DE GRUYTER Adv. Nonlinear Anal. 2019; 8: 73–78

Research Article

Antonio Esposito, Francesco Leonetti* and Pier Vincenzo Petricca

Absence of Lavrentiev gap for non-autonomous functionals with (p, q)-growth

https://doi.org/10.1515/anona-2016-0198
Received September 10, 2016; accepted September 27, 2016

Abstract: We consider non-autonomous functionals of the form $\mathcal{F}(u,\Omega)=\int_\Omega f(x,Du(x))\,dx$, where $u\colon\Omega\to\mathbb{R}^N$, $\Omega\subset\mathbb{R}^n$. We assume that $f(x,z)$ grows at least as $|z|^p$ and at most as $|z|^q$. Moreover, $f(x,z)$ is Hölder continuous with respect to x and convex with respect to z. In this setting, we give a sufficient condition on the density $f(x,z)$ that ensures the absence of a Lavrentiev gap.

Keywords: Variational integrals, non-standard growth, regularity, Lavrentiev gap

MSC 2010: 49N60

1 Introduction

We consider variational integrals of the form

$$\mathcal{F}(u,\Omega)=\int_\Omega f(x,Du(x))\,dx \tag{1.1}$$

where $u\colon\Omega\subset\mathbb{R}^n\to\mathbb{R}^N$, $n\ge 2$, $N\ge 1$, $f\colon\Omega\times\mathbb{R}^{nN}\to\mathbb{R}$ is a Caratheodory function, and Ω is bounded and open. Moreover, we assume that, for exponents $1<p\le q$ and constants $\nu, L\in(0,+\infty)$, $c\in[0,+\infty)$, we have

$$\nu|z|^p-c\le f(x,z)\le L(1+|z|^q). \tag{1.2}$$

In the scalar case $N=1$, when $p=q$, the local minimizers $u\in W^{1,p}(\Omega)$ of (1.1) are locally Hölder continuous, see [13] and [19, p. 361]. If $p<q$ and q is far from p, then the local minimizers might be unbounded, see [12, 15, 17, 18] and [19, Section 5]. We are concerned with higher integrability for the gradient of minimizers. More precisely, assume that $u\colon\Omega\subset\mathbb{R}^n\to\mathbb{R}^N$ makes the energy (1.1) finite. Then the left-hand side of (1.2) implies that $Du\in L^p$. In addition to finite energy, we assume that u is a minimizer of (1.1) and we ask: Does the minimality of u boost the integrability of the gradient Du from L^p to L^q? The answer is given in [9]: We assume that (1.2) holds with $\nu=1$, $c=0$, we require that $z\to f(x,z)\in C^1(\mathbb{R}^{nN})$ and, for constants $\mu\in[0,1]$ and $\alpha\in(0,1]$, we assume that the following hold:

$$L^{-1}(\mu^2+|z_1|^2+|z_2|^2)^{\frac{p-2}{2}}|z_1-z_2|^2\le\left\langle\frac{\partial f}{\partial z}(x,z_1)-\frac{\partial f}{\partial z}(x,z_2);z_1-z_2\right\rangle, \tag{1.3}$$

$$\left|\frac{\partial f}{\partial z}(x,z)-\frac{\partial f}{\partial z}(y,z)\right|\le L|x-y|^\alpha(1+|z|^{q-1}). \tag{1.4}$$

Antonio Esposito, Pier Vincenzo Petricca, Universitá di L'Aquila, DISIM, Via Vetoio snc, 67100 L'Aquila, Italy,
e-mail: antonio.esposito4@hotmail.it, m.joshua1978@libero.it
***Corresponding author: Francesco Leonetti,** Universitá di L'Aquila, DISIM, Via Vetoio snc, 67100 L'Aquila, Italy,
e-mail: leonetti@univaq.it

The exponents p and q must be close enough, i.e.,

$$1 < p \le q < p\Big(\frac{n+\alpha}{n}\Big). \tag{1.5}$$

We recall that n is the dimension of the space where x lives, i.e., $x \in \Omega \subset \mathbb{R}^n$. Let us remark that (1.5) asserts that the smaller α is, the closer p and q must be. In addition, we assume that the Lavrentiev gap on u is zero:

$$\mathcal{L}(u, B_R) = 0 \tag{1.6}$$

for every ball $B_R \subset\subset \Omega$. Such a Lavrentiev gap will be defined in the next section. Under (1.2)–(1.6), the local minimizers u of (1.1) enjoy higher integrability, i.e., $Du \in L^q_{\mathrm{loc}}(\Omega)$. Checking (1.6) is not easy for non-autonomous densities $f(x,z)$; it has been done in [9] for some model functionals using some arguments due to [22]. The aim of the present paper is to give a sufficient condition on the density $f(x,z)$ that ensures the vanishing of the Lavrentiev gap (1.6), see Theorem 3.1 (4).

2 Preliminaries

In the following Ω will be an open, bounded subset of $\mathbb{R}^n$, $n \ge 2$, and we will denote

$$B_R \equiv B_R(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < R\},$$

where, unless differently specified, all the balls considered will have the same center. We assume that $f\colon \Omega \times \mathbb{R}^{nN} \to \mathbb{R}$ is a Caratheodory function verifying the (p,q)-growth (1.2) with $\nu = 1$ and $c = 0$. Moreover, we assume that $z \to f(x,z)$ is convex. Due to the non-standard growth behavior of f, we shall adopt the following notion of local minimizer.

Definition 2.1. A function $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ if and only if $x \mapsto f(x, Du(x)) \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_{\operatorname{supp}\phi} f(x, Du(x))\,dx \le \int_{\operatorname{supp}\phi} f(x, Du(x) + D\phi(x))\,dx,$$

for any $\phi \in W^{1,1}(\Omega;\mathbb{R}^N)$ with $\operatorname{supp}\phi \subset\subset \Omega$.

Let us now explain the Lavrentiev gap. We adopt the viewpoint of [4], see also [3]. Let us set

$$X = W^{1,p}(B_R;\mathbb{R}^N), \quad Y = W^{1,q}_{\mathrm{loc}}(B_R;\mathbb{R}^N) \cap W^{1,p}(B_R;\mathbb{R}^N).$$

We consider functionals $\mathcal{G}\colon X \to [0,+\infty]$ that are sequentially weakly lower semicontinuous (s.w.l.s.c.) on X, and we set

$$\overline{\mathcal{F}}_X = \sup\{\mathcal{G}\colon X \to [0,+\infty] \mid \mathcal{G} \text{ s.w.l.s.c., } \mathcal{G} \le \mathcal{F} \text{ on } X\},$$
$$\overline{\mathcal{F}}_Y = \sup\{\mathcal{G}\colon X \to [0,+\infty] \mid \mathcal{G} \text{ s.w.l.s.c., } \mathcal{G} \le \mathcal{F} \text{ on } Y\}.$$

We have $\overline{\mathcal{F}}_X \le \overline{\mathcal{F}}_Y$, and we define the Lavrentiev gap as follows:

$$\mathcal{L}(v, B_R) = \overline{\mathcal{F}}_Y(v) - \overline{\mathcal{F}}_X(v) \quad \text{for every } v \in X,$$

when $\overline{\mathcal{F}}_X(v) < +\infty$ and $\mathcal{L}(v, B_R) = 0$ if $\overline{\mathcal{F}}_X(v) = +\infty$. Since $f(x,z)$ is convex with respect to z, standard weak lower semicontinuity results give $\overline{\mathcal{F}}_X = \mathcal{F}$ (see, for instance, [14, Chapter 4]).

The following lemma will be used in the proof of the main theorem (see [4]).

Lemma 2.2. *Let $u \in W^{1,p}(B_R;\mathbb{R}^N)$ be a function such that $\mathcal{F}(u, B_R) < +\infty$. Then $\mathcal{L}(u, B_R) = 0$ if and only if there exists a sequence $\{u_m\}_{m\in\mathbb{N}} \subset W^{1,q}_{\mathrm{loc}}(B_R;\mathbb{R}^N) \cap W^{1,p}(B_R;\mathbb{R}^N)$ such that*

$$u_m \rightharpoonup u \quad \text{weakly in } W^{1,p}(B_R;\mathbb{R}^N)$$

and

$$\mathcal{F}(u_m, B_R) \to \mathcal{F}(u, B_R).$$

3 Main section

Theorem 3.1. *Let* $f\colon \Omega \times \mathbb{R}^{nN} \to \mathbb{R}$ *be a function satisfying the following conditions:*

(1) $|z|^p \le f(x,z) \le L(1+|z|^q)$, $1 < p < q < +\infty$,

(2) $|f(x,z) - f(\tilde{x},z)| \le H|x-\tilde{x}|^\alpha(1+|z|^q)$, $0 < \alpha \le 1$,

(3) $z \mapsto f(x,z)$ *is convex for all* x,

(4) *for* $B_R \subset\subset \Omega$, $\varepsilon_0 \in (0,1]$ *such that* $B_{R+2\varepsilon_0} \subset\subset \Omega$, $x \in B_R$ *and* $\varepsilon \in (0,\varepsilon_0)$, *there exists* $\tilde{y} = \tilde{y}(x,\varepsilon) \in \overline{B(x,\varepsilon)}$ *such that for* $z \in \mathbb{R}^{nN}$ *and* $y \in \overline{B(x,\varepsilon)}$, *we have* $f(\tilde{y},z) \le f(y,z)$.

Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ *such that* $x \mapsto f(x,Du(x)) \in L^1_{\mathrm{loc}}(\Omega)$ *and assume that*

$$q \le p\Big(\frac{n+\alpha}{n}\Big). \tag{3.1}$$

Then $\mathcal{L}(u,B_R) = 0$ *for all* $B_R \subset\subset \Omega$.

Remark 3.2. Let us now explain condition (4). For every fixed z, $y \to f(y,z)$ is continuous, so the minimization of $f(y,z)$ when $y \in \overline{B(x,\varepsilon)}$ gives a minimizer y depending on x, ε and z. Condition (4) asks for independence on z, i.e., there exists a minimizer $\tilde{y}$ that works for every z. We will first give the proof of Theorem 3.1, and then we will show examples of densities $f(x,z)$ satisfying condition (4).

Remark 3.3. Let us compare (1.5) with (3.1). When proving the absence of the Lavrentiev gap $\mathcal{L}(u,B_R)=0$, the borderline case $q = p(\frac{n+\alpha}{n})$ is allowed but we need strict inequality (1.5) when proving higher integrability of minimizers, see [9, p. 32].

Proof. Consider $0 < \varepsilon < \varepsilon_0 \le 1$ as in hypothesis (4), then $u \in W^{1,p}(B_{R+2\varepsilon_0};\mathbb{R}^N)$ and

$$\mathcal{F}(u,B_{R+2\varepsilon_0}) = \int_{B_{R+2\varepsilon_0}} f(x,Du(x))\,dx < +\infty.$$

Let us denote $u_\varepsilon(x) := (u * \phi_\varepsilon)(x)$, the usual mollification, where $x \in B_R$, and define

$$f_\varepsilon(x,z) = \min_{y\in\overline{B(x,\varepsilon)}} f(y,z). \tag{3.2}$$

By definition, it follows that

$$|Du_\varepsilon(x)| \le \Big(\int_{B_{R+2\varepsilon_0}} |Du(y)|^p\,dy\Big)^{\frac{1}{p}} \Big(\int_{\mathbb{R}^n} |\phi_\varepsilon(y)|^{p'}\Big)^{\frac{1}{p'}} \le C\varepsilon^{-\frac{n}{p}},$$

where $C = C(\|Du\|_{L^p}) > 1$. Moreover, by the Hölder continuity hypothesis (i.e., hypothesis (2)), we have

$$f_\varepsilon(x,z) \ge f(x,z) - H\varepsilon^\alpha(1+|z|^q). \tag{3.3}$$

Note that the left-hand side of hypothesis (1) gives

$$|z|^p \le f_\varepsilon(x,z). \tag{3.4}$$

Now we observe that is possible to find $K = K(p,q,\|Du\|_{L^p},H) < +\infty$ such that

$$f(x,z) \le Kf_\varepsilon(x,z) + H, \quad x \in B_R,\ |z| \le C\varepsilon^{-\frac{n}{p}}. \tag{3.5}$$

Indeed, let us fix $\delta \in (0,1)$ and observe that, using (3.3), (3.4) and $|z| \le C\varepsilon^{-\frac{n}{p}}$, we get

$$\begin{aligned}
f_\varepsilon(x,z) &= \delta f_\varepsilon(x,z) + (1-\delta)f_\varepsilon(x,z)\\
&\ge \delta f(x,z) - \delta H\varepsilon^\alpha(1+|z|^q) + (1-\delta)|z|^p\\
&= \delta f(x,z) - \delta H\varepsilon^\alpha|z|^q + (1-\delta)|z|^p - \delta H\varepsilon^\alpha\\
&= \delta f(x,z) - \delta H\varepsilon^\alpha|z|^p|z|^{q-p} + (1-\delta)|z|^p - \delta H\varepsilon^\alpha\\
&\ge \delta f(x,z) - \delta C^{q-p}H\varepsilon^{\alpha+(\frac{p-q}{p})n}|z|^p + (1-\delta)|z|^p - \delta H\varepsilon^\alpha\\
&\ge \delta f(x,z) + (1-\delta-\delta C^{q-p}H)|z|^p - \delta H,
\end{aligned}$$

where the last estimate relies on the fact that $\frac{q}{p} \le \frac{n+\alpha}{n}$, $0 < \alpha \le 1$ and $0 < \varepsilon < 1$. Then (3.5) follows choosing $K = \frac{1}{\delta} = 1 + C^{q-p}H$. Now, using hypothesis (4), Jensen's inequality and (3.2), we obtain

$$\begin{aligned} f_\varepsilon(x, Du_\varepsilon(x)) &= f(\tilde{y}, Du_\varepsilon(x)) \\ &\le \int_{B(x,\varepsilon)} f(\tilde{y}, Du(y))\phi_\varepsilon(x-y)\,dy \\ &\le \int_{B(x,\varepsilon)} f(y, Du(y))\phi_\varepsilon(x-y)\,dy \\ &= (f(\cdot\,, Du(\cdot)) * \phi_\varepsilon)(x) \\ &=: f(\cdot\,, Du(\cdot))_\varepsilon(x). \end{aligned} \tag{3.6}$$

Therefore, using (3.5), we have

$$f(x, Du_\varepsilon(x)) \le Kf(\cdot\,, Du(\cdot))_\varepsilon(x) + H.$$

Finally, since $f(\cdot\,, Du(\cdot))_\varepsilon(x) \to f(x, Du(x))$ strongly in $L^1(B_R)$, by recalling that $u_\varepsilon \to u$ in $W^{1,p}(B_R;\mathbb{R}^N)$, and by using a well-known variant of Lebesgue's dominated convergence theorem and Lemma 2.2, the proof is completed. □

Remark 3.4. We note that our assumption (4) is very close to [22, assumption (2.3)]. Our proof is inspired by the one of [9, Lemma 13], which, in turn, is based on some arguments used in [22].

Remark 3.5. Now we give some examples of functions for which Theorem 3.1 is valid.

(1) $f(x,z) = b(z) + a(x)c(z)$ with the following conditions:
  (i) $a \in C^{0,\alpha}(\bar{\Omega})$ and $a(x) \ge 0$ for all x,
  (ii) b and c are convex functions such that

$$|z|^p \le b(z) \le H(|z|^q + 1) \quad \text{for } H \ge 1, \qquad 0 \le c(z) \le L(|z|^q + 1) \quad \text{for } L \ge 1.$$

  For instance, we can consider the following functions:
  - $f(x,z) = b(z)$, independent of x.
  - $f(x,z) = |z|^p + a(x)|z|^q$. This example has been already dealt with in [9]; see also [1, 6–8, 10, 22].
  - $f(x,z) = |z|^p + a(x)|z|^p \ln(e + |z|)$. This example is taken from [1, 2].
  - $f(x,z) = |z|^2 + a(x)[\max\{z_n, 0\}]^q$, where $q > 2$. This example is inspired by [21].
  - $f(x,z) = |z|^p + a(x)[|z_1 - z_2|^q + |z_1|^q]$, where $1 < p < q < +\infty$. This example is inspired by [5].

(2) $f(x,z) = \sum_{i=1}^k [b_i(z) + a_i(x)c_i(z)]$, where $k \in \mathbb{N}$ and a_i, b_i, c_i verify the corresponding conditions of the previous example for all $i \in \{1, 2, \dots, k\}$.

(3) $f(x,z) = h(\sum_{i=1}^k [b_i(z) + a_i(x)c_i(z)])$, where, in addition to the previous conditions, h is increasing, convex, Lipschitz and such that $s \le h(s) \le \alpha s + \beta$.

(4) $f(x,z) = h(a(x), z)$ with the following conditions:
  (i) $t \mapsto h(t,z)$ is increasing,
  (ii) h is convex with respect to the second variable,
  (iii) $a \in C(\bar{\Omega})$,
  (iv) f verifies assumptions (1) and (2) of Theorem 3.1.
  For example,

$$f(x,z) = |z|^p + (e + \tilde{a}(x)|z|)^{a + b\sin(\ln(\ln(e + \tilde{a}(x)|z|)))},$$

  where $\tilde{a} \in C^{0,\alpha}(\bar{\Omega})$, $\tilde{a}(x) \ge 0$ for all x, $a \ge 1 + 2b\sqrt{2}$ and $b > 0$. In order to satisfy the non-standard (p,q)-growth condition, we can consider $1 < p < a + b \le q$. This example is inspired by [11, 20], see also [16].

Remark 3.6. Hypothesis (4) was used during the proof of Theorem 3.1 in order to obtain the second increase in (3.6). Now we want to show an example of a function for which hypothesis (4) fails. Let us consider $\Omega = B(0,1) \subset \mathbb{R}^2$ and the function $f\colon B(0,1) \times \mathbb{R}^{2N} \to \mathbb{R}$ such that

$$f(x,z) = |z|^p + a(x)(|z|^q - 1) + 1,$$

where, for $x = (x_1, x_2)$,

$$a(x) = \begin{cases} x_2 & \text{if } x_2 > 0, \\ 0 & \text{if } x_2 \le 0. \end{cases}$$

In this case the minimum point of the function changes depending on the choice of z. Indeed, let us consider $B_R = B(0, \frac{1}{2})$, $\varepsilon_0 = \frac{1}{8}$, $x = 0$. Then we deal with the two cases: $|z| = 0$ and $|z| = 2$.

When $|z| = 0$, we have

$$f(y, z) = \begin{cases} -y_2 + 1 & \text{if } y_2 > 0, \\ 1 & \text{if } y_2 \le 0, \end{cases}$$

and then the minimum value in $\overline{B(0, \varepsilon)}$ is reached for $\tilde{y} = (0, \varepsilon)$.

If $|z| = 2$, then

$$f(y, z) = \begin{cases} 2^p + y_2(2^q - 1) + 1 & \text{if } y_2 > 0, \\ 2^p + 1 & \text{if } y_2 \le 0, \end{cases}$$

and therefore, in this situation, $\tilde{y}$ is any point (y_1, y_2) such that $y_2 \le 0$.

Corollary 3.7. *Let $h\colon \Omega \times \mathbb{R}^{nN} \to \mathbb{R}$ be a function verifying Theorem 3.1 and let $f\colon \Omega \times \mathbb{R}^{nN} \to \mathbb{R}$ be a function such that*

$$h(x, z) - c_1 \le f(x, z) \le h(x, z) + c_2,$$

where $c_1, c_2 \ge 0$. We consider $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ such that $x \mapsto f(x, Du(x)) \in L^1_{\mathrm{loc}}(\Omega)$ and assume $q \le p(\frac{n+\alpha}{n})$. Then $\mathcal{L}(u, B_R) = 0$ for all $B_R \subset\subset \Omega$.

Proof. We follow the proof of [9, Theorem 6]. By the proof of Theorem 3.1, if $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ is such that $h(x, Du(x)) \in L^1_{\mathrm{loc}}(\Omega)$, then there exists a sequence $\{u_m\}_{m\in\mathbb{N}} \subset W^{1,q}(B_R; \mathbb{R}^N)$ such that $u_m \to u$ strongly in $W^{1,p}(B_R; \mathbb{R}^N)$, $Du_m(x) \to Du(x)$ a.e., $h(x, Du_m(x)) \to h(x, Du(x))$ a.e., and $\int_{B_R} h(x, Du_m(x)) \to \int_{B_R} h(x, Du(x))$. Using a well-known variant of Lebesgue's dominated convergence theorem, we have that

$$\int_{B_R} f(x, Du_m(x)) \to \int_{B_R} f(x, Du(x)),$$

and then, by Lemma 2.2, $\mathcal{L}(u, B_R) = 0$. □

Funding: The authors thank GNAMPA and UNIVAQ for the support.

References

[1] P. Baroni, M. Colombo and G. Mingione, Harnack inequalities for double phase functionals, *Nonlinear Anal.* **121** (2015), 206–222.

[2] P. Baroni, M. Colombo and G. Mingione, Non-autonomous functionals, borderline cases and related functional classes, *St. Petersburg Math. J.* **27** (2016), 347–379.

[3] M. Belloni and G. Buttazzo, A survey of old and recent results about the gap phenomenon in the calculus of variations, in: *Recent Developements in Well-Posed Variational Problems*, Math. Appl. 331, Kluwer Academic, Dordrecht (1995), 1–27.

[4] G. Buttazzo and V. J. Mizel, Interpretation of the Lavrentiev phenomenon by relaxation, *J. Funct. Anal.* **110** (1992), no. 2, 434–460.

[5] A. Cianchi, Local boundedness of minimizers of anisotropic functionals, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **17** (2000), no. 2, 147–168.

[6] F. Colasuonno and M. Squassina, Eigenvalues for double phase variational integrals, *Ann. Mat. Pura Appl. (4)* **195** (2016), no. 6, 1917–1959.

[7] M. Colombo and G. Mingione, Bounded minimisers of double phase variational integrals, *Arch. Ration. Mech. Anal.* **218** (2015), 219–273.

[8] M. Colombo and G. Mingione, Regularity for double phase variational problems, *Arch. Ration. Mech. Anal.* **215** (2015), 443–496.

[9] L. Esposito, F. Leonetti and G. Mingione, Sharp regularity for functionals with (p, q)-growth, *J. Differential Equations* **204** (2004), no. 1, 5–55.

[10] I. Fonseca, G. Mingione and I. Maly, Scalar minimizers with fractal singular sets, *Arch. Ration. Mech. Anal.* **172** (2004), 295–307.
[11] N. Fusco and C. Sbordone, Higher integrability of the gradient of minimizers of functionals with non standard growth conditions, *Comm. Pure Appl. Math.* **43** (1990), 673–683.
[12] M. Giaquinta, Growth conditions and regularity, a counterexample, *Manuscripta Math.* **59** (1987), 245–248.
[13] M. Giaquinta and E. Giusti, On the regularity of the minima of variational integrals, *Acta Math.* **148** (1982), 31–46.
[14] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific, Singapore, 2003.
[15] M. C. Hong, Some remarks on the minimizers of variational integrals with non standard growth conditions, *Boll. Unione Mat. Ital. VII. Ser. A* **6** (1992), 91–101.
[16] F. Leonetti, Weak differentiability for solutions to elliptic systems with p, q-growth conditions, *Ann. Mat. Pura Appl. (4)* **162** (1992), 349–366.
[17] P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions, *Arch. Ration. Mech. Anal.* **105** (1989), 267–284.
[18] P. Marcellini, Regularity and existence of solutions of elliptic equations with p, q-growth conditions, *J. Differential Equations* **90** (1991), 1–30.
[19] G. Mingione, Regularity of minima: An invitation to the dark side of the calculus of variations, *Appl. Math.* **51** (2006), 355–426.
[20] G. Talenti, Buondedness of minimizers, *Hokkaido Math. J.* **19** (1990), 259–279.
[21] Q. Tang, Regularity of minimizers of nonisotropic integrals in the calculus of variations, *Ann. Mat. Pura Appl. (4)* **164** (1993), 77–87.
[22] V. V. Zhikov, On Lavrentiev's phenomenon, *Russ. J. Math. Phys.* **3** (1995), 249–269.