On standard two-intersection sets in $PG(r,q)$

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Abstract

In this paper, we extend and analyze in a finite projective space of any dimension the notion of standard two-intersection sets previously introduced in the projective plane by T.Penttila and G.F.Royle in [7], see also [1]. Moreover, given a pair of suitable distinct standard two-intersection sets in a finite projective space it is possible to get further standard two-intersection sets by applying elementary set-theoretical operations to the elements of the pair.

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1 Introduction and motivation

Let us denote by $PG(r,q)$ the r-dimensional space over the finite field $GF(q)$ with $q=p^h$ a prime power and by Π the pointset of $PG(r,q)$. A k-set K of $PG(r,q)$ is a set of k points of Π. By K^c we denote the set $\Pi \setminus K$, i.e. the complementary set of K. A k-set K is said a *set of type* $(m,n)_d$, with $m<n$, if each subspace of dimension d of $PG(r,q)$ meets K in either m or n points and both values occur. The integers m and n are known as the *intersection numbers* of K with respect to the d-dimensional subspaces. A *two-intersection set* is a set of type $(m,n)_{r-1}$. Such a set is also known as a *two-character set.* It is well known that a necessary condition for the existence of such a set is that $n-m$ divides q^{r-1}, see [8]. In [7] pag. 231 (see also [1] pag. 378) T.Penttila and G.F.Royle said *standard* the parameters of a k-set of type $(m,m+q^{\frac{1}{2}})_1$ in a projective plane of order q a square. Extending their definition, we say *standard* the parameters of a k-set of type $(m,m+q^{\frac{r-1}{2}})_{r-1}$ in $PG(r,q)$ with r odd or q a square. Furthermore, in this paper we will say *standard* a two-intersection set having standard parameters. In [3] the author proved that a standard two-intersection set has size $k_-=m[\theta_{r-1}(q)+q^{\frac{r-1}{2}}]/\theta_{r-2}(q)$ or $k_+=n[\theta_{r-1}(q)-q^{\frac{r-1}{2}}]/\theta_{r-2}(q)$ where $\theta_n(q):=\sum_{i=0}^{n}q^i$. If $k_-=k_+=\theta_r(q)/2$, then both r and q are odd and $m=(\theta_{r-1}(q)-q^{\frac{r-1}{2}})/2$. If a standard two-intersection set has size k_- (respectively k_+) we say that it has *size of type* k_- (respectively *size of type* k_+). Let us note that if two standard two-intersection sets H and K have size of different type, then $|H|\neq\theta_r(q)/2$ and $|K|\neq\theta_r(q)/2$.

In $PG(r,q)$ with r odd, classical examples of standard two-intersection sets are non-singular hyperbolic quadrics (having $m=\theta_{r-2}(q)$ and size of type k_-) and non-singular elliptic quadrics (having $n=\theta_{r-2}(q)$ and size of type k_+). Moreover, in $PG(r,q^2)$, classical examples are Baer subspaces (having $m=\theta_{r-2}(q)$ and size of type k_-) and non-singular Hermitian varieties (having: $m=(q^r+1)\theta_{\frac{r-3}{2}}(q^2)$ and size of type k_- if r

is odd; $n = (q^{r-1}+1)\theta_{\frac{r-2}{2}}(q^2)$ and size of type k_+ if r is even).

A number of people constructed standard two-intersection sets using disjoint unions of standard two-intersection sets having the same type of size, see, for istance, [2], [3], [5], and [6]. In this paper we prove that this is always possible. As a matter of fact, we prove the following three results.

Theorem 1.1. *Let H and K be two standard two-intersection sets in $PG(r,q)$ such that $H \neq K^c$ and $H \cap K = \emptyset$. Then $H \cup K$ is a standard two-intersection set if and only if H and K have size of the same type. Furthermore, H, K, and $H \cup K$ have size of the same type.*

Theorem 1.2. *Let H and K be two standard two-intersection sets in $PG(r,q)$ having size of the same type such that $H \cap K \neq \emptyset$. Then $H \cap K$ is a standard two-intersection set if and only if $H \cup K$ is a standard two-intersection set. Furthermore, H, K, $H \cap K$, and $H \cup K$ have size of the same type.*

Theorem 1.3. *Let H and K be two standard two-intersection sets in $PG(r,q)$ having size of different type such that $H \nsubseteq K$, $K \nsubseteq H$, $H \cap K \neq \emptyset$, $|H \setminus K| \neq \theta_r(q)/2$, and $|K \setminus H| \neq \theta_r(q)/2$. If $H \cap K$ (respectively $H \cup K$) is a standard two-intersection set, then $H \cup K$ (respectively $H \cap K$) is not a standard two-intersection set.*

2 Preliminary results

Let K be a standard k-set of type $(m, m+\delta)_{r-1}$ in $PG(r,q)$ with $\delta := q^{\frac{r-1}{2}}$. For each $i \in \{m, m+\delta\}$, let us denote by:

- t_i the number of hyperplanes meeting K in exactly i points;
- u_i the number of hyperplanes passing through a point not in K and meeting K in exactly i points;
- v_i the number of hyperplanes passing through a point of K and meeting K in exactly i points.

Set $\theta_d(q) := \sum_{i=0}^{d} q^i$. From [8], we get the following result.

Theorem 2.1. *Let K be a standard k-set of type $(m, m+\delta)_{r-1}$ in $PG(r,q)$. Then*

$$\theta_{r-2}(q)k^2 - [(2m+\delta)\theta_{r-1}(q) - \delta^2]k + m(m+\delta)\theta_r(q) = 0 \quad (1)$$

$$\delta t_{m+\delta} = (k-m)\theta_{r-1}(q) - mq\delta^2 \quad (2)$$

$$\delta t_m = (m+\delta)q\delta^2 - (k-m-\delta)\theta_{r-1}(q) \quad (3)$$

$$\delta u_{m+\delta} = (k-m)\theta_{r-2}(q) - m\delta^2 \quad (4)$$

$$\delta u_m = (m+\delta)\delta^2 - (k-m-\delta)\theta_{r-2}(q) \quad (5)$$

$$v_{m+\delta} = u_{m+\delta} + \delta \quad (6)$$

$$v_m = u_m - \delta \quad (7)$$

From [3] we get the following result.

Theorem 2.2. *Let K be a standard k-set of type $(m, m+\delta)_{r-1}$ in $PG(r,q)$. Then either $k = k_- = m[\theta_{r-1}(q) + \delta]/\theta_{r-2}(q)$ or $k = k_+ = (m+\delta)[\theta_{r-1}(q) - \delta]/\theta_{r-2}(q)$.*

Remark 2.3. *If $0 \le a \le b$, then it is easy to see that*

- $\theta_{b+1}(q) = \theta_a(q) + q^{a+1}\theta_{b-a}(q)$;
- $\theta_{b+1}(q) = 1 + q\theta_b(q)$;
- $\theta_{b+1}(q) = \theta_b(q) + q^{b+1}$.

Lemma 2.4. *If $r \ge 2$, then $\theta_r(q)\theta_{r-2}(q) = \theta^2_{r-1}(q) - q^{r-1}$.*

Proof. By Remark 2.3 we have

$$\begin{aligned}
&\theta_r(q)\theta_{r-2}(q) = [q^r + q^{r-1} + \theta_{r-2}(q)]\theta_{r-2}(q) = \\
&= q^r[q^{r-2} + \theta_{r-3}(q)] + q^{r-1}\theta_{r-2}(q) + \theta^2_{r-2}(q) = \\
&= q^{2(r-1)} + q^{r-1}[q\theta_{r-3}(q)] + q^{r-1}\theta_{r-2}(q) + \theta^2_{r-2}(q) = \\
&= q^{2(r-1)} + q^{r-1}[\theta_{r-2}(q) - 1] + q^{r-1}\theta_{r-2}(q) + \theta^2_{r-2}(q) = \\
&= q^{2(r-1)} + 2q^{r-1}\theta_{r-2}(q) + \theta^2_{r-2}(q) - q^{r-1} = \\
&= [q^{r-1} + \theta_{r-2}(q)]^2 - q^{r-1} = \theta^2_{r-1}(q) - q^{r-1}.
\end{aligned}$$

□

Lemma 2.5. *Let K be a standard k-set of type $(m, m+\delta)_{r-1}$ in $PG(r,q)$. Then*

- $(t_{m+\delta}, u_{m+\delta}, v_{m+\delta}) = (k, m, m+\delta)$ *if* $k = k_-$*;*
- $(t_m, u_m, v_m) = (k, m+\delta, m)$ *if* $k = k_+$*.*

Proof. First, let us suppose that $k = k_-$. By Theorem 2.2 we have $k\theta_{r-2}(q) = m[\theta_{r-1}(q)+\delta]$. So $(k-m)\theta_{r-2}(q) = m[\theta_{r-1}(q) - \theta_{r-2}(q)+\delta] = m\delta(\delta+1)$. By equation (2) and Remark 2.4, we get $\delta t_{m+\delta}\theta_{r-2}(q) = m\delta(\delta+1)\theta_{r-1}(q) - mq\delta^2\theta_{r-2}(q) = \delta m[\theta_{r-1}(q) + \delta(\theta_{r-1}(q) - q\theta_{r-2}(q))] = \delta m[\theta_{r-1}(q)+\delta] = \delta k\theta_{r-2}(q)$. So $t_{m+\delta} = k$. By equation (4), we get $\delta u_{m+\delta} = m\delta(\delta+1) - m\delta^2 = m\delta$. So $u_{m+\delta} = m$. By equation (6), we get $v_{m+\delta} = m+\delta$.

Now, let us suppose that $k = k_+$. By using very similar arguments, we get $(t_m, u_m, v_m) = (k, m+\delta, m)$. □

Lemma 2.6. *If K is a standard two-intersection set, then K^c is a standard two-intersection set too. Furhermore, K and K^c have size of the same type.*

Proof. K is a set of type $(m, m+\delta)_{r-1}$. If a hyperplane meets K in m points, then it meets K^c in $\theta_{r-1}(q) - m$ points. If a hyperplane meets K in $m+\delta$ points, then it meets K^c in $\theta_{r-1}(q) - m - \delta$ points. So K^c is a set of type $(d, d+\delta)_{r-1}$ with $d = \theta_{r-1}(q) - \delta - m$. It is clear that $|K^c| = \theta_r(q) - |K|$. If K has size of type k_- (respectively k_+), then by Theorem 2.2 we get $|K|\theta_{r-2}(q) = m[\theta_{r-1}(q)+\delta]$ (respectively $|K|\theta_{r-2}(q) = (m+\delta)[\theta_{r-1}(q) - \delta]$). By Lemma 2.4 we get $\theta_r(q)\theta_{r-2}(q) = [\theta_{r-1}(q) - \delta][\theta_{r-1}(q)+\delta]$. Then $|K^c|\theta_{r-2}(q) = d[\theta_{r-1}(q)+\delta]$ (respectively $|K^c|\theta_{r-2}(q) = (d+\delta)[\theta_{r-1}(q) - \delta]$) easily follows. So K^c has size of type k_- (respectively k_+). □

3 On sets having size of type k_-

In this section by H_m we will denote a standard set of type $(m, m+\delta)_{r-1}$ in $PG(r,q)$ having size of type k_-. Putting $\alpha := [\theta_{r-1}(q)+\delta]/\theta_{r-2}(q)$, by Theorem 2.2, we have $|H_m| = m\alpha$.

Lemma 3.1. *Let H be an H_m and H' be an $H_{m'}$. If $H \cap H' = \emptyset$ and $H' \neq H^c$, then $H \cup H'$ is an $H_{m+m'}$.*

Proof. Since $H \cap H' = \emptyset$ it is clear that $|H \cup H'| = (m + m')\alpha$. Furthermore, $m'|H| = m'(m\alpha) = m(m'\alpha) = m|H'|$. Now let us denote by x the number of the hyperplanes meeting H in $m + \delta$ points and H' in $m' + \delta$ points. If we prove that $x = 0$, then each hyperplane meets $H \cup H'$ in $m + m'$ points or in $m + m' + \delta$ points. So $H \cup H'$ is an $H_{m+m'}$. Let Q be a point not in H' and let us denote by u_i the number of hyperplanes passing through Q and meeting H' in exactly i points, with $i \in \{m', m' + \delta\}$. Since H' has size of type k_-, by Lemma 2.5, we have $u_{m'+\delta} = m'$. Let us denote by y the number of pairs (Q, π) where $Q \in H$ and π is a hyperplane through Q meeting H' in $m' + \delta$ points. Being $Q \notin H'$ we have

$$y = |H|u_{m'+\delta} = |H|m' \tag{8}$$

Now, if we consider the $t_{m'+\delta}$ hyperplanes meeting H' in exactly $m' + \delta$ points, then by Remark 2.5 we have $t_{m'+\delta} = |H'|$. Furthermore,

$$y = x(m + \delta) + (t_{m'+\delta} - x)m \tag{9}$$

since there are x hyperplanes meeting H in $m + \delta$ points and $t_{m'+\delta} - x$ hyperplanes meeting H in m points. By (8) and (9) we get $|H|m' = x\delta + mt_{m'+\delta} = x\delta + m|H'|$. Being $m'|H| = m|H'|$ we get $x\delta = 0$ and hence $x = 0$. □

Lemma 3.2. *Let H be an H_m and H'' be an $H_{m+m'}$. If $H \subset H''$, then $H'' \setminus H$ is an $H_{m'}$.*

Proof. Put $\tau := \theta_{r-1}(q) - \delta$. By Lemma 2.6 $(H'')^c$ is an $H_{\tau-(m+m')}$. Being $H \subset H''$, we have that $H \cap (H'')^c = \emptyset$. So, by Lemma 3.1, $H \cup (H'')^c$ is an $H_{\tau-m'}$. By Lemma 2.6, $(H \cup (H'')^c)^c$ is an $H_{\tau-(\tau-m')}$. Finally, being $H'' \setminus H = H^c \cap H'' = (H \cup (H'')^c)^c$, we have that $H'' \setminus H$ is an $H_{m'}$. □

By Lemmas 3.1 and 3.2 we immediately get the following

Theorem 3.3. *Let H be an H_m and H' be a set such that $H' \neq H^c$ and $H \cap H' = \emptyset$. Now put $H'' := H \cup H'$. Then H' is an $H_{m'}$ if and only if H'' is an $H_{m+m'}$.*

Theorem 3.4. *Let H be an H_m and H' be an $H_{m'}$ such that $H' \neq H^c$ and $H \cap H' \neq \emptyset$. Then $H \cap H'$ is an H_i if and only if $H \cup H'$ is an $H_{m+m'-i}$.*

Proof. First let us suppose that $H \cap H'$ is an H_i. Being $H \cap H' \subset H$, by Lemma 3.2 we have that $H \setminus H' = H \setminus (H \cap H')$ is an H_{m-i}. Now, being $(H \setminus H') \cap H' = \emptyset$, by Lemma 3.1 we have that $(H \setminus H') \cup H'$ is an $H_{m-i+m'}$. So $H \cup H' = (H \setminus H') \cup H'$ is an $H_{m+m'-i}$. Now let us suppose that $H \cup H'$ is an $H_{m+m'-i}$. Being $H' \subset H \cup H'$, by Lemma 3.2 we have that $(H \cup H') \setminus H' = H \setminus H'$ is an H_{m-i}. Now, being $H \setminus H' \subset H$, again by Lemma 3.2, we have that $H \setminus (H \setminus H')$ is an $H_{m-(m-i)}$. So $H \cap H' = H \setminus (H \setminus H')$ is an H_i. □

4 On sets having size of type k_+

For a better reading, we present in a new section the results on standard two-intersection sets having size of type k_+ although they are similar to those ones on sets having size of type k_- and also the proofs run in a very similar way.
In this section by K_m we will denote a standard set of type $(m, m+\delta)_{r-1}$ in $PG(r,q)$ having size of type k_+. Putting $\beta := [\theta_{r-1}(q) - \delta]/\theta_{r-2}(q)$, by Theorem 2.2, we have $|K_m| = (m+\delta)\beta$.

Lemma 4.1. *Let K be a K_m and K' be a $K_{m'}$. If $K \cap K' = \emptyset$ and $K' \neq K^c$, then $K \cup K'$ is a $K_{m+m'+\delta}$.*

Proof. Since $K \cap K' = \emptyset$ it is clear that $|K \cup K'| = \beta[(m + m' + \delta) + \delta]$. Furthermore, $(m' + \delta)|K| = (m' + \delta)[(m + \delta)\beta] = (m + \delta)[(m' + \delta)\beta] = (m + \delta)|K'|$. Now let us denote by x the number of the hyperplanes meeting K in m points and K' in m' points. If we prove that $x = 0$, then each hyperplane meets

$K \cup K'$ in $m+m'+\delta$ points or in $m+m'+2\delta$ points. So $K \cup K'$ is an $K_{m+m'+\delta}$. Let Q be a point not in K' and let us denote by u_i the number of hyperplanes passing through Q and meeting K' in exactly i points, with $i \in \{m', m'+\delta\}$. By Lemma 2.5 we have $u_{m'} = m' + \delta$. Let us denote by w the number of pairs (Q, π) where $Q \in K$ and π is a hyperplane through Q meeting K' in m' points. Being $Q \notin K'$ we have

$$w = |K| u_{m'} = |K|(m' + \delta) \tag{10}$$

Now if we consider the $t_{m'}$ hyperplanes meeting K' in exactly m' points, then by Lemma 2.5 we have $t_{m'} = |K'|$. Furthermore,

$$w = xm + (t_{m'} - x)(m + \delta) \tag{11}$$

since there are x hyperplanes meeting K in m points and $t_{m'} - x$ hyperplanes meeting K in $m+\delta$ points. By (10) and (11) we get $|K|(m'+\delta) = |K'|(m+\delta) - x\delta$. Being $|K|(m'+\delta) = |K'|(m+\delta)$, we get $x\delta = 0$ and so $x = 0$. □

We would like to point out that after submitting the paper we realized that the statement of Lemma 4.1 has already been proved in another way by L.Lane-Harward and T.Penttila, see [6], page 139, Theorem 2.

Lemma 4.2. *Let K be a K_m and K'' be a $K_{m+m'+\delta}$. If $K \subset K''$, then $K'' \setminus K$ is a $K_{m'}$.*

Proof. Put $\tau := \theta_{r-1}(q) - \delta$. By Lemma 2.6 $(K'')^c$ is a $K_{\tau-(m+m'+\delta)}$. Being $K \subset K''$, we have that $K \cap (K'')^c = \emptyset$. So, by Lemma 4.1, $K \cup (K'')^c$ is a $K_{\tau-m'}$. By Lemma 2.6, $(K \cup (K'')^c)^c$ is a $K_{\tau-(\tau-m')}$. Finally, being $K'' \setminus K = K^c \cap K'' = (K \cup (K'')^c)^c$, we have that $K'' \setminus K$ is a $K_{m'}$. □

By Lemmas 4.1 and 4.2 we immediately get the following

Theorem 4.3. *Let K be a K_m and K' a set such that $K \cap K' = \emptyset$ and $K' \neq K^c$. Now put $K'' := K \cup K'$. Then K' is a $K_{m'}$ if and only if K'' is a $K_{m+m'+\delta}$.*

Theorem 4.4. *Let K be a K_m and K' be a $K_{m'}$ such that $K' \neq K^c$ and $K \cap K' \neq \emptyset$. Then $K \cap K'$ is a K_i if and only if $K \cup K'$ is a $K_{m+m'-i}$.*

Proof. First let us suppose that $K \cap K'$ is a K_i. Being $K \cap K' \subset K$, by Lemma 4.2 we have that $K \setminus K' = K \setminus (K \cap K')$ is a $K_{m-i-\delta}$. Now, being $(K \setminus K') \cap K' = \emptyset$, by Lemma 4.1 we have that $(K \setminus K') \cup K'$ is a $K_{(m-i-\delta)+m'+\delta}$. So $K \cup K' = (K \setminus K') \cup K'$ is a $K_{m+m'-i}$.

Now let us suppose that $K \cup K'$ is a $K_{m+m'-i}$. Being $K' \subset K \cup K'$, by Lemma 4.2 we have that $(K \cup K') \setminus K' = K \setminus K'$ is a $K_{(m+m'-i)-m'-\delta} = K_{m-i-\delta}$. Now, being $K \setminus K' \subset K$, again by Lemma 4.2 we have that $K \setminus (K \setminus K')$ is a $K_{m-(m-i-\delta)-\delta} = K_i$. So $K \cap K' = K \setminus (K \setminus K')$ is a K_i. □

5 The proofs of the main results

Here we prove the three theorems claimed in the introduction.

5.1 The proof of Theorem 1.1

Proof. If H and K have size of the same type k_- (respectively k_+), then by Lemma 3.1 (respectively Lemma 4.1) $H \cup K$ is a standard a two-intersection set having size of type k_- (respectively k_+). Now, let us suppose that $H \cup K$ is a standard two-intersection set having size k_- (respectively k_+). So, as seen above, H and K can not have both size of type k_+ (respectively k_-). If both $H \cup K$ and H, or K, have size of type k_- (respectively k_+), then by Lemma 3.3 (respectively Lemma 4.3) K, or H, has size of type k_- (respectively k_+). So H and K have size of the same type k_- (respectively k_+). □

5.2 The proof of Theorem 1.2

Proof. If H and K have size of the same type k_- (respectively k_+), then by Theorem 3.4 (respectively by Theorem 4.4) $H \cap K$

is a standard two-intersection set having size of type k_- (respectively k_+) if and only if $H \cup K$ is a standard two-intersection set having size of type k_- (respectively k_+). □

5.3 The proof of Theorem 1.3

Proof. As we have already seen in the introduction, since H and K have different type of size we have that $|H| \neq \theta_r(q)/2$ and $|K| \neq \theta_r(q)/2$. Moreover, without loosing on generality, we can suppose that H has size of type k_- and K has size of type k_+.

Under the assumption that $H \cap K$ (respectively $H \cup K$) is a standard two-intersection set, we have to prove that $H \cup K$ (respectively $H \cap K$) is not a standard two-intersection set.

On the contrary, let us suppose that $H \cup K$ (respectively $H \cap K$) is a standard two-intersection set. Hence, in both cases, we have that $H \cap K$ and $H \cup K$ are standard two-intersection sets. First, let us suppose that $H \cap K$ has size of type k_-. By Lemma 3.2, we have that $H \setminus K = H \setminus (H \cap K)$ is a standard two-intersection set having size of type k_-. If $H \cup K$ has size of type k_-, then by Lemma 3.2 we have that $K = (H \cup K) \setminus (H \setminus K)$ is a standard two-intersection set having size of type k_-, a contradiction (being $|K| \neq \theta_r(q)/2$). So $H \cup K$ has size of type k_+. By Lemma 3.2 we have that $H \setminus K = (H \cup K) \setminus K$ is a standard two-intersection set having size of type k_+, a contradiction (being $|H \setminus K| \neq \theta_r(q)/2$). Finally. we have that $H \cap K$ has not size of type k_-. In a very similar way, we can prove that $H \cap K$ has not size of type k_+. So $H \cap K$ is not a standard two-intersection set, a contradiction. □

We conclude the paper by studying what happens when H and K are two standard two-intersection sets such that $H \subset K$.

Proposition 5.1. *Let H and K be two standard two-intersection sets such that $H \subset K$. Then $K \setminus H$ is a standard two-intersection set if and only if H and K have size of the same type. Furthermore, H, K, and $K \setminus H$ have size of the same type.*

Proof. If H and K have size of the same type k_- (respectively k_+), then by Lemma 3.2 (respectively Lemma 4.2) $K \setminus H$ is a standard two-intersection set having size of type k_- (respectively k_+). Now, let us suppose that $K \setminus H$ is a standard two-intersection set having size k_- (respectively k_+). So, as seen above, H and K can not have both size of type k_+ (respectively k_-). If both $K \setminus H$ and H have size of type k_- (respectively k_+), then by Lemma 3.1 (respectively Lemma 4.1) $K = (K \setminus H) \cup H$ has size of type k_- (respectively k_+). So H and K have size of the same type k_- (respectively k_+). If both $K \setminus H$ and K have size of type k_- (respectively k_+), then by Lemma 3.2 (respectively Lemma 4.2) $H = K \setminus (K \setminus H)$ has size of type k_- (respectively k_+). So H and K have size of the same type k_- (respectively k_+). □

Corollary 5.2. *Let H and K be two standard two-intersection sets having size of different type. If $H \subset K$, then $K \setminus H$ is a three-intersection set.*

Proof. Let H be a standard set of type $(m, m+\delta)$ and K be a standard set of type $(m', m'+\delta)$. A hyperplane can meet the set $K \setminus H$ in γ points with $\gamma \in \{m'-m-\delta, m'-m, m'-m+\delta\}$. Furthermore, all those values occurr, since by Proposition 5.1 $K \setminus H$ is not a two-intersection set. So $K \setminus H$ is a three-intersection set. □

Let us note that there are standard two-intersection sets H and K having size of different type such that $H \subset K$. Indeed, let $\{\Omega_1, \Omega_2, ..., \Omega_q, \Omega_{q+1}\}$ be an ovoidal fibration of $PG(3,q)$, i.e. a partition of $PG(3,q)$ into $q+1$ ovoids, see [4]. The set $K := \cup_{i=1}^{q} \Omega_i$ is a standard set of type $(q^2, q^2+q)_2$ having size $q(q^2+1) \neq \theta_3(q)/2$ of type k_+. Let A be a point of Ω_{q+1} and let π be the plane tangent to Ω_{q+1} in A. Now, let H be a line of π not through A. The line H is a standard set of type $(1, q+1)_2$ having size $q+1 \neq \theta_3(q)/2$ of type k_-. So H and K have size of different type. Being $H \cap \Omega_{q+1} = \emptyset$, it is $H \subset K$. So, $K \setminus H$ is a three-intersection set.

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