IET Research Journals

ISSN 1751-8644
doi: 0000000000
www.ietdl.org

Internally Positive Representations and Stability Analysis of Linear Differential Systems with Multiple Time-Varying Delays

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Abstract: This work introduces the Internally Positive Representation of linear time-varying delay differential systems, in the general case of multiple time-varying delays. The technique, previously established for the delay-free case and recently extended to various classes of linear delay systems, aims at building a positive representation of systems whose dynamics is, in general, not definite in sign, in order to export results that only hold for positive systems to arbitrary ones. In the special case of constant matrices, this leads to a simple and easy to check condition for the delay-independent stability of differential systems with multiple time-varying delays. The condition is shown to be less conservative than some well known conditions available in the literature. Numerical examples are proposed to validate the theoretical results.

1 Introduction

Positive systems have been extensively studied in the last decades due to their remarkable properties and applications [1, 2]. Recently, also positive delay systems have received a widespread interest from the research community, culminating in a number of insightful results, highlighting a remarkable simplification in the delay-independent stability analysis. Indeed, while studying the stability of delay systems is far from trivial even in the linear case, and more so if time-varying delays are taken into account, when the positivity constraint is satisfied a huge simplification is in order, yielding simple necessary and sufficient conditions for the delay-independent stability. For the case of delay differential systems, the reader is referred to [3–9]. To export the properties of positive systems to not necessarily positive ones (*arbitrary* systems), an useful tool has recently been developed in the linear delay-free case: the Internally Positive Representation (IPR) of systems. The method, first introduced for delay-free discrete-time systems in [10–12] and then for delay-free continuous-time one in [13, 14], allows to build internally positive representations of systems whose dynamics is not definite in sign. Since the method presented in [13], although very easy and straightforward, can produce in some cases an unstable positive system even if the original system is stable, later works on the IPR focused on this issue, showing how to construct IPRs whose stability properties are equivalent to those of the original system [14]. The IPR technique has also been applied to the state estimation problem to achieve an interval observer for both delay-free continuous and discrete-time linear systems [15]. More recently, the method has been applied to delay systems to exploit their aforementioned peculiarities [16–18], in which novel conditions for the delay-independent stability of various classes of arbitrary delay systems have been proposed exploiting the IPR construction. See [19] for a comprehensive overview.

The first part of this paper focuses on the extension of the IPR method to linear differential systems, in the general case of multiple time-varying and distributed delays, in order to systematically build positive representations of such systems. We start from the results presented in [16] and extend them to time-varying delay differential systems. Then, a stability analysis follows in the special case of constant matrices and no distributed delays, in order to exploit the stability results established for positive delay differential systems of this class [3, 6]. We show that only systems that are stable for any set of delays, constant or time-varying, can admit a stable IPR. This property is then used to obtain a simple and easy to check sufficient condition for the delay-independent stability of the original system: we start from the results presented in [16] and see how the condition can be stated directly on the original system matrices, without the need to explicitly compute its IPR. The proposed condition is proved to be less conservative with respect to comparable conditions available in the literature.

The contributions and the structure of this work can be summarized as follows:

- In Section 2 the IPR method for time-varying delay differential systems is introduced, thus extending the results of [16].
- Among the possible applications of the IPR technique, an immediate one is introduced in Section 3, which shows how the method leads to a simple sufficient condition for the delay-independent stability of arbitrary delay differential systems, in the special case of constant matrices. The result shows how the IPR method provides an elegant way to systematically export properties that only hold for positive systems to arbitrary ones.
- A comparison between the aforementioned stability condition and a well known norm-based condition [20–23] is then proposed in Section 4, showing that the former is less conservative than the latter. While the condition does not directly compare to LMI-based results, at least for simplicity and computational complexity, in Section 5 we present examples of systems for which the criterion of this work allows to assess their delay-independent stability, while both delay-independent and delay-dependent LMI conditions respectively fail and introduce conservativeness. Other numerical examples illustrate and validate the theoretical results.
- Concluding remarks and ideas for future work close the paper.

Notations. $\mathbb{R}_+$ is the set of nonnegative real numbers. $\mathbb{C}_-$ and $\mathbb{C}_+$ are the open left-half and right-half complex planes. $\mathbb{R}_+^n$ is the nonnegative orthant of $\mathbb{R}^n$. $\mathbb{R}_+^{m\times n}$ is the cone of positive $m\times n$ matrices. I_n is the $n\times n$ identity matrix. $\Re(z)$ and $\Im(z)$ are the real and imaginary parts of a complex number z. $\mathcal{L}_1^p$ and $\mathcal{L}_{1,+}^p$ are the sets of locally integrable functions with values in $\mathbb{R}^p$ and $\mathbb{R}_+^p$. $\mathcal{C}([a,b],\mathbb{R}^n)$ is the Banach space of all continuous functions on $[a,b]$ with values in $\mathbb{R}^n$, endowed with the uniform convergence norm $\|\cdot\|_\infty$. $A\in\mathbb{R}^{n\times n}$ is *Metzler* if all its off-diagonal elements are nonnegative. $d(A)$ denotes the diagonal matrix extracted from A, i.e. $[d(A)]_{ij}=[A]_{ij}$ for $i=j$ and zero otherwise. $A^{\mathcal{M}}$ is the Metzler matrix defined as $A^{\mathcal{M}}=d(A)+|A-d(A)|$, where $|\cdot|$ denotes the componentwise absolute value of a matrix. $\sigma(A)$ and $\alpha(A)$ denote the spectrum and the spectral abscissa of A, respectively. A is said to be stable or *Hurwitz* if $\sigma(A) \subset \mathbb{C}_-$ or, equivalently, if $\alpha(A) < 0$. $l_i(A), i = 1, \ldots, n$, is the i-th leading principal minor of $A \in \mathbb{R}^{n\times n}$, i.e. $l_i(A) = \det(A_i)$, where A_i is the matrix obtained from A by removing the last $n-i$ rows and columns. Finally, $\underline{m} = \{1, 2, \ldots, m\}$ and $\underline{m}_0 = \{0, 1, \ldots, m\}$.

2 Internally Positive Representation of Delay Differential Systems

2.1 Internally positive delay systems

In this section, we consider time-varying delay differential systems, in the general case of multiple time-varying and distributed delays, denoted by $S_t = \big\{\{A_k(t)\}_0^m, A_d(t,\theta), B(t), C(t), D(t)\big\}_{n,p,q}$ and described by

$$\begin{aligned} \dot{x}(t) &= A_0(t)x(t) + \sum_{k=1}^{m} A_k(t)x(t-\delta_k(t)) \\ &\quad + \int_{-\delta}^{0} A_d(t,\theta)x(t+\theta)d\theta + B(t)u(t), \quad t \geq t_0, \\ y(t) &= C(t)x(t) + D(t)u(t), \\ x(t) &= \phi(t-t_0), \qquad\qquad t \in [t_0-\delta, t_0], \end{aligned} \tag{1}$$

where $u(t) \in \mathbb{R}^p$ is the input, with $u \in \mathcal{L}_1^p$, $y(t) \in \mathbb{R}^q$ is the output, $x(t) \in \mathbb{R}^n$ is the system variable and $\phi \in \mathcal{C}([-\delta, 0], \mathbb{R}^n)$ is the initial state function. The time-delays $\delta_k : \mathbb{R} \to \mathbb{R}_+$ are continuous functions satisfying

$$0 \leq \delta_k(t) \leq \delta, \quad \forall t \geq t_0. \tag{2}$$

Moreover, $B(t) \in \mathbb{R}^{n\times p}$, $C(t) \in \mathbb{R}^{q\times n}$, $D(t) \in \mathbb{R}^{q\times p}$, $A_d(t,\theta) \in \mathbb{R}^{n\times n}$ and $A_k(t) \in \mathbb{R}^{n\times n}$, for $k \in \underline{m}_0$, are continuous matrix-valued functions for $t \geq t_0$, $\theta \in [-\delta, 0]$. It is well known that the delay differential equation in (1) admits a unique solution satisfying a given initial condition ϕ (see e.g. [24]). Throughout the paper, the solution $x(t)$ and the corresponding output trajectory $y(t)$ associated to the system S_t are denoted by

$$\big(x(t), y(t)\big) = \Phi_S\big(t, t_0, \phi, u\big). \tag{3}$$

Following [6, 8, 25], an *internally positive* linear delay differential system is defined as follows.

Definition 1. *A delay differential system $S_t = \big\{\{A_k(t)\}_0^m, A_d(t,\theta), B(t), C(t), D(t)\big\}_{n,p,q}$ is said to be internally positive if*

$$\left\{ \begin{array}{c} \phi \in \mathcal{C}([-\delta, 0], \mathbb{R}_+^n) \\ u \in \mathcal{L}_{1,+}^p \end{array} \right\} \Rightarrow \left\{ \begin{array}{ll} x(t) \in \mathbb{R}_+^n, & \forall t \geq t_0 \\ y(t) \in \mathbb{R}_+^q, & \end{array} \right\}. \tag{4}$$

Stated informally, S_t is internally positive if, once started by nonnegative initial states and forced by nonnegative input functions, its state and output trajectories stay positive at all times. The next result provides necessary and sufficient conditions to fulfil Definition 1 (see [4], [8]).

Lemma 1. *A delay differential system $S_t = \{\{A_k(t)\}_0^m, A_d(t,\theta), B(t), C(t), D(t)\}_{n,p,q}$ is internally positive if and only if $A_0(t)$ is Metzler and $B(t)$, $C(t)$, $D(t)$, $A_d(t,\theta)$ and $A_k(t)$, for $k \in \underline{m}$, are nonnegative, for all $t \geq t_0$ and $\theta \in [-\delta, 0]$.*

2.2 Positive representation of vectors and matrices

For a matrix (or vector) $M \in \mathbb{R}^{m\times n}$, the symbols M^+, M^- denote the componentwise *positive* and *negative* parts of M, i.e.:

$$[M^+]_{ij} = \begin{cases} [M]_{ij} & \text{if } [M]_{ij} \geq 0, \\ 0 & \text{if } [M]_{ij} < 0, \end{cases}$$

and $M^- = (-M)^+$.
Moreover, $|M|$ denotes the componentwise absolute value of M. Clearly, $M = M^+ - M^-$ and $|M| = M^+ + M^-$.

Let $\Delta_n = [I_n \ \ -I_n] \in \mathbb{R}^{n\times 2n}$. The following definitions are taken from [11, 12].

Definition 2. *A positive representation of a vector $x \in \mathbb{R}^n$ is any vector $\tilde{x} \in \mathbb{R}_+^{2n}$ such that*

$$x = \Delta_n \tilde{x}. \tag{5}$$

The min-positive representation of a vector $x \in \mathbb{R}^n$ is the nonnegative vector $\pi(x) \in \mathbb{R}_+^{2n}$ defined as

$$\pi(x) = \begin{bmatrix} x^+ \\ x^- \end{bmatrix}. \tag{6}$$

The min-positive representation of a matrix $M \in \mathbb{R}^{m\times n}$ is the nonnegative matrix $\Pi(M) \in \mathbb{R}_+^{2m\times 2n}$ defined as

$$\Pi(M) = \begin{bmatrix} M^+ & M^- \\ M^- & M^+ \end{bmatrix} \tag{7}$$

while the min-Metzler representation of a matrix $A \in \mathbb{R}^{n\times n}$ is the Metzler matrix $\Gamma(A) \in \mathbb{R}^{2n\times 2n}$ defined as

$$\Gamma(A) = \begin{bmatrix} d(A) + (A - d(A))^+ & (A - d(A))^- \\ (A - d(A))^- & d(A) + (A - d(A))^+ \end{bmatrix}. \tag{8}$$

Of course, if $d(A) \in \mathbb{R}_+^{n\times n}$ then $\Gamma(A) = \Pi(A)$. Moreover, for any $x \in \mathbb{R}^n$ and matrices $M \in \mathbb{R}^{m\times n}$, $A \in \mathbb{R}^{n\times n}$ the following properties hold true:

(a) $x = \Delta_n \pi(x)$;
(b) $\Delta_m \Pi(M) = M\Delta_n$, so that $\Delta_m \Pi(M)\pi(x) = Mx$;
(c) $\Delta_n \Gamma(A) = A\Delta_n$, so that $\Delta_n \Gamma(A)\pi(x) = Ax$.

2.3 Internally Positive Representations

The concept of Internally Positive Representation (IPR) of an arbitrary system has been introduced in [10–12], for delay-free discrete-time systems, and in [13, 14] for delay-free continuous-time systems. The IPR construction presented in [13] can be extended to the case of differential systems with multiple time-varying and distributed delays by the following definition, originally introduced in [16] and here presented for time-varying systems.

Definition 3. *An Internally Positive Representation of a delay differential system $S_t = \{\{A_k(t)\}_0^m, A_d(t,\theta), B(t), C(t), D(t)\}_{n,p,q}$ is an internally positive system $\widetilde{S}_t = \{\{\widetilde{A}_k(t)\}_0^m, \widetilde{A}_d(t,\theta), \widetilde{B}(t), \widetilde{C}(t), \widetilde{D}(t)\}_{\tilde{n},\tilde{p},\tilde{q}}$ together with four continuous transformations $\{T_X^f, T_X^b, T_U, T_Y\}$,*

$$\begin{aligned} T_X^f &: \mathbb{R}^n \mapsto \mathbb{R}_+^{\tilde{n}}, & T_X^b &: \mathbb{R}_+^{\tilde{n}} \mapsto \mathbb{R}^n, \\ T_U &: \mathbb{R}^p \mapsto \mathbb{R}_+^{\tilde{p}}, & T_Y &: \mathbb{R}_+^{\tilde{q}} \mapsto \mathbb{R}^q, \end{aligned} \tag{9}$$

such that $\forall t_0 \in \mathbb{R}$, $\forall(\phi, u) \in \mathcal{C}([-\delta, 0], \mathbb{R}^n) \times \mathcal{L}_1^p$, the following implication holds:

$$\begin{aligned} &\left\{ \begin{array}{l} \tilde{\phi}(\tau) = T_X^f\big(\phi(\tau)\big), \ \forall \tau \in [-\delta, 0] \\ \tilde{u}(t) = T_U\big(u(t)\big), \quad \forall t \geq t_0 \end{array} \right\} \\ &\qquad \Longrightarrow \left\{ \begin{array}{ll} x(t) = T_X^b\big(\tilde{x}(t)\big), & \forall t \geq t_0 \\ y(t) = T_Y\big(\tilde{y}(t)\big), & \end{array} \right\} \end{aligned} \tag{10}$$

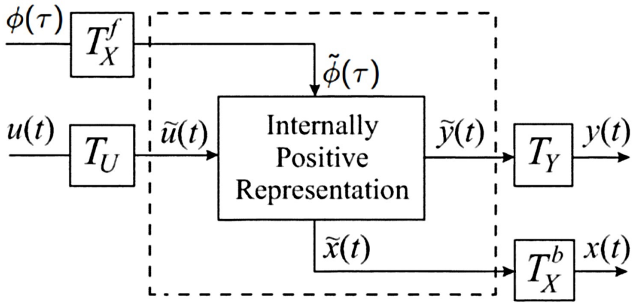

Fig. 1: An Internally Positive Representation of S according to Definition 3.

where

$$\big(x(t), y(t)\big) = \Phi_{S_t}\big(t, t_0, \phi, u\big),$$
$$\big(\tilde{x}(t), \tilde{y}(t)\big) = \Phi_{\tilde{S}_t}\big(t, t_0, \tilde{\phi}, \tilde{u}\big).$$

T_X^f and T_X^b in (9) are the *forward* and *backward* state transformations of the IPR, respectively, while T_U and T_Y are the input and output transformations, respectively. The implication (10) means that if the (nonnegative) initial state function $\tilde{\phi}$ of the IPR is computed as the forward state transformation T_X^f of the initial state ϕ of the *original* system, and the (nonnegative) input $\tilde{u}$ to the IPR is computed as the input transformation T_U of the input u to the *original* system, then the state trajectory of the *original* system is given by the backward transformation T_X^b of the (nonnegative) state $\tilde{x}$ of the IPR, and the output trajectory y of the *original* system is given by the output transformation T_Y of the (nonnegative) output $\tilde{y}$ of the IPR. For consistency, the *backward* map T_X^b must be a left-inverse of the *forward* map T_X^f, i.e. $x = T_X^b\big(T_X^f(x)\big)$, $\forall x \in \mathbb{R}^n$. The IPR of Definition 3 is illustrated in Figure 1.

The following theorem provides a method for the IPR construction of arbitrary time-varying delay differential systems.

Theorem 2. *Consider a delay differential system S_t as in* (1)*, with $S_t = \{\{A_k(t)\}_0^m, A_d(t,\theta), B(t), C(t), D(t)\}_{n,p,q}$. An internally positive system $\bar{S}_t = \{\{\bar{A}_k(t)\}_0^m, \bar{A}_d(t,\theta), \bar{B}(t), \bar{C}(t), \bar{D}(t)\}_{2n,2p,2q}$, with*

$$\begin{gathered}\bar{A}_0(t) = \Gamma(A_0(t)), \quad \bar{A}_d(t,\theta) = \Pi(A_d(t,\theta)), \quad \bar{B}(t) = \Pi(B(t)),\\ \bar{C}(t) = \Pi(C(t)), \qquad \bar{D}(t) = \Pi(D(t)), \qquad \bar{A}_k(t) = \Pi(A_k(t)),\\ \textit{for all } \ k \in \underline{m}, \ \ t \geq t_0, \ \ \theta \in [-\delta, 0]\end{gathered} \tag{11}$$

together with the four continuous transformations

$$\bar{x} = T_X^f(x) = \pi(x), \qquad x = T_X^b(\bar{x}) = \Delta_n \bar{x}, \tag{12}$$
$$\bar{u} = T_U(u) = \pi(u), \qquad y = T_Y(\bar{y}) = \Delta_q \bar{y}, \tag{13}$$

is an IPR of S_t.

Proof. We start noting that, since $\bar{A}_0(t)$ is Metzler at any $t \geq t_0$ and $\bar{B}(t)$, $\bar{C}(t)$, $\bar{D}(t)$, $\bar{A}_d(t,\theta)$, and $\bar{A}_k(t)$, $k \in \underline{m}$, are all nonnegative at any $t \geq t_0$, $\theta \in [-\delta, 0]$, from Lemma 1 it follows that system $\bar{S}_t$ is internally positive. Now, for any initial state function $\phi \in \mathcal{C}([-\delta, 0], \mathbb{R}^n)$, let $\bar{x}(t)$ and $\bar{y}(t)$ denote the state and output trajectories of $\bar{S}_t$

$$\big(\bar{x}(t), \bar{y}(t)\big) = \Phi_{\bar{S}}\big(t, t_0, \bar{\phi}, \bar{u}\big) \tag{14}$$

where $\bar{\phi}(\tau) = T_X^f(\phi(\tau)) = \pi(\phi(\tau))$, $\forall \tau \in [-\delta, 0]$ and $\bar{u}(t) = T_U(u(t)) = \pi(u(t))$, $\forall t \geq t_0$. Thus, (14) solves the system

$$\begin{aligned}\dot{\bar{x}}(t) &= \bar{A}_0(t)\bar{x}(t) + \sum_{k=1}^m \bar{A}_k(t)\bar{x}(t - \delta_k(t))\\ &\quad + \int_{-\delta}^0 \bar{A}_d(t,\theta)\bar{x}(t+\theta)d\theta + \bar{B}(t)\bar{u}(t), \quad t \geq t_0,\\ \bar{y}(t) &= \bar{C}(t)\bar{x}(t) + \bar{D}(t)\bar{u}(t),\\ \bar{x}(t) &= \bar{\phi}(t - t_0), \qquad t \in [t_0 - \delta, t_0].\end{aligned} \tag{15}$$

Consider now the vectors

$$\mathcal{X}(t) = T_X^b(\bar{x}(t)) = \Delta_n \bar{x}(t), \tag{16}$$
$$\mathcal{Y}(t) = T_Y(\bar{y}(t)) = \Delta_q \bar{y}(t). \tag{17}$$

To prove the theorem we have to show that $x(t) = \mathcal{X}(t)$ and $y(t) = \mathcal{Y}(t)$ for all $t \geq t_0 - \delta$.
By virtue of properties (b) and (c), given in Section 2.2, and (16), it follows that, for $t \geq t_0$,

$$\dot{\mathcal{X}}(t) = \Delta_n \dot{\bar{x}}(t) \tag{18}$$
$$= \Delta_n \bar{A}_0(t)\bar{x}(t) + \sum_{k=1}^m \Delta_n \bar{A}_k(t)\bar{x}(t - \delta_k(t)) \tag{19}$$
$$+ \int_{-\delta}^0 \Delta_n \bar{A}_d(t,\theta)\bar{x}(t+\theta)d\theta + \Delta_n \bar{B}(t)\pi\big(u(t)\big)$$
$$= A_0(t)\mathcal{X}(t) + \sum_{k=1}^m A_k(t)\mathcal{X}(t - \delta_k(t)) \tag{20}$$
$$+ \int_{-\delta}^0 A_d(t,\theta)\mathcal{X}(t+\theta)d\theta + B(t)u(t)$$

and for $t \in [t_0 - \delta, t_0]$

$$\mathcal{X}(t) = \Delta_n \bar{x}(t) = \Delta_n \bar{\phi}(t - t_0) = \Delta_n \pi\big(\phi(t - t_0)\big) = \phi(t - t_0) \tag{21}$$

and

$$\begin{aligned}\mathcal{Y}(t) &= \Delta_q \bar{y}(t) = \Delta_q \bar{C}(t)\bar{x}(t) + \Delta_q \bar{D}(t)\pi\big(u(t)\big)\\ &= C(t)\mathcal{X}(t) + D(t)u(t), \qquad t \geq t_0.\end{aligned} \tag{22}$$

Note that $\big(\mathcal{X}(t), \mathcal{Y}(t)\big)$ obey the same equations of (1), with the same initial condition. Then, from the uniqueness of the solution we get $\big(\mathcal{X}(t), \mathcal{Y}(t)\big) = \big(x(t), y(t)\big)$, and this concludes the proof. □

3 Stability Analysis

In this section, we restrict our analysis to delay differential systems with constant matrices and no distributed delays. This restriction is due to the fact that necessary and sufficient conditions for the delay-independent stability of *positive* delay differential systems – which we exploit to infer stability conditions for *arbitrary* systems via the IPR method – have only been reported for time-invariant systems [3, 6]. Moreover, in Section 4 we will see that a vast amount of research has been devoted to the stability of this class of (arbitrary) systems, and this richness naturally suggests interesting comparisons between the results we will present in this section and other notable conditions available in the literature.

Thus, throughout the rest of the paper we consider arbitrary delay differential systems described by:

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{k=1}^{m} A_k x(t - \delta_k(t)) + Bu(t), \qquad & t \geq t_0, \\ y(t) &= Cx(t) + Du(t), \\ x(t) &= \phi(t - t_0), & t \in [t_0 - \delta, t_0], \end{aligned} \tag{23}$$

henceforth denoted by $S = \{\{A_k\}_0^m, B, C, D\}_{n,m,q}$. The definitions of internal positivity and the IPR construction presented for the time-varying case apply to this special case with trivial modifications (see [16]).

Let's now investigate the relationships between the stability of a delay system and that of its IPR. A quite obvious consequence of the continuity of the state transformations $T_X^f(\cdot)$ and $T_X^b(\cdot)$ in (12) is that if an IPR of a system is stable, then the original system is stable as well. We will see that the converse is not always true.

Throughout this paper we use a standard nomenclature about stability. The trivial solution $x(t) \equiv 0$ of a delay system of the type (23) is said to be stable if any solution $x(t)$ for all $t \geq t_0$ satisfies a bound of the type $\|x(t)\| \leq k\|\phi\|_\infty$, for some $k > 0$. If in addition $\lim_{t\to\infty} \|x(t)\| = 0$, the trivial solution is asymptotically stable. If there exist $k > 0$ and $\eta > 0$ such that $\|x(t)\| \leq k\, e^{-\eta\, t}\|\phi\|_\infty$, the trivial solution is said to be exponentially stable.

A delay system as in (23) is said to be *stable* if the trivial solution is asymptotically stable. It is worth recalling that the stability of a delay system of the type (23) depends on the nature of delays (see e.g. [26, 27]): one can have stability for a given set or for any set of constant delays, for commensurate constant delays, for time-varying delays, within a given bound or without a specific bound, fast or slowly varying, etc. For reasons that will soon be clear, in this paper we are mainly concerned with stability for any set of constant or time-varying delays (delay-independent stability).

3.1 Stable IPRs of delay-free systems

For the case of delay-free systems ($A_k = 0, k \in \underline{m}$) in [13] it has been shown that the IPR construction method there presented when applied to stable systems in some cases may produce unstable IPRs. Indeed, the spectrum of $\bar{A}_0 = \Gamma(A_0)$ properly contains the spectrum of A_0, and the additional eigenvalues can be unstable. To see this, we exploit the following property [13]:

$$\sigma\Big(\begin{bmatrix} M & N \\ N & M \end{bmatrix} \Big) = \sigma\left(M - N\right) \cup \sigma\left(M + N\right), \tag{24}$$

where M and N are square matrices. Replacing

$$M = d(A_0) + (A_0 - d(A_0))^+, \quad N = (A_0 - d(A_0))^- \tag{25}$$

in (24) and noting that

$$\begin{aligned} M - N &= A_0, \\ M + N &= d(A_0) + |A_0 - d(A_0)| = A_0^{\mathcal{M}}. \end{aligned} \tag{26}$$

it follows

$$\sigma\big(\Gamma(A_0)\big) = \sigma(A_0) \cup \sigma(A_0^{\mathcal{M}}), \tag{27}$$

and therefore

$$\alpha\big(\Gamma(A_0)\big) = \max\left(\alpha(A_0), \alpha(A_0^{\mathcal{M}})\right). \tag{28}$$

From this it is clear that it can happen that $\alpha\big(\Gamma(A_0)\big) > 0$ (IPR unstable) even if $\alpha(A_0) < 0$ (original system stable).

Note that in general a change of coordinates on the original system can affect the stability of the IPR: defining $A_0^U = U^{-1}A_0U$, for some nonsingular $U \in \mathbb{R}^{n\times n}$, we have, in general

$$\sigma\big(A_0^{\mathcal{M}}\big) \neq \sigma\Big((A_0^U)^{\mathcal{M}}\Big) \tag{29}$$

while, obviously, $\sigma(A_0) = \sigma(A_0^U)$. In [13] it has been proved that if $\sigma(A_0)$ belongs to the sector of $\mathbb{C}_-$ characterized by $\Re(z) + |\Im(z)| < 0$, (sector with amplitude $\pi/2$) then there exists a change of coordinates such that $\alpha\big((A_0^U)^{\mathcal{M}}\big) < 0$, and therefore the resulting IPR is stable. In [14], the IPR construction method of [13] has been suitably extended so that stable IPRs can be constructed for any stable system, without any limitation on the location of the eigenvalues of A_0 within $\mathbb{C}_-$.

3.2 Stability of positive delay-systems

The IPR produced by the method in Theorem 2 is by construction a linear positive delay system. For this reason we recall below the stability conditions for such a class of systems. Consider a system of the type (23) which is internally positive (i.e., A_0 is Metzler and B, C, D, A_k, $k \in \underline{m}$, are nonnegative, Lemma 1). In [4] it has been proved that, when the delays δ_k are constant, a necessary and sufficient stability condition is that there exist strictly positive vectors p and r in $\mathbb{R}^n$ such that

$$\Big(\sum_{k=0}^{m} A_k\Big)^T p + r = 0. \tag{30}$$

Note that, being $\sum_{k=0}^m A_k$ a Metzler matrix, condition (30) is equivalent to $\sum_{k=0}^m A_k$ Hurwitz, i.e.

$$\alpha\Big(\sum_{k=0}^{m} A_k\Big) < 0 \tag{31}$$

(see eg. [1], where many other easy to check conditions equivalent to (30) and (31) can be found that do not require the explicit computation of eigenvalues or of the vectors p and r). Note that these conditions do not depend on the size of the delays. In [3] and in [6] it has been proved that (31) is necessary and sufficient for stability even in the case of time-varying delays $\delta_k(t)$, without limitation on the size of the delays and of their derivatives (a common assumption of Lyapunov-based results).

To summarize, we have the following:

Proposition 3. *If a system S as in* (23)*, with A_0 Metzler and B, C, D, A_k, for $k \in \underline{m}$, nonnegative, is stable for a set of constant delays δ_k, then it is also delay-independent stable, i.e. stable for any arbitrary set of constant or time-varying delays.*

Liu and Lam [7] showed that if a positive delay system is stable for all continuous and bounded delays, then the trivial solution is exponentially stable for all continuous and bounded delays. On the other hand, if the delays are continuous but unbounded, the trivial solution may be asymptotically stable but not exponentially stable.

Remark 1. *We stress that throughout the paper we say that a system is (delay-independent) stable if its trivial solution is (delay-independent) asymptotically stable, as previously defined.*

3.3 Stable IPRs of delay differential systems

Consider the equations (15) of the IPR given in Theorem 2. We have the following:

Theorem 4. *Consider a delay differential system S as in* (23) *and compute its IPR $\bar{S}$. If $\bar{S}$ is delay-independent stable, then S is delay-independent stable.*

Proof. As discussed in the previous paragraph, since the IPR is a positive delay system, a necessary and sufficient condition for its stability is that the Metzler matrix $\sum_{k=0}^{m} \bar{A}_k$ is Hurwitz, and this in turn implies that the IPR is delay-independent stable. The continuity of the state transformations $T_X^f(\cdot)$ and $T_X^b(\cdot)$ defined in (12) trivially implies the delay-independent stability of the original system, since:

$$x(t) = T_X^b(\bar{x}(t)) \implies \|x(t)\| \leq \|T_X^b\|\|\bar{x}(t)\| \tag{32}$$

and then if $\bar{x}(t)$ is stable, $\bar{x}(t)$ is stable as well. □

Stated in another way, Theorem 4 claims that only delay differential systems that are delay-independent stable admit stable Internally Positive Representations. Note that this result, although stated with reference to the IPR of Theorem 2, holds for any IPR satisfying Definition 3. This fact can be used to infer sufficient conditions for the stability of arbitrary systems.

Indeed, Theorem 4 suggests the following sufficient condition of delay-independent stability.

Theorem 5. *Consider a delay system S as in* (23)*. If*

$$\alpha\Big(\Gamma(A_0) + \sum_{k=1}^{m} \Pi(A_k)\Big) < 0, \tag{33}$$

then S is delay-independent stable.

Proof. Note first that the Metzler matrix in (33) coincides with $\sum_{k=0}^{m} \bar{A}_k$, where $\bar{A}_k$ are the matrices of the IPR of Theorem 2. Thus, if (33) is satisfied, then the IPR of S is stable, and thanks to Theorem 4 the original system S is delay-independent stable. □

Remark 2. *Notice that being $\Gamma(A_0) + \sum_{k=1}^{m} \Pi(A_k)$ a Metzler matrix, checking condition* (33) *does not require the explicit computation of the eigenvalues of the matrix, nor its characteristic polynomial. Indeed, an easy equivalent condition just requires checking the positivity of all the leading principal minors of $M = -\big(\Gamma(A_0) + \sum_{k=1}^{m} \Pi(A_k)\big)$. For a number of simple equivalent conditions for the Hurwitz stability of Metzler matrices we refer the reader to [1].*

Going beyond the results presented in [16], we now see how the condition of Theorem 5 can be formulated directly on the matrices of the original system, removing the need of explicitly computing its IPR in order to analyze its stability.

Theorem 6. *Consider a delay differential system S as in* (23)*. If*

$$\alpha\Big(\sum_{k=0}^{m} A_k\Big) < 0, \tag{34}$$

$$\alpha\Big(A_0^{\mathcal{M}} + \sum_{k=1}^{m} |A_k|\Big) < 0. \tag{35}$$

are both satisfied, then S is delay-independent stable.

Proof. Considering that $\Gamma(A_0)$ is

$$\begin{bmatrix} d(A_0) + (A_0 - d(A_0))^+ & (A_0 - d(A_0))^- \\ (A_0 - d(A_0))^- & d(A_0) + (A_0 - d(A_0))^+ \end{bmatrix} \tag{36}$$

and

$$\Pi(A_k) = \begin{bmatrix} A_k^+ & A_k^- \\ A_k^- & A_k^+ \end{bmatrix}, \quad k = 1, \ldots, m, \tag{37}$$

the Metzler matrix in (33) takes the following form

$$\Gamma(A_0) + \sum_{k=1}^{m} \Pi(A_k) = \begin{bmatrix} M & N \\ N & M \end{bmatrix}, \tag{38}$$

with

$$M = d(A_0) + (A_0 - d(A_0))^+ + \sum_{k=1}^{m} A_k^+, \tag{39}$$

$$N = (A_0 - d(A_0))^- + \sum_{k=1}^{m} A_k^-. \tag{40}$$

Thanks to (24), the spectrum of the matrix in (33) is

$$\sigma\Big(\sum_{k=0}^{m} A_k\Big) \cup \sigma\Big(A_0^{\mathcal{M}} + \sum_{k=1}^{m} |A_k|\Big), \tag{41}$$

so that the sufficient condition of Theorem 5 translates into the pair of conditions (34)–(35). □

Remark 3. *Note that condition* (34) *is also necessary for delay-independent stability, because it ensures the system stability in the case of zero-valued delays, and is invariant with respect to changes of coordinates. On the other hand, condition* (35) *is only sufficient and depends on changes of coordinates, a fact that can be exploited if a first test fails (see Example 4). We point out that condition* (35) *is analogous to the one presented in [8] when no distributed delays are considered. Moreover, since $A_0^{\mathcal{M}} + \sum_{k=1}^{m} |A_k|$ is a Metzler matrix, Remark 2 applies and hence checking* (35) *doesn't require the explicit computation of the spectrum of the involved matrix, nor its characteristic polynomial.*

In the next Section we will compare the previous result with similar stability conditions known in the literature. Among these, we will prove that it is less conservative with respect to a very well known norm-based stability condition.

4 Comparison with similar conditions of delay-independent stability

Decades of research have led to a vast amount of stability conditions for differential systems with multiple delays, both in the delay-dependent framework and delay-independent one. These conditions are based on different techniques: frequency sweeping [28], spectral analysis [29], Linear matrix inequalities for Lyapunov-based approaches [26, 30] and others (see [27]). These results refer to different cases such as commensurate or incommensurate delays, constant or time-varying delays, slowly or fast varying delays. Many stability tests rely on numerical computations and some have a not negligible computational complexity (particularly the necessary and sufficient ones). Coming to delay-independent stability, in [31] it has been shown that it is a NP-hard problem even for constant delays. This explains why a great attention has been devoted to sufficient only conditions. Indeed, the first sufficient condition was given in [32] for systems governed by equations of the type

$$\dot{x}_i(t) = a_{ii}x_i(t) + \sum_{j=0,\, j\neq i}^{n} a_{ij}x_j(t - \tau_j^i), \quad i = 1, \ldots, n, \tag{42}$$

(constant delays τ_j^i only on off-diagonal interactions). The stability condition is:

$$\alpha(\widetilde{A}) < 0, \quad \text{where } [\widetilde{A}]_{ij} = \begin{cases} a_{ii} & \text{if } i = j, \\ |a_{ij}| & \text{if } i \neq j \end{cases} \tag{43}$$

(actually, in [32] the condition was the *quasi-diagonal dominance* of the Metzler matrix $\widetilde{A}$, which is equivalent to (43)). Note that condition (35) when particularized to the class of systems (42) coincides with condition (43). In [20] and [21], for the case of single and constant delay, the following sufficient condition for delay-independent stability has been given

$$\mu_p(A_0) + \|A_1\|_p < 0 \tag{44}$$

where $\mu_p(A)$ is the logarithmic norm (or measure) of matrix A induced by the operator norm $\|A\|_p$, defined as:

$$\mu_p(A) = \lim_{\varepsilon \to 0^+} \frac{\|I + \varepsilon A\|_p - 1}{\varepsilon}.$$

The expression of $\mu_p(\cdot)$ can easily be computed for $p = 1, 2, \infty$:

$$\mu_1(A) = \max_{j=1\dots n} \left(a_{jj} + \sum_{i=1,\ i \neq j}^{n} |a_{ij}| \right),$$
$$\mu_2(A) = \frac{1}{2} \lambda_{max}(A^T + A),$$
$$\mu_\infty(A) = \max_{i=1\dots n} \left(a_{ii} + \sum_{j=1,\ j \neq i}^{n} |a_{ij}| \right),$$

while for $p \neq 1, 2, \infty$ its computation is numerically intractable [33]. The condition:

$$\mu_p(A_0) + \sum_{k=1}^{m} \|A_k\|_p < 0 \tag{45}$$

has been shown [34] to be sufficient for the stability of systems with multiple commensurate delays, although only for the case of $p = 2$. In [22] and [23] the same condition has been proven sufficient, for any p, also in the case of incommensurate and time-varying delays of any size, and therefore is a sufficient condition of delay-independent stability of the system (23).

In order to compare condition (35) of Theorem 6 with condition (45) we can exploit the following properties of the logarithmic norm and of the spectral abscissa

$$\begin{aligned} \alpha(M) \leq \mu_p(M) \leq \|M\|_p, \\ \mu_p(M + N) \leq \mu_p(M) + \mu_p(N) \end{aligned} \tag{46}$$

and get

$$\begin{aligned} \alpha\left(A_0^{\mathcal{M}} + \sum_{k=1}^{m} |A_k|\right) &\leq \mu_p\left(A_0^{\mathcal{M}} + \sum_{k=1}^{m} |A_k|\right) \\ &\leq \mu_p(A_0^{\mathcal{M}}) + \sum_{k=1}^{m} \mu_p(|A_k|) \\ &\leq \mu_p(A_0^{\mathcal{M}}) + \sum_{k=1}^{m} \|A_k\|_p. \end{aligned} \tag{47}$$

Since for $p = 1, \infty$ we have $\mu_p(A_0^{\mathcal{M}}) = \mu_p(A_0)$, from (47) we get

$$\alpha\left(A_0^{\mathcal{M}} + \sum_{k=1}^{m} |A_k|\right) \leq \mu_p(A_0) + \sum_{k=1}^{m} \|A_k\|_p, \quad p = 1, \infty. \tag{48}$$

The inequality (48) proves that condition (35) of Theorem 6 is less conservative than condition (45), at least for $p = 1, \infty$.

As a matter of fact, it is rather easy to find delay-independent stable systems which satisfy condition (35) and do not satisfy condition (45) for $p = 2$. Such examples are reported in Section 5.

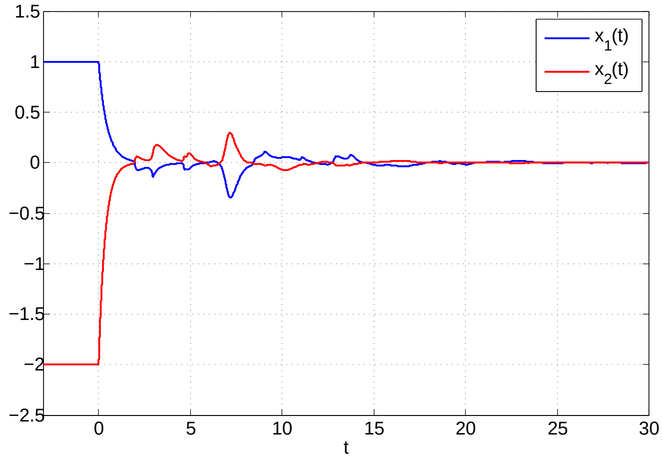

Fig. 2: Example 1: State trajectories of S.

5 Examples

5.1 *Example 1*

Consider an arbitrary delay differential system with three time-varying delays described by $S = \{\{A_k\}_0^3, B, C, D\}_{2,p,q}$, where:

$$A_0 = \begin{bmatrix} -2 & 0.1 \\ -0.3 & -4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.3 & -0.2 \\ 0 & -0.5 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -1 & 0.2 \\ -0.5 & -1 \end{bmatrix}, \; A_3 = \begin{bmatrix} -0.2 & 0 \\ -0.1 & -0.5 \end{bmatrix}.$$

The state trajectories of S with time-varying delays $\delta_1(t) = 4 + 3\sin(2t)$, $\delta_2(t) = 2 + \cos(0.1t)$, $\delta_3(t) = 5 + \sin(t)$, for an initial condition assigned as $x(t) = [1 \ -2]^T$ for $t \in [-3, 0]$ are reported in Figure 2. They illustrate that the system, for this set of delays, is asymptotically stable. Then, we investigate whether the asymptotic stability is delay-independent. First of all:

$$\sigma\left(A_0 + A_1 + A_2 + A_3\right) = \{-3.5365, -5.9635\}$$

and therefore the system S satisfies condition (34), which is necessary for delay-independent stability, and sufficient if completed with (35). The attempt of applying the sufficient conditions (45) for $p = 1, 2, \infty$ gives

$$\begin{aligned} \mu_1(A_0) + \|A_1\|_1 + \|A_2\|_1 + \|A_3\|_1 &= 1.0000 > 0, \\ \mu_2(A_0) + \|A_1\|_2 + \|A_2\|_2 + \|A_3\|_2 &= 0.2799 > 0, \\ \mu_\infty(A_0) + \|A_1\|_\infty + \|A_2\|_\infty + \|A_3\|_\infty &= 0.7000 > 0, \end{aligned}$$

so that no conclusions on the stability can be drawn.

On the other hand, applying the sufficient condition (35) obtained exploiting the IPR construction, one gets

$$\alpha\left(A_0^{\mathcal{M}} + |A_1| + |A_2| + |A_3|\right) = -0.2438 < 0 \tag{49}$$

which implies that S is delay-independent asymptotically stable. The same conclusion could have been attained computing the leading principal minors (see *Remark* 2) of $M = -A_0^{\mathcal{M}} - |A_1| - |A_2| - |A_3|$, yielding:

$$l_1(M) = 0.50 > 0, \quad l_2(M) = \det(M) = 0.55 > 0.$$

5.2 Example 2

In this second example, we study the delay-independent stability of a differential system with two time-varying delays, described by $S = \{\{A_0, A_1, A_2\}, B, C, D\}_{3,p,q}$ with:

$$A_0 = \begin{bmatrix} -9 & 0 & 0 \\ 8.4 & -7 & 11.2 \\ 0 & 0 & -5.8 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2.5 & 0 & 8 \\ 12 & 4.3 & -13 \\ 0.01 & 0 & 3 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 6.4 & 0 & 10.6 \\ -0.1 & 2.6 & 9 \\ 0 & 0 & 0.9 \end{bmatrix}. \tag{50}$$

The attempt of applying the sufficient conditions (45) fails for $p = 1, 2, \infty$, since

$$\begin{aligned} \mu_1(A_0) + \|A_1\|_1 + \|A_2\|_1 &= 49.900 > 0, \\ \mu_2(A_0) + \|A_1\|_2 + \|A_2\|_2 &= 33.938 > 0, \\ \mu_\infty(A_0) + \|A_1\|_\infty + \|A_2\|_\infty &= 58.900 > 0. \end{aligned}$$

Now we apply the sufficient conditions (34)-(35) derived in this work, obtaining:

$$\alpha\Big(A_0 + A_1 + A_2\Big) = -0.002 < 0,$$
$$\alpha\Big(A_0^{\mathcal{M}} + |A_1| + |A_2|\Big) = -0.002 < 0,$$

and this suffices to conclude that S is delay-independent asymptotically stable. Condition (35) could have been equivalently checked computing the leading principal minors of $M = -A_0^{\mathcal{M}} - |A_1| - |A_2|$, obtaining:

$$l_1(M) = 0.1 > 0, \quad l_2(M) = 0.01 > 0,$$
$$l_3(M) = \det(M) = 0.0004 > 0.$$

As stated above, we do not consider the LMI-based stability conditions to be comparable with the conditions proposed in this work, at least for simplicity and in terms of computational complexity, especially in the case of large systems with many delays. However, notice that for the system in this example the simple Lyapunov-Krasovskii LMI condition (see e.g. [26]):

$$\begin{bmatrix} PA_0 + A_0^T P + Q_1 + Q_2 & PA_1 & PA_2 \\ A_1^T P & -Q_1 & 0_{n\times n} \\ A_2^T P & 0_{n\times n} & -Q_2 \end{bmatrix} < 0_{3n\times 3n} \tag{51}$$

with P, Q_1, Q_2 $n \times n$ unknown positive definite matrices, turns out to be *Infeasible* using the CVX package for MATLAB with both *SDPT3* and *SeDuMi* solvers, and hence fails to prove the stability of S even in the case of constant delays.

To sum up, for the system in this example the criteria (45) and (51) fail to assess the stability, which has been proved using conditions (34) and (35).

5.3 Example 3

As a further comparison, consider the single constant delay system $S = \{\{A_0, A_1\}, B, C, D\}_{3,p,q}$, described by

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \delta_1) \tag{52}$$

with A_0, A_1 as in (50).

It is easy to see that S is delay-independent stable by Theorem 6, since:

$$\alpha(A_0 + A_1) = -2.700 < 0$$

and the leading principal minors of $M = -A_0^{\mathcal{M}} - |A_1|$ are all positive: $l_1(M) = 6.500$, $l_2(M) = 17.550$ and $l_3(M) = \det(M) = 48.924$. Moreover, Theorem 6 guarantees that the stability is robust with respect to possibly time-varying delays. We may wonder how a delay-dependent stability condition behaves with respect to system S. Indeed, a very well-known delay-dependent stability condition for delay differential systems as in (23), in the special case of constant delays, has been presented in [35], Corollary 1. The condition, applied to (52), requests that there exist $P_1 = P_1^T > 0$, P_2, P_3, $R_1 = R_1^T$ such that

$$\begin{bmatrix} \mathcal{A}^T P_2 + P_2^T \mathcal{A} & P_1 - P_2^T + \mathcal{A}^T P_3 & \delta_1 P_2^T A_1 \\ P_1 - P_2 + P_3^T \mathcal{A} & -P_3 - P_3^T + \delta_1 R_1 & \delta_1 P_3^T A_1 \\ \delta_1 A_1^T P_2 & \delta_1 A_1^T P_3 & -\delta_1 R_1 \end{bmatrix} < 0_{3n\times 3n} \tag{53}$$

where $\mathcal{A} = A_0 + A_1$. The LMI (53) is satisfied only for $\delta_1 \in (0, 0.2325]$, thus showing how this delay-dependent condition allows to deduce the stability of S only for a restricted interval of possible values of the (constant) delay. On the other hand, we have shown by means of Theorem 6 that system S is stable for all possible values of constant or time-varying delay.

5.4 Example 4

In the Stability Analysis section we have highlighted the fact that condition (35) depends on changes of coordinates. This fact suggests that if a given system does not satisfy (35) then a change of coordinates can be applied to the system in order to see whether, in the new basis, it satisfies (35), thus allowing to assess its delay-independent stability. The idea has been first suggested in [13] for the delay-free framework. This example illustrates such a case.

Consider $S = \{\{A_0, A_1, A_2\}, B, C, D\}_{4,p,q}$ with:

$$A_0 = \begin{bmatrix} -6 & 0.3 & 6 & -9 \\ 0 & -12 & 9 & 3 \\ 3 & 0 & -9 & 0.3 \\ -3 & 0 & 3 & -6 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.19 & 0 & 0.11 & -0.11 \\ 0 & 0.18 & 0 & 0 \\ 0.03 & -0.07 & 0.28 & -0.03 \\ -0.08 & -0.07 & 0.08 & 0.16 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.21 & 0 & 0.1 & -0.1 \\ 0 & 0.01 & 0 & 0 \\ 0.1 & -0.09 & 0.2 & -0.1 \\ 0.01 & -0.09 & -0.01 & 0.11 \end{bmatrix}.$$

We have that

$$\sigma\Big(\sum_{k=0}^{m} A_k\Big) = \{-0.066, -5.808, -14.053, -11.733\}$$

and then S satisfies the necessary condition (34). The sufficient conditions (45) fail for $p = 1, 2, \infty$ since:

$$\begin{aligned} \mu_1(A_0) + \|A_1\|_1 + \|A_2\|_1 &= 9.7900 > 0, \\ \mu_2(A_0) + \|A_1\|_2 + \|A_2\|_2 &= 1.1998 > 0, \\ \mu_\infty(A_0) + \|A_1\|_\infty + \|A_2\|_\infty &= 10.200 > 0, \end{aligned}$$

and so no conclusions on the stability can be drawn with the classical norm-based approach of [20–23].

Also the first attempt of applying the sufficient condition (35) fails to prove the stability

$$\alpha(A_0^{\mathcal{M}} + |A_1| + |A_2|) = 1.7339 > 0.$$

Following the idea in [13] for achieving stable IPRs of delay-free systems, we then apply a diagonalizing change of coordinates to A_0, which has real and distinct eigenvalues, so that $A_0^U = U^{-1}A_0U = d(A_0^U) = \mathrm{diag}_{i=1}^4(\lambda_i)$. Now we get

$$\alpha(A_0^U + |A_1^U| + |A_2^U|) = -0.0600 < 0$$

(note that A_0^U is diagonal, and therefore Metzler) and this is sufficient to prove the delay-independent asymptotic stability of S.

Recomputing the conditions (45) after the change of coordinates we get

$$\begin{aligned}\mu_1(A_0^U) + \|A_1^U\|_1 + \|A_2^U\|_1 &= 0.7233 > 0,\\ \mu_2(A_0^U) + \|A_1^U\|_2 + \|A_2^U\|_2 &= 0.3801 > 0,\\ \mu_\infty(A_0^U) + \|A_1^U\|_\infty + \|A_2^U\|_\infty &= 0.6964 > 0,\end{aligned}$$

and again no conclusions on the stability can be drawn.

Summing up, for the system in this example the criterion (45) fails to assess the stability, which has been proved using conditions (34) and (35) after a change of coordinates.

6 Conclusions and future work

In this paper the Internally Positive Representation of time-varying linear differential systems with multiple time-varying delays has been introduced. Moreover, in the special case of constant matrices, the IPR technique has been exploited to derive an easy to check sufficient condition which is proved to be less conservative with respect to the norm-based delay-independent stability tests presented in [20–23]. Future work will be devoted to the extension of the IPR technique to other classes of delay systems, possibly with similar stability results.

7 References

1 Farina, L., Rinaldi, S.: ‘Positive linear systems: theory and applications’ (John Wiley & Sons, 2011)

2 Luenberger, D.: ‘Introduction to dynamic systems: theory, models, and applications’ (Wiley, 1979)

3 Ait Rami, M.: ‘Stability analysis and synthesis for linear positive systems with time-varying delays’, in Bru, R., Romero-Vivó, S. (Eds.): ’Positive Systems’ (Springer, 2009), pp. 205–215

4 Haddad, W.M., Chellaboina, V.: ‘Stability theory for nonnegative and compartmental dynamical systems with time delay’, *Systems & Control Letters*, 2004, **51**, pp. 355–361

5 Kaczorek, T.: ‘Stability of positive continuous-time linear systems with delays”, *Proc. Eur. Control Conf. (ECC)*, 2009, pp. 1610–1613

6 Liu, X., Yu, W., Wang, L.: ‘Stability analysis for continuous-time positive systems with time-varying delays’, *IEEE Trans. Autom. Control*, 2010, **55**, pp. 1024–1028

7 Liu, X., Lam, J.: ‘Relationships between asymptotic stability and exponential stability of positive delay systems’, *Int. J. General Systems*, 2013, **42**, pp. 224–238

8 Ngoc, P.H.A.: ‘Stability of positive differential systems with delay’, *IEEE Trans. Autom. Control*, 2013, **58**, pp. 203–209

9 Mazenc, F., Malisoff, M.: ‘Stability analysis for time-varying systems with delay using linear Lyapunov functionals and a positive systems approach’, *IEEE Trans. Autom. Control*, 2016, **61**, pp. 771–776

10 Germani, A., Manes, C., Palumbo, P.: ‘State space representation of a class of MIMO systems via positive systems’, *Proc. 46th Conf. Decision and Control (CDC)*, 2007, pp. 476–481

11 Germani, A., Manes, C., Palumbo, P.: ‘Representation of a class of MIMO systems via Internally Positive Realization’, *Eur. J. Control*, 2010, **16**, pp. 291–304

12 Cacace, F., Germani, A., Manes, C., Setola, R.: ‘A new approach to the Internally Positive Representation of linear MIMO systems’, *IEEE Trans. Autom. Control*, 2012, **57**, pp. 119–134

13 Cacace, F., Farina, L., Germani, A., Manes, C.: ‘Internally Positive Representation of a class of continuous time systems’, *IEEE Trans. Autom. Control*, 2012, **57**, pp. 3158–3163

14 Cacace, F., Germani, A., Manes, C.: ‘Stable Internally Positive Representations of continuous time systems’, *IEEE Trans. Autom. Control*, 2014, **59**, pp. 1048–1053

15 Cacace, F., Germani, A., Manes, C.: ‘A new approach to design interval observers for linear systems’, *IEEE Trans. Autom. Control*, 2015, **60**, pp. 1665–1670

16 Conte, F., De Iuliis, V., Manes, C.: ‘Internally Positive Representations and stability analysis of linear delay systems with multiple time-varying delays’, in Cacace, F., et al. (Eds.): ‘Positive Systems: Theory and Applications’ (Springer, 2017), pp. 81–93

17 De Iuliis, V., Germani, A., Manes, C.: ‘Internally Positive Representations and stability analysis of linear difference systems with multiple delays’, *IFAC-PapersOnLine*, 2017, **50**, pp. 3099–3104

18 De Iuliis, V., Germani, A., Manes, C.: ‘Internally Positive Representations and Stability Analysis of Coupled Differential–Difference Systems with Time-Varying Delays’, *IEEE Trans. Autom. Control*, 2018 (online), doi:10.1109/TAC.2018.2866467

19 De Iuliis, V., Germani, A., Manes, C.: ‘Stability analysis of linear delay systems via Internally Positive Representations: an overview’, in Valmorbida, G. et al.: ‘Incorporating constraints on the Analysis of Delay and Distributed Parameter Systems’ (Springer, 2019 (to appear))

20 Mori, T., Fukuma, N., Kuwahara, M.: ‘Simple stability criteria for single and composite linear systems with time delays’, *Int. J. of Control*, 1981, **34**, pp. 1175–1184

21 Mori, T., Fukuma, N., Kuwahara, M.: ‘On an estimate of the decay rate for stable linear delay systems’, *Int. J. of Control*, 1982, **36**, pp. 95–97

22 Schoen, G.M., Geering, H.P.: ‘On stability of time delay systems’, *Proc. 31st Annual Allerton Conf. on Comm., Control and Comp.*, 1993, pp. 1058–1060

23 Schoen, G.M.: ‘Stability and stabilization of time-delay systems’. PhD thesis, Swiss Federal Institute of Technology, Zurich, Switzerland, 1995

24 Hale, J., Lunel, S.M.V.: ‘Introduction to Functional Differential Equations’ (Springer-Verlag, 1993)

25 Kaczorek, T.: ‘Positive 1D and 2D Systems’ (Springer-Verlag, 2001).

26 Fridman, E.: ‘Introduction to Time-Delay Systems’ (Birkhäuser Basel, 2014)

27 Gu, K., Niculescu, S.I.: ‘Survey on recent results in the stability and control of time-delay systems’, *J. of Dynamic Systems, Measurement, and Control*, 2003, **125**, pp. 158–165

28 Chen, J., Latchman, H.A.: ‘Frequency sweeping tests for stability independent of delay’, *IEEE Trans. Autom. Control*, 1995, **40**, pp. 1640–1645

29 Michiels, W., Niculescu, S.I.: ‘Stability, Control, and Computation for Time-Delay Systems: An Eigenvalue-Based Approach’ (SIAM, 2014)

30 Gu, K., Kharitonov, V.L., Chen, J.: ‘Stability of Time-Delay Systems’ (Birkhäuser Basel, 2003)

31 Toker, O., Özba, H.: ‘Complexity issues in robust stability of linear delay-differential systems’, *Mathematics of Control, Signals and Systems*, 1996, **9**, pp. 386–400

32 Lewis, R., Anderson, B.: ‘Necessary and sufficient conditions for delay-independent stability of linear autonomous systems’, *IEEE Trans. Autom. Control*, 1980, **25**, pp. 735–739

33 Hendrickx, J.M., Olshevsky, A.: ‘Matrix p-norms are NP-hard to approximate if $p \neq 1,2,\infty$’, *SIAM J. on Matrix Analysis and Applications*, 2010, **31**, pp. 2802–2812

34 Wang, S.-S., Lee, C.-H., Hung, T.-H.: ‘New stability analysis of system with multiple time delays’, *Proc. Am. Control Conf. (ACC)*, 1991, pp. 1703–1704

35 Fridman, E.: ‘New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems’, *Systems & Control Letters*, 2001, **43**, pp. 309–319