(x, y) \in \mathbb{R}^2 : |x - y| \leq 1\}$ is an expansivity neighbourhood of Δ in X^2. In a similar way, let $X = (\mathbb{R}, \tau_s)$ be the Sorgenfrey line (τ_s is the topology generated by the base consisting of all intervals of the form $[a, b)$) and let $f : X \to X$ be as above. Then (X, f) is a weakly positively expansive dynamical system whose phase space is a nonmetrizable paracompact space.

We end this paper mentioning another variation of topological sensitivity which could be worth considering.

Let (X, f) be a dynamical system and let C be a closed neighbourhood of the diagonal Δ in X^2. Following the definition of weak positive expansiveness, we say that C is a *sensitivity neighbourhood for* f if for every nonempty open subset G of X there are $x, y \in G$ and $n \in \mathbb{N}_0$ such that $(f^n(x), f^n(y)) \notin C$.

Let us say that (X, f) (or simply f) is *strongly topologically sensitive* if there exists a sensitivity neighbourhood for f.

Clearly every weakly positively expansive self-map on a space without isolated points is strongly topologically sensitive.

Observe also that every subsensitive self-map f on a space (X, τ) is strongly topologically sensitive. In fact let d be a metric on X and $\varepsilon > 0$ witnessing the subsensitivity of f. Then the set $C = \{(x, y) \in X^2 : d(x, y) \leq \varepsilon\}$ is a sensitivity neighbourhood for $f : (X, \tau) \to (X, \tau)$.

Moreover the arguments in Theorem 3.2 give also:

(a) Every strongly topologically sensitive map is topologically sensitive.

(b) If (X, f) is topologically sensitive and X^2 is T_4, then (X, f) is strongly topologically sensitive.

Observe that (b) shows, for instance, that every topologically sensitive self-map on $\beta[0, +\infty)$ or $\beta\mathbb{R}$ is an example of a strongly topologically sensitive map which is not subsensitive (recall that $\beta[0, +\infty)$ and $\beta\mathbb{R}$ are compact

Hausdorff spaces which are not metrizable).

Since the phase space of a weakly positively expansive dynamical system (X, f) has a regular G_δ-diagonal, it follows that X must be a Urysohn space. Let us conclude observing that, in a similar way, if (X, f) is strongly topologically sensitive then every nonempty open subset G of X must contain two points that can be separated by disjoint closed neighbourhoods. In fact let C be a sensitivity neighbourhood of the diagonal Δ and let us take $x, y \in G$ and $n \in \mathbb{N}_0$ such that $(f^n(x), f^n(y)) \notin C$. Since C is closed in X^2, there is a basic open neighbourhood $U \times V$ of $(f^n(x), f^n(y))$ in X^2 such that $(U \times V) \cap C = \emptyset$. Now $\Delta \subset \text{Int } C$ and $\overline{U \times V} \cap \text{ Int } C = \emptyset$, therefore $\overline{U} \cap \overline{V} = \emptyset$. So $\underline{f^{-n}(U)}$ and $\underline{f^{-n}(V)}$ are open neighbourhoods of x and y respectively and $\overline{f^{-n}(U)} \cap \overline{f^{-n}(V)} = \emptyset$.

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