Renewal properties of the $d = 1$ Ising model

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Abstract

We consider the $d = 1$ Ising model with Kac potentials at inverse temperature $\beta > 1$ where mean field predicts a phase transition with two possible equilibrium magnetization $\pm m_\beta$, $m_\beta > 0$. We show that when the Kac scaling parameter γ is sufficiently small typical spin configurations are described (via a coarse graining) by an infinite sequence of successive plus and minus intervals where the empirical magnetization is "close" to m_β and respectively $-m_\beta$. We prove that the corresponding marginal of the unique DLR measure is a renewal process.

1 Introduction

In this paper we consider the Ising model with a ferromagnetic Kac potential. The formal hamiltonian is

$$H_\gamma(\sigma) = -\frac{1}{2}\sum_{x\neq y} J_\gamma(x,y)\sigma(x)\sigma(y) \tag{1.1}$$

$\sigma(x) \in \{-1, 1\}$ is the spin at site $x \in \mathbb{Z}^d$ and

$$J_\gamma(x,y) = \gamma c_\gamma J(\gamma|y-x|) \tag{1.2}$$

where $J(r) \geq 0$ is a smooth probability density supported by $r \leq 1$; c_γ a normalization constant such that $\sum_{y\neq x} J_\gamma(x,y) = 1$, $c_\gamma \to 1$ as $\gamma \to 0$.

The mean field version of the model has a free energy

$$f_\beta(m) = -\frac{m^2}{2} - \frac{S(m)}{\beta} \tag{1.3}$$

where $S(m) = -\frac{1-m}{2}\log\frac{1-m}{2} - \frac{1+m}{2}\log\frac{1+m}{2}$ is the entropy when the magnetization is m. When $\beta > 1$, $f_\beta(m)$ is a symmetric two wells function with minima at $\pm m_\beta$, where m_β is the positive solution of the mean field equation

$$m_\beta = \tanh\{\beta m_\beta\}, \quad \beta > 1 \tag{1.4}$$

This suggests that when γ is small the typical spin configurations should be close to m_β or $-m_\beta$. Indeed when $d \geq 2$ it is proved ([9], [5], [4]) that for any γ small enough the plus DLR measure (obained as limit of Gibbs measures with plus boundary conditions) has typical configurations described by a "sea" where the "local magnetization" is close to $+m_\beta$ with small and rare islands where the local magnetization is close to $-m_\beta$. The spin flip of the above picture describes the minus DLR measure.

In this paper we study the $d = 1$ case. In one dimension with finite range interaction and any inverse temperature β there is a unique DLR measure, thus typical configurations cannot be as in $d \geq 2$ predominantly close to m_β or to $-m_\beta$ and therefore they must be close to alternating $+m_\beta$ and $-m_\beta$ intervals. The problem has been first studied in [8]. The original idea in [8] was to see this in the context of metastability, see for instance [15], [3], namely to relate the typical spin configurations to the trajectories of a random walk in the two well potentials $\gamma^{-1}f_\beta(m)$. For γ small the two wells are separated by a very high barrier and typically the random walk stays close to the bottom of a well with small fluctuations and it will very rarely jump to the other well; it will then keep doing that for ever, namely alternating from one well to the other.

Since the barrier height scales as γ^{-1} the waiting time for jumps from one well to the other scales as $e^{c\gamma^{-1}}$ while the time it takes for the actual jump is much smaller as it scales as γ^{-1}.

In a first version of our paper we have considered the problem in the presence of a magnetic field h. The magnetic field modifies the two wells potential so that the intervals with magnetization m_β and $-m_\beta$ get different lengths. The paper however was getting too long and we decided to restrict to $h = 0$. We did not examine the case when the magnetic field is random: the length of the intervals then scales as γ^{-2} (see [6] and [7]) in contrast with the exponential behaviour at $h = 0$. It is an open interesting question whether the renewal properties that we prove in this paper extend to the case with the random magnetic field.

The relation with metastability however is not straightforward because we are studying a Gibbs process while metastability is usually framed in the context of Markov processes. It is true that in $d = 1$ Gibbs processes (with finite range interaction as in our case) are Markov but the transition probability of the latter is not simply related to the Hamiltonian of the Gibbs measure. Indeed to get to the transition probability one needs to know spectral properties of the transfer matrix, in particular the eigenvectors of the maximal eigenvalue. This is the approach used by Kac et al. ([10], [11], [12]) to derive the van der Waals theory from systems with Kac potentials, however to carry out the whole program along these lines looks maybe possible but not easy at all.

An alternative approach, used in [8], and which goes back to Lebowitz and Penrose,

[14], shifts the mathematical context from the spectral analysis of the transfer matrix to a variational problem wth a non local free energy functional. The tunnelling problem for the corresponding non local "penalty functional" has been studied in [1], [2] in the $d \geq 2$ case. Going back to Lebowitz and Penrose, the reduction to a variational problem comes from a coarse graining which gives rise to the free energy functional

$$\mathcal{F}(m) = \int dr \{ f_\beta(m(r)) - f_\beta(m_\beta) \} + \int \int dr dr' J(|r - r'|)[m(r) - m(r')]^2 \tag{1.5}$$

Lebowitz and Penrose used coarse graining to describe spin configurations in terms of a sequence of successive intervals in $\mathbb{Z}$ where a plus interval is followed by an interface interval then by a minus interval, then by another interface interval and this structure is repeated endlessly. In the plus and minus intervals the empirical magnetization of the spins is close to m_β, respectively $-m_\beta$, the interface intervals separate the pluses from the minuses. A precise definition is given in the next section. Calling μ_γ the DLR measure (at inverse temperature $\beta > 1$) in [8] it was proved that:

- The probability $\mu_\gamma[B]$ of the event B that the origin belongs to an interface interval vanishes as $\gamma \to 0$.
- In the set B^c call ℓ_γ the length of the plus or minus interval which contains the origin, then for any $\delta > 0$

  $$\lim_{\gamma \to 0} \mu_\gamma \left[e^{\gamma^{-1}(\bar{f} - \delta)} \leq \ell_\gamma \leq e^{\gamma^{-1}(\bar{f} + \delta)} \middle| B^c \right] = 1 \tag{1.6}$$

  where

  $$\bar{f} = \inf_{m(r) \to \pm m_\beta \; as \; r \to \pm\infty} \mathcal{F}(m) \tag{1.7}$$

- Properly normalized the distribution of ℓ_γ converges to an exponential distribution of mean 1 as $\gamma \to 0$ and the lengths of successive intervals become independent.

Purpose of this paper is to investigate the structure of the plus, minus and interface intervals when γ is small but without taking the limit $\gamma \to 0$. We call Ω the space of sequences of such intervals and P_γ the measure on Ω induced by μ_γ. Namely P_γ is obtained from μ_γ by integrating over all spin configurations which give rise to the same sequence and since we are just interested in the configurations in Ω P_γ retains exactly the information we are interested in. The disadvantage when going from μ_γ to P_γ is that we loose the nice property of μ_γ that its conditional probabilities have finite range dependence on the conditioning.

P_γ however has a very nice structure, in fact (and this is the main result in this paper) (Ω, P_γ) is a renewal process. More precisely we show that it is possible to add to the configurations in Ω sequences of "renewal marks" so that the new space (Ω^*, P^*_γ) has for all γ small enough the following properties:

- (Ω^*, P^*_γ) is a renewal process where the renewal property occurs when a renewal mark appears.

- The marginal of P^*_γ on Ω, namely disregarding the marks, is P_γ.

Thus if we want to compute the probability of a given finite sequence of intervals we go to the space (Ω^*, P^*_γ) and look for the first mark appearing before our sequence, what happens earlier is not relevant in computing the probability.

The paper is organized as follows. In Section 2 we define the parameters relevant to our analysis and in Section 3 we present our main results and discuss their physical interpretation. Section 4 presents an outline of how the proof of the statements of Section 3 is organized. The subsequent sections and appendices are devoted to detail the full proofs.

2 Plus, minus and interface intervals

In this section we make precise the definition of plus, minus and interface intervals. We will use throughout the paper four main lengths: γ^{-1}, which is the interaction length, $\ell^-_\gamma = \delta\gamma^{-1}$, $\delta \in (0,1)$, and $\ell^+_\gamma = \gamma^{-(1+\alpha)}$, $\alpha \in (0, 1/2)$, which are the lengths used in the definition of plus and minus phases, finally $\gamma^{-1/2}$ which is the coarse graining length. We will also use a parameter $\zeta > 0$ in the definition of the plus and minus phases. The relation between δ and ζ is as follows: ζ can be any positive number $\le \zeta^*$, $\zeta^* > 0$ suitably small; then for any such ζ there is $\delta^* = \delta^*(\zeta)$ positive and we can take any $\delta \le \delta^*$, (all that independently of γ). For the definition of δ^* and ζ^* we refer to Chapter 6 of [16].

We want the four lengths commensurable, namely ℓ^+_γ an integer multiple of γ^{-1} which in turns should be an integer multiple of ℓ^-_γ which should be an integer multiple of $\gamma^{-1/2}$. This could be achieved by taking integer parts but to have simpler notation we suppose $\alpha = 1/4$, $\gamma \in \{2^{-4n}, n \in \mathbb{N}\}$ and $\delta \in \{2^{-k}, k \in \mathbb{N}\}$. For γ small enough the above commensurability requests are fulfilled and we will tacitly suppose that γ is as small as needed.

We use the standard notation in lattice systems, namely if $\Delta \subset \mathbb{Z}$ then $\sigma_\Delta = \{\sigma(x), x \in \Delta\}$, if $\Delta \cap \Lambda = \emptyset$, $\sigma_\Delta, \sigma_\Lambda = \{\sigma(x), x \in \Delta \cup \Lambda\}$ and with Δ and Λ as above, Δ finite,

$$H_\gamma(\sigma_\Delta|\sigma_\Lambda) = -\sum_{\{x,y\}\in\Delta\cup\Lambda:\{x,y\}\cap\Delta\neq\emptyset} J_\gamma(x,y)\sigma(x)\sigma(y) \tag{2.1}$$

which is the energy of the spins in Δ in interaction with those in Λ.

We call

$$C^\pm_i = \{x \in [i\ell^\pm_\gamma, (i+1)\ell^\pm_\gamma)\} \tag{2.2}$$

and shorthand

$$s_i = \sigma_{C^+_i}, \quad \underline{s} = \{s_i, i \in \mathbb{Z}\} \tag{2.3}$$

s_i will be called the i-th block spin. Given a spin configuration σ we first define

$$\eta_i = \pm 1 \text{ if } \Big|\frac{1}{\ell_\gamma^-}\sum_{y\in C_i^-}(\sigma(y)\mp m_\beta)\Big| \le \zeta, \quad \eta_i = 0 \text{ otherwise} \tag{2.4}$$

and then

$$\theta_i = \pm 1 \text{ if } \eta_j \equiv \pm 1 \text{ for all } C_j^- \subset C_i^+, \quad \theta_i = 0 \text{ otherwise} \tag{2.5}$$

$$\Theta_i = \pm 1 \text{ if } \theta_j \equiv \pm 1,\ j = i-1, i, i+1, \quad \Theta_i(\underline{s}) = 0 \text{ otherwise} \tag{2.6}$$

The plus [minus] phase in a configuration σ is the set of points where $\Theta = 1$ [$\Theta = -1$]. As we are going to see plus and minus intervals are defined by allowing fluctuations in the plus and minus phases.

Definition. [Plus, minus and interface intervals]

- An interval $[m, n]$ is a $+-$ interface if $\Theta_{m-1} = 1$, $\Theta_{n+1} = -1$ and $\Theta_i = 0$ for all $i \in [m, n]$.
- An interval $[m, n]$ is a $-+$ interface if $\Theta_{m-1} = -1$, $\Theta_{n+1} = 1$ and $\Theta_i = 0$ for all $i \in [m, n]$.
- $[m, n]$ is a plus interval if there is a $+-$ interface which starts at $n+1$ and a $-+$ interface which ends at $m-1$.
- $[m, n]$ is a minus interval if there is a $-+$ interface which starts at $n+1$ and a $+-$ interface which ends at $m-1$.

Thus a plus interval $[m, n]$ starts at m with $\Theta_m = 1$ and ends at n with $\Theta_n = 1$ while $\Theta_i \ge 0$ at all $i \in (m, n)$, moreover $[m, n]$ is maximal with such properties. There could be plus intervals made of a singleton, i.e. with $m = n$. Minus intervals are defined symmetrically.

A spin configuration determines a partition of $\mathbb{Z}$ whose elements are the plus, minus and interface intervals. The partition is denoted by ω, the atoms of ω are denoted by $\omega_{\ell,m}$, $\ell \in \mathbb{Z}$, $m \in \{1, .., 4\}$, using the following convention. $\{\omega_{\ell,1}, \ell \in \mathbb{Z}\}$, is the collection of all the plus intervals, $\{\omega_{\ell,2}, \ell \in \mathbb{Z}\}$, of the $+-$ interfaces, $\{\omega_{\ell,3}, \ell \in \mathbb{Z}\}$, of the minus intervals and finally $\{\omega_{\ell,4}, \ell \in \mathbb{Z}\}$, of the $-+$ interfaces. The atoms are ordered from left to right in the sense that $\omega_{\ell,m} < \omega_{\ell',m'}$ if $\ell < \ell'$ or $\ell = \ell', m < m'$. The same partition ω can arise from two sequences $\omega_{\ell,m}$ and $\omega'_{\ell,m}$ if there is ℓ_0 so that for all ℓ and m, $\omega_{\ell,m} = \omega'_{\ell+\ell_0,m}$, we then call $\omega_{\ell,m}$ and $\omega'_{\ell,m}$ equivalent and denote by ω classes of equivalence and by Ω the space of all such equivalent classes.

It is sometimes convenient to describe a configuration ω as the collection $(x_{\ell,m})$ of the positions of the left points of the atoms $\omega_{\ell,m}$ of ω. Of course we are interested in equivalent classes, where $(x_{\ell,m})$ is equivalent to $(x'_{\ell,m})$ if there is $n \in \mathbb{Z}$ so that $x_{\ell,m} = x'_{\ell+n,m}$ for all

ℓ and m. Local sets in Ω are denoted by $X^* = \{x^*_{\ell,m}, (\ell',m') \le (\ell,m) \le (\ell'',m'')\}$ and are the set of all $(x_{\ell,m})$ such that (for a suitable choice in the equivalence class of $(x_{\ell,m})$) $x_{\ell,m} = x^*_{\ell,m}, (\ell',m') \le (\ell,m) \le (\ell'',m'')$.

Calling ψ the map from the space of spin configurations $\{-1,1\}^{\mathbb{Z}}$ (such that $\omega_{\ell,m}$ is well defined) to the space Ω, we define on the local sets X^*

$$P_\gamma[X^*] = \mu_\gamma\big[\underline{s} : \psi(\underline{s}) \in X^*\big] \tag{2.7}$$

where μ_γ is the DLR measure with hamiltonian (1.1) at inverse temperature $\beta > 1$. P_γ is then extended to the σ-algebra generated by the local events.

3 Main results

The main results in this paper are (1) the proof that the DLR process of the plus, minus and interface intervals is a renewal process and (2) a characterization of the thermodynamics of the system. We state here the main theorems which will then be proved in the next sections, leaving the more technical details to the appendices.

3.1 The renewal process

The space Ω is made of quadruples $(\omega_{\ell,1},..,\omega_{\ell,4})$ one after the other and indexed by $\ell \in \mathbb{Z}$. We will prove that their distribution P_γ can be realized by suitably clustering together finitely many quadruples and giving independent weights to each cluster. Thus the process starts anew every time that a new cluster appears.

We start by looking only at the lengths of the quadruples without caring about their location. We denote by $\underline{u}$ finite sequences of quadruples of integers: $\underline{u} = \{u_{\ell,m}, \ell = 1,..,k, m = 1,..,4\}$, $k \in \mathbb{N}$, and define

$$\mathcal{R} = \{\underline{u} : u_{1,1} \ge 3;\ u_{\ell,1} \ge 1,\ u_{\ell,2} \ge 2,\ u_{\ell,3} \ge 1,\ u_{\ell,4} \ge 2\} \tag{3.1}$$

The length of $\underline{u}$ is defined as

$$|\underline{u}| = \sum_{\ell,m} u_{\ell,m} \tag{3.2}$$

and by (3.1) if $\underline{u} \in \mathcal{R}$ then $|\underline{u}| \in [8,\infty)$.

In Section 8 we will introduce a probability on $\mathcal{R}$ (denoted by w_{λ_γ} for reasons explained in the Remark below) with the following properties (see Theorem 10 and Theorem 12):

Properties of w_{λ_γ}.

- $w_{\lambda_\gamma}(\underline{u}) > 0$ for all $\underline{u} \in \mathcal{R}$ and $\sum_{\underline{u}} w_{\lambda_\gamma}(\underline{u}) = 1$ (being a probability).

- There are $c > 0$ and $\delta_\gamma > 0$ so that for all $R > 0$

$$\sum_{\underline{u}:|\underline{u}|\geq R} w_{\lambda_\gamma}(\underline{u}) \leq c e^{-\delta_\gamma R} \tag{3.3}$$

- The first moment is finite and denoted by:

$$\alpha_\gamma^{-1} := \sum_{\underline{u}\in\mathcal{R}} w_{\lambda_\gamma}(\underline{u})|\underline{u}| \tag{3.4}$$

Remark. Observe that the convergence of the series in (3.4) follows from (3.3) because

$$\sum_{\underline{u}\in\mathcal{R}} w_{\lambda_\gamma}(\underline{u})|\underline{u}| = \sum_{n\geq 1} \sum_{\underline{u}:|\underline{u}|=n} w_{\lambda_\gamma}(\underline{u}) \tag{3.5}$$

We will see that $\log \alpha_\gamma$ and $\log \delta_\gamma$ are proportional to $-\gamma^{-1}$.

We will first define weights $w(\underline{u})$ on $\mathcal{R}$ which are determined by the statistical weight of the plus, minus and interface intervals. We will then call $w_\lambda(\underline{u}) := e^{-\lambda|\underline{u}|}w(\underline{u})$, $\lambda > 0$, and prove that there is a unique value λ_γ of λ for which $w_\lambda(\underline{u})$ is a probability. We will see that $w_{\lambda_\gamma}(\underline{u})$ satisfies the properties listed above and that λ_γ is related to the thermodynamic pressure of the system.

We next denote by W_γ the probability on $\mathcal{R}^{\mathbb{Z}}$ product of the w_{λ_γ}. The elements of $\mathcal{R}^{\mathbb{Z}}$ are denoted by $(\underline{u}_i, i \in \mathbb{Z})$. We will often use in the sequel the following classical theorem (which, for the readers convenience, is proved in Appendix G):

Theorem 1. *There are c' and δ'_γ positive so that for any positive integer n*

$$\Big| W_\gamma \Big[\{ (\underline{u}_i)_{i\in\mathbb{Z}} : \text{ there is } k \text{ so that } \sum_{i=1}^{k} |\underline{u}_i| = n \} \Big] - \alpha_\gamma \Big| \leq c' e^{-\delta'_\gamma n} \tag{3.6}$$

where α_γ is defined in (3.4).

We may regard the elements of $\mathcal{R}^{\mathbb{Z}}$ as sequences of rods of lengths $|\underline{u}_i|$ with internal structure $\underline{u}_i$: our next step is "to put them" on $\mathbb{Z}$.

Definitions and notation. The space Ω^* and the map $\phi : \Omega^* \to \Omega$.

Consider the sequence of pairs

$$(\underline{u}_i, x_i)_{i\in\mathbb{Z}} : \underline{u}_i \in \mathcal{R}, x_i \in \mathbb{Z}, x_{i+1} - x_i = |\underline{u}_i| \tag{3.7}$$

x_i is interpreted as the position of the rod $\underline{u}_i$, more precisely of its left endpoint; the rods are placed consecutively, one after the other by the last condition in (3.7). The labelling is not important and we call equivalent $(\underline{u}_i, x_i)_{i\in\mathbb{Z}}$ and $(\underline{u}'_i, x'_i)_{i\in\mathbb{Z}}$ if there is $n \in \mathbb{Z}$ such that $(\underline{u}'_i, x'_i) = (\underline{u}_{i+n}, x_{i+n})$ for all i. Notice that an element $(\underline{u}_i, x_i)_{i\in\mathbb{Z}}$ is determined by the sequence of the lengths of the rod and by the position of only one of the rods, as the other positions are fixed by the constraints $x_{i+1} - x_i = |\underline{u}_i|$. We call Ω^* the space of all $(\underline{u}_i, x_i)_{i\in\mathbb{Z}}$ (identifying equivalent elements).

There is a "natural map" $\phi : \Omega^* \to \Omega$ defined by looking in an element of Ω^* only at the sequence $\omega_{\ell,m}$ of the internal structures of the rods in $(\underline{u}_i, x_i)_{i\in\mathbb{Z}}$. The range of ϕ is actually a subset $\Omega^{\geq 3}$ of Ω of all ω such that $\{\ell : |\omega_{\ell,1}| \geq 3\}$ is a doubly infinite sequence (the DLR measure of the complement is equal to 0). Alternatively we may recover the elements of Ω^* from those of $\Omega^{\geq 3}$ by putting "marks" in the set $\{\omega_{\ell,1} : |\omega_{\ell,1}| \geq 3\}$: then a mark at $(\ell, 1)$ selects a site x defined as the left endpoint of $\omega_{\ell,1}$. The sets $\{x_i\}$ of such marks are then identified with the sets $\{x_i\}$ in $(\underline{u}_i, x_i)_{i\in\mathbb{Z}}$, the specification $\underline{u}_i$ being then the lengths of the $\omega_{\ell,m}$ between x_i and x_{i+1}.

In agreement with the notation of Section 2 given an element $(\underline{u}_i, x_i)_{i\in\mathbb{Z}} \in \Omega^*$ we denote by $\{x_{\ell,m}, \ell \in \mathbb{Z}, m = 1, .., 4\}$ the set of all the left endpoints of the partition $\omega = \phi((\underline{u}_i, x_i)_{i\in\mathbb{Z}})$.

We define a "renewal probability" P^*_γ on Ω^* by specifying first the probability of the "cylinders" and then extending this to the minimal σ-algebra generated by the cylinders.

The specification of a cylinder set is a sequence $(\underline{v}_i, y_i)_{i\in[1,k]}$, $k \geq 1$, such that $\underline{v}_i \in \mathcal{R}$, $i = 1, .., k$ and $y_{i+1} - y_i = |\underline{v}_i|$, $i = 1, .., k-1$. The cylinder with such specification is:

$$C_{(\underline{v}_i, y_i)_{i\in[1,k]}} = \Big\{ (\underline{u}_i, x_i)_{i\in\mathbb{Z}} \in \Omega^* : \ (\underline{u}_i, x_i) = (\underline{v}_i, y_i), i = 1, .., k \Big\} \tag{3.8}$$

Since Ω^* is defined modulo equivalence this is the same as

$$C_{(\underline{v}_i, y_i)_{i\in[1,k]}} = \Big\{ (\underline{u}_i, x_i)_{i\in\mathbb{Z}} \in \Omega^* : \ (\underline{u}_{n+i}, x_{n+i}) = (\underline{v}_i, y_i), i = 1, .., k \Big\} \tag{3.9}$$

for any $n \in \mathbb{Z}$. Physically $C_{(\underline{v}_i, y_i)_{i\in[1,k]}}$ is the event where there is a rod $\underline{v}_1$ at y_1 and the next $k-1$ rods have specification $\underline{v}_2, .., \underline{v}_k$.

We next define the P^*_γ probability of a cylinder as

$$P^*_\gamma \Big[C_{(\underline{v}_i, y_i)_{i\in[1,k]}} \Big] = \alpha_\gamma \prod_{i=1}^{k} w_{\lambda_\gamma}(\underline{u}_i) \tag{3.10}$$

The probability P^*_γ on Ω^* is finally defined as the probability which extends (3.10) to the minimal σ-algebra generated by the cylinders.

In Section 9 we will prove the following theorem about the renewal property of the distribution of the plus, minus and interface intervals:

Theorem 2 (Renewal property of the DLR measure)**.** *The inverse image under ϕ of any local set X^* in Ω is a countable union of disjoint cylinders C_i in Ω^*. It is therefore in the σ-algebra where P^*_γ is defined and*

$$P_\gamma[X^*] = \sum_i P^*_\gamma[C_i], \quad \phi^{-1} X^* = \bigcup_i C_i \tag{3.11}$$

As a consequence P_γ (which is defined on the σ-algebra generated by the local sets) satisfies:

$$P_\gamma = P^*_\gamma \, \phi^{-1} \tag{3.12}$$

3.2 Thermodynamics of the model

Besides the thermodynamic pressure p_γ (of our Ising model with hamiltonian (1.1)) we will introduce several other pressures. In particular we consider here the pressure p^+_γ relative to plus intervals (namely obtained by restricting to plus intervals), see the next section for a precise definition. We will prove at the end of Section 8 that:

Theorem 3. *The thermodynamic pressure p_γ is equal to*

$$p_\gamma = p^+_\gamma + \frac{\lambda_\gamma}{\beta \ell^+_\gamma} \tag{3.13}$$

where λ_γ is the positive parameter introduced in the Remark at the beginning of this section (existence of λ_γ and properties of $w_{\lambda_\gamma}(\underline{u})$ are proved in Section 8). Moreover if $Z^{\rm pbc}_{\Lambda_n}$ is the partition function in the region $\Lambda_n = [-n, n]$ (in block spin variables) with periodic boundary conditions, then

$$\Big| \frac{Z^{\rm pbc}_{\Lambda_n}}{e^{\beta p_\gamma (2n+1)\ell^+_\gamma}} - 1 \Big| \le c'' e^{-\delta''_\gamma n} \tag{3.14}$$

where c'' and δ''_γ are positive constants.

Thus by Theorem 3 the renewal property which is strictly related to λ_γ has a thermodynamic meaning in terms of the pressure difference $p_\gamma - p^+_\gamma$. λ_γ however has also a thermodynamic interpretation in terms of "surface tension". In fact in (8.5) it is shown that $\big|\lambda_\gamma - \epsilon_\gamma\big| \le c_\gamma \epsilon^2_\gamma$ and in (8.8) that $\big|\alpha_\gamma - \frac{\epsilon_\gamma}{2}\big| \le c' \epsilon^2_\gamma$ where $\epsilon_\gamma \approx e^{-c\gamma^{-1}}$, $c > 0$, while $c_\gamma \approx e^{c'\gamma^{-b}}$, $b \in (\frac{1}{2}, 1)$; thus for γ small λ_γ is "essentially" equal to ϵ_γ. It is ϵ_γ which has a thermodynamic meaning in terms of "surface tension". Since there is no phase transition this cannot be taken literally and what we mean by surface tension here is the free energy cost of replacing a plus interval by a plus and a minus interval separated in the middle by an interface interval. The precise definition is given in (7.12) and it is proved in (7.15) that the surface tension is equal to $-\frac{1}{\beta}\log \epsilon_\gamma$. The physical reason behind the fact that the difference $p_\gamma - p^+_\gamma$ is related to the surface tension is that the "true" pressure p_γ is larger than the plus pressure p^+_γ $(= p^-_\gamma)$ as it gets an entropic contribution by alternating plus and minus intervals of all possible lengths and the cost of inserting interface intervals is given by ϵ_γ.

The limit $\gamma \to 0$ is also interesting, in fact we have

$$\lim_{\gamma\to 0} \gamma \log \lambda_\gamma = \lim_{\gamma \to 0} \gamma \log \epsilon_\gamma = -\beta \bar f \tag{3.15}$$

$\bar f$, which has been defined in (1.7), is the surface tension of the mesoscopic system derived in the limit $\gamma \to 0$ and described by the free energy functional $\mathcal{F}(m)$, see (1.5). $\bar f$ is the free energy of the instanton profile which optimizes the free energy of profiles which are asymptotically $\pm m_\beta$.

4 Outline of proofs

In this section we give an outline of how the proof of the statements of Section 3 is organized.

- In Section 5 we prove that to leading order the dependence from the boundary conditions of the partition function restricted to plus intervals factorizes into a product $e^{F_\gamma(s_0)}e^{F_\gamma(s_{n+1})}$ where s_0 is the (block spin) left boundary condition and s_{n+1} the right boundary condition. By the spin flip symmetry the analogous statement holds for the minus intervals. The precise statement is given in Theorem 4 whose proof is quite technical: it is outlined in Appendix A while the details are given in Appendix B and Appendix C.

- In Section 6 we use the analysis on the plus and minus intervals to show that the partition function (with periodic boundary conditions) can be written as a sum of products of weights (denoted by $w(\underline{u})$) plus remainders which are negligible, as proved in Appendix D.

- The next step is to control the weights $w(\underline{u})$ which involve the partition functions restricted to interface intervals. This is done in Section 7, more technical details are proved in Appendix E.

- In Section 8 we go back to the expression for the partition function in terms of products of weights $w(\underline{u})$ obtained in Section 6 which is estimated using Laplace transform as in a Tauberian theorem. We thus write $w_\lambda(\underline{u}) := e^{-\lambda|\underline{u}|}w(\underline{u})$ and prove that there is a unique value of λ, denoted by λ_γ, such that $w_{\lambda_\gamma}(\underline{u})$ has the properties stated in Section 3, see Theorem 12. Technical details of its proof are given in Appendix F. We then conclude the proof of Theorem 3 at the end of Section 8.

- The proof of the renewal property of the plus, minus and interface interval as stated in Theorem 2 is given in Section 9.

5 Statistical weight of plus and minus intervals

As already mentioned $w_{\lambda_\gamma}(\underline{u})$ will be defined in terms of the statistical weight of plus, minus and interface intervals. In this section we estimate the statistical weight of the plus and minus intervals.

The statistical weight $Z^+_{\gamma,n}$ of a plus interval of length $u = n + 2 \geq 3$ is

$$Z^+_{\gamma,n}(s_0, s_{n+1}) := \sum_{s_1,..,s_n} \mathbf{1}_{\theta_1=\theta_n=1}\mathbf{1}_{\Theta_i\geq 0, i=2,..,n-1}e^{-\beta H_\gamma(s_1,..,s_n|s_0,s_{n+1})} \tag{5.1}$$

where $\theta(s_0) = \theta(s_{n+1}) = 1$. It is the statistical weight of a plus interval whose endpoints are 0 and $n+1$, by translation invariance the definition applies to all plus intervals and by the spin flip symmetry also to the minus intervals.

Theorem 4. *For all γ small enough there are p^+_γ, $F^{(k)}_\gamma(s)$, $k = 1,2$, and $G^{(1)}_{\gamma,n}(s', s'')$, so that*

$$Z^+_{\gamma,n}(s_0, s_{n+1}) = e^{\beta(\ell^+_\gamma n)p^+_\gamma + F^{(1)}_\gamma(s_0) + F^{(2)}_\gamma(s_{n+1}) + G^{(1)}_{\gamma,n}(s_0,s_{n+1})} \tag{5.2}$$

where $n \geq 1$ and

$$F^{(k)}_\gamma(s) \leq c\gamma^{-1}, k = 1,2; \quad |G^{(1)}_{\gamma,n}(s_0, s_{n+1})| \leq ae^{-b_0\gamma\ell^+_\gamma n} \tag{5.3}$$

c, a, b positive constants independent of γ.

Remarks.

- By the spin flip symmetry the statistical weight of a minus interval is equal to

$$Z^-_{\gamma,n}(s_0, s_{n+1}) = e^{\beta(\ell^+_\gamma n)p^-_\gamma + F^{(3)}_\gamma(s_0) + F^{(4)}_\gamma(s_{n+1}) + G^{(3)}_{\gamma,n}(s_0,s_{n+1})} \tag{5.4}$$

  where $\theta(s_0) = \theta(s_{n+1}) = -1$ and

$$\begin{aligned} &p^-_\gamma = p^+_\gamma, \quad F^{(3)}_\gamma(s_0) = F^{(1)}_\gamma(-s_0), \quad F^{(4)}_\gamma(s_{n+1}) = F^{(2)}_\gamma(-s_{n+1}), \\ &G^{(3)}_{\gamma,n}(s_0, s_{n+1}) = G^{(1)}_{\gamma,n}(-s_0, -s_{n+1}) \end{aligned} \tag{5.5}$$

- The renewal property is a consequence of (5.2). In fact if we neglect $G^{(1)}_{\gamma,n}(s_0, s_{n+1})$ the dependence of the partition function on s_0 and s_{n+1} factorizes. With this in mind we will write

$$e^{G^{(1)}_{\gamma,n}(s_0,s_{n+1})} = \{e^{G^{(1)}_{\gamma,n}(s_0,s_{n+1})} - e^{A_{\gamma,n}}\} + e^{A_{\gamma,n}}, \quad A_{\gamma,n} = \min_{s_0,s_{n+1}} G^{(1)}_{\gamma,n}(s_0, s_{n+1}) \tag{5.6}$$

  When we take the term $e^{A_{\gamma,n}}$ we decouple right and left, when we take the curly bracket we get a small contribution (as n will typically be large).

- While n is the length of the interval in the block spin representation $\ell^+_\gamma n$ is the length of the same interval in the original spin variables ($\ell^+_\gamma n = \gamma^{-1-\alpha} n$) so that p^+_γ is the pressure in the plus ensemble. As we shall see the true pressure has an additional contribution of entropic origin, due to the alternating presence of plus and minus intervals.

- The equality (5.2) reminds of the expression obtained for the partition function using the transfer matrix method. The role of our boundary terms $F^{(i)}_\gamma$ are played in the transfer matrix approach by the right and left eigenvectors of the transfer matrix relative to the maximal eigenvalue. The latter is identified to the exponential of βp,

p the pressure, and it gives rise (as in our case) to a term $e^{-\beta p|\Lambda|}$, $|\Lambda|$ the volume of the region where the partition function is computed. Besides these terms there is an exponentially small correction due to the spectral gap of the transfer matrix which in our case is played by $e^{G^{(i)}_\gamma}$.

The use of the transfer matrix method with Kac potentials has been central in the analysis of the van der Waals phase transition, see the original papers by Kac et al., [10],[11],[12]. Its application is however very delicate because in the limit $\gamma \to 0$ the maximal eigenvalue becomes degenerate. In our case however this does not happen because of the restriction to plus intervals, nonetheless the use of transfer matrix techniques is not straightforward in our case and we will instead use the Lebowitz-Penrose coarse graining technique.

- The equality (5.4) also reminds of the cluster expansion estimates where the influence of the boundaries comes from the clusters which touch the boundaries. The main contribution is due to the small clusters which give rise to surface corrections as our terms $F^{(i)}_\gamma$. Clusters which connect the two boundaries are long and thus exponentially small, in our case they correspond to the term $e^{G^{(i)}_\gamma}$. We do not know whether cluster expansion works in our case, we have therefore followed a different route based on the Dobrushin uniqueness approach.

In Appendix A we outline the main steps of the proof of Theorem 4, details are then given in Appendix B and Appendix C.

6 The weights w_{λ_γ}

The pressure p^+_γ defined in Theorem 4 is clearly a lower bound to the true pressure p_γ because it is the pressure of restricted partition functions (to plus intervals). The true pressure p_γ gets instead an additional entropic contribution by alternating plus and minus intervals of all possible lengths. However each transition from a plus to a minus interval (and viceversa) involves an interface interval whose statistical weight is very small (as $\gamma \to 0$) as we shall see in the next section. We will indeed prove that $p_\gamma - p^+_\gamma = \frac{\lambda_\gamma}{\beta \ell^+_\gamma} > 0$, with "the correction" λ_γ to the pressure being exponentially small in γ^{-1}. λ_γ is the same parameter introduced in Section 3, this is due to the fact that the computation of the partition functions will involve the weights $w_{\lambda_\gamma}(\underline{u})$.

In general the thermodynamic pressure is better approximated by partitions functions with periodic boundary conditions. Let then Λ_n be the interval $[-n,n]$, in block-spin variables, while in the original spin variables it is the interval $[-n\ell^+_\gamma,(n+1)\ell^+_\gamma)$, $\underline{s} = (s_{-n},...,s_n)$ a block spin configuration in Λ_n and $H^{\rm pbc}_\gamma(\underline{s})$ the hamiltonian with Λ_n-periodic boundary conditions, our aim is to study the partition function

$$Z_{\Lambda_n}^{\rm pbc} := \sum_{\underline{s}} e^{-\beta H_\gamma^{\rm pbc}(\underline{s})} \tag{6.1}$$

We shall prove that $Z_{\Lambda_n}^{\rm pbc} = e^{\beta p_\gamma^+(2n+1)\ell_\gamma^+ + \lambda_\gamma(2n+1)}$ plus corrections which are exponentially small in the sense of (3.14).

As mentioned above the key point will be to reduce the computation of $Z_{\Lambda_n}^{\rm pbc}$ to a partition function with the weights $w_{\lambda_\gamma}(\underline{u})$, to this end we need to re-introduce the partitions into plus-minus and interface intervals in the context of the "torus" Λ_n.

Given $\underline{s}$ we denote by $\underline{s}'$ its Λ_n-periodic extension to $\mathbb{Z}$, i.e. $s'_{i+k(2n+1)} = s_i$, $i \in \Lambda_n$, $k \in \mathbb{Z}$. By restricting i to Λ_n, $\theta(i, \underline{s}')$ and $\Theta(i, \underline{s}')$ are the phase indicators in Λ_n. As a difference with the infinite volume, the sets $\mathcal{X}^0$ and $\mathcal{X}^\pm$ of configurations $\underline{s}$, where for all $i \in \Lambda_n$: $\Theta_i = 0$, respectively $\Theta_i \geq 0$ (and somewhere $= 1$) and $\Theta_i \leq 0$ (and somewhere $= -1$), have non zero probability. We denote by g the complement of $\mathcal{X}^0 \cup \mathcal{X}^+ \cup \mathcal{X}^-$ and write

$$Z_{\Lambda_n}^{\rm pbc} = Z_{\Lambda_n}^{\rm pbc,g} + Z_{\Lambda_n}^{{\rm pbc},\mathcal{X}^0} + Z_{\Lambda_n}^{{\rm pbc},\mathcal{X}^+} + Z_{\Lambda_n}^{{\rm pbc},\mathcal{X}^-} \tag{6.2}$$

where on the right hand side we have written the partition functions restricted to configurations in g and respectively $\mathcal{X}^0$, $\mathcal{X}^+$ and $\mathcal{X}^-$. We will prove in Appendix D that $Z_{\Lambda_n}^{{\rm pbc},\mathcal{X}^0}$, $Z_{\Lambda_n}^{{\rm pbc},\mathcal{X}^+}$ and $Z_{\Lambda_n}^{{\rm pbc},\mathcal{X}^-}$ are bounded by $c_\gamma n e^{\beta p_\gamma^+ \ell_\gamma^+(2n+1)}$ and are therefore negligible with respect to the estimate $e^{\beta p_\gamma^+(2n+1)\ell_\gamma^+ + \lambda_\gamma(2n+1)}$ that we want to prove for $Z_{\Lambda_n}^{\rm pbc}$. We will thus study in the sequel of this section $Z_{\Lambda_n}^{\rm pbc,g}$. Since $\underline{s}'$ is periodic with period $2n+1$ if $\underline{s} \in g$ then $\underline{s}'$ has infinitely many sites where $\Theta(i) = \pm 1$ and therefore it defines a partition ω' of $\mathbb{Z}$ into finite atoms $\omega'_{\ell,m}$. Like $\underline{s}'$ also the partition ω' is periodic, namely if $\omega'_{\ell,m}$ is an atom of ω' then also its translates by $2n+1$ are atoms of ω'.

Call $\omega'_\ell = \omega'_{\ell,1} \cup \cdots \cup \omega'_{\ell,4}$ and let ℓ_1 be such that $\omega'_{\ell_1} \ni -n$, let x' be the leftmost point of ω'_{ℓ_1} and $\omega'_{\ell_1} \dots \omega'_{\ell_k}$ the successive atoms of ω' which cover the interval $[x', x'')$, $x'' = x' + 2n + 1$ (by periodicity in fact at x'' starts the atom of ω' which is ω'_{ℓ_1} shifted by $2n+1$). We then denote by $\omega = \{\omega_{i,m} = f(\omega'_{\ell_i,m}), m = 1,..,4; i = 1,..,k\}$ the partition of Λ_n image of the atoms $\omega'_{\ell_i,m}$ and call $x = f(x')$ its "starting point" (the map $f : \mathbb{Z} \to \Lambda_n$ has been defined above). We finally denote by $\underline{u} = \{u_{\ell,m}, \ell = 1,..,k; m = 1,..,4\}$ the lengths of the intervals $\omega_{\ell,m}$ and observe that the pairs $(x, \underline{u})$ are in one to one correspondence with the partitions ω. Calling $u_\ell = u_{\ell,1} + \cdots + u_{\ell,4}$, $|\underline{u}| = \sum_\ell u_\ell$, the set of possible values of $(x, \underline{u})$ is:

$$A = \{(x, \underline{u}) :\; x \in [-n, n],\; |\underline{u}| = 2n+1; x + u_1 - 1 \geq n+1, \text{ when x} > -\text{n}\} \tag{6.3}$$

the condition $x + u_1 - 1 \geq n+1$, when x $> -$n corresponds to the condition that $\omega'_{\ell_1} \ni -n$. We remark that the sequences $\underline{u} \in A$ are not necessarily in $\mathcal{R}$ namely $u_{1,1}$ could be smaller than 3, see (3.1).

Recalling that

$$Z_{\Lambda_n}^{\rm pbc,g} := \sum_{\underline{s} \in g} e^{-\beta H_\gamma^{\rm pbc}(\underline{s})} \tag{6.4}$$

we then have:

$$Z^{\rm pbc,g}_{\Lambda_n} := \sum_{(x,\underline{u})\in A} \sum_{\underline{s}\to(x,\underline{u})} e^{-\beta H^{\rm pbc}_\gamma(\underline{s})} \tag{6.5}$$

where for $(x,\underline{u}) \in A$ we denote by $\underline{s} \to (x,\underline{u})$ the set of all $\underline{s} \in g$ such that the induced partition ω gives $(x,\underline{u})$. We sum over the interiors of the plus and minus intervals using (5.2) and then over the interface intervals, and get

$$Z^{\rm pbc,g}_{\Lambda_n} = e^{\beta p^+_\gamma(2n+1)\ell^+_\gamma} \sum_{(x,\underline{u})\in A} \sum_{\{s_{\ell,m}\}} \prod_{\ell=1}^{k(\underline{u})} \prod_{m=1}^{4} e^{V^m_{u_{\ell,m}}(s_{\ell,m},s_{\ell,m+1})} \tag{6.6}$$

where $k(\underline{u}) = k$ if $\underline{u} = (u_{\ell,m}, \ell = 1,..,k)$. In (6.6) $\{s_{\ell,m}\}$ are block-spins and the sum is over all $s_{\ell,m}, \ell = 1,..,k(\underline{u}), m = 1,..,4$, such that $\theta(s_{\ell,m}) = 1$, $m = 1,2$; $\theta(s_{\ell,m}) = -1$, $m = 3,4$; If $u_{\ell,1} = 1$ then $s_{\ell,1} = s_{\ell,2}$ and $s_{\ell,3} = s_{\ell,4}$ if $u_{\ell,3} = 1$. Moreover $s_{\ell,5} := s_{\ell+1,1}$ and $s_{k(\underline{u})+1,1} = s_{1,1}$. $s_{\ell,m}$, $m = 1,2$, are the block-spins of the leftmost and rightmost blocks in $\omega_{\ell,1}$ whose length is $u_{\ell,1}$. Analogously $s_{\ell,m}$, $m = 3,4$, are the block-spins of the leftmost and rightmost blocks in $\omega_{\ell,3}$ whose length is $u_{\ell,3}$. It remains to define the two body potentials $V^m_u(s,s')$.

The potentials $V^m_u(s,s')$, $m = 2,4$, $u \geq 2$, take into account the contribution of the interfaces. Let $\theta(s_0) = 1$, $\theta(s_{u+1}) = -1$ and

$$\begin{aligned} Z^{+,-}_{\gamma,u}(s_0,s_{u+1}) &= \sum_{s_{[1,u]}} \mathbf{1}_{\theta_1=1,\theta_u=-1}\mathbf{1}_{\Theta_i=0,i=1,..,u} e^{-\beta H_\gamma(s_{[1,u]}|s_0,s_{u+1})} \\ Z^{-,+}_{\gamma,u}(-s_0,-s_{u+1}) &= \sum_{s_{[1,u]}} \mathbf{1}_{\theta_1=-1,\theta_u=1}\mathbf{1}_{\Theta_i=0,i=1,..,u} e^{-\beta H_\gamma(s_{[1,u]}|-s_0,-s_{u+1})} \end{aligned} \tag{6.7}$$

Then with $F^{(i)}_\gamma(s)$ as in Theorem 4

$$e^{V^2_u(s,s')} = \frac{Z^{+,-}_{\gamma,u}(s,s')e^{-\beta[H_\gamma(s)+H_\gamma(s')]+F^{(2)}_\gamma(s)+F^{(3)}_\gamma(s')}}{e^{\beta p^+_\gamma(u+2)\ell^+_\gamma}} \tag{6.8}$$

Similarly

$$e^{V^4_u(s,s')} = \frac{Z^{-,+}_{\gamma,u}(s,s')e^{-\beta[H_\gamma(s)+H_\gamma(s')]+F^{(4)}_\gamma(s)+F^{(1)}_\gamma(s')}}{e^{\beta p^+_\gamma(u+2)\ell^+_\gamma}} \tag{6.9}$$

When the length u of a plus interval is ≥ 3 the sum over the spins in its interior is bounded by the r.h.s. of (5.2). The factor $e^{\beta p^+_\gamma \ell^+_\gamma(u-2)}$ contributes to the first factor on the r.h.s. of (6.6); the factor $e^{\beta(F^{(1)}(s)+F^{(2)}(s))}$ is put in $V^m_u(\cdot,\cdot)$, $m = 2,4$. Then $V^1_u(s,s') = G^{(1)}_{\gamma,u}(s,s')$.

When $u = 1,2$ (5.2) does not apply and we need to subtract in $V^1_u(s,s')1_{u\leq 2}$ the terms $F^{(1)}_\gamma(s)$, $F^{(2)}_\gamma(s)$ that we have put in $V^m_u(\cdot,\cdot)$, $m = 2,4$. In this case in fact there is no interior in the plus interval: when $u = 1$ the contribution of the plus interval is 1, when $u = 2$ it is $e^{-\beta W_\gamma(s|s')}$ where $W_\gamma(s|s')$ is the interaction between two contiguous block spins.

The definition of $V_u^1(s,s')$ is therefore:

$$\begin{aligned} V_u^1(s,s') &= \Big(\beta H_\gamma(s) + \beta p_\gamma^+ \ell_\gamma^+ - F_\gamma^{(1)}(s) - F_\gamma^{(2)}(s)\Big)\mathbf{1}_{u=1,s=s'} \\ &+ \Big(-\beta W_\gamma(s|s') - F_\gamma^{(1)}(s) - F_\gamma^{(2)}(s')\Big)\mathbf{1}_{u=2} + G_{\gamma,u}^{(1)}(s,s')\mathbf{1}_{u\ge 3} \end{aligned} \tag{6.10}$$

where the extra terms in (6.10) compensate with the corresponding factors in the definition of $V_u^2(s,s')$ and $V_u^4(s,s')$.

Similarly

$$\begin{aligned} V_u^3(s,s') &= \Big(\beta H_\gamma(s) + \beta p_\gamma^+ \ell_\gamma^+ - F_\gamma^{(3)}(s) - F_\gamma^{(4)}(s)\Big)\mathbf{1}_{u=1,s=s'} \\ &+ \Big(-\beta W_\gamma(s|s') - F_\gamma^{(3)}(s) - F_\gamma^{(4)}(s')\Big)\mathbf{1}_{u=2} + G_{\gamma,u}^{(3)}(s,s')\mathbf{1}_{u\ge 3} \end{aligned} \tag{6.11}$$

Recalling the strategy outlined in the second remark after Theorem 4 whenever $u \ge 3$ we write each $e^{V_u^1(s,s')}$ as

$$e^{V_u^1(s,s')} = \{e^{G_{\gamma,u-2}^{(1)}(s,s')} - e^{A_{\gamma,u-2}^{(1)}}\} + e^{A_{\gamma,u-2}^{(1)}} \tag{6.12}$$

By the definition of $A_{\gamma,u}^{(1)}$ (see (5.6)) the curly bracket is non negative. We have

$$Z_{\Lambda_n}^{\mathrm{pbc,g}} = Z_{\Lambda_n}^{\mathrm{pbc,gb}} + Z_{\Lambda_n}^{\mathrm{pbc,gg}} \tag{6.13}$$

where $Z_{\Lambda_n}^{\mathrm{pbc,gb}}$ comes from the case where for all ℓ either $u_{\ell,1} < 3$ or we take the curly bracket. Thus

$$Z_{\Lambda_n}^{\mathrm{pbc,gb}} = e^{\beta p_\gamma^+(2n+1)\ell_\gamma^+} \sum_{(x,\underline{u})\in A} w^{(b)}(\underline{u}) \tag{6.14}$$

where

$$\begin{aligned} w^{(b)}(\underline{u}) &= \sum_{\{s_{\ell,m}\}} \{\prod_{\ell=1}^{k(\underline{u})} \prod_{m=2}^{4} e^{V^m_{u_{\ell,m}}(s_{\ell,m},s_{\ell,m+1})}\} \prod_{\ell=1}^{k(\underline{u})} K_{u_{\ell,1}}(s_{\ell,1},s_{\ell,2}) \\ K_u(s,s') &= e^{V_u^1}\mathbf{1}_{u\le 2} + \{e^{G_{\gamma,u-2}^{(1)}(s,s')} - e^{A_{\gamma,u-2}^{(1)}}\}\mathbf{1}_{u\ge 3} \end{aligned} \tag{6.15}$$

The sum over $\{s_{\ell,m}\}$ in (6.15) is as in (6.6), $s_{k(\underline{u}),5} = s_{1,1}$. We will prove in Appendix D, see (D.12), that there exists $\zeta_\gamma > 0$ so that:

$$\lim_{n\to\infty} \frac{Z_{\Lambda_n}^{\mathrm{pbc,gb}}}{e^{\beta p_\gamma(2n+1)\ell_\gamma^+ - \zeta_\gamma(2n+1)}} = 0$$

so that also the contribution of $Z_{\Lambda_n}^{\mathrm{pbc,gb}}$ is negligible in the proof of (3.14), we can therefore restrict to $Z_{\Lambda_n}^{\mathrm{pbc,gg}}$.

To write explicitly $Z_{\Lambda_n}^{\mathrm{pbc,gg}}$ we replace A by A^*, where

$$A^* = \{(x,\underline{u}_1,..,\underline{u}_k),\ i=1,\dots,k:\ \underline{u}_i \in \mathcal{R}; \sum_i |\underline{u}_i| = 2n+1; x+|\underline{u}_1|-1 \ge n+1,\ \text{when x} > -\text{n}\} \tag{6.16}$$

and $\mathcal{R}$ is defined in (3.1). Namely we start from a sequence $\underline{u}$ as in (6.3), and call $\mathcal{L}$ the set of labels ℓ s.t. $u_{\ell,1} \geq 3$ and the last term in (6.12) is taken. We then group together the $u_{\ell',m}$ with ℓ' in between successive values of ℓ in $\mathcal{L}$ and each of these groups is a $\underline{u}_i$ in (6.16). With these notation

$$Z_{\Lambda_n}^{\text{pbc,gg}} = e^{\beta p_\gamma^+ (2n+1)\ell_\gamma^+} \sum_k \sum_{(x,\underline{u}_1,,,\underline{u}_k)\in A^*} \prod_{i=1}^k w(\underline{u}_i) \tag{6.17}$$

where, similarly to (6.15),

$$w(\underline{u}) = \sum_{\{s_{\ell,m}\}} \{\prod_{\ell=1}^{k(\underline{u})} \prod_{m=2}^{4} e^{V^m_{u_{\ell,m}}(s_{\ell,m},s_{\ell,m+1})}\} e^{A^{(1)}_{\gamma,u_{1,1}}} \prod_{\ell=2}^{k(\underline{u})} K_{u_{\ell,1}}(s_{\ell,1},s_{\ell,2}) \tag{6.18}$$

For any $\lambda > 0$ we define:

$$w_\lambda(\underline{u}) = e^{-\lambda|\underline{u}|} w(\underline{u}), \quad |\underline{u}| = \sum_{\ell,m} u_{\ell,m} \tag{6.19}$$

and get

$$Z_{\Lambda_n}^{\text{pbc,gg}} = e^{(2n+1)[\beta p_\gamma^+ \ell_\gamma^+ + \lambda]} \sum_k \sum_{(x,\underline{u}_1,,,\underline{u}_k)\in A^*} \prod_{i=1}^k w_\lambda(\underline{u}_i) \tag{6.20}$$

While (6.20) holds for all $\lambda > 0$ it greatly simplifies if we choose $\lambda = \lambda_\gamma$, $\lambda_\gamma > 0$ such that $w_{\lambda_\gamma}(\underline{u})$ satisfies the properties listed in Section 3. We will thus continue the estimate of $Z_{\Lambda_n}^{\text{pbc,gg}}$ at the end of Section 8, after proving the existence of λ_γ and properties of $w_{\lambda_\gamma}(\underline{u})$.

7 Statistical weight of interfaces

In Theorem 4 we have estimated the statistical weight of a plus interval defined as the partition function $Z^+_{\gamma,n}(s,s')$. Analogously the statistical weight of a $+-$ interface is defined as the partition function $Z^{+,-}_{\gamma,n}(s,s')$ of (6.7) ($-+$ interfaces are just $+-$ interfaces after spin flip, so that in the sequel we just refer to the former). In Appendix E we will prove the following bound:

Theorem 5. *There are $c_0 > 0$, a positive integer n_0, $b \in (\frac{1}{2}, 1)$ and $c_b > 0$ so that for any s_0 and s_{n+1} such that $\theta(s_0) = 1 = -\theta(s_{n+1})$*

$$Z^{+,-}_{\gamma,n}(s_0, s_{n+1}) \leq Z^{++}_{\gamma,n}(s_0, -s_{n+1})\, e^{-\gamma^{-1}\beta \bar{f} + c_b\gamma^{-b} - \gamma^{-1}c_0(n-n_0)\mathbf{1}_{n\geq n_0}} \tag{7.1}$$

$\bar{f}$ is the free energy of an instanton, see (1.7), and $Z^{++}_{\gamma,n}$ is defined in (B.2).

Moreover

$$Z^{+,-}_{\gamma,2}(s_0, s_3) \geq e^{-\gamma^{-1}\beta\bar{f} - c_b\gamma^{-b}} Z^{++}_{\gamma,2}(s_0, -s_3) \tag{7.2}$$

Observe that $Z^{++}_{\gamma,n}$ has still a "dangerous" dependence on the boundary conditions because the interaction with the boundaries is exponential in γ^{-1}, this will be settled in the next subsection. In a final subsection we will relate the statistical weight of the interface to the surface tension introduced at the end of Section 3.

7.1 Bounds on the potentials $V_u^m(s,s')$, $m=2,4$

We will first bound

$$\epsilon_\gamma(n) := \sum_{s:\theta(s)=1}\sum_{s':\theta(s')=-1} e^{V_n^2(s,s')} \tag{7.3}$$

and

$$\epsilon_\gamma := \sum_n \epsilon_\gamma(n) \tag{7.4}$$

Recalling (6.8)

$$\epsilon_\gamma(n) = \sum_{s:\theta(s)=1}\sum_{s':\theta(s')=-1} e^{-\beta p_\gamma^+(n+2)\ell_\gamma^+} Z^{+,-}_{\gamma,n}(s,s') e^{-\beta[H_\gamma(s)+H_\gamma(s')]+F^{(2)}(s)+F^{(3)}(s')} \tag{7.5}$$

ϵ_γ will be referred to as the statistical weight of the interface.

Theorem 6. *There are $b\in(\frac{1}{2},1)$ and $c_b>0$ so that for all γ small enough*

$$\epsilon_\gamma(n) \le e^{-\gamma^{-1}\beta\bar f + c_b\gamma^{-b} - \gamma^{-1}c_0(n-n_0)\mathbf{1}_{n\ge n_0}} \tag{7.6}$$

$$e^{-\gamma^{-1}\beta\bar f - c_b\gamma^{-b}} \le \epsilon_\gamma \le e^{-\gamma^{-1}\beta\bar f + c_b\gamma^{-b}} \tag{7.7}$$

Proof. Using (7.1) we get from (7.5)

$$\begin{aligned}
&\epsilon_\gamma(n) \le \omega_{\gamma,n}\epsilon^*_\gamma(n), \quad \omega_{\gamma,n} = e^{-\gamma^{-1}\beta\bar f + c_b\gamma^{-b} - \gamma^{-1}c_0(n-n_0)\mathbf{1}_{n\ge n_0}}\\
&\epsilon^*_\gamma(n) := e^{-\beta p_\gamma^+(n+2)\ell_\gamma^+}\Big\{\sum_{\theta(s)=\theta(s')=1} Z^{++}_{\gamma,n}(s,s') e^{-\beta[H_\gamma(s)+H_\gamma(s')]+F^{(2)}(s)+F^{(1)}(s')}\Big\}
\end{aligned} \tag{7.8}$$

We are going to prove that

$$\epsilon^*_\gamma(n) \le 1 \tag{7.9}$$

which then proves (7.6).

Let $m>1$, eventually $m\to\infty$, then by Theorem 4

$$\begin{aligned}
1 &= Z^+_{\gamma,m}(s_{-m-1},s)e^{-\{\beta p_\gamma^+\ell_\gamma^+ m + F_\gamma^{(1)}(s_{-m-1}) + F_\gamma^{(2)}(s) + G^{(1)}_{\gamma,m}(s_{-m-1},s\}}\\
&= Z^+_{\gamma,m}(s',s_{n+m+2})e^{-\{\beta p_\gamma^+\ell_\gamma^+ m + F_\gamma^{(1)}(s') + F_\gamma^{(2)}(s_{n+m+2}) + G^{(1)}_{\gamma,m}(s',s_{n+m+2})\}}
\end{aligned}$$

hence

$$
\begin{aligned}
\epsilon^*_\gamma(n) \quad &\le \quad e^{-\beta p^+_\gamma (n+2m+2)\ell^+_\gamma - F^{(2)}_\gamma(s_{n+m+2}) - F^{(1)}_\gamma(s_{-m-1}) + 2\|G^{(1)}_{\gamma,m}\|_\infty} \\
&\times \quad \sum_{s_{[-m,n+m+1]}} e^{-\beta H_\gamma(s_{[-m,n+m+1]}|s_{-m-1},s_{m+n+2})} \\
&\times \quad \mathbf{1}_{\Theta_i\ge 0, i\in[-m,n+m+1]} \mathbf{1}_{\theta_{-m}=\theta_{n+m+1}=1} \mathbf{1}_{\Theta_i=1,i\in[0,n+1]}
\end{aligned}
$$

The last factor is bounded by $Z^+_{\gamma,n+2m+2}(s_{-m-1}, s_{m+n+2})$, (having dropped the characteristic function $\mathbf{1}_{\Theta_i=1,i\in[0,n+1]}$). Then using again Theorem 4:

$$
\epsilon^*_\gamma(n) \le e^{2\|G^{(1)}_{\gamma,m}\|_\infty + \|G^{(1)}_{\gamma,2m+n+2}\|_\infty}
$$

Letting $m \to \infty$ we prove (7.9) so that the upper bound in (7.6) and (7.7) is proved.

By (7.2)

$$
\epsilon_\gamma(2) \ge e^{-\beta p^+_\gamma 4\ell^+_\gamma} \omega'_{\gamma,2} \times \{ \sum_{\theta(s)=\theta(s')=1} Z^{++}_{\gamma,2}(s,s') e^{-\beta[H_\gamma(s)+H_\gamma(s')]+F^{(2)}(s)+F^{(1)}(s')} \}
$$

with $\omega'_{\gamma,2}$ obtained from $\omega_{\gamma,2}$ by changing the sign of the term $c_b\gamma^{-b}$. As before and with $n \equiv 2$ below,

$$
\begin{aligned}
\epsilon_\gamma(2) \quad &\ge \quad \omega'_{\gamma,2} e^{-\beta p^+_\gamma (n+2m+2)\ell^+_\gamma - F^{(2)}_\gamma(s_{n+m+2}) - F^{(1)}_\gamma(s_{-m-1}) - 2\|G^{(1)}_{\gamma,m}\|_\infty} \\
&\times \quad \sum_{s_{[-m,n+m+1]}} e^{-\beta H_\gamma(s_{[-m,m+n+1]}|s_{-m-1},s_{m+n+2})} \mathbf{1}_{\Theta_i\ge 0,i\in[-m,n+m+1]} \mathbf{1}_{\theta_{-m}=\theta_{n+m+1}=1} \mathbf{1}_{\Theta_i=1,i\in[0,3]}
\end{aligned}
$$

Call $\mu^+(s_{[-m,m+n+1]}|s_{-m-1}, s_{m+n+2})$ the Gibbs measure in the interval $[-m, m+n+1]$ conditioned to $\{\Theta_i \ge 0, i \in [-m, n+m+1]; \theta_{-m} = \theta_{n+m+1} = 1\}$ and with boundary conditions s_{-m-1}, s_{m+n+2}. Then the last sum is equal to

$$
Z^+(s_{-m-1}, s_{m+n+2}) \mu^+ \Big[\{\Theta_i = 1, i \in [0,3]\} \Big]
$$

and

$$
\mu^+ \Big[\{\Theta_i = 1, i \in [0,3]\} \Big] \ge 1 - 4e^{-3b'\gamma^{-1}}, \quad b' > 0
$$

which follows from (B.8). The lower bound is proved because $\epsilon_\gamma \ge \epsilon_\gamma(2)$. □

We shall see, in Appendix F (see (F.13)–(F.14) and Subsection F.3), that the factors $e^{V^3_u(s,s')}$ and $K_u(s,s')$ in (6.15) can be bounded uniformly in s and s' if $u \ge 3$ (but not when $u < 3$). Thus if the intervals to the right and left of an interface have both $u \ge 3$, then, after using the above uniform bounds, we are in the setup of Theorem 6 and get an estimate for the contribution of the interface. In the other cases we proceed as follows.

We call two interfaces "connected" if they are separated by an interval of length $u < 3$. We then consider a maximal sequence of connected interfaces which starts on the left from the interface interval (ℓ', m') and ends on the right with (ℓ'', m''), calling (ℓ, m) the intervals (interface or not) in between the two extremal ones. Thus the lengths $u_{\ell,m}$, $m \in \{1, 3\}$, of the intervals between (ℓ', m') and (ℓ'', m'') have all length < 3 while the two intervals one to the left of (ℓ', m') and the other to the right of (ℓ'', m'') have both length ≥ 3. By using the above uniform bounds for these latter we are then left with the case considered in the following theorem:

Theorem 7. *There is $c > 0$ so that for all γ small enough the following holds. Fix any maximal connected sequence of interfaces with (ℓ', m'), (ℓ'', m'') and $\{u_{\ell,m}\}$ as above. Call k the number of the interfaces in the sequence, then*

$$\sum_{\{u_{\ell,m}\}} \sum_{\{s_{\ell,m}\}} \prod_{\ell,m} e^{V^m_{u_{\ell,m}}(s_{\ell,m}, s_{\ell,m+1})} \leq c^k \epsilon^k_\gamma e^{2kc_b\gamma^{-b}} \tag{7.10}$$

where $\theta(s_{\ell,m}) = 1$ when $m = 1, 2$ and $\theta(s_{\ell,m}) = -1$ when $m = 3, 4$.

Proof. Fix $\{u_{\ell,m}\}$ supposing for the sake of definiteness that $m' = 2$ and $m'' = 4$. Call

$$n := \sum_{\ell,m} u_{\ell,m}$$

and shorthand $s_0 = s_{\ell',m'}$, $s_{n+1} = s_{\ell'',m''}$. We write $s_{[1,n]} \in \mathcal{S}_{[1,n]}(\{u_{\ell,m}\})$ for the configurations which give $\{u_{\ell,m}\}$, then

$$\begin{aligned} \sum_{\{s_{\ell,m}\}} \prod_{\ell,m} e^{V^m_{u_{\ell,m}}(s_{\ell,m}, s_{\ell,m+1})} &= e^{-\beta p^+_\gamma (n+2)\ell^+_\gamma} \sum_{s_{[0,n+1]}} e^{-\beta H_\gamma(s_{[0,n+1]})} e^{F^{(2)}_\gamma(s_0) + F^{(1)}_\gamma(s_{n+1})} \\ &\times \mathbf{1}_{s_{[1,n]} \in \mathcal{S}_{[1,n]}(\{u_{\ell,m}\})} \mathbf{1}_{\theta(s_0) = \theta(s_{n+1}) = 1} \end{aligned} \tag{7.11}$$

Noticing that for $n \leq 2$ $Z^+_{\gamma,n} = Z^{++}_{\gamma,n}$, by the spin flip symmetry, by (7.1) and (7.8) this is bounded by

$$[e^{-\gamma^{-1}\beta \bar{f} + c_b\gamma^{-b}}]^k e^{-\gamma^{-1}c_0 \sum_\ell \sum_{m \in \{2,4\}} (u_{\ell,m} - n_0) \mathbf{1}_{u_{\ell,m} \geq n_0}} \epsilon^*_\gamma(n)$$

where $\epsilon^*_\gamma(n)$ is defined in (7.8) and it is bounded by 1 (see (7.9)). By (7.7):

$$e^{-\gamma^{-1}\beta \bar{f} + c_b\gamma^{-b}} \leq \epsilon_\gamma e^{2c_b\gamma^{-b}}$$

hence (7.10) recalling that $u_{\ell,m} \leq 2$ when $m \in \{1, 3\}$. □

7.2 Free energy of an interface

We define the free energy ϕ_γ of an interface as the free energy cost of splitting a plus interval into a plus and a minus interval with an interface in between them. More precisely

$$\phi_\gamma := -\frac{1}{\beta} \log \sum_{n\geq 2} \liminf_{m\to\infty} \frac{A^{+-}_{m,n}(s_{-m-1}, s_{m+n+2})}{Z^{+}_{\gamma,2m+n}(s_{-m-1}, -s_{m+n+2})} \tag{7.12}$$

where s_{-m-1} and s_{m+n+2} are arbitrary with the only condition that $\theta(s_{-m-1}) = -\theta(s_{m+n+2}) = 1$. In (7.12)

$$\begin{aligned} A^{+-}_{m,n}(s_{-m-1}, s_{m+n+2}) &= \sum_{s_{[-m,n+m+1]}} e^{-\beta H_\gamma(s_{[-m,n+1]}|s_{-m-1},s_{m+n+2})} \mathbf{1}_{\Theta_i\geq 0, i\leq 0} \mathbf{1}_{\Theta_i\leq 0, i\geq n+1} \\ &\times \ \mathbf{1}_{\Theta_i=0, i\in[1,n]} \mathbf{1}_{\Theta_0=1, \Theta_{n+1}=-1} \end{aligned} \tag{7.13}$$

is the partition function with the constraint that $[-m-1, 0]$ is a plus interval, $[1, n]$ is a $+-$ interface and $[n+1, n+m+2]$ is a minus interval. The liminf in (7.12) is actually a limit:

Theorem 8. *For any sequence s_{-m-1} and s_{m+n+2} as above*

$$\lim_{m\to\infty} \frac{A^{+-}_{m,n}(s_{-m-1}, s_{m+n+2})}{Z^{+}_{\gamma,2m+n}(s_{-m-1}, -s_{m+n+2})} = \epsilon_\gamma(n) \tag{7.14}$$

where $\epsilon_\gamma(n)$ is defined in (7.3)

Proof. Let $\theta(s) = 1 = -\theta(s')$ then by Theorem 4

$$\begin{aligned} A^{+-}_{m,n}(s_{-m-1}, s_{m+n+2}) &= \{ \sum_{s:\theta(s)=1} \sum_{s':\theta(s')=-1} e^{V^2_n(s,s')} e^{G^{(1)}_m(s_{-m-1},s) + G^{(3)}_m(s', s_{m+n+2})} \} \\ &\times \ e^{\beta p^+_\gamma (2m+n+2)\ell^+_\gamma} e^{F^{(1)}(s_{-m-1}) + F^{(4)}(s_{m+n+2}))} \end{aligned}$$

The denominator $Z^{+}_{\gamma,2m+n}(s_{-m-1}, -s_{m+n+2})$ in (7.12) is equal to

$$Z^{+}_{\gamma,2m+n}(s_{-m-1}, -s_{m+n+2}) = e^{\beta p^+_\gamma (2m+n+2)\ell^+_\gamma + F^{(1)}(s_{-m-1}) + F^{(2)}(-s_{m+n+2}) + G^{(1)}_{2m+2}(s_{-m-1}, -s_{m+n+2})}$$

Since $F^{(2)}(-s_{m+n+2}) = F^{(4)}(s_{m+n+2})$

$$\frac{A^{+-}_{m,n}(s_{-m-1}, s_{m+n+2})}{Z^{+}_{\gamma,2m+n}(s_{-m-1}, -s_{m+n+2})} \leq \{ e^{\|G^{(1)}_m\|_\infty + \|G^{(3)}_m\|_\infty + \|G^{(1)}_{2m+2}\|_\infty} \} \epsilon_\gamma(n)$$

Analogously

$$\frac{A^{+-}_{m,n}(s_{-m-1}, s_{m+n+2})}{Z^{+}_{\gamma,2m+n}(s_{-m-1}, -s_{m+n+2})} \geq \{ e^{-\|G^{(1)}_m\|_\infty - \|G^{(3)}_m\|_\infty - \|G^{(1)}_{2m+2}\|_\infty} \} \epsilon_\gamma(n)$$

By (5.3) the curly bracket converges to 1 when $m \to \infty$ and (7.14) then follows. □

By Theorem 8 and by Theorem 6 we then have:

Theorem 9. *The free energy ϕ_γ defined in (7.12) is well defined, the liminf on the right hand side is a limit and*

$$e^{-\beta\phi_\gamma} = \epsilon_\gamma \tag{7.15}$$

Then, recalling (7.7),

$$|\phi_\gamma - \gamma^{-1}\bar{f}| \leq c_b\gamma^{-b} \tag{7.16}$$

Thus

$$\lim_{\gamma\to 0} \gamma\phi_\gamma = \bar{f} \tag{7.17}$$

8 Normalization of the weights $w(\underline{u})$

In this section we prove the existence of λ_γ and other properties of $w_{\lambda_\gamma}(\underline{u})$ defined in (6.19), including those stated in Section 3. The following theorem is proved in Appendix F:

Theorem 10. *Taking c_b as in in Theorem 6, there is a constant $c > 0$ such that for all γ small enough and $\frac{1}{2}\epsilon_\gamma \leq \lambda \leq \frac{3}{2}\epsilon_\gamma$*

$$\Big|\sum_{\underline{u}} w_\lambda(\underline{u}) - (\frac{\epsilon_\gamma}{\lambda})^2\Big| \leq c_\gamma\, \epsilon_\gamma, \quad c_\gamma = ce^{8c_b\gamma^{-b}} \tag{8.1}$$

Corollary 11. *In the context of Theorem 10 for γ small enough there is $c > 0$ so that for any $\lambda \in (\frac{3}{4}\epsilon_\gamma, \frac{5}{4}\epsilon_\gamma)$ and any $R > 0$*

$$\sum_{\underline{u}:|\underline{u}|\geq R} w_\lambda(\underline{u}) \leq ce^{-\epsilon_\gamma R/4}, \quad \sum_{\underline{u}\in\mathcal{R}} w_\lambda(\underline{u})|\underline{u}| \leq c\frac{1}{1-e^{-\epsilon_\gamma/4}} \tag{8.2}$$

As a consequence the function $\lambda \to \sum_{\underline{u}\in\mathcal{R}} w_\lambda(\underline{u})$ is in C^1 when $\lambda \in (\frac{3}{4}\epsilon_\gamma, \frac{5}{4}\epsilon_\gamma)$.

Proof. We obviously have for any $\lambda \in (\frac{3}{4}\epsilon_\gamma, \frac{5}{4}\epsilon_\gamma)$:

$$w_\lambda(\underline{u}) \leq e^{-|\underline{u}|\frac{\epsilon_\gamma}{4}} w_{\frac{1}{2}\epsilon_\gamma}(\underline{u}) \tag{8.3}$$

and the first inequality in (8.2) follows from (8.3) with $c = \sum_{\underline{u}} w_{\frac{1}{2}\epsilon_\gamma}(\underline{u})$. The second inequality follows from (3.5) and the first inequality. □

Theorem 12. *For any γ small enough there is a unique $\lambda_\gamma > 0$ such that*

$$\sum_{\underline{u}} w_{\lambda_\gamma}(\underline{u}) = 1 \tag{8.4}$$

Moreover there is c so that for all γ small enough

$$\big|\lambda_\gamma - \epsilon_\gamma\big| \le c\epsilon_\gamma^2 \tag{8.5}$$

Finally the probability $w_{\lambda_\gamma}(\underline{u})$ on $\mathcal{R}$ satisfies the properties listed in Section 3.

Proof. Uniqueness follows because $\sum_{\underline{u}} w_\lambda(\underline{u})$ is a strictly decreasing function of λ when it is finite. We will next prove (8.4). We postpone the (elementary) proof that

$$\Big(\frac{\epsilon_\gamma}{\epsilon_\gamma - c_\gamma\epsilon_\gamma^2}\Big)^2 \ge 1 + c_\gamma\epsilon_\gamma, \quad \Big(\frac{\epsilon_\gamma}{\epsilon_\gamma + c_\gamma\epsilon_\gamma^2}\Big)^2 \le 1 - c_\gamma\epsilon_\gamma \tag{8.6}$$

Then, by (8.1)

$$\sum_{\underline{u}} w_{\epsilon_\gamma - c_\gamma\epsilon_\gamma^2}(\underline{u}) \ge \Big(\frac{\epsilon_\gamma}{\epsilon_\gamma - c_\gamma\epsilon_\gamma^2}\Big)^2 - c_\gamma\epsilon_\gamma \ge 1$$

Similarly

$$\sum_{\underline{u}} w_{\epsilon_\gamma + c_\gamma\epsilon_\gamma^2}(\underline{u}) \le \Big(\frac{\epsilon_\gamma}{\epsilon_\gamma + c_\gamma\epsilon_\gamma^2}\Big)^2 + c_\gamma\epsilon_\gamma \le 1$$

By Corollary 11 $\sum_{\underline{u}} w_\lambda(\underline{u})$ is a C^1 function of λ in $(\frac{3}{4}\epsilon_\gamma, \frac{5}{4}\epsilon_\gamma)$, hence by continuity there is a value λ_γ in $[\epsilon_\gamma - c_\gamma{\epsilon_\gamma}^2, \epsilon_\gamma + c_\gamma{\epsilon_\gamma}^2]$ where it is equal to 1.

The proof of (8.6) follows from the inequalities:

$$\Big(\frac{1}{1 - c_\gamma\epsilon_\gamma}\Big)^2 \ge (1 + c_\gamma\epsilon_\gamma)^2 \ge 1 + c_\gamma\epsilon_\gamma$$

$$\Big(\frac{1}{1 + c_\gamma\epsilon_\gamma}\Big)^2 \le (1 - c_\gamma\epsilon_\gamma + (c_\gamma\epsilon_\gamma)^2)^2 \le 1 - c_\gamma\epsilon_\gamma$$

In the last inequality we have used that $4c_\gamma\epsilon_\gamma \le 1$.

The proof of (8.5) follows from (8.1) with $\lambda = \lambda_\gamma$. Finally (3.3) is proved by (8.2). The existence of the first moment, as stated in (3.4), follows also from (8.2).

□

By (8.1) $\sum_{\underline{u}} w_\lambda(\underline{u})$ as a function of λ is to first order approximated by $(\frac{\epsilon_\gamma}{\lambda})^2$, it is actually true that also its derivative with respect to λ is well approximated by the derivative of $(\frac{\epsilon_\gamma}{\lambda})^2$. Indeed the following theorem holds (we omit for brevity its proof which is based on the estimates obtained in Appendix F):

Theorem 13. *There is $c > 0$ such that for all γ small enough*

$$\Big| \sum_{\underline{u}} w_{\lambda_\gamma}(\underline{u})|\underline{u}| - \frac{2}{\epsilon_\gamma} \Big| \le c \tag{8.7}$$

As a consequence there is $c' > 0$ so that

$$|\alpha_\gamma - \frac{\epsilon_\gamma}{2}| \le c'\epsilon_\gamma^2 \tag{8.8}$$

Proof of Theorem 3.

We have shown in Section 6 that

$$Z_{\Lambda_n}^{\rm pbc} = Z_{\Lambda_n}^{\rm pbc,gg} + Z_{\Lambda_n}^{\rm pbc,\mathcal{X}^0} + Z_{\Lambda_n}^{\rm pbc,\mathcal{X}^+} + Z_{\Lambda_n}^{\rm pbc,\mathcal{X}^-} + Z_{\Lambda_n}^{\rm pbc,gb} \tag{8.9}$$

In Appendix D it is proved that there exists $\zeta_\gamma > 0$ so that

$$\lim_{n\to\infty} \frac{1}{e^{\beta p_\gamma(2n+1)\ell_\gamma^+ - \zeta_\gamma(2n+1)}} \Big(Z_{\Lambda_n}^{\rm pbc,\mathcal{X}^0} + Z_{\Lambda_n}^{\rm pbc,\mathcal{X}^+} + Z_{\Lambda_n}^{\rm pbc,\mathcal{X}^-} + Z_{\Lambda_n}^{\rm pbc,gb} \Big) = 0 \tag{8.10}$$

(6.20) with $\lambda = \lambda_\gamma$ becomes

$$Z_{\Lambda_n}^{\rm pbc,gg} = e^{(2n+1)[\beta p_\gamma^+ \ell_\gamma^+ + \lambda_\gamma]} \sum_{(x,\underline{u}_1)\in A'} w_{\lambda_\gamma}(\underline{u}_1) \sum_k \sum_{\underline{u}_2,,,\underline{u}_k : \sum |\underline{u}_i| = 2n+1-|\underline{u}_1|} \prod_{i=2}^k w_{\lambda_\gamma}(\underline{u}_i) \tag{8.11}$$

where $A' = \{(x,\underline{u}_1) : x + |\underline{u}_1| - 1 \ge n+1 \text{ when } x > -n, |\underline{u}_1| \le 2n+1\}$ (if $|\underline{u}_1| = 2n+1$ then the sum over k in (8.11) is absent). We have:

$$\sum_{(x,\underline{u}_1)\in A', |\underline{u}_1|\le n/2} w_{\lambda_\gamma}(\underline{u}_1) = \sum_{|\underline{u}_1|\le n/2} |\underline{u}_1| w_{\lambda_\gamma}(\underline{u}_1) = \frac{1}{\alpha_\gamma} - \sum_{|\underline{u}_1| > n/2} |\underline{u}_1| w_{\lambda_\gamma}(\underline{u}_1) \tag{8.12}$$

with the last term exponentially small in n (by the properties of λ_γ stated in Section 3). By (3.6) for any $|\underline{u}_1| \le n/2$ the quantity

$$\sum_k \sum_{\underline{u}_2,,,\underline{u}_k : \sum |\underline{u}_i| = 2n+1-|\underline{u}_1|} \prod_{i=2}^k w_{\lambda_\gamma}(\underline{u}_i) - \alpha_\gamma \tag{8.13}$$

decays exponentially in n. This proves (3.14) and therefore Theorem 3.

9 Proof of Theorem 2

In this section we will prove (3.11) and hence Theorem 2. We thus fix a local event $X^* = \{x^*_{\ell,m}, (\ell',m') \le (\ell,m) \le (\ell'',m'')\}$ (see the end of Section 2). As a general result for one dimensional Gibbs measure we have

$$\mu_\gamma[\underline{s} : \psi(\underline{s}) \in X^*] = \lim_{n\to\infty} \mu^{\rm pbc}_{\gamma,\Lambda_n}[\underline{s}' : \psi(\underline{s}') \in X^*] \tag{9.1}$$

where (using the notation in Section 6) $\Lambda_n = [-n,n]$, $\underline{s}'$ is the Λ_n-periodic extension of the configuration in Λ_n and $\mu^{\rm pbc}_{\gamma,\Lambda_n}$ is the Gibbs measure with periodic boundary conditions.

$$\mu^{\rm pbc}_{\gamma,\Lambda_n}[\underline{s}' : \psi(\underline{s}') \in X^*] = \frac{Z^{\rm pbc}_{\Lambda_n}[\underline{s}' : \psi(\underline{s}') \in X^*]}{Z^{\rm pbc}_{\Lambda_n}} = \frac{Z^{\rm pbc}_{\Lambda_n}[\underline{s}' : \psi(\underline{s}') \in X^*]}{e^{\beta p_\gamma (2n+1)\ell^+_\gamma}} \frac{e^{\beta p_\gamma (2n+1)\ell^+_\gamma}}{Z^{\rm pbc}_{\Lambda_n}} \tag{9.2}$$

where $Z^{\rm pbc}_{\Lambda_n}[\underline{s}' : \psi(\underline{s}') \in X^*]$ is the partition function restricted to $\{\underline{s}' : \psi(\underline{s}') \in X^*\}$.

By (3.14) the last factor converges to 1 as $n \to \infty$. To estimate the first one we proceed as in Section 6 with the role of the point $-n$ replaced by $x^*_{\ell',1}$. We then have

$$\frac{Z^{\rm pbc}_{\Lambda_n}[\underline{s}' : \psi(\underline{s}') \in X^*]}{e^{\beta p_\gamma (2n+1)\ell^+_\gamma}} = Z^{\rm pbc,*}_{\Lambda_n} + \frac{Z^{\rm pbc,gb}_{\Lambda_n}}{e^{\beta p_\gamma (2n+1)\ell^+_\gamma}} \tag{9.3}$$

where $Z^{\rm pbc,gb}_{\Lambda_n}$ is defined in (6.13) and the last term in (9.3) vanishes by (D.12) below ; $Z^{\rm pbc,*}_{\Lambda_n}$ is:

$$Z^{\rm pbc,*}_{\Lambda_n} := \sum_{(x,k,\underline{u}_1,\dots\underline{u}_k)\in A^*} \mathbf{1}_{\{(x,k,\underline{u}_1,\dots\underline{u}_k)\to X^*\}} \prod_{i=1}^k w_{\lambda_\gamma}(\underline{u}_i) \tag{9.4}$$

where $A^* = \{(x,k,\underline{u}_1,\dots\underline{u}_k) : x^*_{\ell',1} \in [x, x+|\underline{u}_1|-1]; |\underline{u}_1| + \dots + |\underline{u}_k| = 2n+1\}$. Moreover $(x,k,\underline{u}_1,\dots\underline{u}_k)$ defines a sequence $x_{\ell,m}, \ell = 1,..,k; m = 1,..,4$ where $x_{1,1} = x$, $x_{1,2} = x+u_{1,1}$, $x_{\ell,m} = x_{\ell,m-1}+u_{\ell,m-1}$ (with $x_{\ell,0} = x_{\ell-1,4}$). With these notation $\{(x,k,\underline{u}_1,\dots\underline{u}_k) \to X^*\}$ is the set of elements in A^* such that the corresponding $(x_{\ell,m})$ verifies $x_{\ell+\ell_0,m} = x^*_{\ell,m}, \ell = \ell',..,\ell''$ for some ℓ_0.

Let $C_1 = [x, x+|\underline{u}_1|-1] =: [x'_1, x''_1]$, $C_i = [x''_{i-1}+1, x''_{i-1}+|\underline{u}_i|] =: [x'_i, x''_i]$, $i \le k$. By the definition of A^* the point $x^*_{\ell',1} \in C_1$ and we call C_h the last set C_i which has non empty intersection with $[x^*_{\ell',1}, x^*_{\ell'',4}]$ so that $C_1,...,C_h$ is the collection of all C_i which have non empty intersection with $[x^*_{\ell',1}, x^*_{\ell'',4}]$. Since the C_i are disjoint their number $h \le x^*_{\ell'',4} - x^*_{\ell',1} + 1$.

We fix a positive number $R > x^*_{\ell'',4} - x^*_{\ell',1} + 1$, and take n so large that $R < 2n+1$. We then split the sum in (9.4), over configurations $(x,k,\underline{u}_1,\dots\underline{u}_k)$ such that $|C_i| \le R, i = 1,..,h$ and the other ones. The contribution of the latter ones is bounded by:

$$2\sum_{|\underline{u}|>R} w_{\lambda_\gamma}(\underline{u})|\underline{u}| \tag{9.5}$$

In fact if $|C_i| > R$ then necessarily either $i = 1$ or $i = h$ (or both) which are bounded in the same way, hence the factor 2 in (9.5). The quantity in (9.5) is bounded by:

$$e^{-R\frac{\epsilon_\gamma}{4}} \sum_{|\underline{u}|>R} w_{\frac{1}{2}\epsilon_\gamma}(\underline{u})|\underline{u}| \le c''_\gamma e^{-R\frac{\epsilon_\gamma}{4}} \tag{9.6}$$

where we have used (8.3)

It remains to estimate

$$Z^{\text{pbc},**}_{\Lambda_n} := \sum_{(x,k,\underline{u}_1,\dots\underline{u}_k)\in A^*} \mathbf{1}_{\{(x,k,\underline{u}_1,\dots\underline{u}_k)\to X^*\}} \mathbf{1}_{\{|C_i|\le R, i=1,..,h\}} \prod_{i=1}^{k} w_{\lambda_\gamma}(\underline{u}_i) \tag{9.7}$$

This can be written as (see (3.6) for notation):

$$\begin{aligned} &\sum_{(x,h,\underline{u}_1,\dots\underline{u}_h)} \mathbf{1}_{\{|C_i|\le R, i=1,..,h\}} \mathbf{1}_{\{(x,h,\underline{u}_1,\dots\underline{u}_h)\to X^*\}} \prod_{i=1}^{h} w_{\lambda_\gamma}(\underline{u}_i) \\ &\times W_\gamma\Big[\text{ there is } k \text{ so that } \sum_{i=1}^{k} |\underline{u}_i| = 2n+1 - \big(|C_1| + \dots + |C_h|\big)\Big] \end{aligned} \tag{9.8}$$

We next let $n \to \infty$, since $|C_1| + \dots + |C_h| \le Rh$ (recall that $h \le x^*_{\ell'',4} - x^*_{\ell',1} + 1$), then by (3.6) the last factor converges to α_γ. Then we let $R \to \infty$ and finally get

$$\alpha_\gamma \sum_{(x,h,\underline{u}_1,\dots\underline{u}_h)\in B} \mathbf{1}_{\{(x,h,\underline{u}_1,\dots\underline{u}_h)\to X^*\}} \prod_{i=1}^{h} w_{\lambda_\gamma}(\underline{u}_i) \tag{9.9}$$

where B is the collection of all $(x, h, \underline{u}_1, \dots \underline{u}_h)$ such that the corresponding sets $C_1, .., C_h$ cover $[x^*_{\ell',1}, x^*_{\ell'',4}]$ and each one has non empty intersection with $[x^*_{\ell',1}, x^*_{\ell'',4}]$.

A Outline of the proof of Theorem 4

The advantage of describing the spin configurations in terms of plus-minus and interface intervals is to split difficulties. In fact in a plus (or minus) interval the system sees only one phase and therefore one can exploit the local stability of single phases. The interface intervals are dealt with large deviation estimates as in Theorem 5. What we said above about the plus (or minus) interval is not entirely true because the definition of the plus interval allows for values of Θ which are not always equal to 1, as $\Theta = 0$ is also allowed, this spoils a direct applications of the previous stability statements.

A.1 Reduction to the restricted ensemble

The problem is settled in the next theorem where we show that the computation of $Z^+_{\gamma,n}$ can be reduced to that of a partition function in the "restricted plus ensemble" where $\Theta \equiv 1$ (i.e. without fluctuations). The price is an additional hamiltonian which however is proved to be "small".

Theorem 14. *There are γ^*, c and b all positive so that for all $\gamma \le \gamma^*$ and $n \ge 1$.*

$$Z^+_{\gamma,n}(s_0, s_{n+1}) = \sum_{(s_1,..,s_n):\theta(s_i)=1, i=1,..,n} e^{-\beta[H_\gamma(s_1,..,s_n|s_0,s_{n+1})+U_\gamma(s_1,..,s_n|s_0,s_{n+1})]} \tag{A.1}$$

$$U_\gamma(s_1, .., s_n|s_0, s_{n+1}) = \sum_{\Delta\subset[0,n+1]:N_\Delta\ge 5} u_{\Delta,\gamma}(s_\Delta)$$

The latter is the energy in $[1, n]$ in interaction with $\{0, n + 1\}$ of the hamiltonian U_γ with many body potentials $\{u_{\Delta,\gamma}(s_\Delta)\}$ which are bounded by

$$|u_{\Delta,\gamma}(s_\Delta)| \le ce^{-b\gamma^{-1}N_\Delta}, \quad N_\Delta \text{ the cardinality of } \Delta \tag{A.2}$$

when $n = 1, 2$ the additional hamiltonian U_γ is absent.

The proof of Theorem 14 is reported in Appendix B. The strategy of the proof is like in a typical step of the Pirogov-Sinai analysis of Ising systems at low temperatures where the energy is reduced to the ground state energy (i.e. with all plus, or all minus) and the partition function becomes the partition function of a gas of polymers whose activities are the weights of the contours. In our case we reduce to the restricted ensemble and the contribution of the contours is expressed (using cluster expansion) in terms of the additional hamiltonian U_γ.

A.2 An interpolation hamiltonian

It is now convenient to go back to the original spins $\sigma(x)$ and, following the proof in Chapter 11 in [16], we introduce a reference hamiltonian

$$H^{\text{free}} = -\sum_x m_\beta \sigma(x) \tag{A.3}$$

where m_β is defined in (1.4).

Recalling from (2.2) that $C_i^{\pm} = [\ell_\gamma^{\pm} i, \ell_\gamma^{\pm}(i+1))$, we write

$$\Lambda_n = \bigcup_{i=1}^{n} C_i^+, \quad \Lambda_{n,-} = \bigcup_{i=-\infty}^{n} C_i^+, \quad \Lambda_{n,+} = \bigcup_{i=1}^{\infty} C_i^+ \tag{A.4}$$

Namely $\Lambda_{n,-}$ and $\Lambda_{n,+}$ are obtained from Λ_n by adding all the intervals C_i^+ to the left and respectively to the right of Λ_n. We also write:

$$\Lambda_n^* = \bigcup_{i=0}^{n+1} C_i^+, \quad \Lambda_{n,-}^* = \bigcup_{i=-\infty}^{n+1} C_i^+, \quad \Lambda_{n,+}^* = \bigcup_{i=0}^{\infty} C_i^+ \tag{A.5}$$

We denote by $\mu^0_{t,\Lambda_n,\sigma_{\Lambda_n^c}}$ the Gibbs measure on the plus ensemble $\Theta \equiv 1$ with the interpolating hamiltonian

$$\begin{aligned} & t[H_\gamma(\sigma_{\Lambda_n}|\sigma_{\Lambda_n^c}) + U_\gamma(\sigma_{\Lambda_n}|\sigma_{\Lambda_n^*\setminus\Lambda_n})] + (1-t)H_\Lambda^{\text{free}}(\sigma_{\Lambda_n}), \quad t \in [0,1] \\ & U_\gamma(\sigma_{\Lambda_n}|\sigma_{\Lambda_n^*\setminus\Lambda_n}) = \sum_{\Delta \subset \Lambda_n^*, \Delta\cap\Lambda_n \neq \emptyset} u_{\Delta,\gamma} \end{aligned} \tag{A.6}$$

$U_\gamma(\sigma_{\Lambda_n}|\sigma_{\Lambda_n^*\setminus\Lambda_n})$ is equal to the expression in (A.1) once written in the original spins $\sigma(x)$, the condition $\Delta \cap \Lambda_n \neq \emptyset$ is redundant because $\Delta \subset \Lambda_n^*$ and $N_\Delta \geq 5$. Then

$$\frac{1}{\beta}\log\frac{Z^+_{\gamma,\Lambda_n}(\sigma_{\Lambda_n^c})}{[2\cosh(\beta m_\beta)]^{|\Lambda_n|}} = \int_0^1 dt E_{\mu_{t,\Lambda_n,\sigma_{\Lambda_n^c}}}\Big[H_{\Lambda_n}^{\text{free}}(\sigma_{\Lambda_n}) - H_\gamma(\sigma_{\Lambda_n}|\sigma_{\Lambda_n^c}) - U_\gamma(\sigma_{\Lambda_n}|\sigma_{\Lambda_n^*\setminus\Lambda_n})\Big] \tag{A.7}$$

$|\Lambda_n| = n\ell_\gamma^+$, i.e. the number of original spins in Λ_n.

The key point in what follows is that the interpolating hamiltonian restricted to the plus ensemble satisfies the Dobrushin conditions for uniqueness and the corresponding Gibbs measures have exponential decay of correlations. We will exploit all that in the analysis of the right hand side of (A.7). We will write the integrand as sum of terms which are divided into three categories: those localized away from the boundaries, those close to the right boundary and finally those close to the left boundary. The first ones will contribute to the pressure, the second ones to $F_\gamma^{(2)}$ and the last ones to $F_\gamma^{(1)}$. The term $G_{\gamma,n}^{(1)}$ takes into account the small dependence on the far away boundaries. We outline below how we implement this strategy.

A.3 The infinite volume measures μ_t and the pressure p_γ^+

.

We will prove in Appendix C that for γ small enough and any $t \in [0,1]$ there is a unique DLR measure μ_t on the plus ensemble $\Theta \equiv 1$ with hamiltonian $t[H_\gamma + U_\gamma] + (1-t)H^{\rm free}$ where $U_\gamma = \{u_{\Delta,\gamma}\}$. We will prove Theorem 4 with

$$p_\gamma^+ = \frac{1}{\beta}\log\{2\cosh(\beta m_\beta)\} + \int_0^1 dt \frac{1}{\ell_\gamma^+} \sum_{x\in[0,\ell_\gamma^+)} E_{\mu_t}[k_x] \tag{A.8}$$

where

$$k_x = -m_\beta \sigma(x) + \frac{1}{2}\sum_y J_\gamma(x,y)\sigma(x)\sigma(y) - \sum_{\Delta\ni x}\frac{1}{|\Delta|}u_{\Delta,\gamma}(\sigma_\Delta) \tag{A.9}$$

By (A.2) the series in the last term is absolutely convergent.

(A.7) can be written in a similar way

$$\frac{1}{\beta}\log\frac{Z^+_{\gamma,\Lambda_n}(\sigma_{\Lambda_n^c})}{[2\cosh(\beta m_\beta)]^{|\Lambda_n|}} = \sum_{x\in\Lambda_n^*}\int_0^1 dt E_{\mu^0_{t,\Lambda_n,\sigma_{\Lambda_n^c}}}\Big[k_{\Lambda_n,x}\Big] \tag{A.10}$$

where for $x \in \Lambda_n^*$

$$k_{\Lambda_n,x} = -m_\beta\sigma(x)\mathbf{1}_{x\in\Lambda_n} + \frac{1}{2}\sum_{y:\{x,y\}\cap\Lambda_n\neq\emptyset} J_\gamma(x,y)\sigma(x)\sigma(y) - \sum_{\Delta\subseteq\Lambda_n^*,\Delta\ni x}\frac{1}{|\Delta|}u_{\Delta,\gamma}(\sigma_\Delta) \tag{A.11}$$

Observe that if above $\Delta\subseteq\Lambda_n^*$ then necessarily $\Delta\cap\Lambda_n\neq\emptyset$ because $N_\Delta \geq 5$. Thus

$$\frac{1}{\beta}\log Z^+_{\gamma,\Lambda_n}(\sigma_{\Lambda_n^c}) - |\Lambda_n|p_\gamma^+ = \int_0^1 dt\{\sum_{x\in\Lambda_n^*} E_{\mu^0_{t,\Lambda_n,\sigma_{\Lambda_n^c}}}\Big[k_{\Lambda_n,x}\Big] - \sum_{x\in\Lambda_n} E_{\mu_t}\Big[k_x\Big]\} \tag{A.12}$$

A.4 Semi-infinite measures and the surface terms $F_\gamma^{(k)}$

.

We will prove in Appendix C that for γ small enough and any $t \in [0,1]$ for any configuration $\sigma_{\Lambda^c_{n,-}}$ in $\Theta \equiv 1$ there is a unique semi-infinite DLR measure $\mu^0_{t,\Lambda_{n,-},\sigma_{\Lambda^c_{n,-}}}$ on the plus ensemble $\Theta \equiv 1$ with hamiltonian $t[H_\gamma + U'_\gamma] + (1-t)H^{\rm free}$ where $U'_\gamma = \{u_{\Delta,\gamma}, \Delta \subset \Lambda^*_{n,-}\}$, analogous statement holding for $\mu^0_{t,\Lambda_{n,+},\sigma_{\Lambda^c_{n,+}}}$.

For any $x \in \Lambda^*_{n,\mp}$ we first define

$$k_{\Lambda_{n,\mp},x} = -m_\beta\sigma(x)\mathbf{1}_{x\in\Lambda_{n,\mp}} + \frac{1}{2}\sum_{y:\{x,y\}\cap\Lambda_{n,\mp}\neq\emptyset} J_\gamma(x,y)\sigma(x)\sigma(y) - \sum_{\Delta\subseteq\Lambda^*_{n,\mp},\Delta\ni x}\frac{1}{|\Delta|}u_{\Delta,\gamma}(\sigma_\Delta) \tag{A.13}$$

and then

$$F^{(2)}_\gamma(\sigma_{\Lambda^c_{n,-}}) = \int_0^1 dt\{\sum_{x\in\Lambda_{n,-}} \Big(E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}\Big[k_{\Lambda_{n,-},x}\Big] - E_{\mu_t}\Big[k_x\Big]\Big) + \sum_{x\in\Lambda^*_{n,-}\setminus\Lambda_{n,-}} E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_n}}}\Big[k_{\Lambda_{n,-},x}\Big]\} \tag{A.14}$$

Notice that $F^{(2)}_\gamma(\cdot)$ does not depend on n. Analogously

$$F^{(1)}_\gamma(\sigma_{\Lambda^c_{n,+}}) = \int_0^1 dt\{\sum_{x\in\Lambda_{n,+}} \Big(E_{\mu^0_{t,\Lambda_{n,+};\sigma_{\Lambda^c_{n,+}}}}\Big[k_{\Lambda_{n,+},x}\Big] - E_{\mu_t}\Big[k_x\Big]\Big) + \sum_{x\in\Lambda^*_{n,+}\setminus\Lambda_n} E_{\mu^0_{t,\Lambda_n;\sigma_{\Lambda^c_n}}}\Big[k_{\Lambda_{n,+},x}\Big]\} \tag{A.15}$$

A.5 The terms $G^{(i)}_{\gamma,n}(s_0, s_{n+1})$

We are going to show that the difference between the right hand side of (A.12) and the sum $F^{(1)}_\gamma(\sigma_{\Lambda^c_{n,+}}) + F^{(2)}_\gamma(\sigma_{\Lambda^c_{n,-}})$, which thus defines the remainder term $G^{(1)}_{\gamma,n}(s_0, s_{n+1})$, is

$$G^{(1)}_{\gamma,n}(s_0, s_{n+1}) = \sum_{i=1}^6 \Psi^{(i)} \tag{A.16}$$

where, calling $x^* := \ell^+_\gamma[\frac{n}{2}]$, $[\frac{n}{2}]$ the integer part of $\frac{n}{2}$,

$$\Psi^{(1)} = \int_0^1 dt \sum_{x\in\Lambda_n, x<x^*} \Big(E_{\mu^0_{t,\Lambda_n;\sigma_{\Lambda^c_n}}}\Big[k_{\Lambda_n,x}\Big] - E_{\mu^0_{t,\Lambda_{n,+};\sigma_{\Lambda^c_{n,+}}}}\Big[k_{\Lambda_{n,+},x}\Big]\Big) \tag{A.17}$$

$$\Psi^{(2)} = \int_0^1 dt \sum_{x\in\Lambda_n, x\ge x^*} \Big(E_{\mu^0_{t,\Lambda_n;\sigma_{\Lambda^c_n}}}\Big[k_{\Lambda_n,x}\Big] - E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}\Big[k_{\Lambda_{n,-},x}\Big]\Big) \tag{A.18}$$

$$\Psi^{(3)} = -\int_0^1 dt \sum_{x\in\Lambda_n, x\ge x^*} \Big(E_{\mu^0_{t,\Lambda_{n,+};\sigma_{\Lambda^c_{n,+}}}}\Big[k_{\Lambda_{n,+},x}\Big] - E_{\mu_t}\Big[k_x\Big]\Big) \tag{A.19}$$

$$\Psi^{(4)} = -\int_0^1 dt \sum_{x\in\Lambda_{n,-}, x<x^*} \Big(E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}\Big[k_{\Lambda_{n,-},x}\Big] - E_{\mu_t}\Big[k_x\Big]\Big) \tag{A.20}$$

$$\Psi^{(5)} = \int_0^1 dt \sum_{x\in\Lambda^*_{n,+}\setminus\Lambda_{n,+}} \{E_{\mu^0_{t,\Lambda_n;\sigma_{\Lambda^c_n}}}\Big[k_{\Lambda_n,x}\Big] - E_{\mu^0_{t,\Lambda_{n,+};\sigma_{\Lambda^c_n}}}\Big[k_{\Lambda_{n,+},x}\Big]\} \tag{A.21}$$

$$\Psi^{(6)} = \int_0^1 dt \sum_{x\in\Lambda^*_{n,-}\setminus\Lambda_{n,-}} \{E_{\mu^0_{t,\Lambda_n;\sigma_{\Lambda^c_n}}}\Big[k_{\Lambda_n,x}\Big] - E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_n}}}\Big[k_{\Lambda_{n,-},x}\Big]\} \tag{A.22}$$

In Appendix C we will prove that the series in the definition of the $\Psi^{(i)}$ are absolutely convergent. To prove (A.16) we observe that

$$\begin{aligned} F^{(1)}_\gamma(\sigma_{\Lambda^c_{n,+}}) + \Psi^{(1)} + \Psi^{(3)} + \Psi^{(5)} &= \int_0^1 dt\Big(\sum_{x\in\Lambda_n, x<x^*} \{E_{\mu^0_{t,\Lambda_n,\sigma_{\Lambda^c_n}}}\Big[k_{\Lambda_n,x}\Big] - E_{\mu_t}\Big[k_x\Big]\} \\ &+ \sum_{x\in\Lambda^*_{n,+}\setminus\Lambda_n} E_{\mu^0_{t,\Lambda_n;\sigma_{\Lambda^c_n}}}\Big[k_{\Lambda_n,x}\Big]\Big) \end{aligned}$$

$$\begin{aligned} F^{(2)}_\gamma(\sigma_{\Lambda^c_{n,+}}) + \Psi^{(2)} + \Psi^{(4)} + \Psi^{(6)} &= \int_0^1 dt\Big(\sum_{x\in\Lambda_n, x\geq x^*} \{E_{\mu^0_{t,\Lambda_n,\sigma_{\Lambda^c_n}}}\Big[k_{\Lambda_n,x}\Big] - E_{\mu_t}\Big[k_x\Big]\} \\ &+ \sum_{x\in\Lambda^*_{n,-}\setminus\Lambda_n} E_{\mu^0_{t,\Lambda_n;\sigma_{\Lambda^c_n}}}\Big[k_{\Lambda_n,x}\Big]\Big) \end{aligned}$$

Summing the last two equations we see that $F^{(1)}_\gamma + F^{(2)}_\gamma + \Psi^{(1)} + \cdots + \Psi^{(6)}$ is equal to the right hand side of (A.12), hence (A.16).

As we shall see in Appendix C, the logic behind the above manipulations is that the terms in the $\Psi^{(i)}$ can be reduced to difference of conditional Gibbs expectations of cylindrical functions localized away from where the conditioning of the Gibbs measures differ from each other and thus exploit the exponential decay of correlations.

B Contours, cluster expansion and proof of Theorem 14

(A.1) shows that $Z^+_{\gamma,n}(s_0, s_{n+1})$ can be written as a partition function where the spin configurations are restricted to the plus ensemble with all $\Theta_i = 1$. This is automatically true when $n = 1, 2$, when $n \geq 3$ it can still be done but the energy gets an additional term. This is what we will prove in the sequel using contours. Contours are denoted by $\Gamma = \{\text{sp}(\Gamma), \theta_\Gamma\}$, where $\text{sp}(\Gamma)$ is the spatial support of Γ, namely an interval of length $N_\Gamma \geq 3$ ($N_\Gamma \geq 3$ tacitly in the sequel); θ_Γ, called the specification of Γ, is a function on $\text{sp}(\Gamma)$, namely a sequence $\theta_1, .., \theta_n$, $n = N_\Gamma$, with values $0, \pm 1$ and such that, setting $\theta_0 = \theta_{n+1} = 1$,

$$\theta_1 = \theta_n = 1, \;\; \Theta_i = 0, \;\; i = 1, .., n \tag{B.1}$$

The weight of a contour Γ with $\mathrm{sp}(\Gamma)=\{1,..,\mathrm{n}\}$ and specification $(\theta_1,..,\theta_n)$ is

$$\begin{aligned} W_\gamma(\Gamma|s_0,s_{n+1}) &= \frac{\sum_{s_1,..,s_n} \mathbf{1}_{\theta(s_i)=\theta_i\ i=1,2,..,n} e^{-\beta H_\gamma(s_1,..,s_n|s_0,s_{n+1})}}{Z^{++}_{\gamma,n}(s_0,s_{n+1})} \\ Z^{++}_{\gamma,n}(s_0,s_{n+1}) &= \sum_{s_1,..,s_n} \mathbf{1}_{\theta_i=1,\ i=1,2,..,n} e^{-\beta H_\gamma(s_1,..,s_n|s_0,s_{n+1})} \end{aligned} \tag{B.2}$$

with $\theta(s_0)=\theta(s_{n+1})=1$. By translations the definition is then extended to all contours Γ. Denoting by $\underline{\Gamma}$ any collection of "compatible" contours Γ we have:

$$Z^{+}_{\gamma,n}(s_0,s_{n+1}) = \sum_{(s_1,..,s_n):\theta(s_i)=1, i=1,..,n} \Xi_{\gamma,[1,n],s_0,..,s_{n+1}} e^{-\beta H_\gamma(s_1,..,s_n|s_0,s_{n+1})} \tag{B.3}$$

where

$$\Xi_{\gamma,[1,n],s_0,..,s_{n+1}} := \sum_{\underline{\Gamma}:\mathrm{sp}(\Gamma)\subseteq\{1,..,\mathrm{n}\}} \prod_{\Gamma\in\underline{\Gamma}} W_\gamma(\Gamma|s_0,..,s_{n+1}) \tag{B.4}$$

$W_\gamma(\Gamma|s_0,..,s_{n+1})$ actually depends only on s_m and s_{m+k+1} if $\mathrm{sp}(\Gamma)=(m+1,..,m+k)$. Theorem 5.2 will follow by proving that

$$\log \Xi_{\gamma,[1,n],s_0,..,s_{n+1}} = -\beta \sum_{\Delta\subset[0,n+1]:N_\Delta\ge 5} u_{\Delta,\gamma}(s_\Delta) \tag{B.5}$$

(N_Δ the number of sites in the interval Δ) with $u_{\Delta,\gamma}$ satisfying (A.2). $\Xi_{\gamma,[1,n],s_0,..,s_{n+1}}$ is the partition function of a gas of polymers with weights $W_\gamma(\Gamma|s_0,..,s_{n+1})$ and (B.5) will follow from cluster expansion whose validity depends on the smallness of the weights $W_\gamma(\Gamma|s_0,..,s_{n+1})$.

Theorem 15. *There are c and γ^* positive so that for all $\gamma\le\gamma^*$ and $n\ge 3$,*

$$W_\gamma(\Gamma|s_0,s_{n+1}) \le w_\gamma(\Gamma) := \exp\{-c\delta\zeta^2\gamma^{-1}N_\Gamma\}, \quad N_\Gamma = n \tag{B.6}$$

Proof. In Theorem 9.2.5.1 of [16] it is proved that

$$W_\gamma(\tilde\Gamma|s_0,s_{n+1}) \le \exp\{-c'\delta\zeta^2\gamma^{-1}N_\Gamma\} \tag{B.7}$$

where the contour $\tilde\Gamma$ is defined by specifying the values of η on $\mathrm{sp}(\Gamma)$ rather than the values of θ as in our case (under the smallness conditions on δ and ζ that we have required). It should also be remarked that δ and ζ in Theorem 9.2.5.1 depend suitably on γ but the proof works also in our case. To use (B.7) we observe that the number of specifications η which give a same specification θ is bounded by $3^{N_\Gamma \ell^+_\gamma/\ell^-_\gamma}$, $\ell^-_\gamma := \delta\gamma^{-1}$, so that

$$W_\gamma(\Gamma|s_0,s_{n+1}) \le 3^{N_\Gamma \ell^+_\gamma/\ell^-_\gamma} e^{-c'\delta\zeta^2\gamma^{-1}N_\Gamma}$$

which for α small yields (B.6) (for γ small enough).

□

The proof of (B.5) follows from cluster expansion whose validity relies on the Peierls estimates proved in Theorem 15. An immediate consequence of (B.6) is the validity of a strengthened K-P (Kotecki-Preiss) condition: there is $b' > 0$ so that for all γ small enough

$$\sum_{\Gamma:\mathrm{sp}(\Gamma)\ni x} e^{b'\gamma^{-1}N_\Gamma} W_\gamma(\Gamma) \le 1 \tag{B.8}$$

where N_Γ is the number of ℓ_γ^+-blocks in the spatial support of Γ.

The K-P condition allows to exponentiate $\Xi_{\gamma,[1,n],s}$ in a convergent series. Referring to the literature for a proof we just state the result in Theorem 16 below. We need some extra notation: we regard the space of all contours Γ (denoted by $\{\Gamma\}$) as a graph with nodes Γ and connections (Γ, Γ') if $\mathrm{sp}(\Gamma) \cap \mathrm{sp}(\Gamma') \neq \emptyset$. $I(\Gamma)$ denotes any integer valued function on $\{\Gamma\}$ such that $\{\Gamma : I(\Gamma) > 0\}$ is a connected set.

Theorem 16 (Cluster expansion)**.** *Let γ be so small that the K-P condition (see [13]) (B.8) holds. Then the sum on the right hand side of (B.9) below (over all functions I as above) is absolutely convergent and*

$$\log \Xi_{\gamma,[1,n],s} = \sum_I a_I W_\gamma^I, \quad W_\gamma^I := \prod_\Gamma W_\gamma(\Gamma)^{I(\Gamma)} \tag{B.9}$$

The coefficients a_I are combinatorial (signed) factors whose explicit definition is not needed here. Moreover given any Γ and any subset $\mathcal{I}$ in $\{I\}$ such that $I(\Gamma) \ge 1$ for all $I \in \mathcal{I}$ and a non negative function $f(I)$ on $\mathcal{I}$,

$$\sum_{I\in\mathcal{I}} f(I)I(\Gamma)|a_I|W_\gamma^I \le W_\gamma(\Gamma)e^{(1+\gamma^{-1}b')N_\Gamma} \sup_{I\in\mathcal{I}} f(I)e^{-\gamma^{-1}b'\|I\|}, \quad \|I\| = \sum_\Gamma N_\Gamma I(\Gamma) \tag{B.10}$$

The convergence of the series on the right hand side of (B.9) follows from (B.10). Indeed, calling $\mathcal{I}_\Gamma = \{I : I(\Gamma) \ge 1\}$ and using (B.10) and (B.8)

$$\begin{aligned}\sum_I |a_I|W_\gamma^I &\le \sum_{i=1}^n \sum_{\Gamma:\mathrm{sp}(\Gamma)\ni i} \sum_{I\in\mathcal{I}_\Gamma} |a_I|W_\gamma^I \\ &\le \sum_{i=1}^n \sum_{\Gamma:\mathrm{sp}(\Gamma)\ni i} W_\gamma(\Gamma)e^{(1+\gamma^{-1}b')N_\Gamma} \le en\end{aligned}$$

We can now complete the proof of Theorem 14. Calling Δ_0 the interior of an interval Δ, (A.1) holds with

$$-\beta u_{\Delta,\gamma} = \sum_{I\in\mathcal{I}_\Delta} a_I W_\gamma^I, \quad \mathcal{I}_\Delta = \{I : \bigcup_{\Gamma:I(\Gamma)\ge 1} \mathrm{sp}(\Gamma) = \Delta_0\} \tag{B.11}$$

and by (B.10) with $\mathcal{I} = \mathcal{I}_\Gamma \cap \mathcal{I}_\Delta$, for any $i \in \Delta_0$

$$\begin{aligned}
|\beta u_{\Delta,\gamma}| &\leq \sum_{\Gamma:\mathrm{sp}(\Gamma)\ni i,\mathrm{sp}(\Gamma)\subset\Delta_0} \sum_{I\in\mathcal{I}_\Gamma\cap\mathcal{I}_\Delta} |a_I| W_\gamma^I \\
&\leq \sum_{\Gamma:\mathrm{sp}(\Gamma)\ni i,\mathrm{sp}(\Gamma)\subset\Delta_0} W_\gamma(\Gamma) e^{(1+\gamma^{-1}b')|\Gamma|} e^{-\gamma^{-1}b' N_{\Delta_0}} \\
&\leq e^{1-\gamma^{-1}b' N_{\Delta_0}}
\end{aligned}$$

because $\|I\| = \sum_\Gamma N_\Gamma I(\Gamma) \leq N_{\Delta_0}$ when $I \in \mathcal{I}_\Gamma \cap \mathcal{I}_\Delta$. Observe that the minimal value of N_Δ is achieved when $N_{\Delta_0} = 3$ is minimal and therefore by (B.8) $N_\Delta \geq 5$.

C Dobrushin uniqueness and proof of Theorem 4

In the proof of Theorem 4 we will need to compare expectations of Gibbs measures where we change both the boundary conditions and the potential U_γ. With this in mind we call U' and U'' energies with many body potentials $\{u'_\Delta\}$ and $\{u''_\Delta\}$ which verify the bound (A.2). Denote by A an interval union of intervals C_i^+ and by $\mu'_{t,A,\sigma'_{A^c}}$ the Gibbs measure with hamiltonian

$$t[H_\gamma(\sigma_A|\sigma'_{A^c}) + U'(\sigma_A|\sigma'_{A^c})] + (1-t)H_\Lambda^{\mathrm{free}}(\sigma_A), \quad t \in [0,1] \tag{C.1}$$

$$U'(\sigma_A|\sigma'_{A^c}) = \sum_{\Delta:\Delta\cap A\neq\emptyset} u'_\Delta \tag{C.2}$$

restricted to the plus ensemble where $\Theta \equiv 1$ ($\Theta_x(\sigma_A, \sigma'_{A^c}) = 1$ identically). $\mu''_{t,A,\sigma''_{A^c}}$ is defined analogously with U' replaced by U''. The key ingredient in the sequel is that for all γ small enough and for all $t \in [0,1]$ these hamiltonians satisfy the Dobrushin uniqueness criterion with respect to a suitable Vaserstein distance as stated below.

Theorem 17 (Dobrushin uniqueness)**.** *There are γ^*, c_1 and b_1 all positive so that for all $\gamma \leq \gamma^*$, $t \in [0,1]$, A, $B \supseteq A$, (unions of C_i^+ intervals) σ'_{A^c}, σ''_{A^c}, U' and U'' as above and with $u'_\Delta = u''_\Delta$ when $\Delta \subseteq B$ the following holds. There is a joint representation $\mathcal{P}$ (its expectation denoted by $\mathcal{E}$) of $\mu'_{t,A,\sigma'_{A^c}}$ and $\mu''_{t,A,\sigma''_{A^c}}$ such that for all $C_i^- \subset A$:*

$$\mathcal{E}\Big[d_{C_i^-}(\sigma'_A, \sigma''_A)\Big] \leq \sum_{C_j^-\subset B\setminus A} c_1 e^{-b_1\,\gamma\,\mathrm{dist}(C_i^-,C_j^-)} d_{C_j^-}(\sigma'_{A^c}, \sigma''_{A^c}) + \sum_{C_j^-\subset B^c} c_1 e^{-b_1\,\gamma\,\mathrm{dist}(C_i^-,C_j^-)} \tag{C.3}$$

where

$$d_{C_i^-}(\sigma'_A, \sigma''_A) = \sum_{x\in C_i^-} |\sigma'_A(x) - \sigma''_A(x)| \tag{C.4}$$

The theorem is proved in Chapter 11.5 of [16], it would take too long to enter into its proof and we just refer to [16].

Recalling (A.4) and (A.5) for notation we state and prove a first corollary of Theorem 17:

Corollary 18. *For any $\gamma \le \gamma^*$ (see Theorem 17) and $t \in [0,1]$ there is a unique DLR measure μ_t on $\Theta \equiv 1$ with hamiltonian $t[H_\gamma + U] + (1-t)H_\gamma^{\rm free}$, $U = \{u_{\Delta,\gamma}\}$. Moreover for any $\sigma_{\Lambda^c_{n,\pm}}$ with $\Theta \equiv 1$ there is a unique semi-infinite DLR measures $\mu^0_{t,\Lambda_{n,\pm},\sigma_{\Lambda^c_{n,\pm}}}$ on the plus ensemble $\Theta \equiv 1$.*

Proof. We will first prove the statement relative to μ_t. Let f be any cylindrical function so that there exists a constant $c_f \le 2\|f\|_\infty$ and a set B_f (union of intervals C_i^-) such that

$$|f(\sigma') - f(\sigma'')| \le c_f \sum_{C_i^- \subset B_f} d_{C_i^-}(\sigma', \sigma'') \tag{C.5}$$

Then uniqueness of μ_t follows from

$$\lim_{A \nearrow \mathbb{Z}} \sup_{\sigma'_{A^c}, \sigma''_{A^c}} |E_{\mu_{t,A,\sigma'_{A^c}}}[f] - E_{\mu_{t,A,\sigma''_{A^c}}}[f]| = 0$$

the sup being on configurations with $\Theta \equiv 1$ and $\mu_{t,A,\sigma_{A^c}}$ the Gibbs measure in A, boundary conditions σ_{A^c} and energy $U = \{u_{\Delta,\gamma}\}$. Let $A = [-n\ell_\gamma^+, n\ell_\gamma^+)$ with n so large that $B_f \subset A$, then by Theorem 17

$$E_{\mu_{t,A,\sigma'_{A^c}}}[f] - E_{\mu_{t,A,\sigma''_{A^c}}}[f] = \mathcal{E}[f(\sigma') - f(\sigma'')]$$

and by (C.5)

$$|E_{\mu_{t,A,\sigma'_{A^c}}}[f] - E_{\mu_{t,A,\sigma''_{A^c}}}[f]| \le c_f \sum_{C_i^- \subset B_f} \sum_{C_j^- \subset A^c} c_1 e^{-b_1\, \gamma\, {\rm dist}(C_i^-, C_j^-)} 2\delta\gamma^{-1} \tag{C.6}$$

which vanishes as $A \nearrow \mathbb{Z}$. Translation invariance by $k\ell_\gamma^+$ follows from uniqueness.

The proof for $\mu^0_{t,\Lambda_{n,-}}$ is similar. In this case $A = A_k = [-k\ell_\gamma^+, n\ell_\gamma^+)$, $U' = U''$ with many body potentials $\{u_{\Delta,\gamma}\mathbf{1}_{\Delta \subseteq \Lambda^*_{n,-}}\}$, $u_{\Delta,\gamma}$ as in (A.1). We then have to bound

$$E_{\mu^0_{t,A_k,\sigma'_{(-\infty,-k\ell_\gamma^+)},\sigma_{[n\ell_\gamma^+,(n+1)\ell_\gamma^+)}}}[f] - E_{\mu^0_{t,A_k,\sigma''_{(-\infty,-k\ell_\gamma^+)},\sigma_{[n\ell_\gamma^+,(n+1)\ell_\gamma^+)}}}[f]$$

The important point is that the boundary spins differ only when $x \le -k\ell_\gamma^+$ as they coincide for $x \ge n\ell_\gamma^+$. Then the same argument used for μ_t applies also in this case. The proof for $\mu^0_{t,\Lambda_{n,+}}$ is completely analogous and the corollary is proved. $\square$

We will prove that the series defining $F_\gamma^{(2)}(\sigma_{\Lambda^c_{n,-}})$ is absolutely convergent and that $F_\gamma^{(2)}(\sigma_{\Lambda^c_{n,-}})$ satisfies the inequality (5.3), the proof of the same statement for $F_\gamma^{(1)}(\sigma_{\Lambda^c_{n,-}})$ is analogous and omitted. Let $C_i^- \subset \Lambda_n$ and

$$g_i(\sigma) = \sum_{x \in C_i^-} \sigma(x), \quad h_i(\sigma) = \frac{1}{2} \sum_{x \in C_i^-, y \in \Lambda_{n,-}} J_\gamma(x,y)\sigma(x)\sigma(y) \tag{C.7}$$

Recalling the definition of k_x and $k_{\Lambda_n,x}$, (see (A.9), (A.11)) in (A.14), we define

$$\begin{aligned} R_{1,t} &= \sum_{C_i^- \subset \Lambda_{n,-}} \{|E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[g_i] - E_{\mu_t}[g_i]| + |E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[h_i] - E_{\mu_t}[h_i]|\} \\ &+ \sum_{\Delta \subseteq \Lambda_{n,-}} |E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[u_{\Delta,\gamma}] - E_{\mu_t}[u_{\Delta,\gamma}]| \end{aligned} \tag{C.8}$$

$$R_{2,t} = \sum_{x \in \Lambda_{n,-}, y \notin \Lambda_{n,-}} J_\gamma(x,y)\{|E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[\sigma(x)\sigma(y)]| + |E_{\mu_t}[\sigma(x)\sigma(y)]|\} \tag{C.9}$$

$$R_{3,t} = \sum_{\Delta:\Delta \subset \Lambda^*_{n,-};\Delta \cap \Lambda^*_{n,-} \setminus \Lambda_{n,-} \neq \emptyset} \|u_{\Delta,\gamma}\|_\infty + \sum_{\Delta:\Delta \cap \Lambda_{n,-} \neq \emptyset;\Delta \cap \Lambda^c_{n,-} \neq \emptyset} \|u_{\Delta,\gamma}\|_\infty \tag{C.10}$$

We will prove that

$$|F^{(2)}(\sigma_{\Lambda^c_{n,-}})| \le \int_0^1 dt \sum_{i=1}^3 R_{i,t} < \infty \tag{C.11}$$

The first inequality follows directly from the definition (A.14) of $F^{(2)}(\sigma_{\Lambda^c_{n,-}})$ and the definition of the terms $R_{i,t}$, we thus have to bound the latter.

We have

$$R_{2,t} \le 2 \sum_{x \in \Lambda_n, y \notin \Lambda_n} J_\gamma(x,y) \le c\gamma^{-1}$$

By (A.2) $R_{3,t}$ is bounded by

$$2 \sum_{\Delta:\Delta \cap \Lambda_{n,-} \neq \emptyset;\Delta \cap \Lambda^c_{n,-} \neq \emptyset} |u_{\Delta,\gamma}| \le 2ce^{-(b/2)\gamma^{-1}5} 2 \sum_{k>0,k'>0} e^{-(b/2)\gamma^{-1}(k+k')} \le c' e^{-(b/2)\gamma^{-1}5}$$

To bound $R_{1,t}$ we will use the following lemma:

Lemma 19. *Let f be a cylindrical function on $\Lambda_{n,-}$ and let c_f and B_f as in (C.5). Then:*

$$|E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[f] - E_{\mu_t}[f]| \le c_f \sum_{C_i^- \subset B_f} \sum_{C_j^- \subset \Lambda^c_{n,-}} 2\delta\gamma^{-1} c_1 e^{-b_1\gamma \mathrm{dist}(C_i^-, C_j^-)} \tag{C.12}$$

Proof. Let $A_k = [-k\ell_\gamma^+, n\ell_\gamma^+)$ (eventually $k \to \infty$), then

$$E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[f] - E_{\mu_t}[f] = \int \mu_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}(d\sigma')\mu_t(d\sigma'') \times\Big(E_{\mu'_{t,A_k,\sigma'_{A^c_k}}}[f] - E_{\mu''_{t,A_k,\sigma''_{A^c_k}}}[f]\Big)$$

where μ' is defined with the hamiltonian $U' = \{u_{\Delta,\gamma}\mathbf{1}_{\Delta\subset\Lambda^*_{n,-}}\}$ while μ'' is defined with the hamiltonian $U'' = \{u_{\Delta,\gamma}\}$. Let $\mathcal{E}$ be the expectation relative to the coupling of Theorem 17, then

$$E_{\mu'_{t,A_k,\sigma'_{A^c_k}}}[f] - E_{\mu''_{t,A_k,\sigma''_{A^c_k}}}[f] = \mathcal{E}\Big[f(\sigma'_{A_k}) - f(\sigma''_{A_k})\Big]$$

Since $d_{C_i^-} \le 2\delta\gamma^{-1}$, by Theorem 17

$$|E_{\mu'_{t,\Lambda_{n,-};\sigma'_{A^c_k}}}[f] - E_{\mu''_{t,\Lambda_{n,-};\sigma''_{A^c_k}}}[f]| \le c_f \sum_{C_i^-\subset B_f}\sum_{C_j^-\subset A^c_k} 2\delta\gamma^{-1}c_1 e^{-b_1\gamma \mathrm{dist}(C_i^-,C_j^-)}$$

which proves (C.12) by letting $k\to\infty$. □

We will apply Lemma 19 with f equal to g_i, h_i (see (C.7)) and Δ.

Lemma 20. *For $f = g_i$ the coefficient c_f is equal to 1 and $B_f = C_i^-$. For $f = h_i$, $c_f = 1$ and*

$$B_f = \bigcup_{C_j^-\subset\Lambda_{n,-},\mathrm{dist}(C_j^-,C_i^-)\le\gamma^{-1}} C_j^-$$

For $f = u_{\Delta,\gamma}$, $c_f = 2\|u_{\Delta,\gamma}\|_\infty$ and $B_f = \Delta$.

Proof. The statements for g_i and u_Δ are obviously true. The proof for h_i is as follows. By (C.7)

$$h_i(\sigma') - h_i(\sigma'') = \frac{1}{2}\sum_{x\in C_i^-,y\in\Lambda_{n,-}} J_\gamma(x,y)\Big(\sigma'(x)[\sigma'(y)-\sigma''(y)] + [\sigma'(x)-\sigma''(x)]\sigma''(y)\Big)$$

Since $J_\gamma(x,y) = 0$ if $|x-y| > \gamma^{-1}$ and $\sum_x J_\gamma(x,y) = 1$

$$|\sum_{x\in C_i^-,y\in\Lambda_{n,-}} J_\gamma(x,y)\sigma'(x)[\sigma'(y)-\sigma''(y)]| \le \sum_{C_j^-\subset\Lambda_{n,-}} \mathbf{1}_{\mathrm{dist}(C_j^-,C_i^-)\le\gamma^{-1}} d_{C_j^-}(\sigma',\sigma'')$$

Thus

$$|h_i(\sigma') - h_i(\sigma'')| \le \frac{1}{2}\Big(d_{C_i^-}(\sigma',\sigma'') + \sum_{C_j^-\subset\Lambda_{n,-}} \mathbf{1}_{\mathrm{dist}(C_j^-,C_i^-)\le\gamma^{-1}} d_{C_j^-}(\sigma',\sigma'')\Big)$$

hence the thesis. □

As a corollary of Lemma 19 and Lemma 20 we have:

Corollary 21. *There are c' and b' so that for any $\gamma \le \gamma^*$ (see Theorem 17) and $t \in [0,1]$*

$$|E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[g_i] - E_{\mu_t}[g_i]| \le 2\delta\gamma^{-1} \sum_{C^-_j \subset \Lambda^c_{n,-}} c_1 e^{-b_1\gamma\,\mathrm{dist}(C^-_i,C^-_j)} \tag{C.13}$$

$$|E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[h_i] - E_{\mu_t}[h_i]| \le \sum_{C^-_{i'} \subset \Lambda_{n,-}:\mathrm{dist}(C^-_{i'},C^-_i)\le\gamma^{-1}} 2\delta\gamma^{-1} \sum_{C^-_j \subset \Lambda^c_{n,-}} c_1 e^{-b_1\gamma\,\mathrm{dist}(C^-_{i'},C^-_j)}$$

where h_i and g_i are defined in (C.7). Moreover for any $\Delta \subseteq \Lambda_{n,-}$

$$|E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[u_{\Delta,\gamma}] - E_{\mu_t}[u_{\Delta,\gamma}]| \le 2\|u_{\Delta,\gamma}\|_\infty \sum_{C^-_i \subset \Delta} 2\delta\gamma^{-1} \sum_{C^-_j \subset \Lambda^c_{n,-}} c_1 e^{-b_1\gamma\,\mathrm{dist}(C^-_i,C^-_j)} \tag{C.14}$$

Proof of the first inequality in (5.3).

We will next prove that $R_{1,t} \le c\gamma^{-1}$ which together with the analogous bounds already proved for $R_{2,t}$ and $R_{3,t}$ yields via (C.11) that $|F^{(2)}(\sigma_{\Lambda^c_{n,-}})| \le c\gamma^{-1}$.

Let m be the integer such that $(n+1)\ell^+_\gamma = m\ell^-_\gamma$. Then in (C.13) $i \le m-1$, $j \ge m$ and $\mathrm{dist}(C^-_i, C^-_j) = \delta\gamma^{-1}(j-i-1)$ so that

$$\sum_{C^-_i \subset \Lambda_{n,-}} |E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[g_i] - E_{\mu_t}[g_i]| \le 2\delta\gamma^{-1} \sum_{j\ge m, i\le m-1} c_1 e^{-b_1\delta(j-i-1)} \le c'\gamma^{-1}$$

The sum over $C^-_i \subset \Lambda_{n,-}$ of the right hand side in the second inequality in (C.13) is bounded by

$$2\delta\gamma^{-1} \sum_{i'\le m-1} [c''\delta^{-1}] \sum_{j\ge m} c_1 e^{-b_1\delta(j-i'-1)} \tag{C.15}$$

where $c''\delta^{-1}$ bounds the number of intervals C^-_i which have distance $\le \gamma^{-1}$ from $C^-_{i'}$. Thus

$$\sum_{C^-_i \subset \Lambda_{n,-}} |E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[h_i] - E_{\mu_t}[h_i]| \le c'\gamma^{-1}$$

Finally by (C.14)

$$\sum_{\Delta\subseteq\Lambda_{n,-}} |\{E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[u_{\Delta,\gamma}] - E_{\mu_t}[u_{\Delta,\gamma}]\}| \le K2\delta\gamma^{-1} \sum_{i\le m-1, j\ge m} c_1 e^{-b_1\delta(j-i-1)} \tag{C.16}$$

where, given C^-_i, $K \ge 2\sum_{\Delta\supset C^-_i} \|u_{\Delta,\gamma}\|_\infty$. We are going to show that K is finite: by (A.2)

$$2\sum_{\Delta\supset C^-_i} \|u_{\Delta,\gamma}\|_\infty \le 2\sum_{n\ge 5} nce^{-b\gamma^{-1}n}$$

The sum is finite and vanishing as $\gamma \to 0$ hence the existence of K, so that by (C.16)

$$\sum_{\Delta\subseteq\Lambda_{n,-}} |\{E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[u_{\Delta,\gamma}] - E_{\mu_t}[u_{\Delta,\gamma}]\}| \le c'\gamma^{-1}$$

$\square$

We shall next prove that $G^{(1)}_{\gamma,n}(s_0,s_{n+1})$ as given in (A.16) satisfies the bound (5.3) and thus complete the proof of Theorem 5.1. We will prove the bound for $\Psi^{(2)}+\Psi^{(4)}$, the proof for $\Psi^{(1)}+\Psi^{(3)}$ is similar and omitted. As in the proof of $F^{(2)}$ we write

$$\Psi^{(2)} \le \int_0^1 dt \sum_{i=1}^3 S_{i,t} \tag{C.17}$$

where recalling that $x^* = \ell_\gamma^+ \left[\frac{n}{2}\right]$, (see after (A.16)).

$$\begin{aligned}
S_{1,t} = &\sum_{C_i^- \subset \Lambda_n, i\ell_\gamma^- \ge x^*} \{|E_{\mu^0_{t,\Lambda_n;\sigma_{\Lambda_n^c}}}[g_i] - E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[g_i]| + |E_{\mu^0_{t,\Lambda_n;\sigma_{\Lambda_n^c}}}[h_i] - E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[h_i]|\} \\
&+ \sum_{\Delta \subseteq \Lambda_n, \Delta\cap[x^*,\infty)\neq\emptyset} |E_{\mu_{t,\Lambda_n;\sigma_{\Lambda_n^c}}}[u_{\Delta,\gamma}] - E_{\mu_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[u_{\Delta,\gamma}]| \qquad \text{(C.18)} \\
S_{2,t} = &\sum_{x\in\Lambda_n, y\ge(n+1)\ell_\gamma^+} J_\gamma(x,y)|E_{\mu^0_{t,\Lambda_n;\sigma_{\Lambda_n^c}}}[\sigma(x)] - E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[\sigma(x)]| \\
&+ \sum_{\Delta\subseteq\Lambda_n^*, \Delta\cap[(n+1)\ell_\gamma^+,(n+2)\ell_\gamma^+)\neq\emptyset} |E_{\mu^0_{t,\Lambda_n;\sigma_{\Lambda_n^c}}}[u_{\Delta,\gamma}] - E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[u_{\Delta,\gamma}]| \qquad \text{(C.19)} \\
S_{3,t} = &\, 2 \sum_{\Delta:\Delta\subset\Lambda_n^*; \Delta\cap[0,\ell_\gamma^+)\neq\emptyset, \Delta\cap[x^*,\infty)\neq\emptyset} \|u_{\Delta,\gamma}\|_\infty + \sum_{\Delta\subset\Lambda^*_{n,-}:\Delta\cap[-\ell_\gamma^+,0)\neq\emptyset,\Delta\cap[x^*,\infty)\neq\emptyset} \|u_{\Delta,\gamma}\|_\infty \qquad \text{(C.20)}
\end{aligned}$$

Let f satisfy (C.12), then proceeding as in the proof of Lemma 19 we write

$$\begin{aligned}
E_{\mu^0_{t,\Lambda_n;\sigma_{\Lambda_n^c}}}[f] - E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[f] = &\int \mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}(d\sigma''_{(-\infty,\ell_\gamma^+)}) \\
&\times\Big(E_{\mu'_{t,\Lambda_n;\sigma_{(-\infty,\ell_\gamma^+)},\sigma_{[(n+1)\ell_\gamma^+,\infty)}}}[f] - E_{\mu''_{t,\Lambda_n;\sigma''_{(-\infty,\ell_\gamma^+)},\sigma_{[(n+1)\ell_\gamma^+,\infty)}}}[f]\Big)
\end{aligned}$$

where μ' is defined with the hamiltonian $U' = \{u_{\Delta,\gamma}\mathbf{1}_{\Delta\subseteq\Lambda_n^*}\}$ while μ'' is defined with the hamiltonian $U'' = \{u_{\Delta,\gamma}1_{\Delta\subset\Lambda^*_{n,-}}\}$. To simplify notation we just write

$$\mu'_t \equiv \mu'_{t,\Lambda_n;\sigma_{(-\infty,\ell_\gamma^+)},\sigma_{[(n+1)\ell_\gamma^+,\infty)}}, \quad \mu''_t \equiv \mu''_{t,\Lambda_n;\sigma''_{(-\infty,\ell_\gamma^+)},\sigma_{[(n+1)\ell_\gamma^+,\infty)}}$$

By (C.12) and (C.3)

$$|E_{\mu'_t}[f] - E_{\mu''_t}[f]| \le c_f \sum_{C_i^-\subset B_f} \sum_{C_j^-\subset(-\infty,\ell_\gamma^+)} 2\delta\gamma^{-1}c_1 e^{-b_1\gamma\,\mathrm{dist}(C_i^-,C_j^-)} \tag{C.21}$$

Then, by Lemma 20,

$$|E_{\mu'_t}[g_i] - E_{\mu''_t}[g_i]| \le 2\delta\gamma^{-1} \sum_{C_j^- \subset (-\infty, \ell_\gamma^+)} c_1 e^{-b_1\gamma\, \mathrm{dist}(C_i^-, C_j^-)} \tag{C.22}$$

$$|E_{\mu'_t}[h_i] - E_{\mu''_t}[h_i]| \le \sum_{C_{i'}^- \subset \Lambda_n : \mathrm{dist}(C_{i'}^-, C_i^-) \le \gamma^{-1}} 2\delta\gamma^{-1} \sum_{C_j^- \subset (-\infty, \ell_\gamma^+)} c_1 e^{-b_1\gamma\, \mathrm{dist}(C_{i'}^-, C_j^-)} \tag{C.23}$$

Let m^*, m and m' be the integers such that $m^*\ell_\gamma^- = x^* = \ell_\gamma^+ \left[\frac{n}{2}\right]$, $m\ell_\gamma^- = (n+1)\ell_\gamma^+$ and $m'\ell_\gamma^- = \ell_\gamma^+$. Then

$$\sum_{i\in[m^*, m-1]} |E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[g_i] - E_{\mu_t}[g_i]| \le 2\delta\gamma^{-1} \sum_{j<m', i\in[m^*, m-1]} c_1 e^{-b_1\delta(i-j-1)}$$

Calling $i = m^* + k$, $k \ge 0$, and $j = m' + k'$ the right hand side is bounded by

$$2\delta\gamma^{-1} c_1 e^{-b_1\delta(m^*-m')} \sum_{k'<0, k\ge 0} e^{-b_1\delta(k-k'-1)} \le c' e^{-b_1\gamma[(n/2)-2]\ell_\gamma^+}$$

because $\delta(m^* - m') = \gamma(\delta\gamma^{-1})(m^* - m') = \gamma(x^* - \ell_\gamma^+) \ge \gamma[(n/2) - 2]\ell_\gamma^+$.

The values of i' in (C.23) are bounded from below by $i' \ge m^* - k_0$, $k_0 = \delta^{-1} + 1$ (because of the condition $\mathrm{dist}(C_{i'}^-, C_i^-) \le \gamma^{-1}$). Thus calling $i' = m^* + k$, $k \ge -k_0$, $j = m' + k'$,

$$\begin{aligned} \sum_{i\in[m^*, m-1]} |E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[h_i] - E_{\mu_t}[h_i]| &\le 2\delta\gamma^{-1}[c''\delta^{-1}] e^{-b_1\delta(m^*-m')} \sum_{k'<0, k\ge -k_0} c_1 e^{-b_1\delta(k-k'-1)} \\ &\le c\gamma^{-1} e^{-b_1\gamma[(n/2)-2]\ell_\gamma^+} \end{aligned}$$

see (C.15) for the term $[c''\delta^{-1}]$.

We split the sum over Δ in the last term of (C.18) by distinguishing whether Δ is or is not in $[x^*, \infty)$. In the former case we proceed as for $F_\gamma^{(2)}$ and get

$$\sum_{\Delta \subset \Lambda_n \cap [x^*, \infty)} |E_{\mu^0_{t,\Lambda_n;\sigma_{\Lambda^c_n}}}[u_{\Delta,\gamma}] - E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda^c_{n,-}}}}[u_{\Delta,\gamma}]| \le K 2\delta\gamma^{-1} \sum_{i\ge m^*, j<0} c_1 e^{-b_1\delta(j-i-1)}$$

which is then bounded by

$$K 2\delta\gamma^{-1} c_1 e^{-b_1\delta m^*} \sum_{k\ge 0, j<0} e^{-b_1\delta(k-j-1)}$$

see (C.16). It remains to consider the case where $\Delta = [k\ell_\gamma^+, k'\ell_\gamma^+)$, $0 \le k < [\frac{n}{2}]$, $[\frac{n}{2}] \le k' < n$. We tacitly suppose below that k and k' satisfy the above bounds, then calling $\Delta = [k\ell_\gamma^+, k'\ell_\gamma^+)$

$$\begin{aligned} &\sum_{k,k',\Delta} |E_{\mu'_t}[u_{\Delta,\gamma}] - E_{\mu''_t}[u_{\Delta,\gamma}]| \\ &\qquad \le \sum_{k,k'} c e^{-b\gamma^{-1}|k'-k|} \sum_{C_i^- \subset [k\ell_\gamma^+, k'\ell_\gamma^+)} 2\delta\gamma^{-1} \sum_{j<0} c_1 e^{-b_1\delta(i-j-1)} \end{aligned}$$

which is bounded by

$$\{\sum_{k,k'} c e^{-(b/2)\gamma^{-1}|k'-k|} e^{-(b/2)\gamma^{\alpha}(x^*-k\ell_\gamma^+)}\}\{\frac{|k'-k|\ell_\gamma^+}{\ell_\gamma^-}\}2\delta\gamma^{-1}\sum_{j<0} c_1 e^{-b_1[\gamma k\ell_\gamma^+ +\delta(-j-1)]}$$

For γ small enough $(b/2)\gamma^{\alpha}(x^*-k\ell_\gamma^+) > b_1\gamma(x^*-k\ell_\gamma^+)$, so that the above is bounded by

$$2\delta\gamma^{-1}c_1 e^{-b_1\gamma x^*}\{\sum_{k,k'}\frac{c|k'-k|\ell_\gamma^+}{\ell_\gamma^-}e^{-(b/2)\gamma^{-1}|k'-k|}\}\sum_{j<0}c_1 e^{-b_1\delta(-j-1)]} \le c'\gamma^{-1}e^{-b_1\gamma x^*}$$

Calling $\Omega(x) := |E_{\mu^0_{t,\Lambda_n;\sigma_{\Lambda_n^c}}}[\sigma(x)] - E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda_{n,-}^c}}}[\sigma(x)]|$ we bound

$$\Omega(x) \le \mathcal{E}\Big[d_{C_i^-}(\sigma'_A,\sigma''_A)\Big]$$

Recalling that $\sigma_{\Lambda_n^c}(x) = \sigma_{\Lambda_{n,-}^c}(x)$ for $x \in \Lambda_{n,-}^c$ we get from (C.3)

$$\sum_{x\in\Lambda_n, x\ge n\ell_\gamma^+ -\gamma^{-1}} J_\gamma(x,y)\Omega(x) \le \sum_{i=m-k_0}^{m} + \sum_{C_j^-\subset B^c} c_1 e^{-b_1\,\gamma\,\mathrm{dist}(C_i^-,C_j^-)} \le c' e^{-b_1\gamma(n\ell_\gamma^+)}$$

For the second sum we use (C.21) with $f = u_{\Delta,\gamma}$. We have

$$|u_{\Delta,\gamma}(\sigma') - u_{\Delta,\gamma}(\sigma'')| \le \mathbf{1}_{\sigma'_{\Delta\cap\Lambda_n}\ne\sigma''_{\Delta\cap\Lambda_n}} c e^{-b\gamma|\Delta|} \le c e^{-b\gamma|\Delta|}\sum_{C_i^-\subset\Delta\cap\Lambda_n} d_{C_i^-}(\sigma',\sigma'')$$

so that

$$\begin{aligned}
\sum_{\Delta\subseteq[0,(n+1)\ell_\gamma^+),\Delta\cap[n\ell_\gamma^+,(n+1)\ell_\gamma^+)\ne\emptyset} & |E_{\mu^0_{t,\Lambda_n;\sigma_{\Lambda_n^c}}}[u_{\Delta,\gamma}] - E_{\mu^0_{t,\Lambda_{n,-};\sigma_{\Lambda_{n,-}^c}}}[u_{\Delta,\gamma}]| \\
&\le \sum_{k\in[0,n)} 2\delta\gamma^{-1}c_1 c\{e^{-(b/2)\gamma^{-1}|n-k|-b_1\gamma k\ell_\gamma^+}\}e^{-(b/2)\gamma^{-1}|n-k|}\frac{|n+1-k|\ell_\gamma^+}{\ell_\gamma^-} \\
&\quad\times\sum_{j<0}c_1 e^{-b_1\delta(-j-1)]} \le c'\delta\gamma^{-1}e^{-b_1\gamma^{\alpha}n\ell_\gamma^+}
\end{aligned}$$

Finally

$$S_{3,t} \le 2\sum_{k'\ge n;k<0} c e^{-b\gamma^{-1}|k'-k|} \le 2ce^{-b\gamma^{\alpha}n\ell_\gamma^+}\{\sum_{k'>0,k<0} e^{-b\gamma^{-1}(k'-k)}\} \le c' e^{-b\gamma^{\alpha}n\ell_\gamma^+}$$

$\Psi^{(4)}$ is like $F^{(2)}$ but with the sum restricted to $x < x^*$. The analysis of the many terms which contribute to $\Psi^{(4)}$ are therefore like the corresponding ones for $F^{(2)}$ but we need to exploit the fact that the conditioning where the Gibbs measures differ is far away. This is done with the same ideas used so for $\Psi^{(2)}$, we omit the details. Also the analysis of $\Psi^{(6)}$ is similar to that of terms already considered and it is omitted.

D Bounds on restricted partition functions

In this appendix we will bound the restricted partition functions introduced in Section 6 namely: $Z_{\Lambda_n}^{\mathrm{pbc},\mathcal{X}^0}$, $Z_{\Lambda_n}^{\mathrm{pbc},\mathcal{X}^+}$, $Z_{\Lambda_n}^{\mathrm{pbc},\mathcal{X}^-}$ see (6.2) and $Z_{\Lambda_n}^{\mathrm{pbc,gb}}$ see (6.14). We start from $Z_{\Lambda_n}^{\mathrm{pbc},\mathcal{X}^+}$.

D.1 Bounds on $Z_{\Lambda_n}^{\mathrm{pbc},\mathcal{X}^\pm}$

By the spin flip symmetry it is sufficient to bound $Z_{\Lambda_n}^{\mathrm{pbc},\mathcal{X}^+}$. Let

$$Z_{\Lambda_n;0}^{\mathrm{pbc}} := \sum_{\Theta_i \geq 0, \Theta_0 = 1} \sum_{s_{\Lambda_n}} e^{-\beta H_\gamma^{\mathrm{pbc}}(s_{\Lambda_n})} \tag{D.1}$$

then

$$Z_{\Lambda_n}^{\mathrm{pbc},\mathcal{X}^+} \leq (2n+1) Z_{\Lambda_n;0}^{\mathrm{pbc}} \tag{D.2}$$

Recalling that $H_\gamma^{\mathrm{pbc}}(s_{\Lambda_n})$ is defined with periodic boundary conditions so that $s_0 = s_{2n+1}$

$$Z_{\Lambda_n;0}^{\mathrm{pbc}} := \sum_{s_0:\theta(s_0)=1} e^{-\beta H_\gamma^{\mathrm{pbc}}(s_0)} \sum_{s_{[1,2n]}} \mathbf{1}_{\theta_1=\theta_{2n}=1;\Theta_i\geq 0} e^{-\beta H_\gamma^{\mathrm{pbc}}(s_{[1,2n]}|s_0,s_{2n+1})} \tag{D.3}$$

By Theorem 4 there is a constant $c > 0$ so that

$$Z_{\Lambda_n;0}^{\mathrm{pbc}} \leq \sum_{s_0:\theta(s_0)=1} e^{-\beta H_\gamma^{\mathrm{pbc}}(s_0)} e^{\beta p_\gamma^+(2n)\ell_\gamma^+ + c\gamma^{-1}} \tag{D.4}$$

so that $Z_{\Lambda_n;0}^{\mathrm{pbc}} \leq e^{\beta p_\gamma^+(2n)\ell_\gamma^+ + c'\ell_\gamma^+}$ (c' a constant). In conclusion:

$$Z_{\Lambda_n}^{\mathrm{pbc},\mathcal{X}^+} \leq \{(2n+1)e^{c'\ell_\gamma^+}\} e^{\beta p_\gamma^+(2n+1)\ell_\gamma^+} \tag{D.5}$$

D.2 Bound on $Z_{\Lambda_n}^{\mathrm{pbc},\mathcal{X}^0}$

We will reduce the bound of $Z_{\Lambda_n}^{\mathrm{pbc},\mathcal{X}^0}$ to that of $Z_{\Lambda_n}^{\mathrm{pbc},\mathcal{X}^+}$. A better bound could be derived by using the Lebowitz-Penrose coarse graining techniques and estimates on the associated free energy functional but the bound below is faster and simpler.

We call Λ_n' the set Λ_n without the points $-1, 0, 1$. We then have

$$Z_{\Lambda_n}^{\mathrm{pbc},\mathcal{X}^0} = \sum_{s_0',s_{\pm 1}'} e^{-\beta H_\gamma^{\mathrm{pbc}}(s_0',s_{\pm 1}')} \sum_{s_{\Lambda_n'}} e^{-\beta H_\gamma^{\mathrm{pbc}}(s_{\Lambda_n'}|s_0',s_{\pm 1}')} \mathbf{1}_{\Theta_i=0, i=1,..,2n} \tag{D.6}$$

with Θ_i computed on the configuration $s_{\Lambda_n'}, s_0', s_{\pm 1}'$. Let $s_0, s_{\pm 1}$ be a configuration with $\theta_i = 1$, $i = 0, \pm 1$. We have

$$|H_\gamma^{\mathrm{pbc}}(s_{\Lambda_n'}|s_0', s_{\pm 1}') - H_\gamma^{\mathrm{pbc}}(s_{\Lambda_n'}|s_0, s_{\pm 1})| \leq c_1 \gamma^{-1} \tag{D.7}$$

$$H_\gamma^{\mathrm{pbc}}(s_0, s_{\pm 1}) \geq -c_2\ell_\gamma^+, \qquad \sum_{s_0', s_{\pm 1}'} e^{-\beta H_\gamma^{\mathrm{pbc}}(s_0', s_{\pm 1}')} \leq e^{c_3 \ell_\gamma^+} \tag{D.8}$$

We thus get from (D.6)

$$Z_{\Lambda_n}^{\mathrm{pbc},\mathcal{X}^0} \leq e^{c_3\ell_\gamma^+} e^{\beta c_1 \gamma^{-1}} e^{\beta c_2 \ell_\gamma^+} Z_{\Lambda_n}^{\mathrm{pbc},\mathcal{X}^+} \tag{D.9}$$

and using the bound for $Z_{\Lambda_n}^{\mathrm{pbc},\mathcal{X}^+}$ proved before, we get for a suitable constant c'':

$$Z_{\Lambda_n}^{\mathrm{pbc},\mathcal{X}^0} \leq e^{c''\ell_\gamma^+} e^{\beta p_\gamma^+ (2n+1)\ell_\gamma^+} \tag{D.10}$$

D.3 Bound on $Z_{\Lambda_n}^{\mathrm{pbc,gb}}$

Recalling (6.14):

$$\frac{Z_{\Lambda_n}^{\mathrm{pbc,gb}}}{e^{\beta p_\gamma (2n+1)\ell_\gamma^+}} = \frac{1}{e^{\lambda_\gamma (2n+1)}} \sum_{(x,\underline{u})\in A} w^{(b)}(\underline{u}) \tag{D.11}$$

We will next prove that that there exists $\zeta_\gamma > 0$ so that:

$$\lim_{n\to\infty} \frac{1}{e^{\lambda_\gamma(2n+1)-\zeta_\gamma(2n+1)}} \sum_{(x,\underline{u})\in A} w^{(b)}(\underline{u}) = 0 \tag{D.12}$$

where $w^{(b)}(\underline{u})$ is defined in (6.15) and A in (6.3) (A depends on n)

We write:

$$\sum_{(x,\underline{u})\in A} w^{(b)}(\underline{u}) = \sum_{(x,\underline{u})\in A^{\geq 3}} w^{(b)}(\underline{u}) + \sum_{(x,\underline{u})\in A^{<3}} w^{(b)}(\underline{u}) \tag{D.13}$$

where:

$$A^{\geq 3} := \{(x,\underline{u}) : |\underline{u}| = 2n+1;\ \exists \bar\ell : u_{\bar\ell,1} \geq 3;\ x + u_1 - 1 \geq n+1,\ \text{when x} > -\text{n}\} \tag{D.14}$$

and $A^{<3}$ the complementary set.

Let $u \geq 3$, recalling (5.3) and notation in (6.15) we write:

$$\begin{aligned} \frac{K_u(s,s') e^{A^{(1)}_{\gamma,u-2}}}{e^{A^{(1)}_{\gamma,u-2}}} \mathbf{1}_{u\geq 3} &\leq e^{A^{(1)}_{\gamma,u-2}} \Big(\{ e^{G^{(1)}_{\gamma,u-2}(s,s')} - e^{A^{(1)}_{\gamma,u-2}} \} e^{-A^{(1)}_{\gamma,u-2}} \Big) \mathbf{1}_{u\geq 3} &\text{(D.15)} \\ &\leq 4a e^{-b_0\gamma\ell_\gamma^+} \cdot e^{A^{(1)}_{\gamma,u-2}} \mathbf{1}_{u\geq 3} &\text{(D.16)} \end{aligned}$$

We then use this estimate in the first sum in (D.13) on $u_{\bar\ell,1}$ where $\bar\ell$ is defined in (D.14):

$$w^{(b)}(\underline{u}) \leq 4a e^{-b_0\gamma\ell_\gamma^+} w(\underline{u}) \tag{D.17}$$

Substituting in the first term of (D.13), we get by (3.3):

$$\frac{1}{e^{\lambda_\gamma(2n+1)}}\sum_{(x,\underline{u})\in A^{\geq 3}} w^{(b)}(\underline{u}) \leq 4ae^{-b_0\gamma\ell_\gamma^+}(2n+1)\sum_{\underline{u}:|\underline{u}|\geq 2n+1} w_{\lambda_\gamma}(\underline{u}) \leq 4ae^{-b_0\gamma\ell_\gamma^+}(2n+1)e^{-\delta_\gamma(2n+1)} \tag{D.18}$$

which vanishes exponentially when $n\to\infty$.

Let consider other term of (D.13). In this case $u_{\ell,1}<3$ for any ℓ. We consider $u_{1,1}$ and bound $K_{u_{1,1}}(s,s')$ as follows:

$$\frac{K_u(s,s')e^{A^{(1)}_{\gamma,u-2}}}{e^{A^{(1)}_{\gamma,1}}}\mathbf{1}_{u<3} \leq e^{A^{(1)}_{\gamma,1}}\left(e^{V^1_u}e^{-A^{(1)}_{\gamma,1}}\right)\mathbf{1}_{u\leq 2} \tag{D.19}$$

$$\leq e^{-b_0\gamma\ell_\gamma^+}\cdot e^{A^{(1)}_{\gamma,1}}\mathbf{1}_{u\leq 2} \tag{D.20}$$

Since $\sum_{u\leq 2}\leq 2$

$$\frac{1}{e^{\lambda_\gamma(2n+1)}}\sum_{(x,\underline{u})\in A^{<3}} w^{(b)}(\underline{u}) \leq 2e^{-b_0\gamma\ell_\gamma^+}\cdot 2\cdot\sum_{\underline{u}:|\underline{u}|\geq 2n+1} w_{\lambda_\gamma}(\underline{u}) \leq 4e^{-b_0\gamma\ell_\gamma^+}e^{-\delta_\gamma(2n+1)} \tag{D.21}$$

Collecting (D.21) and (D.18) we get (D.12).

E Instanton and proof of Theorem 5

E.1 Coarse graining

We define the coarse graining maps $\phi_{\rm cg}:\{-1,1\}^{\mathbb{Z}}\to[-1,1]^{\mathbb{Z}}$ and $\psi_{\rm cg}:[-1,1]^{\mathbb{Z}}\to L^\infty(\mathbb{R})$ by setting

$$\phi_{\rm cg}(\sigma)=\underline{m}=(m_i)_{i\in\mathbb{Z}},\quad m_i=\gamma^{1/2}\sum_{x\in[i\gamma^{1/2},(i+1)\gamma^{1/2})}\sigma(x) \tag{E.1}$$

$$\psi_{\rm cg}(\underline{m})(r)=m_i,\quad \text{when } r\in[i\gamma^{1/2},(i+1)\gamma^{1/2}) \tag{E.2}$$

We denote by $\mathcal{X}_\Lambda$ the subspace of $L^\infty(\Lambda,[-1,1])$ made of functions on Λ which are constant on the intervals $[i\gamma^{1/2},(i+1)\gamma^{1/2})$, we tacitly suppose that Λ is union of such intervals and by default any function m that we will consider in the sequel is (unless otherwise stated) in $\mathcal{X}_\Lambda$ for some Λ. When $\Lambda=\mathbb{R}$ we simply write $\mathcal{X}$.

Let $\underline{m}$ be in the range of $\phi_{\rm cg}$ and, recalling (2.3) for notation,

$$Z_{n,\underline{m},s_{[1,n]^c}}=\sum_{s_{[1,n]}}\mathbf{1}_{\phi_{\rm cg}(s_{[1,n]},s_{[1,n]^c})=\underline{m}}e^{-\beta H_\gamma(s_{[1,n]}|s_{[1,n]^c})} \tag{E.3}$$

Then (see for instance Theorem 4.2.2.2 in [16]) there is a constant c so that

$$\begin{aligned}\log Z_{n,\underline{m},s_{[1,n]^c}} &\le -\gamma^{-1}\beta F(m_{[\gamma^{-\alpha},(n+1)\gamma^{-\alpha})}|m_{[\gamma^{-\alpha},(n+1)\gamma^{-\alpha})^c}) \\ &+ cn\ell_\gamma^+\gamma^{1/2}\log\gamma^{-1}\end{aligned} \tag{E.4}$$

where $m(\cdot)=\psi_{\rm cg}(\underline{m})$ and $F(m_\Lambda|m_\Delta)$, $\Lambda\cap\Delta=\emptyset$, is the Lebowitz-Penrose functional

$$F(m_\Lambda|m_\Delta)=F(m_\Lambda)-\int_\Lambda dr\int_\Delta dr' J(|r-r'|)m(r)m(r') \tag{E.5}$$

$$F(m_\Lambda)=\int_\Lambda \frac{-1}{\beta}S(m)dr-\frac{1}{2}\int_\Lambda dr\int_\Lambda dr' J(|r-r'|)m(r)m(r')$$

The entropy $S(m)$ is defined after (1.3). The proof of (E.4) simply follows from the smoothness of the interaction kernel $J(|r-r'|)$ and the Stirling formula.

In analogy with (2.4) we define new phase indicators for functions $m\in L^\infty(\mathbb{R},[-1,1])$ by setting for any $i\in\mathbb{Z}$:

$$\eta_i^*=\pm 1 \ \text{ if } \ \Big|\delta^{-1}\int_{i\delta}^{(i+1)\delta} dr\,[m(r)\mp m_\beta]\Big|\le\zeta, \quad \eta_i^*=0 \text{ otherwise} \tag{E.6}$$

and then, like in (2.5)–(2.6),

$$\begin{aligned}\theta_i^*=\pm 1 \ \text{ if } \ \eta_j^*\equiv\pm 1 \ \text{ for all } \ j:[\delta j,\delta(j+1))\subset[\gamma^{-\alpha}i,\gamma^{-\alpha}(i+1))\\ \theta_i^*=0 \text{ otherwise}\end{aligned} \tag{E.7}$$

$$\Theta_i^*=\pm 1 \ \text{ if } \ \theta_j^*\equiv\pm 1,\ j=i-1,i,i+1, \quad \Theta_i^*(\underline{s})=0 \text{ otherwise} \tag{E.8}$$

Since η_i^*, θ_i^* and Θ_i^* are respectively equal to η_i, θ_i and Θ_i when $m=\psi_{\rm cg}(\phi_{\rm cg}(\sigma))$, then the statistical weight $Z_{\gamma,n}^{-,+}$ of a $-+$ interface is bounded by

$$\begin{aligned}&\log Z_{\gamma,n}^{-,+}(s_0,s_{n+1})\\ &\quad\le -\gamma^{-1}\beta \inf_{\theta_n^*=1=-\theta_1^*;\Theta_i^*=0,i=1,..,n} F(m_{[\gamma^{-\alpha},(n+1)\gamma^{-\alpha})}|m_{[0,\gamma^{-\alpha})},m_{[(n+1)\gamma^{-\alpha},(n+2)\gamma^{-\alpha})})\\ &\quad+\frac{n\ell_\gamma^+}{\gamma^{-1/2}}\log(2\gamma^{-1/2}+1)+cn\ell_\gamma^+\gamma^{1/2}\log\gamma^{-1}\end{aligned} \tag{E.9}$$

the inf being on functions in $\mathcal{X}_{[\gamma^{-\alpha},(n+1)\gamma^{-\alpha})}$ (defined at the beginning of this appendix) and which verify the constraint stated in the argument of the inf. To get (E.9) we have used that the cardinality of all $\underline{m}$ is $(2\gamma^{-1/2}+1)^{n\ell_\gamma^+\gamma^{1/2}}$

E.2 An equilibrium variational problem

Since the inf in (E.9) is over m which have $\theta_1^*=-1$ and $\theta_n^*=1$ in the intervals $[\gamma^{-\alpha},2\gamma^{-\alpha})$ and respectively $[n\gamma^{-\alpha},(n+1)\gamma^{-\alpha})$, then m in the above intervals is in the minus, plus, equilibrium phase. In Theorem 6.3.3.1 in [16] it is proved that in an equilibrium phase the minimizer of the free energy approaches exponentially fast (away from the boundary conditions) the equilibrium value $\pm m_\beta$, respectively. Thus

Theorem. There are a and c positive so that

$$\inf_{\theta_n^*=1=-\theta_1^*;\Theta_i^*=0,i=1,..,n} F(m_{[\gamma^{-\alpha},(n+1)\gamma^{-\alpha})}|m_{[0,\gamma^{-\alpha})},m_{[(n+1)\gamma^{-\alpha},(n+2)\gamma^{-\alpha})})$$
$$\geq \inf {}^* F(m_{[\gamma^{-\alpha},(n+1)\gamma^{-\alpha})}|m_{[0,\gamma^{-\alpha})},m_{[(n+1)\gamma^{-\alpha},(n+2)\gamma^{-\alpha})}) - ce^{-a\gamma^{-\alpha}} \quad \text{(E.10)}$$

where $\inf^*$ shorthands the inf over $m \in L^\infty([\gamma^{-\alpha},(n+1)\gamma^{-\alpha}),[-1,1])$ such that:

- $\theta_1^* = -1, \theta_n^* = 1; \Theta_i^* = 0, i = 1,..,n;$
- $m(r) = -m_\beta$ for all $r \in [\frac{3}{2}\gamma^{-\alpha} - 1, \frac{3}{2}\gamma^{-\alpha} + 1];$
- $m(r) = m_\beta$ for all $r \in [(n+\frac{1}{2})\gamma^{-\alpha} - 1, (n+\frac{1}{2})\gamma^{-\alpha} + 1].$

We split the interval $[\gamma^{-\alpha},(n+1)\gamma^{-\alpha})$ into three intervals:

$$A = [\gamma^{-\alpha}, \frac{3}{2}\gamma^{-\alpha}), \quad A' = [(n+\frac{1}{2})\gamma^{-\alpha}, (n+1)\gamma^{-\alpha})$$
$$\Lambda = [\frac{3}{2}\gamma^{-\alpha}, (n+\frac{1}{2})\gamma^{-\alpha}) \quad \text{(E.11)}$$

Lemma 22. *Let $m_{[\gamma^{-\alpha},(n+1)\gamma^{-\alpha})}(r)$ be equal to $-m_\beta$ when $r \in [\frac{3}{2}\gamma^{-\alpha}, \frac{3}{2}\gamma^{-\alpha}+1]$ and to m_β when $r \in [(n+\frac{1}{2})\gamma^{-\alpha}, (n+\frac{1}{2})\gamma^{-\alpha}+1]$. Then*

$$F(m_{[\gamma^{-\alpha},(n+1)\gamma^{-\alpha})}|m_{[0,\gamma^{-\alpha})},m_{[(n+1)\gamma^{-\alpha},(n+2)\gamma^{-\alpha})}) = F_1 + F_2 + F_3$$
$$+2m_\beta^2 \int_0^1 dr \int_{-1}^0 dr' J(|r-r'|) \quad \text{(E.12)}$$

where

$$F_1 := F(m_A|m_{[0,\gamma^{-\alpha})}, -m_\beta \mathbf{1}_{r\in[\frac{3}{2}\gamma^{-\alpha}-1,\frac{3}{2}\gamma^{-\alpha}]})$$
$$F_2 := F(m_{A'}|m_{[(n+1)\gamma^{-\alpha},(n+2)\gamma^{-\alpha})}, m_\beta \mathbf{1}_{[(n+\frac{1}{2})\gamma^{-\alpha},(n+\frac{1}{2})\gamma^{-\alpha}+1]})$$
$$F_3 := F(m_\Lambda| - m_\beta \mathbf{1}_{r\in[\frac{3}{2}\gamma^{-\alpha}-1,\frac{3}{2}\gamma^{-\alpha}]}, m_\beta \mathbf{1}_{[(n+\frac{1}{2})\gamma^{-\alpha},(n+\frac{1}{2})\gamma^{-\alpha}+1]}) \quad \text{(E.13)}$$

Proof. The l.h.s. of (E.12) differs from $F_1+F_2+F_3$ because the interactions across $\frac{3}{2}\gamma^{-\alpha}$ and across $(n+\frac{1}{2})\gamma^{-\alpha}$ are counted twice. The last term on the r.h.s. of (E.12) corrects such overcounting. □

Since F_1 and F_2 do not depend on m_Λ we can minimize separately the three F_i.

Minimization of F_1. We need to minimize F_1 over functions m_A such that η_i^* is equal to -1 for all $i : [\delta i, \delta(i+1)) \subset A$ and with boundary conditions which have the same property (in particular the boundary condition at $[\frac{3}{2}\gamma^{-\alpha} - 1, \frac{3}{2}\gamma^{-\alpha}]$ is identically $-m_\beta$. In Theorem 6.3.3.1 in [16] it is proved that there is a unique minimizer m_A^* which is the unique solution of

$$m_A^*(r) = \tanh\{J * m(r)\}, \quad r \in A \tag{E.14}$$

where m in the argument of tanh is equal to m_A^* in A, to $m_{[0,\gamma^{-\alpha})}$ in $[0, \gamma^{-\alpha})$ and to $-m_\beta$ in $[\frac{3}{2}\gamma^{-\alpha} - 1, \frac{3}{2}\gamma^{-\alpha}]$. By the symmetry under change of sign:

$$\inf_{\eta_i^* \equiv -1} F_1 = F(-m_A^* | - m_{[0,\gamma^{-\alpha})}, m_\beta \mathbf{1}_{r \in [\frac{3}{2}\gamma^{-\alpha}-1, \frac{3}{2}\gamma^{-\alpha}]}) \tag{E.15}$$

where $\eta_i^*(-m_A^*) \equiv 1$.

Minimization of F_2. The argument is completely analogous to that for F_1, hence there is $m_{A'}^*$ with $\eta_i^*(m_{A'}^*) \equiv 1$ in A' which satisfies the analogue of (E.14) in A' and such that

$$\inf_{\eta_i^* \equiv 1} F_2 = F(m_{A'}^* | m_{[(n+1)\gamma^{-\alpha}, (n+2)\gamma^{-\alpha})}, m_\beta \mathbf{1}_{r \in [\mathbf{1}_{[(n+\frac{1}{2})\gamma^{-\alpha}, (n+\frac{1}{2})\gamma^{-\alpha}+1]}}) \tag{E.16}$$

Minimization of F_3. Let $\mathcal{G}$ be the set of $m \in L^\infty(\mathbb{R}, [-1, 1])$ with the following properties:

- $m(r) = m_\beta$ for $r > (n + 1/2)\gamma^{-\alpha}$ and $m(r) = -m_\beta$ for $r \leq 3/2\gamma^{-\alpha}$.
- $\theta_1^*(m) = -1$, $\theta_n^*(m) = 1$.
- $\Theta_i^*(m) = 0$, $i = 1, .., n$.

Call $\mathcal{G}_\Lambda$ the set of functions which are the restriction to Λ of functions in $\mathcal{G}$.

Lemma 23. *With the above notation*

$$\inf_{m \in \mathcal{G}_\Lambda} F_3 = \inf_{m \in \mathcal{G}} \mathcal{F}(m) + |\Lambda| f_\beta(m_\beta) + m_\beta^2 \int_{r>0} dr \int_{r'<0} dr' J(|r - r'|) \tag{E.17}$$

Proof. Let $m \in \mathcal{G}$, then

$$\begin{aligned} \mathcal{F}(m) &= F(m_\Lambda | m_{\Lambda^c}) - |\Lambda| f_\beta(m_\beta) + \int_{-\infty}^{3/2\gamma^{-\alpha}} dr \Big(\frac{1}{2} m_\beta^2 - \frac{1}{2} \int_{-\infty}^{3/2\gamma^{-\alpha}} dr' m_\beta^2 J(|r - r'|) \Big) \\ &+ \int_{(n+1/2)\gamma^{-\alpha}}^{\infty} dr \Big(\frac{1}{2} m_\beta^2 - \frac{1}{2} \int_{(n+1/2)\gamma^{-\alpha}}^{\infty} dr' m_\beta^2 J(|r - r'|) \Big) \end{aligned}$$

hence (E.17). □

The $\inf_{m\in\mathcal{G}}\mathcal{F}(m)$ will be studied in the following two subsections, here we proceed by analyzing the other terms. Going back to (E.10) we get

$$\begin{aligned}&\inf{}^*F(m_{[\gamma^{-\alpha},(n+1)\gamma^{-\alpha})}|m_{[0,\gamma^{-\alpha})},m_{[(n+1)\gamma^{-\alpha},(n+2)\gamma^{-\alpha})})\geq\inf_{m\in\mathcal{G}}\mathcal{F}(m)\\&+F(-m^*_A|-m_{[0,\gamma^{-\alpha})},m_\beta\mathbf{1}_{r\in[\frac{3}{2}\gamma^{-\alpha}-1,\frac{3}{2}\gamma^{-\alpha}]})\\&+F(m^*_{A'}|m_{[(n+1)\gamma^{-\alpha},(n+2)\gamma^{-\alpha})},m_\beta\mathbf{1}_{r\in[\mathbf{1}_{[(n+\frac{1}{2})\gamma^{-\alpha},(n+\frac{1}{2})\gamma^{-\alpha}+1]}})\\&+|\Lambda|f_\beta(m_\beta)+3m_\beta^2\int_{r>0}dr\int_{r'<0}dr'J(|r-r'|)\end{aligned}\tag{E.18}$$

We claim that

$$\begin{aligned}&\inf{}^*F(m_{[\gamma^{-\alpha},(n+1)\gamma^{-\alpha})}|m_{[0,\gamma^{-\alpha})},m_{[(n+1)\gamma^{-\alpha},(n+2)\gamma^{-\alpha})})\geq\inf_{m\in\mathcal{G}}\mathcal{F}(m)\\&+F(\hat{m}_{[\gamma^{-\alpha},(n+1)\gamma^{-\alpha})}|-m_{[0,\gamma^{-\alpha})},m_{[(n+1)\gamma^{-\alpha},(n+2)\gamma^{-\alpha})})\end{aligned}\tag{E.19}$$

where $\hat{m}_{[\gamma^{-\alpha},(n+1)\gamma^{-\alpha})}$ is equal to $-m^*_A$ and $m^*_{A'}$ in A and A' and elsewhere it is equal to m_β.

To prove the claim we need to show that the sum of the last four terms in (E.18) is equal to the last term in (E.19). To this end we define $\mathcal{G}^*$ as the set of $m\in L^\infty(\mathbb{R},[-1,1])$ with the following properties:

- $m(r)=m_\beta$ for $r>(n+1/2)\gamma^{-\alpha}$ and for $r\leq 3/2\gamma^{-\alpha}$.
- $\theta^*_1(m)=1$, $\theta^*_n(m)=1$.
- $\Theta^*_i(m)=0$, $i=1,..,n$.

proceed as before with the $\inf^*$ over $m\in\mathcal{G}^*$ we get the following analogue of (E.18)

$$\begin{aligned}&\inf{}^*F(\hat{m}_{[\gamma^{-\alpha},(n+1)\gamma^{-\alpha})}|-m_{[0,\gamma^{-\alpha})},m_{[(n+1)\gamma^{-\alpha},(n+2)\gamma^{-\alpha})})\geq\inf_{m\in\mathcal{G}^*}\mathcal{F}(m)\\&+F(-m^*_A|-m_{[0,\gamma^{-\alpha})},m_\beta\mathbf{1}_{r\in[\frac{3}{2}\gamma^{-\alpha}-1,\frac{3}{2}\gamma^{-\alpha}]})\\&+F(m^*_{A'}|m_{[(n+1)\gamma^{-\alpha},(n+2)\gamma^{-\alpha})},m_\beta\mathbf{1}_{r\in[\mathbf{1}_{[(n+\frac{1}{2})\gamma^{-\alpha},(n+\frac{1}{2})\gamma^{-\alpha}+1]}})\\&+|\Lambda|f_\beta(m_\beta)+3m_\beta^2\int_{r>0}dr\int_{r'<0}dr'J(|r-r'|)\end{aligned}$$

and the claim follows because $\inf_{m\in\mathcal{G}^*}\mathcal{F}(m)=0$ as $m(r)\equiv m_\beta$ is in $\mathcal{G}^*$.

By the smoothness in $A\cup A'$ it follows that the last term in (E.19) is bounded from below by

$$F(\tilde{m}_{[\gamma^{-\alpha},(n+1)\gamma^{-\alpha})}|-m_{[0,\gamma^{-\alpha})},m_{[(n+1)\gamma^{-\alpha},(n+2)\gamma^{-\alpha})})-cn\gamma^{-\alpha}$$

where $\tilde{m} = \psi(\underline{m})$ with $\underline{m}$ in the range of ϕ (see the notation at the beginning of this appendix). The converse of (E.4) holds namely (see for instance Theorem 4.2.2.2 in [16])

$$\begin{aligned} -\gamma^{-1}\beta F(\tilde{m}_{[\gamma^{-\alpha},(n+1)\gamma^{-\alpha})}| - m_{[0,\gamma^{-\alpha})}, m_{[(n+1)\gamma^{-\alpha},(n+2)\gamma^{-\alpha})}) \\ \le \log Z^{+}_{n,\underline{m},(-s_0,s_{n+1})} + cn\ell^{+}_{\gamma}\gamma^{-1/2}\log\gamma^{-1} \\ \le \log Z^{+}_{\gamma,n}(-s_0,s_{n+1}) + cn\ell^{+}_{\gamma}\gamma^{-1/2}\log\gamma^{-1} \end{aligned} \tag{E.20}$$

Thus (7.1)will be proved once we show that

$$\inf_{m\in\mathcal{G}} \mathcal{F}(m) \ge \bar{f} + \gamma^{-1}c_0(n-n_0)\mathbf{1}_{n\ge n_0} \tag{E.21}$$

E.3 The instanton

We will prove (E.21) separately for "small and large n", the former in this subsection, latter in the next one.

Recalling the definition (1.7) of $\bar{f}$ we have

$$\inf_{m\in\mathcal{G}} \mathcal{F}(m) = \bar{f} = \int dr\, \bar{m}(r) \tag{E.22}$$

where $\bar{m}(r)$, called the instanton, solves

$$\bar{m}(r) = \tanh\Big(\int dr' J(|r-r'|)\bar{m}(r')\Big)$$

with $\bar{m}(r) \to \pm m_\beta$ as $r \to \pm\infty$. $\bar{m}(r)$ is modulo translations the only function with such properties, see for instance Theorem 8.1.2.1 in [16].

(E.22) proves (E.21) for n small, i.e. $n < n_0$ (which will be defined in the next subsection).

E.4 A large deviation estimate

The bound (E.22) being independent of n becomes inadequate when n is large, for that we use a large deviation estimate proved in Theorem 6.4.2.3 in [16]. Let $\mathcal{I}_0$ be the collection of intervals $[i\delta,(i+1)\delta)$ where $\eta*_i = 0$ and $\mathcal{I}_{\neq}$ a collection of consecutive pairs of such intervals with $\eta*_i = -\eta*_{i+1}$ and such that no interval appears twice in $\mathcal{I}_{\neq}$. One can check that there is $b > 0$ so that $|\mathcal{I}_0| + |\mathcal{I}_{\neq}| \ge bn$. Then by Theorem 6.4.2.3 in [16] there is a positive constant c so that

$$\inf F_3 \ge c\zeta^2\delta bn \tag{E.23}$$

(E.22) and (E.23) yield

$$\inf F_3 \ge \bar{f} + c\zeta^2\delta b(n-n_0)\mathbf{1}_{n>n_0}, \quad n_0 : c\zeta^2\delta bn_0 > \bar{f} \tag{E.24}$$

thus completing the proof of (E.21) and hence (7.1).

F Proof of Theorem 10

We split

$$\sum_{\underline{u}\in\mathcal{R}} w_\lambda(\underline{u}) = \sum_{k\geq 1}\psi_k(\lambda), \quad \psi_k(\lambda) = \sum_{\underline{u}\in\mathcal{R}:k(\underline{u})=k} w_\lambda(\underline{u}) \tag{F.1}$$

and define

$$\psi_0(\lambda) := \sum_{\underline{u}\in\mathcal{R}:k(\underline{u})=1;u_3\geq 3} w(\underline{u})e^{-\lambda|\underline{u}|} \tag{F.2}$$

We will prove that there is c so that for all γ small enough and all $\lambda\in[\frac{\epsilon_\gamma}{2},\frac{3\epsilon_\gamma}{2}]$,

$$\begin{aligned}&|\psi_0(\lambda)-(\frac{\epsilon_\gamma}{\lambda})^2|\leq c\epsilon_\gamma e^{2c_b\gamma^{-b}}, \quad |\phi_0(\lambda)|\leq c\epsilon_\gamma e^{4c_b\gamma^{-b}}, \quad \psi_0(\lambda)+\phi_0(\lambda)\equiv\psi_1(\lambda)\\ &\psi_2(\lambda)\leq c\epsilon_\gamma e^{8c_b\gamma^{-b}}, \quad \sum_{k>2}\psi_k(\lambda)\leq c\epsilon_\gamma\end{aligned} \tag{F.3}$$

By (7.7) $\epsilon_\gamma e^{8c_b\gamma^{-b}}$ is infinitesimal as $\gamma\to 0$ because $b\in(1/2,1)$. We start by proving the first inequality in (F.3):

F.1 The term ψ_0

We rewrite $\psi_0(\lambda)$ more explicitly; recalling (6.18),

$$\psi_0(\lambda) = \sum_{u_1,u_3\geq 3}\sum_{u_2,u_4\geq 2} e^{-\lambda(u_1+\dots+u_4)}e^{A^{(1)}_{\gamma,u_1}}\sum_{\{s_1,s_2,s_3,s_4\}} e^{G^{(3)}_{\gamma,u_3}(s_2,s_4)}e^{V^2_{u_2}(s_2,s_3)}e^{V^4_{u_4}(s_4,s_1)} \tag{F.4}$$

where V^2 and V^4 are defined in (6.8) and (6.9) and the sum over $\{s_1,s_2,s_3,s_4\}$ is restricted to $\theta(s_1)=\theta(s_2)=1$, $\theta(s_3)=\theta(s_4)=-1$.

We write

$$\psi_0(\lambda) = \sum_{u_1,u_3\geq 3}\sum_{u_2,u_4\geq 2} e^{-\lambda(u_1+\dots+u_4)}\sum_{\{s_1,s_2,s_3,s_4\}} e^{V^2_{u_2}(s_2,s_3)}e^{V^4_{u_4}(s_4,s_1)}+\psi_0'(\lambda) \tag{F.5}$$

By (5.3)

$$\begin{aligned}|\psi_0'(\lambda)|\leq \sum_{u_1,u_3\geq 3}\sum_{u_2,u_4\geq 2} e^{-\lambda(u_1+\dots+u_4)}\Big(&|1-e^{ae^{-b_0\gamma\ell_\gamma^+u_3}}|\sum_{\{s_1,s_2,s_3,s_4\}} e^{V^2_{u_2}(s_2,s_3)}e^{V^4_{u_4}(s_4,s_1)}\\ &+|1-e^{ae^{-b_0\gamma\ell_\gamma^+u_1}}|e^{ae^{-b_0\gamma\ell_\gamma^+u_3}}\sum_{\{s_1,s_2,s_3,s_4\}} e^{V^2_{u_2}(s_2,s_3)}e^{V^4_{u_4}(s_4,s_1)}\Big)\end{aligned} \tag{F.6}$$

We then have

$$\begin{aligned}|\psi_0'(\lambda)|\leq&\frac{1}{1-e^{-\lambda}}\sum_{u_3\geq 3}|1-e^{ae^{-b_0\gamma\ell_\gamma^+u_3}}|\sum_{u_2,u_4\geq 2}\sum_{\{s_1,s_2,s_3,s_4\}} e^{V^2_{u_2}(s_2,s_3)}e^{V^4_{u_4}(s_4,s_1)}\\ &+\{\sum_{u_1\geq 3}|1-e^{ae^{-b_0\gamma\ell_\gamma^+u_1}}|\}\{\frac{1}{1-e^{-\lambda}}e^{ae^{-b_0\gamma\ell_\gamma^+3}}\}\sum_{u_2,u_4\geq 2}\sum_{\{s_1,s_2,s_3,s_4\}} e^{V^2_{u_2}(s_2,s_3)}e^{V^4_{u_4}(s_4,s_1)}\end{aligned} \tag{F.7}$$

By (7.3) and (7.15)

$$\sum_{u_2,u_4\geq 2}\sum_{\{s_1,s_2,s_3,s_4\}} e^{V^2_{u_2}(s_2,s_3)}e^{V^4_{u_4}(s_4,s_1)} = \epsilon_\gamma^2 \tag{F.8}$$

so that

$$|\psi_0'(\lambda)| \leq c\lambda^{-1}\epsilon_\gamma^2 = c'\epsilon_\gamma \tag{F.9}$$

when $\lambda \in [\frac{\epsilon_\gamma}{2}, \frac{3\epsilon_\gamma}{2}]$.

Call $\psi_0^*(\lambda)$ the first term on the right hand side of (F.5), then

$$\psi_0^*(\lambda) = \sum_{u_1,u_3\geq 3} e^{-\lambda(u_1+u_3)}\Big(\epsilon_\gamma^2 + \sum_{u_2,u_4\geq 2}[e^{-\lambda(u_2+u_4)}-1]\epsilon_\gamma(u_2)\epsilon_\gamma(u_4)\Big) \tag{F.10}$$

We have

$$|\sum_{u_1,u_3\geq 3} e^{-\lambda(u_1+u_3)}\epsilon_\gamma^2 - (\frac{\epsilon_\gamma}{\lambda})^2| \leq c\epsilon_\gamma \tag{F.11}$$

In the last term in (F.10) we write

$$|e^{-\lambda(u_2+u_4)}-1| \leq |e^{-\lambda u_2}-1| + |e^{-\lambda u_4}-1|$$

Then by (7.6)-(7.7) the last term in (F.10) is bounded by

$$(\frac{1}{1-e^{-\lambda}})^2 2\epsilon_\gamma[\epsilon_\gamma c\lambda e^{2c_b\gamma^{-b}}] \leq c'\epsilon_\gamma e^{2c_b\gamma^{-b}}$$

which completes the proof of the first inequality in (F.3).

F.2 The term ϕ_0

By (F.3) $\phi_0 = \psi_1 - \psi_0$, hence

$$\phi_0(\lambda) = \sum_{u_1\geq 3, u_3\leq 2}\sum_{u_2,u_4\geq 2} e^{-\lambda(u_1+\dots+u_4)}e^{A^{(1)}_{\gamma,u_1}}\sum_{\{s_1,s_2,s_3,s_4\}} e^{V^3_{\gamma,u_3}(s_2,s_4)}e^{V^2_{u_2}(s_2,s_3)}e^{V^4_{u_4}(s_4,s_1)} \tag{F.12}$$

thus $\phi_0 > 0$. We bound

$$e^{-\lambda(u_2+\dots+u_4)}e^{A^{(1)}_{\gamma,u_1}} \leq 2$$

so that

$$\phi_0(\lambda) \leq \frac{2}{1-e^{-\lambda}}\sum_{u_3\leq 2; u_2,u_4\geq 2}\sum_{\{s_1,s_2,s_3,s_4\}} e^{V^3_{\gamma,u_3}(s_2,s_4)}e^{V^2_{u_2}(s_2,s_3)}e^{V^4_{u_4}(s_4,s_1)} \tag{F.13}$$

By (7.10) and (6.11)

$$\phi_0(\lambda) \leq \frac{2}{1-e^{-\lambda}}c^2\epsilon_\gamma^2 e^{4c_b\gamma^{-b}} \tag{F.14}$$

which proves the second inequality in (F.3).

F.3 The terms $\psi_k, k \geq 2$

Fix $k \geq 2$. In the sum over $u_{\ell,m}$, $\ell = 1,..,k$, $m = 1,..,4$ we distinguish the cases when $u_{\ell,1}$ and $u_{\ell,3}$ are ≥ 3 or ≤ 2, here $\ell \geq 2$ because $u_{1,1} \geq 3$ by the definition of $\psi_k(\lambda)$. We thus have 2^{2k-1} terms and when $u \equiv u_{\ell,1} \geq 3$ we bound $e^{-\lambda u} \leq 1$ and use (5.3) to bound (for γ small enough)

$$\sup_{s,s'} |e^{G^{(1)}_{\gamma,u}(s,s')} - e^{A^{(1)}_{\gamma,u}}| \leq 2\{\sup_{s,s'} |G^{(1)}_{\gamma,u}(s,s')| + |A^{(1)}_{\gamma,u}|\} \leq 4ae^{-b\gamma\ell_\gamma^+ u}$$

Thus the sum over any such $u_{\ell,1}$ is bounded by a constant. When $u \equiv u_{\ell,3} \geq 3$ we bound

$$\sup_{s,s'} e^{G^{(3)}_{\gamma,u}(s,s')} \leq 2$$

and the sum over any such $u_{\ell,3}$ is bounded by $\frac{2}{1-e^{-\lambda}}$. For the remaining terms we use (7.10) and get the bound, (d below is a suitable constant)

$$\psi_k(\lambda) \leq c^k (\frac{1}{1-e^{-\lambda}})^{k+1} \epsilon_\gamma^{2k} e^{4kc_b\gamma^{-b}} \leq d^k \epsilon_\gamma^{k-1} e^{4kc_b\gamma^{-b}} \tag{F.15}$$

(recall that $\epsilon_\gamma/\lambda \leq 3/2$). When $k = 2$:

$$\psi_2(\lambda) \leq \epsilon_\gamma d^2 e^{8c_b\gamma^{-b}} \tag{F.16}$$

while (for γ small enough)

$$\sum_{k\geq 3} \psi_k(\lambda) \leq \epsilon_\gamma \Big(\sum_{k\geq 3} d^k \epsilon_\gamma^{k/3} e^{4kc_b\gamma^{-b}} \Big) \leq c'\epsilon_\gamma \tag{F.17}$$

because $k - 2 \geq k/3$ for $k \geq 3$.

G Proof of Theorem 1

In this appendix we will prove the exponential bound in Theorem 1. The proof will use properties of the following Markov chain.

G.1 An auxiliary Markov chain

The auxiliary Markov chain is defined as follows. The state space is $\{0, 8, 9, ...\}$ namely all the non negative integers minus those from 1 to 7. Recall that the length of a quadruple in $\mathcal{R}$ is ≥ 8. We suppose that the transition probability denoted by $p(z, z')$ is such that:

- $p(0, 0) = 1$, $p(z, z') > 0$ for all $z \neq 0$ and all z' (in the state space).
- There are c and δ positive so that $p(z, z') \leq ce^{-\delta z'}$ for all $z \neq 0$ and z'.
- There is $\zeta > 0$ so that $p(z, 0) \geq \zeta$ for all $z \neq 0$.

We call P_{z_0} the law of the Markov chain $\{z_n\}$ starting from z_0 and with transition probability $p(z, z')$. We denote by n_+ the stopping time of the chain at 0, namely n_+ is such that $z_n > 0$ for all $n < n_+$ and $z_{n_+} = 0$. Notice that $z_n = 0$ for all $n \geq n_+$.

Theorem 24. *There are C and a positive so that for any $t > 0$ and $z_0 \neq 0$:*

$$P_{z_0}\Big[\sum_{n=1}^{n_+} z_n > t\Big] \leq Ce^{-at} \tag{G.1}$$

Proof. The proof is a straight consequence of the following two bounds.

$$P_{z_0}\Big[n_+ > n\Big] \leq (1 - \zeta)^n \tag{G.2}$$

which is proved using the third property of the transition probability. There are κ, C' and a' positive so that for all N

$$P_{z_0}\Big[\sum_{n=1}^{N} z_n > \kappa N\Big] \leq C'e^{-a'N} \tag{G.3}$$

which follows using the Chebishev exponential inequality and the exponential bound for the transition probability. □

G.2 Exponential decay

We go back to the proof of Theorem 1. We define for any integer $u \geq 8$

$$q(u) := \sum_{\underline{u} \in \mathcal{R}} \mathbf{1}_{|\underline{u}|=u} w_{\lambda_\gamma}(\underline{u}) \tag{G.4}$$

$q(u)$ is a probability on $\{u \in \mathbb{N} : u \geq 8\}$, $q(u) > 0$ for all u and

$$q(u) \leq c_\omega e^{-\omega u}, \quad \omega > 0 \tag{G.5}$$

Let $x_0 \in \mathbb{Z}$, set $X_0 = x_0$ and for $n \geq 1$

$$X_n = x_0 + u_1 + \cdots + u_n, \quad \underline{X} = (X_n)_{n \geq 0} \tag{G.6}$$

where the variables u_i are i.i.d. with law $q(u)$. We denote by $\mathbb{P}_{x_0}$ the law of $\underline{X}$. We want to prove that

$$\lim_{x_0 \to -\infty} \mathbb{P}_{x_0}\big[0 \in \underline{X}\big] = \alpha^{-1}, \quad \alpha = \sum_{u \geq 8} u q(u) \tag{G.7}$$

and that the convergence is exponentially fast. The existence of the limit (in a much more general setup) is well known in the literature as the Erdös, Feller and Pollard theorem. For completeness we will prove it in the remark at the end of this appendix, but for the time being we take it for proved so that we only need to show that the convergence in (G.7) is exponential.

We will first prove that there is $\zeta > 0$ such that

$$\mathbb{P}_{x_0}\big[0 \in \underline{X}\big] \geq \zeta \quad \text{for all } x_0 \leq -8 \tag{G.8}$$

(this is one of the assumptions used in the previous subsection).
Proof of (G.8). Let $n > 15$ then for any $x_0 < -n$

$$\mathbb{P}_{x_0}\big[\underline{X} \cap [-n, -8] = \emptyset\big] \leq \sum_{y < -n} \mathbb{P}_{x_0}\big[y \in \underline{X}\big] \sum_{u > |y| - 8} q(u) \leq \sum_{u > n+1-8} q(u) < 1 - q(8)$$

Then, recalling that $q(8) > 0$,

$$\mathbb{P}_{x_0}\big[0 \in \underline{X}\big] \geq q(8) \min_{u \in [8,n]} q(u) =: \zeta$$

We call $\underline{Y}$ the sequence defined in (G.6) with x_0 replaced by y_0. By (G.7) it will be enough to prove that there are b and c positive so that for all $x_0 \leq -8$:

$$\lim_{y_0 \to -\infty} \Big| \mathbb{P}_{x_0}\big[0 \in \underline{X}\big] - \mathbb{P}_{y_0}\big[0 \in \underline{Y}\big] \Big| \leq c e^{-b|x_0|} \tag{G.9}$$

Given $y_0 < x_0 < 0$ with $|x_0|$ large enough, we are going to define a coupling Q_{x_0,y_0} of $\underline{X}$ and $\underline{Y}$ so that denoting by E_{x_0,y_0} its expectation, (G.9) becomes

$$\lim_{y_0 \to -\infty} \Big| E_{x_0,y_0}\big[\mathbf{1}_{0 \in \underline{X}} - \mathbf{1}_{0 \in \underline{Y}}\big] \Big| \leq c e^{-b|x_0|} \tag{G.10}$$

To define Q_{x_0,y_0} we introduce "overshooting" variables $z_n, n \geq 0$ which are functions of $\underline{X}$ and $\underline{Y}$, setting $z_0 := x_0 - y_0$. Think of x_0 as a "target" for $\underline{Y}$ which is "shooting" from y_0: call then n_1 the first integer n such that $Y_{n_1} \geq x_0$ and define $z_1 = Y_{n_1} - x_0$ if this is not in $\{1, .., 7\}$ otherwise we set $z_1 = Y_{n_1+1} - x_0$. z_1 says how much we missed the target x_0, which instead has been hit if $z_1 = 0$.

If $z_1 > 0$ then $\underline{X}$, which is shooting from x_0, has the new target $x_0 - z_1$. Iterating this procedure by alternating targets for $\underline{Y}$ and $\underline{X}$ we get a sequence z_n which is stopped as soon as the target has been hit. Call n_+ such a hitting time: i.e. the first n such that $z_n = 0$. This means that there exist integers $\bar{n}$ and $\bar{m}$ such that

$$X_{\bar{n}} = Y_{\bar{m}}, \quad X_{\bar{n}} - x_0 = z_1 + \cdots + z_{n_+} \tag{G.11}$$

We denote $X_n = x_0 + \sum u_i$ and $Y_n = y_0 + \sum v_i$ so that the coupling Q_{x_0,y_0} is defined by stating how the u_i and v_i are coupled. The u_i and v_i are taken independently till $\bar{n}$ and $\bar{m}$ respectively, after that $u_{\bar{n}+i} = v_{\bar{m}+i}$. We also set $z_n = 0$ for $n > n_+$. We have

$$\Big| E_{x_0,y_0}\big[\mathbf{1}_{0 \in \underline{X}} - \mathbf{1}_{0 \in \underline{Y}}\big]\Big| \leq Q_{x_0,y_0}\big[z_1 + \cdots z_{n_+} > |x_0|\big] \tag{G.12}$$

The variables z_i under the law Q_{x_0,y_0} define a Markov chain which satisfies the properties stated in the previous subsection (the condition of exponential decay follows from (G.5)), then (G.10) follows from (G.1).

Remark. The above argument proves that the sequence $\mathbb{P}_{x_0}\big[0 \in \underline{X}\big]$ is Cauchy and it thus converges to some limit θ as $x_0 \to -\infty$. It is then easy to prove that $\theta = 1/\alpha$. Indeed, since

$$\mathbb{P}_{x_0}\big[\underline{X} \cap \mathbb{N} \neq \emptyset\big] = 1$$

then

$$1 = \mathbb{P}_{x_0}\big[\text{there exists } n\text{: } X_n < 0 \text{ and } X_{n+1} \geq 0\ \big] \tag{G.13}$$

hence

$$1 = \sum_{n>0} \mathbb{P}_{x_0}\big[-n \in \underline{X}\big] \sum_{u \geq n} q(u) \tag{G.14}$$

By taking the limit $x_0 \to -\infty$

$$1 = \theta \sum_{n>0} \sum_{u \geq n} q(u) = \theta\alpha \tag{G.15}$$

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