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A simple testing procedure for unit root and model specification

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ABSTRACT

Tests for the joint null hypothesis of a unit root based on the components representation of a time series are developed. The proposed testing procedure is designed to detect a unit root as well as guide the practitioner regarding the specification of trend component of a time series. The limiting null distributions of the newly developed F-statistics are derived. Finite sample simulation evidence shows that the F-statistics maintain their size, and have power against the trend-break stationary alternative. The use of our methodology is illustrated through an empirical examination of the US–UK real exchange rate, the UK industrial production, and the UK CPI series.

1. Introduction

There is considerable literature regarding the statistical theory and application of unit root tests in time series, see Patterson (2011, 2012). Unit root tests have been routinely used in the empirical analysis to evaluate the dynamics of various economic time series such as aggregate output, industrial production, interest rates, and consumption. Knowledge of whether a time series contains a unit root or not provides guidance as to how the underlying trend in the series should be modeled as well as determine the degree of persistence in the economic variable. Since the publication of the seminal papers by Dickey and Fuller (1979, 1981), there has been a large literature devoted to devising unit root tests for different specification of the trend. For instance, Perron (1989) argued that inference drawn from the Dickey-Fuller unit root tests may be misleading if the underlying model ignores a break in the mean or trend of the time series that may result from major events such as the oil price shock or the Great Depression.

While the tests by Perron (1989), as well as further extensions, do account for the presence of structural breaks, the practitioner cannot ascertain whether inferences drawn by these tests are affected by the possible mis-specification of the underlying model. Therefore, in this paper, we propose a methodology that allows the practitioner to test for the presence of a unit root and, at the same time, assess the validity of the underlying model. We focus on the class of unit root

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tests that allow for one structural break under both the null and the alternative hypothesis. As in Perron (1989, 1990b), we consider three different characterizations of the break under the trend-stationary alternative hypothesis: (a) the level shift model that allows for a one-time shift the mean; (b) the crash model that allows for a one-time shift in the intercept of the underlying trend; and (c) the mixed model that allows for a simultaneous break in the intercept and the slope.[1]

We follow the approach of Schmidt and Phillips (1992) used by Popp (2008) to develop a new class of Perron-type unit root tests.[2] We propose F-statistics for the joint null hypothesis of a unit root and model specification based on the reduced form regressions implied by the conventional components representation of the underlying data generating process. Model specification is tested using restrictions on the mean or trend function coefficients implied by the corresponding reduced form regressions. Specifically, the F-statistics for the level shift model and the crash model are based on the joint null hypothesis that there is a unit root and that the coefficients of the intercept and the intercept break dummy are both equal to zero. The F-statistic for the mixed model is based on the joint null hypothesis that there is a unit root and that the coefficients of the time-trend and the lagged trend-break dummy are both equal to zero.

Tests for the joint null hypothesis of a unit root are also considered by Dickey and Fuller (1981), Hall (1992), Perron (1990a), Carrion-i-Silvestre and Sanso (2006), Sen (2007), and Pitarakis (2014).[3] While our approach is similar to that of Carrion-i-Silvestre and Sanso (2006), we differ in two important respects. First, we use the conventional components representation of the underlying data generating process as suggested by Schmidt and Phillips (1992). As argued by Schmidt and Phillips (1992), the interpretation of the mean and the time trend coefficients is the same under both the unit root null hypothesis as well as the trend stationarity alternative hypothesis. Second, we propose a testing procedure that simultaneously tests for both unit root and model mis-specification. It is also worth noting that while our test is designed to assess the validity of the underlying trend specification, the test of Pitarakis (2014) is designed to test for changes in the level of persistence.

Our testing procedure for unit root and model mis-specification is based on using both Popp's (2008) version of the Perron-type statistics along with our newly developed F-statistics. We derive the asymptotic distribution of the new F-tests under the corresponding joint null hypothesis, and tabulate their finite sample critical values. Under the null hypothesis of unit root, if the model is correctly specified, we would expect that both the Perron-type statistics and the F-statistics will be insignificant. Further, we would expect the Perron-type statistics to be insignificant and the F-tests to be significant if the model is mis-specified in the presence of a unit root. So, the practitioner can distinguish the case when the underlying model is mis-specified or not for a series that contains a unit root. On the other hand, under the trend-break stationary alternative, we would expect both the Perron-type statistics and the F-statistics to be significant, irrespective of whether the model is mis-specified or not. However, in this case, the practitioner can use conventional testing procedures that assume stationarity to ascertain the appropriate specification of the trend component before using the series for further modeling and/or forecasting purposes. We argue, therefore, that our F-test is a useful supplement to the Perron-type unit root tests. Simulation evidence presented in this paper shows that the F-statistics maintain their size in finite samples, and exhibit power that increases with the sample size.

We illustrate the use of our statistics by examining the real exchange rate between the US Dollar and the UK Pound series (1971Q1–2012Q4), the UK industrial production series (1957Q1–2012Q2), and the UK CPI series (1990Q1–2012Q4). We use the level shift model for the real exchange rate series, the crash model for the industrial production series, and the mixed model for the CPI series. The estimated break-date is 1987Q1 for the real exchange rate series, 1974Q1 for the UK industrial production series, and 2008Q2 for the UK CPI series. For all three series, we reject the joint null of a unit root. However, Popp's (2008) statistic is only significant for the UK CPI series. Therefore, we conclude that the US/UK real exchange rate series and the UK industrial production series contain a unit root, but the trend component of these series are mis-specified. The UK CPI series, on the other hand, is trend-break stationary, and so the correct specification of its trend component can be found using conventional testing procedures designed for stationary processes.

The rest of the paper is organized as follows. In Section 2, we discuss the data generating process, and define the statistics for the joint unit root null hypotheses. In Section 3, we derive the asymptotic null distribution of the new test statistics, and tabulate their finite sample critical values. We discuss the size and power properties of our tests using finite sample simulations in Section 4. In Section 5, we illustrate the use of our statistics by examining three time series, namely, real exchange rate between the US Dollar and the UK Pound, UK industrial production, and UK CPI, and we offer some concluding remarks in Section 6. All proofs are relegated to an Appendix.

[1] While Perron (1989, 1990b) treats the break-date as known, several studies have extended the Perron unit root tests to endogenize the choice of the break-date so as to avoid possible correlations between the choice of the break-date and the data, see Perron and Vogelsang (1992), Zivot and Andrews (1992), Christiano (1992), Banerjee et al. (1992), Perron (1997), Vogelsang and Perron (1998), and Carrion-i-Silvestre et al. (2009).

[2] These new Perron-type unit root tests of Popp (2008) have the desirable property that the implied break-date estimator accurately identifies a break if it exists under either the null or alternative hypothesis.

[3] It should be noted that we propose an F-test following Dickey and Fuller (1981), Perron (1990a), Carrion-i-Silvestre and Sanso (2006), and Sen (2007), while a Wald test version is proposed by Hall (1992) and Pitarakis (2014). The difference between the F-test and the Wald test is a factor of normalization by the appropriate degrees of freedom. So, both versions have a non-standard limiting null distribution, and their critical values have to be calculated using simulations.

2. New unit root tests based on the joint null hypothesis

In this section, we use the conventional components representation of a time series as discussed in Schmidt and Phillips (1992) and Popp (2008). The data generating process of the time series $\{y_t\}_{t=1}^{T}$ is given by:

$$y_t = d_t + u_t, \tag{1}$$
$$u_t = \rho u_{t-1} + \varepsilon_t, \tag{2}$$
$$\varepsilon_t = \Psi^*(L) e_t, \tag{3}$$

where d_t is the deterministic component, u_t is the stochastic component, $e_t \sim i.i.d.(0, \sigma^2)$, the lag polynomial $\Psi^*(L)$ is factorized as $\Psi^*(L) = A^*(L)^{-1} B(L)$, where $A^*(L)$ and $B(L)$ are lag polynomials of order p and q, respectively. It is assumed that all the roots of $A^*(L)$ and $B(L)$ are outside the unit circle.

We consider three different specifications of the deterministic component:

$$\mathrm{M}_0: \ d_t = \alpha + \Psi^*(L)\,[\theta DU_t^0], \tag{4}$$
$$\mathrm{M}_1: \ d_t = \alpha + \beta t + \Psi^*(L)\,[\theta DU_t^0], \tag{5}$$
$$\mathrm{M}_2: \ d_t = \alpha + \beta t + \Psi^*(L)\,[\theta DU_t^0 + \gamma DT_t^0], \tag{6}$$

where the parameters θ and γ measure the magnitude of the possible intercept and slope breaks, DU_t^0 is a dummy variable, $DU_t^0 = 1_{(t>T_B^0)}$, $DT_t^0 = 1_{(t>T_B^0)}\,(t - T_B^0)$, $1(\cdot)$ is the indicator function, and T_B^0 is the true break-date. M_0 is the level shift model considered by Perron and Vogelsang (1992), M_1 refers to the crash model that allows for a one-time break in the intercept of the underlying trend function, and M_2 indicates the mixed model that allows for a simultaneous break in the intercept and slope of the underlying trend function. M_1 and M_2 were originally considered by Perron (1989). We should point out that the data generating process does not allow for the dominant auto-regressive root to affect the dynamics of the break under the alternative hypothesis. For instance, consider the mixed model. In Perron's (1989) specification, the dynamics of the break under the alternative evolve according to $(1 - \rho L)\,\Psi^*(L)[\theta\, DU_t^0 + \gamma\, DT_t^0]$. However, here, the dynamics of the break evolve according to $\Psi^*(L)[\theta\, DU_t^0 + \gamma\, DT_t^0]$, and so the Perron-type unit root test based on (1)–(6) will collapse into the Additive Outlier unit root tests proposed by Perron (1989) in the eventuality that $\Psi^*(L) = 1$. Therefore, the Perron-type unit root tests based on (1)–(6) lie in between the Innovation Outlier tests and the Additive Outlier tests by Perron (1989).

The reduced form regressions implied by the structural model in (1)–(3) and the form of breaks in (4)–(6) are:

$$\mathrm{M}_0: \ y_t = \alpha_0^* + \delta DU_{t-1}(T_B) + \theta D_t(T_B) + \rho y_{t-1} + \sum_{j=1}^{k} c_j\, \Delta y_{t-j} + e_t, \tag{7}$$

$$\mathrm{M}_1: \ y_t = \alpha_1^* + \beta_1^*\, t + \delta DU_{t-1}(T_B) + \theta D_t(T_B) + \rho y_{t-1} + \sum_{j=1}^{k} c_j\, \Delta y_{t-j} + e_t, \tag{8}$$

$$\mathrm{M}_2: \ y_t = \alpha_2^* + \beta_2^*\, t + \xi\, D_t(T_B) + \kappa DU_{t-1}(T_B) + \zeta\, DT_{t-1}(T_B) + \rho y_{t-1} + \sum_{j=1}^{k} c_j\, \Delta y_{t-j} + e_t, \tag{9}$$

where $D_t(T_B) = 1_{(t=T_B+1)}$, $\phi = (\rho - 1)$, $\delta = -\phi\,\theta$, $\xi = (\gamma + \theta)$, $\kappa = (\gamma + \delta)$, $\zeta = -\phi\,\gamma$, $\beta_1^* = \beta_2^* = \Psi^*(1)^{-1}(1 - \rho)\,\beta$, $\alpha_0^* = \Psi^*(1)^{-1}(1 - \rho)\,\alpha$, $\alpha_1^* = \alpha_2^* = \Psi^*(1)^{-1}\,[(1 - \rho)\,\alpha + \rho\,\beta]$. We should note that the break parameter is the coefficient of the impulse dummy variable $D_t(T_B)$ in the reduced form regressions (7)–(9). The first lag differences of the dependent variable are included in regressions (7)–(9) in order to account for any additional correlation in the error term, and the appropriate value of the lag-truncation parameter, k, is determined using a data-dependent method, see Zivot and Andrews (1992) for further details. When the break-date is known to the practitioner, the unit root test is based on the t-statistics for $H_0: \rho = 1$ in regressions (7)–(9). Under the unit root null hypothesis, $\delta = 0$ in regressions (7) and (8), and $\zeta = 0$ in regression (9).

When the true location of the break-date is unknown, regressions (7)–(9) are estimated for all possible break-dates $T_B = [\lambda\, T]$ corresponding to $\lambda \in [\lambda^*, 1 - \lambda^*]$. The break-date estimator is given by:

$$\hat{T}_B = \begin{cases} arg\ max_{T_B}\ |t_{\hat{\theta}}(T_B)| & \text{for } \mathrm{M}_0 \text{ and } \mathrm{M}_1 \\ arg\ max_{T_B}\ |t_{\hat{\xi}}(T_B)| & \text{for } \mathrm{M}_2 \end{cases} \tag{10}$$

where $t_{\hat{\theta}}(T_B)$ is the t-statistic for θ in regressions (7) and (8) for model M_0 and model M_1 respectively, and $t_{\hat{\xi}}(T_B)$ is the t-statistic for ξ in regression (9) for model M_2. Harvey and Mills (2004) argue that the estimated break-fraction implied by (10) is super-consistent for the true break-fraction.[4]

[4] See also the simulation evidence provided in Popp (2008) regarding the break-date estimator defined in Eq. (10).

We consider the following joint unit root null hypotheses:

$$H_0^{\mathrm{M}_0}: \alpha_0^* = 0, \quad \delta = 0, \;\; \rho = 1, \tag{11}$$

$$H_0^{\mathrm{M}_1}: \beta_1^* = 0, \quad \delta = 0, \;\; \rho = 1, \tag{12}$$

$$H_0^{\mathrm{M}_2}: \beta_2^* = 0, \quad \zeta = 0, \;\; \rho = 1. \tag{13}$$

For a given break-date (T_B), the F-statistic $F_i(T_B)$ for the null hypothesis $\mathrm{H}_0^{\mathrm{M}_i}$ $(i = 0, 1, 2)$ is defined as follows:

$$F_i(T_B) = \frac{\left(R_i\,\hat{\mu}_i(T_B) - r_i\right)' \left[R_i \left\{\sum_{t=1}^{T} x_t^i(T_B)\, x_t^i(T_B)'\right\}^{-1} R_i'\right]^{-1} \left(R_i\,\hat{\mu}_i(T_B) - r_i\right)}{3\,\hat{\sigma}_i^2(T_B)}, \tag{14}$$

where R_i and r_i are the matrices corresponding to the null hypotheses $H_0^{M_i}$ $(i = 0, 1, 2)$, that is:

$$R_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0_k' \\ 0 & 1 & 0 & 0 & 0_k' \\ 0 & 0 & 0 & 1 & 0_k' \end{bmatrix}, \qquad R_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0_k' \\ 0 & 0 & 1 & 0 & 0 & 0_k' \\ 0 & 0 & 0 & 0 & 1 & 0_k' \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0_k' \\ 0 & 0 & 0 & 0 & 1 & 0 & 0_k' \\ 0 & 0 & 0 & 0 & 0 & 1 & 0_k' \end{bmatrix},$$

and $r_0 = r_1 = r_2 = (0\,0\,1)'$, $x_t^i(T_B)$ $(i = 0, 1, 2)$ are the explanatory variable vectors corresponding to regressions (7)–(9):

$$x_t^0(T_B) = [1\; DU_{t-1}(T_B)\; D_t(T_B)\; \Delta y_{t-1}\; \Delta y_{t-2}\; \ldots\; \Delta y_{t-k}]',$$
$$x_t^1(T_B) = [1\; t\; DU_{t-1}(T_B)\; D_t(T_B)\; \Delta y_{t-1}\; \Delta y_{t-2}\; \ldots\; \Delta y_{t-k}]',$$
$$x_t^2(T_B) = [1\; t\; DU_{t-1}(T_B)\; D_t(T_B)\; DT_{t-1}(T_B)\; \Delta y_{t-1}\; \Delta y_{t-2}\; \ldots\; \Delta y_{t-k}]',$$

$\hat{\mu}_i(T_B)$ $(i = 0, 1, 2)$ are the estimated parameter vectors corresponding to regressions (7)–(9):

$$\hat{\mu}_0(T_B) = (\hat{\alpha}_0^*\; \hat{\delta}\; \hat{\theta}\; \hat{\rho}\; \hat{c}')',$$
$$\hat{\mu}_1(T_B) = (\hat{\alpha}_1^*\; \hat{\beta}_1^*\; \hat{\delta}\; \hat{\theta}\; \hat{\rho}\; \hat{c}')',$$
$$\hat{\mu}_2(T_B) = (\hat{\alpha}_2^*\; \hat{\beta}_2^*\; \hat{\kappa}\; \hat{\xi}\; \hat{\zeta}\; \hat{\rho}\; \hat{c}')',$$

and $\hat{\sigma}_i^2(T_B) = (T - m_i - k)^{-1} \sum_{t=1}^{T} \left[y_t - x_t^i(T_B)'\,\hat{\mu}_i(T_B)\right]^2$ for $i = 0, 1, 2$ are the estimated mean squared errors from regressions (7)–(9) respectively, with $m_0 = 4$, $m_1 = 5$, and $m_2 = 6$. Let $F_0(T_B)$, $F_1(T_B)$, and $F_2(T_B)$ respectively be the F-statistics for $\mathrm{H}_0^{\mathrm{M}_0}$, $\mathrm{H}_0^{\mathrm{M}_1}$, and $\mathrm{H}_0^{\mathrm{M}_2}$ evaluated at the estimated break-date $\hat{T}_B$ defined in (10). The F-statistics $F_0(\hat{T}_B)$, $F_1(\hat{T}_B)$, and $F_2(\hat{T}_B)$ are designed to have power against the alternative hypotheses $H_A^{\mathrm{M}_0}: \alpha_0^* \neq 0$ and/or $\delta \neq 0$ and/or $\rho \neq 1$, $H_A^{M_1}: \beta_1^* \neq 0$ and/or $\delta \neq 0$ and/or $\rho \neq 1$, and $H_A^{M_2}: \beta_2^* \neq 0$ and/or $\zeta \neq 0$ and/or $\rho \neq 1$, respectively.[5]

3. Limiting null distributions

In this section, we derive the limiting distribution of the F-statistics, $F_0(\hat{T}_B)$, $F_1(\hat{T}_B)$, and $F_2(\hat{T}_B)$ for models M_0, M_1, and M_2, respectively. The asymptotic results are derived under the assumption that the errors are i.i.d. $(0, \sigma^2)$, so that $\Psi^*(L) = 1$ and $k = 0$, see also Vogelsang and Perron (1998).[6]

First, we consider the unit root statistic for the level shift model, denoted by $F_0(\hat{T}_B)$, where $\hat{T}_B$ is the estimated break-date that maximizes the absolute value of the t-statistics for $H_0: \theta = 0$ in regression (7). The data generating process under the unit root null hypothesis is:

$$y_t = \theta D_t(T_B^0) + y_{t-1} + e_t, \tag{15}$$

where T_B^0 is the true location of the break-date, $e_t \sim i.i.d.(0, \sigma^2)$, and $y_0 = 0$.

Theorem 1. *Consider a time series $\{y_t\}$ that evolves according to the data generating process given in (15) with the corresponding true break-fraction denoted by λ_0. The estimated break-date is defined by (10), that is $\hat{T}_B = \arg\max_{T_B} |t_{\hat{\theta}}(T_B)|$, where $T_B = [\lambda T]$*

[5] The joint null hypotheses given in (11)–(13) can be rejected even if the series contains a unit root ($\rho = 1$). Consider, for instance, the case when the trend component is mis-specified as: $d_t = \mu + \alpha_t + \beta t^2$. It follows that $y_t = \rho(\alpha - \beta) + [(1-\rho)\alpha + 2\beta\rho]t + (1-\rho)\beta t^2 + \rho y_{t-1} + e_t$, and under the unit root null hypothesis $y_t = (\alpha - \beta) + 2\beta t + y_{t-1} + e_t$. In this case, we would expect that the F-statistic will be significant owing to model mis-specification, but the t-test will be insignificant owing to the presence of a unit root.

[6] Vogelsang and Perron (1998, pp. 1084) argue that the results hold for more general errors as long as $k = 0$.

is chosen for all $\lambda \in \Lambda$ *and* Λ *is a closed subset of the interval* $(0, 1)$. *The following describes the limiting distribution of the unit root statistic* $F_0(\hat{T}_B)$ *based on the level shift model regression* (7):

$$F_0(\hat{T}_B) \Rightarrow \frac{\tilde{E}_0(\lambda_0)'\, \tilde{V}_0(\lambda_0)^{-1}\, \tilde{E}_0(\lambda_0)}{3\,\sigma^2}, \tag{16}$$

where $\tilde{E}_0(\lambda_0) = \left[\sigma\, W(1), \sigma\, [W(1) - W(\lambda_0)], \frac{1}{2}\,\sigma^2\, [W(1)^2 - 1]\right]'$, *and* $\tilde{V}_0(\lambda_0)$ *is a 3x3 symmetric matrix with* $\tilde{V}_0(\lambda_0)[1,1] = 1, \tilde{V}_0(\lambda_0)[1,2] = 1 - \lambda_0, \tilde{V}_0(\lambda_0)[1,3] = \sigma \int_0^1 W(r)\,dr, \tilde{V}_0(\lambda_0)[2,2] = 1 - \lambda_0, \tilde{V}_0(\lambda_0)[2,3] = \sigma \int_{\lambda_0}^1 W(r)\,dr, \tilde{V}_0(\lambda_0)[3,3] = \sigma^2 \int_0^1 W(r)^2\,dr$, *and* $W(r)$ *is a standard Wiener process. The proof is outlined in the Appendix.*

Next, we consider the unit root statistic for the crash model, denoted by $F_1(\hat{T}_B)$, where $\hat{T}_B$ is the estimated break-date that maximizes the absolute value of the t-statistics for $H_0 : \theta = 0$ in regression (8). The data generating process under the unit root null hypothesis is:

$$y_t = \alpha_1^* + \theta D_t(T_B^0) + y_{t-1} + e_t, \tag{17}$$

where T_B^0 is the location of the true break-date, $e_t \sim i.i.d.(0, \sigma^2)$, and $y_0 = 0$.

Theorem 2. *Consider a time series* $\{y_t\}$ *that evolves according to the data generating process given in* (17) *with the corresponding true break-fraction denoted by* λ_0. *The estimated break-date is defined by* (10), *that is* $\hat{T}_B = arg\ max_{T_B} |t_{\hat{\theta}}(T_B)|$, *where* $T_B = [\lambda\, T]$ *is chosen for all* $\lambda \in \Lambda$ *and* Λ *is a closed subset of the interval* $(0, 1)$. *The following describes the limiting distribution of the unit root statistic* $F_1(\hat{T}_B)$ *based on the crash model regression* (8):

$$F_1(\hat{T}_B) \Rightarrow \frac{\left[\tilde{R}_1\, \tilde{V}_1(\lambda_0)^{-1}\, \tilde{E}_1(\lambda_0)\right]' \left[\tilde{R}_1\, \tilde{V}_1(\lambda_0)^{-1}\, \tilde{R}_1'\right]^{-1} \left[\tilde{R}_1\, \tilde{V}_1(\lambda_0)^{-1}\, \tilde{E}_1(\lambda_0)\right]}{3\,\sigma^2}, \tag{18}$$

where $\tilde{E}_1(\lambda_0) = \left[\sigma\, W(1), \sigma\, [W(1) - \int_0^1 W(r)\, dr], \sigma\, [W(1) - W(\lambda_0)], \frac{1}{2}\,\sigma^2\, [W(1)^2 - 1]\right]'$, *and* $\tilde{V}_1(\lambda_0)$ *is a* 4×4 *symmetric matrix with* $\tilde{V}_1(\lambda_0)[1,1] = 1, \tilde{V}_1(\lambda_0)[1,2] = \frac{1}{2}, \tilde{V}_1(\lambda_0)[1,3] = 1 - \lambda_0, \tilde{V}_1(\lambda_0)[1,4] = \sigma \int_0^1 W(r)\,dr, \tilde{V}_1(\lambda_0)[2,2] = \frac{1}{3}, \tilde{V}_1(\lambda_0)[2,3] = \frac{1}{2}(1 - \lambda_0^2), \tilde{V}_1(\lambda_0)[2,4] = \sigma \int_0^1 r\, W(r)\,dr, \tilde{V}_1(\lambda_0)[3,3] = (1 - \lambda_0), \tilde{V}_1(\lambda_0)[3,4] = \sigma \int_{\lambda_0}^1 W(r)\,dr, \tilde{V}_1(\lambda_0)[4,4] = \sigma^2 \int_0^1 W(r)^2\,dr$, $W(r)$ *is a standard Wiener process, and*

$$\tilde{R}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The proof is outlined in the Appendix.

Finally, we consider the unit root statistic for the mixed model, denoted by $F_T^2(\hat{T}_B)$, where $\hat{T}_B$ is the estimated break-date that maximizes the absolute value of the t-statistics for $H_0 : \xi = 0$ in regression (9). The data generating process under the unit root null hypothesis is:

$$y_t = \alpha_2^* + \xi\, D_t(T_B^0) + \gamma\, DU_{t-1}(T_B^0) + y_{t-1} + e_t, \tag{19}$$

where T_B^0 is the true location of the break-date, DU_{t-1}^0 is the dummy variable defined using the true break-date T_B^0, $e_t \sim i.i.d.(0, \sigma^2)$, and $y_0 = 0$.

Theorem 3. *Consider a time series* $\{y_t\}$ *that evolves according to the data generating process given in* (19) *with the corresponding true break-fraction denoted by* λ_0. *The estimated break-date is defined by* (10), *that is* $T_B(\hat{\xi}) = arg\ max_{T_B} |t_{\hat{\xi}}(T_B)|$, *where* $T_B = [\lambda\, T]$ *is chosen for all* $\lambda \in \Lambda$ *and* Λ *is a closed subset of the interval (0,1). The following describes the limiting distribution of the unit root statistic* $F_2(\hat{T}_B)$ *based on the mixed model regression* (9):

$$F_2(\hat{T}_B) \Rightarrow \frac{\left[\tilde{R}_2\, \tilde{V}_2(\lambda_0)^{-1}\, \tilde{E}_2(\lambda_0)\right]' \left[\tilde{R}_2\, \tilde{V}_2(\lambda_0)^{-1}\, \tilde{R}_2'\right]^{-1} \left[\tilde{R}_2\, \tilde{V}_2(\lambda_0)^{-1}\, \tilde{E}_2(\lambda_0)\right]}{3\,\sigma^2}, \tag{20}$$

where $\tilde{E}_2(\lambda_0)$ *is a 5x1 vector with* $\tilde{E}_2(\lambda_0)[1,1] = \sigma\ W(1), \tilde{E}_2(\lambda_0)[2,1] = \sigma\ [W(1) - \int_0^1 W(r)\ dr], \tilde{E}_2(\lambda_0)[3,1] = \sigma\, [W(1) - W(\lambda_0)], \tilde{E}_2(\lambda_0)[4,1] = \sigma\ W(1) - \sigma\ \lambda_0\ [W(1) - W(\lambda_0)] - \sigma\ \int_0^1 W(r)\ dr$, *and* $\tilde{E}_2(\lambda_0)[5,1] = \frac{1}{2}\,\sigma^2\, [W(1)^2 - 1]$, *and* $\tilde{V}_2(\lambda_0)$ *is a 5x5 symmetric matrix with* $\tilde{V}_2(\lambda_0)[1,1] = 1, \tilde{V}_2(\lambda_0)[1,2] = \frac{1}{2}, \tilde{V}_2(\lambda_0)[1,3] = 1 - \lambda_0, \tilde{V}_2(\lambda_0)[1,4] = \frac{1}{2}(1 - \lambda_0^2) - \lambda_0(1 - \lambda_0), \tilde{V}_2(\lambda_0)[1,5] = \sigma \int_0^1 W(r)\,dr, \tilde{V}_2(\lambda_0)[2,2] = \frac{1}{3}, \tilde{V}_2(\lambda_0)[2,3] = \frac{1}{2}(1 - \lambda_0^2), \tilde{V}_2(\lambda_0)[2,4] =$

$\frac{1}{6}\lambda_0^3-\frac{1}{2}\lambda_0+1$, $\tilde{V}_2(\lambda_0)[2,5]=\sigma\int_0^1 r\,W(r)\,dr$, $\tilde{V}_2(\lambda_0)[3,3]=(1-\lambda_0)$, $\tilde{V}_2(\lambda_0)[3,4]=\frac{1}{2}(1-\lambda_0^2)-\lambda_0(1-\lambda_0)$, $\tilde{V}_2(\lambda_0)[3,5]=\sigma\int_{\lambda_0}^1 W(r)\,dr$, $\tilde{V}_2(\lambda_0)[4,4]=\frac{1}{3}(1-\lambda_0)^3$, $\tilde{V}_2(\lambda_0)[4,5]=\sigma\int_{\lambda_0}^1(r-\lambda_0)W(r)\,dr$, $\tilde{V}_2(\lambda_0)[5,5]=\sigma^2\int_0^1 W(r)^2\,dr$, *$W(r)$ is a standard Wiener process, and*

$$\tilde{R}_2=\begin{bmatrix}0&1&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}.$$

The proof is outlined in the Appendix.

The asymptotic distribution of our tests, $F_i(\hat{T}_B)$ $(i=0,1,2)$, has been derived under the assumption that the errors are independently and identically distributed. In the presence of additional correlation in the data generating process, the asymptotic distribution of $F_i(\hat{T}_B)$ $(i=0,1,2)$ will be the same as that derived in Theorems 1–3, if we include additional first difference lags of the data as shown in the regression equations (7)–(9), see Perron (1989) for further details. The number of lags to be included in the estimation regression will be determined using some data dependent algorithm as in Vogelsang and Perron (1998).

We calculate the finite sample critical values of the unit root statistics $F_i(\hat{T}_B)$ for $i=0,1,2$ using the following data generated process:

$$y_t=y_{t-1}+e_t,\quad y_0=0. \tag{21}$$

Without loss of generality, y_0 is set equal to 0. We specify the sequence of innovations e_t to be *i.i.d.* $N(0,1)$, since the asymptotic distributions are invariant to additional correlation structure of the data, see Perron and Vogelsang (1992). In practice, a general-to-specific algorithm such as the $k(t-sig)$ suggested by Perron and Vogelsang (1992) is used to determine the appropriate number of lagged first differences that should be included in the estimation regressions (7)–(9). We calculate the finite sample critical values for $kmax=0,5$, and six different sample sizes, $T=50,100,150,200,250,500$, using 5000 replications. The critical values of $F_i(\hat{T}_B)$ for $i=0,1,2$ are reported in Tables 1–3.

4. Finite sample size and power

In this section, we present evidence regarding the finite sample performance of the new statistics for the joint null hypothesis of a unit root, namely, $F_0(\hat{T}_B)$, $F_1(\hat{T}_B)$, and $F_2(\hat{T}_B)$. We generate $\{y_t\}$ according to the data generating process given in (1)–(6) assuming that the correlation structure of the innovation process is given by $\Psi^*(L)=(1-a_1L)^{-1}(1+b_1L)$. We consider all cases corresponding to the true break-fractions $\lambda_0=0.3,0.5$, and 0.7. For models M_0 and M_1, we use the intercept-break magnitude $\theta=0,5,10$, and for model M_2, we use combinations of the following values for intercept-break magnitude and the slope-break magnitude: $\theta=0,5,10$ and $\gamma=0,5,10$.[7] We consider five different error specifications corresponding to (a_1,b_1) equal to $(0,0)$, $(0.6,0)$, $(-0.6,0)$, $(0,0.5)$, and $(0,-0.5)$. The first case implies that the errors are independently and identically distributed. The second case allows for positively correlated errors within an AR(1) framework, and the third case allows for negatively correlated errors within an AR(1) framework. The last two cases correspond to MA(1) errors with a positive and a negative moving average component in order to determine how the $k(t-sig)$ procedure handles processes with moving average errors. The maximum number of the lag augmentation terms, *kmax*, is set to 5 and is reduced when the coefficient of the last augmentation term is not significant at the 5% level. The error process e_t follows a standard normal process, i.e. $e_t\sim N(0,1)$. We consider two different sample sizes: $T=100$ and $T=200$. Furthermore, the trimming factor is $\lambda^*=0.1$, so that we search for the break-date implied by the interval $[\lambda^*,1-\lambda^*]$. All simulations are based on 5000 replications of $\{y_t\}$ and were carried out in GAUSS. We evaluate the size and power of all statistics using the corresponding 5% finite sample critical values.

The size and power of the level shift model statistic $(F_0(\hat{T}_B))$, the crash model statistic $(F_1(\hat{T}_B))$, and the mixed model statistic $(F_2(\hat{T}_B))$ are given in Tables 4–7.[8] The size of $F_0(\hat{T}_B)$ is fairly close to the nominal size in all cases, except when there is a negative moving average component in the error process. There are some size distortions when $\theta=0$, but these distortions disappear as the sample size increases. For instance, with $\theta=0$, $\lambda_0=0.5$, and $(a_1,b_1)=(0,0)$, the size of $F_0(\hat{T}_B)$ is 0.065 with $T=100$ and 0.058 with $T=200$. However, with $\theta=5$, $\lambda_0=0.5$, and $(a_1,b_1)=(0,0)$, the size of $F_0(\hat{T}_B)$ is 0.055 with $T=100$ and 0.050 with $T=200$. The empirical size of $F_0(\hat{T}_B)$ is considerably higher with (a_1,b_1) equal to $(0,-0.5)$. In this case, for instance, with $\theta=5$ and $\lambda_0=0.5$, the size of $F_0(\hat{T}_B)$ is 0.109 when $T=100$ and 0.075 when $T=200$. A very similar pattern emerges for $F_2(\hat{T}_B)$. The only difference is that the size distortions when there is a negative moving average component are more pronounced. Previous studies such as Schwert (1989) and Vogelsang and Perron (1998)

[7] We should note that the mixed model (M_2) is only appropriate when $\theta\neq0$. We feel that practitioners will seldom expect $\theta=0$ in empirical applications when a break in the slope of the trend-function is suspected. If a break in the slope is not expected, then we recommend that practitioners use the crash model (M_1) characterization of the break for trending data or the level shift model (M_0) for non-trending data.

[8] The size and power of our statistics corresponding to $\theta=0$ (no break) do not depend on the location of break (λ).

Table 1
Critical values for $F_0(\hat{T}_B)$ with $kmax = 0, 5$.

T	$kmax$		λ_0								
			0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
50	0	1%	5.88	6.20	6.40	6.59	6.29	6.49	6.36	6.09	5.89
		2.5%	4.90	5.24	5.35	5.49	5.39	5.37	5.34	5.13	4.93
		5%	4.21	4.46	4.52	4.72	4.71	4.66	4.57	4.37	4.22
		10%	3.47	3.72	3.78	3.91	3.98	3.86	3.84	3.65	3.41
100	0	1%	5.56	5.60	6.02	5.92	5.89	5.97	6.03	5.81	5.64
		2.5%	4.70	4.92	5.01	5.09	5.09	5.17	5.08	4.97	4.76
		5%	4.06	4.26	4.34	4.42	4.44	4.49	4.36	4.27	4.06
		10%	3.37	3.60	3.66	3.79	3.75	3.82	3.67	3.60	3.37
150	0	1%	5.33	5.66	5.83	5.90	6.06	5.93	5.94	5.65	5.37
		2.5%	4.55	4.89	4.97	5.17	5.10	5.10	5.02	4.81	4.57
		5%	3.92	4.19	4.34	4.49	4.44	4.45	4.35	4.24	3.95
		10%	3.29	3.54	3.68	3.75	3.78	3.77	3.67	3.54	3.31
200	0	1%	5.24	5.50	5.83	5.81	5.80	5.73	5.76	5.55	5.37
		2.5%	4.46	4.74	4.97	5.05	5.00	4.93	4.90	4.86	4.59
		5%	3.88	4.16	4.31	4.40	4.37	4.33	4.27	4.26	4.02
		10%	3.25	3.51	3.67	3.75	3.72	3.74	3.65	3.57	3.32
250	0	1%	5.27	5.61	5.74	5.62	5.74	5.78	5.79	5.63	5.52
		2.5%	4.55	4.82	4.87	4.93	5.07	4.94	5.01	4.88	4.65
		5%	3.94	4.23	4.22	4.31	4.45	4.38	4.33	4.17	3.96
		10%	3.31	3.54	3.61	3.66	3.79	3.73	3.67	3.55	3.34
500	0	1%	5.27	5.47	5.84	5.59	5.63	5.66	5.57	5.50	5.21
		2.5%	4.59	4.76	5.03	4.92	4.88	4.91	4.83	4.76	4.50
		5%	3.96	4.15	4.32	4.28	4.33	4.28	4.23	4.15	3.86
		10%	3.35	3.52	3.68	3.69	3.68	3.64	3.58	3.51	3.29
50	5	1%	6.79	7.10	7.37	7.31	7.34	7.33	7.43	7.02	6.57
		2.5%	5.68	5.99	6.13	6.26	6.28	6.27	6.02	5.78	5.49
		5%	4.86	5.10	5.24	5.42	5.46	5.32	5.23	4.78	4.72
		10%	3.94	4.19	4.35	4.49	4.49	4.43	4.34	4.00	3.93
100	5	1%	6.00	6.20	6.25	6.57	6.40	6.47	6.34	6.15	6.05
		2.5%	5.15	5.31	5.46	5.54	5.38	5.54	5.35	5.23	5.02
		5%	4.38	4.57	4.73	4.81	4.76	4.81	4.65	4.48	4.29
		10%	3.63	3.81	3.98	4.06	4.04	4.06	3.92	3.75	3.56
150	5	1%	5.92	5.99	6.17	6.24	6.28	6.02	6.32	6.11	5.75
		2.5%	4.95	5.06	5.29	5.33	5.35	5.25	5.33	5.16	4.83
		5%	4.23	4.43	4.60	4.61	4.60	4.60	4.49	4.39	4.17
		10%	3.53	3.70	3.90	3.93	3.92	3.91	3.83	3.70	3.49
200	5	1%	5.76	6.01	6.00	6.04	5.89	6.05	6.22	5.86	5.54
		2.5%	4.83	5.16	5.12	5.19	5.16	5.13	5.20	4.97	4.70
		5%	4.11	4.47	4.49	4.58	4.52	4.52	4.48	4.34	4.04
		10%	3.41	3.73	3.78	3.88	3.83	3.80	3.80	3.66	3.38
250	5	1%	5.34	5.75	5.95	5.89	5.90	6.08	5.76	5.75	5.47
		2.5%	4.65	4.97	5.06	5.14	5.06	5.20	5.00	4.91	4.77
		5%	4.08	4.34	4.41	4.52	4.48	4.52	4.38	4.25	4.12
		10%	3.38	3.62	3.73	3.87	3.84	3.87	3.73	3.56	3.43
500	5	1%	5.39	5.57	5.77	5.92	5.91	5.84	5.67	5.63	5.44
		2.5%	4.58	4.90	4.93	5.04	5.05	5.16	4.94	4.86	4.62
		5%	3.86	4.27	4.37	4.40	4.35	4.45	4.33	4.26	4.03
		10%	3.30	3.57	3.75	3.73	3.70	3.76	3.65	3.60	3.38

have also found size distortions in unit root tests when there is a negative moving average component in the time series. While the empirical size of $F_1(\hat{T}_B)$ follows a similar pattern as that of $F_0(\hat{T}_B)$, the size distortions with $\theta = 0$ are more severe.

We should point out that some size distortions are expected in the absence of a break given that the limiting null distributions of $F_i(\hat{T}_B)$ $(i = 0, 1, 2)$ given in Theorems 1–3 are derived under the assumption that there is a break under the null hypothesis. If there were no break under the null hypothesis ($\theta = 0$ and $\gamma = 0$), then the limiting null distributions of $F_i(\hat{T}_B)$ $(i = 0, 1, 2)$ would be based on the expressions given in Theorems 1–3, evaluated at the break-date estimator derived in Costantini and Sen (2012).[9] A number of studies have noted this discontinuity in the limiting distribution depending on whether there is a break under the null hypothesis or not, see for instance Carrion-i-Silvestre et al. (2009). The discontinuity

[9] Critical values corresponding to the limiting null distribution in the absence of a break are available from the authors upon request.

Table 2
Critical values for $F_1(\hat{T}_B)$ with $kmax = 0, 5$.

T	$kmax$		λ_0								
			0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
50	0	1%	7.53	7.52	7.68	7.74	7.67	7.53	7.71	7.63	7.55
		2.5%	6.39	6.49	6.58	6.48	6.38	6.55	6.64	6.61	6.48
		5%	5.61	5.62	5.75	5.61	5.61	5.68	5.81	5.74	5.56
		10%	4.73	4.81	4.84	4.82	4.74	4.81	4.91	4.90	4.63
100	0	1%	7.10	7.21	7.26	7.18	6.82	7.02	7.27	7.11	7.07
		2.5%	6.12	6.32	6.30	6.15	6.03	6.13	6.26	6.25	6.07
		5%	5.35	5.58	5.56	5.36	5.35	5.48	5.47	5.47	5.37
		10%	4.52	4.79	4.72	4.57	4.56	4.67	4.69	4.71	4.63
150	0	1%	6.92	7.08	7.05	6.99	6.76	6.87	7.12	6.85	7.07
		2.5%	5.95	6.13	6.19	6.10	5.90	5.98	6.08	6.01	6.06
		5%	5.26	5.42	5.41	5.35	5.24	5.31	5.40	5.39	5.29
		10%	4.51	4.70	4.62	4.57	4.58	4.58	4.68	4.63	4.55
200	0	1%	6.71	7.23	6.99	6.80	7.01	6.85	6.98	6.87	6.96
		2.5%	5.87	6.20	6.04	5.94	5.97	6.06	6.08	6.10	5.94
		5%	5.17	5.44	5.31	5.24	5.22	5.35	5.40	5.42	5.24
		10%	4.50	4.69	4.65	4.54	4.56	4.54	4.68	4.68	4.52
250	0	1%	6.89	6.92	6.79	6.85	6.91	6.98	6.92	7.02	6.75
		2.5%	5.92	5.96	6.00	5.94	5.97	6.03	5.97	6.06	5.96
		5%	5.28	5.31	5.39	5.31	5.33	5.33	5.34	5.39	5.21
		10%	4.54	4.62	4.64	4.61	4.56	4.58	4.65	4.68	4.47
500	0	1%	6.74	6.80	6.79	6.86	6.90	6.90	6.83	6.80	6.56
		2.5%	5.93	5.98	5.88	6.01	5.92	5.99	5.99	5.91	5.79
		5%	5.24	5.30	5.25	5.29	5.23	5.27	5.29	5.20	5.15
		10%	4.49	4.63	4.53	4.59	4.48	4.54	4.59	4.60	4.43
50	5	1%	8.65	8.89	8.91	8.61	8.83	8.87	8.88	8.93	8.63
		2.5%	7.39	7.64	7.66	7.43	7.63	7.47	7.70	7.50	7.21
		5%	6.45	6.58	6.64	6.42	6.54	6.46	6.71	6.46	6.23
		10%	5.46	5.60	5.62	5.47	5.51	5.55	5.71	5.57	5.31
100	5	1%	7.54	7.73	7.73	7.53	7.75	7.95	7.64	7.63	7.52
		2.5%	6.54	6.70	6.73	6.56	6.69	6.71	6.64	6.65	6.46
		5%	5.75	5.89	5.85	5.80	5.94	5.84	5.88	5.85	5.60
		10%	4.91	5.07	5.00	4.95	5.04	5.04	5.10	4.99	4.82
150	5	1%	7.27	7.41	7.35	7.37	7.27	7.21	7.41	7.51	7.45
		2.5%	6.36	6.46	6.45	6.39	6.33	6.43	6.45	6.61	6.35
		5%	5.61	5.71	5.72	5.63	5.53	5.62	5.68	5.79	5.55
		10%	4.80	4.88	4.97	4.83	4.78	4.76	4.86	4.99	4.72
200	5	1%	7.19	7.36	7.36	7.18	7.12	7.33	7.19	7.08	6.95
		2.5%	6.26	6.42	6.39	6.27	6.18	6.24	6.26	6.32	6.04
		5%	5.45	5.66	5.62	5.46	5.45	5.50	5.55	5.61	5.34
		10%	4.68	4.88	4.83	4.72	4.68	4.74	4.81	4.78	4.64
250	5	1%	6.99	7.14	7.02	7.09	7.03	7.16	7.24	7.17	7.09
		2.5%	6.14	6.34	6.16	6.20	6.16	6.21	6.33	6.30	6.13
		5%	5.36	5.62	5.46	5.46	5.47	5.48	5.52	5.60	5.32
		10%	4.63	4.86	4.78	4.75	4.68	4.73	4.76	4.81	4.60
500	5	1%	6.71	6.98	7.08	6.95	6.85	6.67	6.87	7.09	6.98
		2.5%	5.93	6.13	6.09	6.03	5.98	5.95	6.02	6.08	5.94
		5%	5.25	5.41	5.45	5.32	5.30	5.32	5.37	5.41	5.21
		10%	4.54	4.72	4.70	4.64	4.57	4.56	4.66	4.69	4.44

in the limiting null distribution can introduce size distortions when one uses the critical values based on the limiting null distribution with break when there is, in fact, no break under the unit root null. We do not find evidence of significant size distortions in models M_0 and M_2, but some size distortions in model M_1. In this case, we suggest that, when using model M_1, practitioners may apply the pre-test methodology outlined by Carrion-i-Silvestre et al. (2009).

The power of all statistics increases with the sample size as well as the magnitude of departure from the unit root null hypothesis, as measured by the distance of the parameter ρ from one. For instance, with $(a_1, b_1) = (0.6, 0)$, $\lambda_0 = 0.5$, and $\theta = 5$, the power of $F_0(\hat{T}_B)$ is 0.110 when $\rho = 0.9$ and $T = 100$, 0.293 when $\rho = 0.8$ and $T = 100$, 0.358 when $\rho = 0.9$ and $T = 200$, and 0.821 when $\rho = 0.8$ and $T = 200$. A similar pattern emerges with models M_1 and M_2. We should note that there is usually a drop in power from $\theta = 0$ to $\theta > 0$, but the power of our statistics increases as the break magnitude (θ) increases. The relatively high power when $\theta = 0$, similar to the case of size, is a consequence of using critical values that assume that there is a break. Therefore, our results indicate that the empirical size of all statistics is close to the nominal size, except when there is a negative moving average root in the error process. The empirical size of our tests gets closer

Table 3
Critical values for $F_2(\hat{T}_B)$ with $kmax = 0, 5$.

T	$kmax$		λ_0 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
50	0	1%	7.85	8.44	8.96	9.20	9.44	9.28	8.65	8.57	7.53
		2.5%	6.54	7.29	7.83	7.94	8.19	7.95	7.43	7.21	6.36
		5%	5.59	6.34	6.80	7.07	7.13	7.00	6.64	6.20	5.53
		10%	4.70	5.34	5.81	6.05	6.09	5.96	5.66	5.21	4.68
100	0	1%	7.32	7.95	8.31	8.38	8.53	8.28	8.18	7.96	7.37
		2.5%	6.26	6.92	7.23	7.42	7.59	7.30	7.23	6.83	6.20
		5%	5.47	6.15	6.39	6.57	6.72	6.52	6.40	6.03	5.44
		10%	4.65	5.28	5.51	5.74	5.79	5.74	5.50	5.17	4.59
150	0	1%	7.07	7.67	8.17	8.34	8.32	8.47	8.19	7.78	7.00
		2.5%	6.08	6.77	7.21	7.31	7.34	7.44	7.22	6.69	6.00
		5%	5.38	6.01	6.45	6.48	6.56	6.55	6.41	5.93	5.26
		10%	4.66	5.18	5.59	5.71	5.76	5.70	5.52	5.12	4.53
200	0	1%	7.11	7.78	8.09	8.29	8.18	8.11	8.00	7.58	6.97
		2.5%	6.14	6.74	7.04	7.30	7.27	7.18	7.05	6.65	6.13
		5%	5.40	5.97	6.27	6.48	6.53	6.42	6.33	5.90	5.40
		10%	4.66	5.14	5.47	5.71	5.72	5.64	5.48	5.13	4.65
250	0	1%	7.01	7.74	8.02	7.93	8.21	8.02	7.82	7.69	6.89
		2.5%	6.09	6.67	7.08	7.15	7.26	7.00	6.95	6.61	6.06
		5%	5.39	5.88	6.29	6.48	6.49	6.35	6.16	5.84	5.36
		10%	4.66	5.15	5.48	5.64	5.69	5.59	5.44	5.02	4.58
500	0	1%	6.98	7.65	8.15	7.90	8.08	8.29	7.70	7.57	6.91
		2.5%	5.97	6.61	6.96	7.00	7.14	7.17	6.81	6.58	6.06
		5%	5.34	5.89	6.11	6.35	6.45	6.33	6.15	5.86	5.27
		10%	4.63	5.05	5.42	5.62	5.73	5.54	5.43	5.01	4.52
50	5	1%	8.81	9.85	10.10	10.82	11.07	10.36	9.83	9.21	8.99
		2.5%	7.70	8.45	8.80	9.29	9.47	9.18	8.64	7.96	7.31
		5%	6.70	7.42	7.79	8.21	8.40	8.11	7.57	6.88	6.42
		10%	5.61	6.25	6.64	7.05	7.23	6.96	6.49	5.82	5.41
100	5	1%	7.73	8.90	9.23	9.19	9.20	9.20	8.98	8.57	7.68
		2.5%	6.84	7.55	7.89	7.99	8.08	8.03	7.83	7.34	6.69
		5%	5.84	6.56	7.03	7.12	7.25	7.14	6.94	6.49	5.81
		10%	5.00	5.64	6.03	6.25	6.32	6.27	6.02	5.55	4.89
150	5	1%	7.42	8.28	8.52	8.96	8.86	8.91	8.91	8.34	7.32
		2.5%	6.41	7.15	7.57	7.82	7.75	7.83	7.63	7.24	6.25
		5%	5.67	6.33	6.72	6.90	6.98	6.98	6.74	6.28	5.53
		10%	4.82	5.47	5.84	5.99	6.10	6.00	5.86	5.35	4.76
200	5	1%	7.52	8.13	8.42	8.47	8.75	8.57	8.47	8.21	7.38
		2.5%	6.45	7.09	7.32	7.47	7.78	7.66	7.45	7.12	6.24
		5%	5.64	6.23	6.53	6.73	6.87	6.82	6.58	6.22	5.49
		10%	4.81	5.41	5.69	5.90	6.01	5.95	5.69	5.34	4.66
250	5	1%	7.32	8.03	8.47	8.38	8.72	8.42	8.44	7.92	7.28
		2.5%	6.36	6.91	7.36	7.43	7.75	7.30	7.30	6.84	6.38
		5%	5.58	6.16	6.58	6.61	6.87	6.64	6.44	6.03	5.49
		10%	4.74	5.28	5.74	5.75	5.99	5.80	5.67	5.22	4.62
500	5	1%	7.16	7.65	8.26	8.18	8.24	8.18	8.04	7.89	7.03
		2.5%	6.04	6.75	7.33	7.20	7.30	7.23	7.19	6.69	6.07
		5%	5.38	5.91	6.47	6.53	6.53	6.61	6.27	5.91	5.41
		10%	4.66	5.11	5.60	5.78	5.79	5.71	5.50	5.16	4.64

to the nominal size as the break magnitude increases and the sample size becomes larger. Further, our tests maintain their size quite well in the absence of a break under the unit root null hypothesis. Finally, the power of our tests increases as the sample size increases and the distance from the unit root null hypothesis widens.

5. Empirical application

In this section, we illustrate the application of our newly developed tests for the joint null hypothesis of a unit root. In particular, we use the level shift model for the real exchange rate between the US Dollar and the UK Pound (1971Q1–2012Q4), the crash model for the UK industrial production (1957Q1–2012Q2), and the mixed model for the UK CPI (1990Q1–2012Q4). The data was obtained from the Main Economic Indicators, Organization of Economic Development and Cooperation. A plot of each series is shown in Figs. 1–3. Based on the plots of these series, we choose model M_0 for the US/UK real exchange rate series, M_1 for the UK industrial production series, and M_2 for the UK CPI series. The results are summarized in Table 8.

Table 4
5% Rejection frequency for $F_0(\hat{T}_B)$.

T	a_1	b_1	θ	λ_0								
				0.3	0.5	0.7	0.3	0.5	0.7	0.3	0.5	0.7
				$\rho = 1$			$\rho = 0.9$			$\rho = 0.8$		
100	0	0	0	0.077	0.065	0.063	0.199	0.204	0.209	0.538	0.540	0.557
			5	0.053	0.055	0.052	0.133	0.144	0.151	0.457	0.452	0.474
			10	0.049	0.052	0.048	0.135	0.134	0.135	0.471	0.461	0.482
100	0.6	0	0	0.071	0.069	0.075	0.154	0.164	0.162	0.321	0.318	0.340
			5	0.053	0.055	0.063	0.116	0.110	0.128	0.302	0.293	0.313
			10	0.056	0.054	0.052	0.125	0.122	0.127	0.382	0.368	0.394
100	−0.6	0	0	0.063	0.060	0.068	0.191	0.187	0.191	0.524	0.528	0.540
			5	0.044	0.048	0.054	0.131	0.131	0.136	0.451	0.443	0.473
			10	0.046	0.048	0.045	0.121	0.126	0.132	0.449	0.440	0.466
100	0	0.5	0	0.066	0.071	0.073	0.193	0.179	0.199	0.446	0.441	0.459
			5	0.057	0.060	0.059	0.147	0.136	0.155	0.378	0.384	0.392
			10	0.053	0.058	0.053	0.135	0.137	0.140	0.413	0.410	0.429
100	0	−0.5	0	0.131	0.122	0.142	0.447	0.446	0.455	0.767	0.762	0.761
			5	0.102	0.109	0.117	0.381	0.397	0.396	0.722	0.723	0.744
			10	0.108	0.113	0.113	0.361	0.374	0.387	0.709	0.708	0.722
				$\rho = 1$			$\rho = 0.9$			$\rho = 0.8$		
200	0	0	0	0.056	0.058	0.054	0.527	0.510	0.525	0.933	0.930	0.935
			5	0.049	0.050	0.046	0.477	0.475	0.474	0.931	0.918	0.922
			10	0.053	0.053	0.051	0.486	0.483	0.491	0.966	0.963	0.966
200	0.6	0	0	0.060	0.057	0.064	0.391	0.385	0.390	0.800	0.809	0.800
			5	0.055	0.053	0.055	0.366	0.358	0.355	0.824	0.821	0.813
			10	0.051	0.054	0.047	0.405	0.407	0.402	0.913	0.911	0.922
200	−0.6	0	0	0.055	0.053	0.060	0.518	0.502	0.525	0.946	0.941	0.945
			5	0.045	0.046	0.048	0.463	0.461	0.470	0.931	0.938	0.930
			10	0.049	0.049	0.049	0.472	0.482	0.475	0.970	0.965	0.966
200	0	0.5	0	0.064	0.058	0.061	0.426	0.410	0.427	0.859	0.852	0.852
			5	0.055	0.055	0.049	0.378	0.376	0.384	0.846	0.851	0.849
			10	0.055	0.057	0.051	0.407	0.416	0.418	0.925	0.922	0.920
200	0	−0.5	0	0.094	0.084	0.088	0.699	0.684	0.709	0.967	0.964	0.966
			5	0.077	0.075	0.077	0.675	0.647	0.667	0.949	0.959	0.959
			10	0.078	0.082	0.071	0.642	0.648	0.671	0.982	0.982	0.980

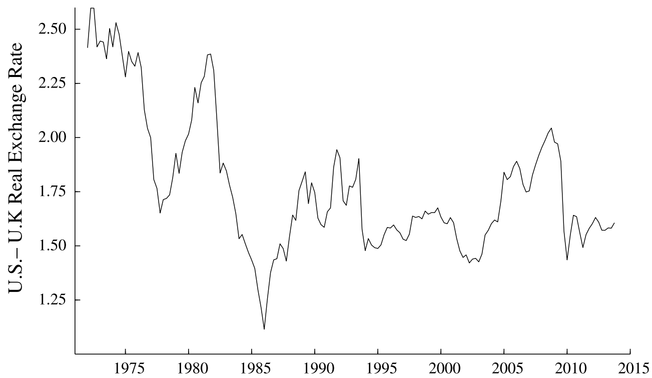

Fig. 1. US–UK Real Exchange Rate, 1971Q1–2012Q4.

For the US/UK real exchange rate series, we reject the joint null hypothesis of a unit root based on $F_0(\hat{T}_B)$ at the 1% level, but fail to reject the unit root null hypothesis based on Popp's (2008) statistic $t_0(\hat{T}_B)$, and the estimated break-date is 1987Q1. For the UK industrial production series, we reject the joint null hypothesis of a unit root based on $F_1(\hat{T}_B)$ at the 1% level, but fail to reject the unit root null based on Popp's (2008) statistic $t_1(\hat{T}_B)$. The estimated break-date for UK industrial production

Table 5
5% Rejection frequency for $F_1(\hat{T}_B)$.

T	a_1	b_1	θ	λ_0								
				0.3	0.5	0.7	0.3	0.5	0.7	0.3	0.5	0.7
				$\rho = 1$			$\rho = 0.9$			$\rho = 0.8$		
100	0	0	0	0.102	0.116	0.107	0.211	0.228	0.224	0.498	0.536	0.521
			5	0.068	0.075	0.074	0.153	0.163	0.164	0.421	0.449	0.434
			10	0.064	0.066	0.072	0.140	0.158	0.150	0.407	0.457	0.425
100	0.6	0	0	0.071	0.069	0.075	0.154	0.164	0.162	0.321	0.318	0.340
			5	0.053	0.055	0.063	0.116	0.110	0.128	0.302	0.293	0.313
			10	0.056	0.054	0.052	0.125	0.122	0.127	0.382	0.368	0.394
100	−0.6	0	0	0.063	0.060	0.068	0.191	0.187	0.191	0.524	0.528	0.540
			5	0.044	0.048	0.054	0.131	0.131	0.136	0.451	0.443	0.473
			10	0.046	0.048	0.045	0.121	0.126	0.132	0.449	0.440	0.466
100	0	0.5	0	0.066	0.071	0.073	0.193	0.179	0.199	0.446	0.441	0.459
			5	0.057	0.060	0.059	0.147	0.136	0.155	0.378	0.384	0.392
			10	0.053	0.058	0.053	0.135	0.137	0.140	0.413	0.410	0.429
100	0	−0.5	0	0.131	0.122	0.142	0.447	0.446	0.455	0.767	0.762	0.761
			5	0.102	0.109	0.117	0.381	0.397	0.396	0.722	0.723	0.744
			10	0.108	0.113	0.113	0.361	0.374	0.387	0.709	0.708	0.722
				$\rho = 1$			$\rho = 0.9$			$\rho = 0.8$		
200	0	0	0	0.085	0.089	0.085	0.478	0.495	0.462	0.911	0.921	0.908
			5	0.058	0.067	0.058	0.423	0.440	0.407	0.909	0.913	0.896
			10	0.066	0.060	0.059	0.424	0.422	0.406	0.946	0.942	0.939
200	0.6	0	0	0.060	0.057	0.064	0.391	0.385	0.390	0.800	0.809	0.800
			5	0.054	0.053	0.055	0.366	0.358	0.355	0.824	0.821	0.813
			10	0.051	0.054	0.047	0.405	0.407	0.402	0.913	0.911	0.922
200	−0.6	0	0	0.055	0.053	0.060	0.518	0.502	0.525	0.946	0.941	0.945
			5	0.045	0.046	0.048	0.463	0.461	0.470	0.931	0.938	0.930
			10	0.049	0.049	0.049	0.472	0.482	0.475	0.970	0.965	0.966
200	0	0.5	0	0.064	0.058	0.061	0.426	0.410	0.427	0.859	0.853	0.852
			5	0.055	0.055	0.049	0.378	0.376	0.384	0.846	0.851	0.849
			10	0.055	0.057	0.051	0.407	0.416	0.418	0.925	0.922	0.920
200	0	−0.5	0	0.094	0.084	0.088	0.699	0.684	0.709	0.967	0.964	0.966
			5	0.077	0.075	0.077	0.675	0.647	0.667	0.949	0.959	0.959
			10	0.078	0.082	0.071	0.642	0.648	0.671	0.982	0.982	0.980

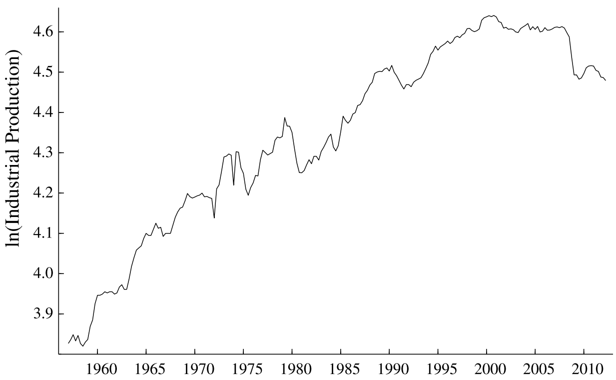

Fig. 2. ln(UK Industrial Production), 1957Q1–2012Q2.

occurs at 1974Q1.[10] Based on our results for the US/UK real exchange rate series and the UK industrial production series, the failure of $t_0(\hat{T}_B)$ and of $t_1(\hat{T}_B)$ to reject the null hypothesis confirms that there is a unit root in each series. The significance of $F_0(\hat{T}_B)$ and $F_1(\hat{T}_B)$ alerts the practitioner that the specification of the trend component is not valid for either of these series. However, we cannot ascertain the source of model mis-specification based on the significance of the F-statistics. The source

[10] It has been suggested in the literature that practitioners use the mixed model specification in empirical analysis to guard against mis-specification in the form of break, see Sen (2003). When we used model M_2 for the UK industrial production series, the results did not change significantly.

Table 6
5% Rejection frequency for $F_2(\hat{T}_B)$ with $T = 100$.

a_1	b_1	θ	γ	λ_0 0.3	0.5	0.7	0.3	0.5	0.7	0.3	0.5	0.7
				$\rho = 1$			$\rho = 0.9$			$\rho = 0.8$		
0	0	0	0	0.070	0.064	0.081	0.147	0.131	0.147	0.351	0.307	0.346
			5	0.048	0.048	0.044	0.088	0.077	0.091	0.248	0.206	0.221
			10	0.050	0.054	0.055	0.087	0.085	0.094	0.267	0.244	0.264
		5	0	0.053	0.054	0.052	0.098	0.095	0.114	0.268	0.256	0.277
			5	0.051	0.052	0.055	0.107	0.095	0.106	0.279	0.262	0.296
			10	0.056	0.059	0.050	0.092	0.084	0.103	0.269	0.241	0.282
		10	0	0.052	0.053	0.056	0.094	0.084	0.098	0.276	0.249	0.281
			5	0.052	0.048	0.054	0.100	0.099	0.110	0.286	0.266	0.291
			10	0.049	0.057	0.053	0.100	0.095	0.099	0.274	0.252	0.284
0.6	0	0	0	0.071	0.061	0.072	0.123	0.103	0.127	0.208	0.189	0.223
			5	0.059	0.054	0.055	0.094	0.079	0.079	0.184	0.162	0.162
			10	0.063	0.057	0.050	0.095	0.089	0.096	0.255	0.225	0.234
		5	0	0.056	0.058	0.062	0.088	0.092	0.096	0.177	0.168	0.189
			5	0.059	0.055	0.050	0.087	0.080	0.081	0.183	0.159	0.157
			10	0.045	0.037	0.028	0.072	0.056	0.060	0.175	0.149	0.145
		10	0	0.056	0.055	0.051	0.097	0.083	0.089	0.215	0.194	0.226
			5	0.053	0.060	0.049	0.075	0.064	0.079	0.151	0.142	0.148
			10	0.053	0.059	0.046	0.094	0.092	0.086	0.248	0.222	0.254
−0.6	0	0	0	0.067	0.061	0.077	0.133	0.115	0.138	0.339	0.305	0.346
			5	0.053	0.051	0.045	0.088	0.082	0.088	0.226	0.211	0.217
			10	0.048	0.047	0.049	0.088	0.075	0.092	0.246	0.242	0.262
		5	0	0.047	0.050	0.048	0.085	0.085	0.095	0.251	0.245	0.274
			5	0.052	0.050	0.043	0.095	0.085	0.087	0.241	0.210	0.221
			10	0.041	0.033	0.035	0.072	0.067	0.068	0.194	0.173	0.174
		10	0	0.050	0.043	0.041	0.087	0.077	0.088	0.260	0.242	0.259
			5	0.046	0.052	0.048	0.092	0.092	0.097	0.267	0.250	0.270
			10	0.046	0.044	0.046	0.086	0.077	0.093	0.239	0.217	0.258
0	0.5	0	0	0.084	0.074	0.092	0.151	0.140	0.163	0.308	0.292	0.328
			5	0.065	0.066	0.050	0.102	0.102	0.105	0.224	0.198	0.218
			10	0.062	0.062	0.054	0.088	0.082	0.091	0.219	0.195	0.219
		5	0	0.066	0.075	0.061	0.125	0.112	0.124	0.260	0.250	0.269
			5	0.061	0.061	0.054	0.110	0.102	0.101	0.224	0.202	0.225
			10	0.052	0.043	0.048	0.084	0.070	0.068	0.197	0.158	0.182
		10	0	0.062	0.061	0.062	0.108	0.105	0.116	0.257	0.247	0.275
			5	0.060	0.065	0.059	0.108	0.101	0.110	0.244	0.225	0.255
			10	0.053	0.062	0.053	0.096	0.081	0.090	0.201	0.189	0.218
0	−0.5	0	0	0.268	0.253	0.268	0.459	0.432	0.454	0.692	0.686	0.712
			5	0.236	0.263	0.226	0.422	0.418	0.402	0.655	0.643	0.649
			10	0.267	0.304	0.257	0.461	0.473	0.449	0.711	0.701	0.704
		5	0	0.218	0.241	0.221	0.389	0.407	0.399	0.674	0.659	0.681
			5	0.232	0.273	0.227	0.398	0.431	0.414	0.659	0.652	0.649
			10	0.247	0.290	0.231	0.426	0.433	0.409	0.662	0.642	0.639
		10	0	0.205	0.245	0.204	0.371	0.396	0.390	0.651	0.652	0.669
			5	0.216	0.238	0.216	0.399	0.407	0.412	0.680	0.694	0.695
			10	0.272	0.319	0.260	0.464	0.483	0.461	0.709	0.705	0.698

of the model mis-specification, for instance, may result from time varying parameters in the deterministic component of the time series.

Finally, for the UK CPI series, the mixed model statistic rejects both the joint null hypothesis based on $F_2(\hat{T}_B)$ and the unit root null based on Popp's (2008) statistic, $t_2(\hat{T}_B)$, at the 5% level, with an estimated break-date at 2008Q2. Here, we can infer that the UK CPI series is stationary, and the practitioner can determine the correct specification of the trend component using conventional testing methodologies developed for stationary processes.

6. Conclusions

We use the conventional components representation of a time series to devise unit root tests for the joint null hypothesis. This representation allows us to preserve the interpretation of the mean and time trend parameters under both the unit root null hypothesis and the trend-break stationary alternative hypothesis. A one-time break is allowed under the unit root null hypothesis, and so our tests guard against spurious rejection where there is in fact a break under the null hypothesis. We

Table 7
5% Rejection frequency for $F_2(\hat{T}_B)$ with $T = 200$.

a_1	b_1	θ	γ	λ_0 0.3	0.5	0.7	0.3	0.5	0.7	0.3	0.5	0.7
				$\rho = 1$			$\rho = 0.9$			$\rho = 0.8$		
0	0	0	0	0.068	0.050	0.061	0.337	0.288	0.334	0.843	0.823	0.842
			5	0.049	0.042	0.044	0.260	0.210	0.212	0.731	0.652	0.684
			10	0.052	0.051	0.048	0.297	0.236	0.279	0.843	0.793	0.817
		5	0	0.050	0.052	0.058	0.304	0.273	0.303	0.829	0.802	0.831
			5	0.052	0.049	0.051	0.299	0.266	0.290	0.843	0.817	0.838
			10	0.052	0.053	0.052	0.291	0.248	0.275	0.853	0.819	0.844
		10	0	0.054	0.052	0.048	0.292	0.256	0.285	0.867	0.817	0.860
			5	0.051	0.051	0.048	0.320	0.278	0.301	0.861	0.821	0.855
			10	0.049	0.055	0.053	0.297	0.271	0.289	0.849	0.816	0.842
0.6	0	0	0	0.069	0.045	0.067	0.255	0.212	0.243	0.617	0.544	0.606
			5	0.062	0.051	0.046	0.213	0.171	0.188	0.582	0.477	0.520
			10	0.058	0.049	0.049	0.257	0.214	0.243	0.770	0.693	0.740
		5	0	0.058	0.053	0.054	0.236	0.203	0.224	0.607	0.569	0.605
			5	0.062	0.054	0.050	0.213	0.165	0.191	0.580	0.476	0.520
			10	0.046	0.042	0.036	0.211	0.157	0.178	0.679	0.578	0.625
		10	0	0.056	0.054	0.051	0.241	0.200	0.244	0.726	0.672	0.705
			5	0.052	0.049	0.050	0.192	0.166	0.197	0.594	0.536	0.587
			10	0.052	0.055	0.047	0.274	0.217	0.239	0.773	0.700	0.738
−0.6	0	0	0	0.065	0.043	0.059	0.345	0.279	0.320	0.856	0.816	0.853
			5	0.050	0.040	0.040	0.254	0.199	0.224	0.739	0.637	0.658
			10	0.052	0.047	0.045	0.282	0.239	0.257	0.843	0.797	0.828
		5	0	0.052	0.051	0.048	0.302	0.254	0.293	0.835	0.801	0.832
			5	0.045	0.043	0.047	0.249	0.199	0.215	0.726	0.647	0.665
			10	0.048	0.038	0.035	0.237	0.190	0.221	0.766	0.683	0.721
		10	0	0.053	0.049	0.049	0.285	0.243	0.278	0.863	0.822	0.860
			5	0.051	0.047	0.049	0.298	0.257	0.280	0.862	0.811	0.861
			10	0.047	0.045	0.047	0.286	0.236	0.273	0.844	0.793	0.807
0	0.5	0	0	0.070	0.053	0.070	0.283	0.250	0.290	0.671	0.619	0.675
			5	0.065	0.045	0.049	0.241	0.190	0.214	0.602	0.528	0.546
			10	0.059	0.046	0.053	0.237	0.193	0.234	0.738	0.675	0.707
		5	0	0.054	0.054	0.059	0.250	0.217	0.247	0.662	0.612	0.644
			5	0.057	0.046	0.051	0.227	0.190	0.203	0.593	0.524	0.553
			10	0.048	0.036	0.046	0.215	0.167	0.185	0.650	0.551	0.584
		10	0	0.051	0.052	0.046	0.260	0.219	0.255	0.739	0.682	0.733
			5	0.053	0.057	0.050	0.242	0.203	0.221	0.636	0.600	0.633
			10	0.051	0.052	0.047	0.245	0.210	0.228	0.733	0.680	0.693
0	−0.5	0	0	0.164	0.135	0.158	0.603	0.564	0.598	0.907	0.894	0.914
			5	0.141	0.130	0.124	0.542	0.494	0.514	0.865	0.835	0.837
			10	0.148	0.162	0.143	0.545	0.525	0.537	0.903	0.882	0.889
		5	0	0.135	0.142	0.137	0.563	0.541	0.555	0.906	0.895	0.904
			5	0.135	0.138	0.122	0.532	0.488	0.519	0.865	0.828	0.833
			10	0.140	0.143	0.119	0.540	0.522	0.506	0.867	0.826	0.840
		10	0	0.126	0.130	0.121	0.534	0.508	0.543	0.926	0.905	0.927
			5	0.134	0.152	0.138	0.588	0.560	0.568	0.920	0.907	0.914
			10	0.143	0.154	0.141	0.563	0.524	0.556	0.902	0.874	0.890

propose a simple testing procedure for unit root and model mis-specification based on Popp's (2008) t-statistics and our newly proposed F-statistics. When both statistics are insignificant, we can conclude that the model is correctly specified and that the series has a unit root. When the t-statistic is insignificant and the F-statistic is significant, we can infer that the time series has a unit root, but the trend component of the series is mis-specified. On the other hand, if both the t-statistic and the F-statistic are significant, the practitioner can infer that the series is stationary. In this case, the practitioner would have to use additional conventional testing methodologies designed for stationary processes to determine the correct specification of the trend component of the series. Therefore, our testing methodology is a 'diagnostic test' designed to help detect possible mis-specification in the trend component of time series, which does not lead the practitioner to the correct specification, but it provides valuable information regarding the suitability of the specification for subsequent modeling and forecasting purposes.

For each model specification and the corresponding joint null hypothesis of a unit root, we derived the limiting null distribution of the proposed statistics, and tabulated their finite sample critical values. We evaluated the performance of the new statistics using simulations. Our simulations indicate that the new tests maintain the size fairly well, though the

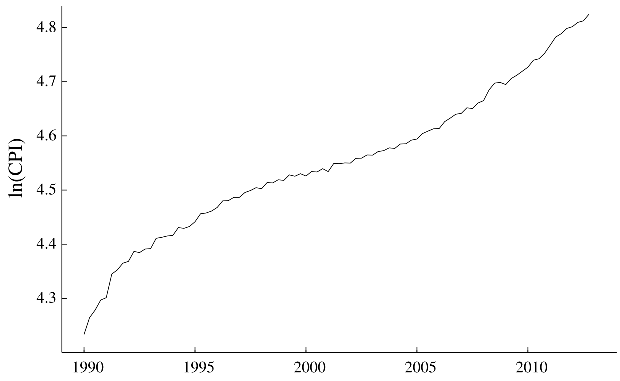

Fig. 3. ln(UK CPI), 1990Q1–2012Q4.

Table 8
Empirical results.

Series	T	$\hat{T}_B$	$\hat{\lambda}$	k^*	$\hat{\rho}$	$t_i(\hat{T}_B)$	$F_i(\hat{T}_B)$	$\hat{\sigma}$
Model M_0								
RER	168	1987Q1	0.39	1	0.943	−2.82	10.32***	0.074
Model M_1								
ln(*IP*)	222	1974Q1	0.31	4	0.977	−1.49	12.13***	0.016
Model M_2								
ln(*CPI*)	92	2008Q2	0.80	4	0.814	−4.43**	9.27**	0.004

Note: For the US/UK real exchange rate (RER), we use the finite sample critical values corresponding to $T = 150$ and $kmax = 5$ given in Table 1. For the industrial production series, we use the finite sample critical values corresponding to $T = 200$ and $kmax = 5$ given in Table 2. For the CPI series, we use the finite sample critical values corresponding to $T = 100$ and $kmax = 5$ given in Table 3. $\hat{\sigma}$ is the estimated mean square error from the corresponding regression. The critical values for Popp's statistics ($t_i(\hat{T}_B)$, $i = 0, 1, 2$) are taken from Sen (2015).

*** Denote significance at the 1% level.
** Denote significance at the 5% level.
* Denote significance at the 10% level.

tests are under-sized when the intercept break magnitude is relatively large. The power of the tests increases with the magnitude of the departure from the unit root null hypothesis as well as the sample size. We illustrated the use of our statistics by examining the real exchange rate between the US Dollar and the UK Pound, the UK industrial production, and the UK CPI series. Our findings indicate that the UK CPI series should be modeled as a trend stationary process with a break in 2008Q2. In addition, while we fail to reject the unit root null hypothesis for both the real exchange rate between the US Dollar and the UK Pound and the UK industrial production based on Popp's (2008) statistics, we find evidence of model mis-specification in each of these series.

In future research, we will focus on extending our testing procedure to the case of multiple structural breaks. The ability for practitioners to assess the validity of the model will provide invaluable guidance in the empirical analysis of time series data.

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Appendix. Mathematical proofs

All results are based on the functional weak convergence result $\sigma^{-1} T^{-1/2} \sum_{t=1}^{[\tau T]} e_t \Rightarrow W(\tau) \, \forall \, \tau \in [0, 1]$ where $W(r)$ is the Wiener process defined on the unit interval, and "$\Rightarrow$" denotes weak convergence. Let Y denote the vector of observed time series, Y_{-1} denote the first lag vector of the time series, 1 denote a vector of ones, t denote the vector of time trend, DU_{t-1} denote the vector implied by the variable $DU_{t-1}(T_B)$, DT_{t-1} denote the vector implied by the variable $DT_{t-1}(T_B)$, D_t denote the vector implied by the variable $D_t(T_B)$, and e denote the vector of residuals. The true break-date is denoted by T_B^0, and the true break-fraction is denoted by λ_0 where $T_B^0 = [\lambda_0 T]$.

We will use the property that a regression of the form $Y = X^1 \hat{\mu}^1 + X^2 \hat{\mu}^2 + \hat{e}$ yields a numerically identical estimate of μ^2 as obtained from a regression of $\tilde{Y} = \tilde{X}^2 \hat{\mu}^2 + \hat{e}$ where $\tilde{Y}$ and $\tilde{X}^2$ are projections respectively of Y and the columns of X^2 on the space spanned by the columns of X^1.

Proof of Theorem 1. Consider the data generating process given by Eq. (15). For a given break-date, T_B, the regression model (7) can be written as:

$$Y = X_0(T_B)\,\mu_0 + e = X_0^1(T_B)\,\mu_0^1 + X_0^2(T_B)\,\mu_0^2 + e, \tag{A.1}$$

where $X_0(T_B) = [1\ DU_{t-1}\ D_t\ Y_{-1}]$, $X_0^1(T_B) = [D_t]$, $X_0^2(T_B) = [1\ DU_{t-1}\ Y_{-1}]$, $\mu_0 = (\alpha_0^*\ \delta\ \theta\ \rho)'$, $\mu_0^1 = \theta$, and $\mu_0^2 = (\alpha_0^*\ \delta\ \rho)'$. Let $\tilde{Y}$ and $\tilde{X}_0^2(T_B)$ denote the projections of Y and $X_0^2(T_B)$ on the spanned by the vector $X_0^1(T_B)$, and consider the regression:

$$\tilde{Y} = \tilde{X}_0^2(T_B)\,\mu_0^2 + e. \tag{A.2}$$

The F-statistic $F_0(T_B)$ defined in Eq. (14) for the null hypothesis $H_0^{M_0}$: $\alpha_0^* = 0, \delta = 0, \rho = 1$ in regression (7) is identical to the F-statistic for this joint null hypothesis based on regression (A.2). The null hypothesis $H_0^{M_0}$ based on regression (A.2) can be expressed as $\tilde{R}_0\,\mu_0^2(T_B) = \tilde{r}_0$ where $\tilde{r}_0 = (0\ 0\ 1)'$ and $\tilde{R}_0$ is a 3x3 identity matrix. Therefore, the F-statistic for $H_0^{M_0}$ based on regression (A.2) is given by:

$$F_0(T_B) = \frac{\left(\tilde{R}_0\,\hat{\mu}_0^2(T_B) - \tilde{r}_0\right)' \left[\tilde{R}_0\,\left(\tilde{X}_0^2(T_B)'\tilde{X}_0^2(T_B)\right)^{-1}\tilde{R}_0'\right]^{-1}\left(\tilde{R}_0\,\hat{\mu}_0^2(T_B) - \tilde{r}_0\right)}{3\tilde{\sigma}_0^2(T_B)}, \tag{A.3}$$

where $\hat{\mu}_0^2(T_B)$ is the OLS estimator of μ_0^2 based on regression (A.2), and $\tilde{\sigma}_0^2(T_B)$ is the mean square error from regression (A.2). Define $\tilde{N}_0 = diag[T^{-1/2}, T^{-1/2}, T^{-1}]$. It follows that:

$$\tilde{N}_0^{-1}\,(\hat{\mu}_0^2(T_B) - \mu_0^2) = \left(\tilde{N}_0\,\tilde{X}_0^2(T_B)'\,\tilde{X}_0^2(T_B)\,\tilde{N}_0\right)^{-1}\left(\tilde{N}_0\tilde{X}_0^2(T_B)'e\right).$$

Using the results: $T^{-3/2}\sum_{t=1}^T y_{t-1} - y_{T_B} \Rightarrow \sigma\int_0^1 W(r)\,dr$, $T^{-3/2}\sum_{t=T_B+1}^T y_{t-1} \Rightarrow \sigma\int_\lambda^1 W(r)\,dr$, $T^{-2}\sum_{t=1}^T y_{t-1}^2 - y_{T_B}^2 \Rightarrow \sigma^2\int_0^1 W(r)^2\,dr$, $T^{-1/2}\sum_{t=1}^T e_t - e_{T_B+1} \Rightarrow \sigma W(1)$, $T^{-1/2}\sum_{t=T_B+1}^T e_t \Rightarrow \sigma[W(1) - W(\lambda)]$, and $T^{-1}\sum_{t=1}^T y_{t-1}e_t - y_{T_B}e_{T_B+1} \Rightarrow \frac{1}{2}\sigma^2\left[W(1)^2 - 1\right]$, it can be easily shown that:

$$\tilde{N}_0^{-1}\,(\hat{\mu}_0^2(T_B) - \mu_0^2) \Rightarrow \tilde{V}_0(\lambda)^{-1}\,\tilde{E}_0(\lambda), \tag{A.4}$$

where $\tilde{V}_0(\lambda)$ and $\tilde{E}_0(\lambda)$ are defined in Theorem 1. Since $\tilde{R}_0\,\hat{\mu}_0^2(T_B) - \tilde{r}_0 = \hat{\mu}_0^2(T_B) - \mu_0^2$, and using (A.3) and (A.4), it follows that:

$$F_0(T_B) \Rightarrow \frac{\tilde{E}_0(\lambda)'\,\tilde{V}_0(\lambda)^{-1}\,\tilde{E}_0(\lambda)}{3\,\sigma^2}. \tag{A.5}$$

Therefore, the distribution of $F_0(\hat{T}_B)$ follows from the limiting distribution of $F_0(T_B)$ given in (A.5) and from the property that the break-fraction estimator, $\hat{T}_B/T$, is a T-consistent estimator of the true break-fraction, λ^0, see Harvey and Mills (2004, page 869 and 876). □

Proof of Theorem 2. Consider the data generating process given by Eq. (17). Without loss of generality, we assume that β is equal to zero given that, under the unit root null hypothesis, $F_1(T_B)$ is invariant to the value of β. For a given break-date, T_B, the regression model (8) can be written as:

$$Y = X_1(T_B)\,\mu_1 + e = X_1^1(T_B)\,\mu_1^1 + X_1^2(T_B)\,\mu_1^2 + e, \tag{A.6}$$

where $X_1(T_B) = [1\ t\ DU_{t-1}\ D_t\ Y_{-1}]$, $X_1^1(T_B) = [D_t]$, $X_1^2(T_B) = [1\ t\ DU_{t-1}\ Y_{-1}]$, $\mu_1 = (\alpha_1^*\ \beta_1^*\ \delta\ \theta\ \rho)'$, $\mu_1^1 = \theta$, and $\mu_1^2 = (\alpha_1^*\ \beta_1^*\ \delta\ \rho)'$. Let $\tilde{Y}$ and $\tilde{X}_1^2(T_B)$ denote the projections of Y and $X_1^2(T_B)$ on the spanned by the vector $X_1^1(T_B)$, and consider the regression:

$$\tilde{Y} = \tilde{X}_1^2(T_B)\,\mu_1^2 + e. \tag{A.7}$$

The F-statistic $F_1(T_B)$ defined in Eq. (14) for the null hypothesis $H_0^{M_1}$: $\beta_1^* = 0, \delta = 0, \rho = 1$ in regression (8) is identical to the F-statistic for this joint null hypothesis based on regression (A.7). The null hypothesis $H_0^{M_1}$ based on regression (A.7) can be expressed as $\tilde{R}_1\,\mu_1^2(T_B) = \tilde{r}_1$ where $\tilde{r}_1 = (0\ 0\ 1)'$ and $\tilde{R}_1$ is a 3×4 matrix given by:

$$\tilde{R}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore, the F-statistic for $H_0^{M_1}$ based on regression (A.7) is given by:

$$F_1(T_B) = \frac{\left(\tilde{R}_1\,\hat{\mu}_1^2(T_B) - \tilde{r}_1\right)' \left[\tilde{R}_1\,\left(\tilde{X}_1^2(T_B)'\tilde{X}_1^2(T_B)\right)^{-1}\,\tilde{R}_1'\right]^{-1}\left(\tilde{R}_1\,\hat{\mu}_1^2(T_B) - \tilde{r}_1\right)}{3\tilde{\sigma}_1^2(T_B)}, \tag{A.8}$$

where $\hat{\mu}_1^2(T_B)$ is the OLS estimator of μ_1^2 based on regression (A.7), and $\tilde{\sigma}_1^2(T_B)$ is the mean square error from regression (A.7). Define $\tilde{N}_1 = diag[T^{-1/2}, T^{-3/2}, T^{-1/2}, T^{-1}]$. It follows that:

$$\tilde{N}_1^{-1}\,(\hat{\mu}_1^2(T_B) - \mu_1^2) = \left(\tilde{N}_1\,\tilde{X}_1^2(T_B)'\,\tilde{X}_1^2(T_B)\,\tilde{N}_1\right)^{-1}\left(\tilde{N}_1\tilde{X}_1^2(T_B)'e\right).$$

Using the results: $T^{-3/2}\sum_{t=1}^T y_{t-1} - y_{T_B} \Rightarrow \sigma\int_0^1 W(r)\,dr$, $T^{-5/2}\sum_{t=1}^T ty_{t-1} - (T_B+1)y_{T_B} \Rightarrow \sigma\int_0^1 rW(r)\,dr$, $T^{-3/2}\sum_{t=T_B+1}^T y_{t-1} \Rightarrow \sigma\int_\lambda^1 W(r)\,dr$, $T^{-2}\sum_{t=1}^T y_{t-1}^2 - y_{T_B}^2 \Rightarrow \sigma^2\int_0^1 W(r)^2\,dr$, $T^{-1/2}\sum_{t=1}^T e_t - e_{T_B+1} \Rightarrow \sigma W(1)$, $T^{-3/2}\sum_{t=1}^T te_t - (T_B+1)e_{T_B+1} \Rightarrow \sigma\left[W(1) - \int_0^1 W(r)\,dr\right]$, $T^{-1/2}\sum_{t=T_B+1}^T e_t \Rightarrow \sigma[W(1) - W(\lambda)]$, $T^{-1}\sum_{t=1}^T y_{t-1}e_t - y_{T_B}e_{T_B+1} \Rightarrow \frac{1}{2}\sigma^2\left[W(1)^2 - 1\right]$, $T^{-2}\sum_1^T t - (T_B+1) \to \frac{1}{2}$, $T^{-3}\sum_1^T t^2 - (T_B+1)^2 \to \frac{1}{3}$, and $T^{-2}\sum_{T_B+2}^T t \to \frac{1}{2}(1-\lambda^2)$, it can be easily shown that:

$$\tilde{N}_1^{-1}\,(\hat{\mu}_1^2(T_B) - \mu_1^2) \Rightarrow \tilde{V}_1(\lambda)^{-1}\,\tilde{E}_1(\lambda), \tag{A.9}$$

where $\tilde{V}_1(\lambda)$ and $\tilde{E}_1(\lambda)$ are defined in Theorem 2. Since $\tilde{R}_1\,\tilde{N}_1^{-1}\,\left(\hat{\mu}_1^2(T_B) - \mu_1^2\right) = \tilde{N}_1^{*-1}\left[\tilde{R}_1\hat{\mu}_1^2 - \tilde{r}_1\right]$, where $\tilde{N}_1^* = diag[T^{-3/2}, T^{-1/2}, T^{-1}]$, it follows that:

$$\begin{aligned}\tilde{N}_1^{*-1}\left[\tilde{R}_1\hat{\mu}_1^2 - \tilde{r}_1\right] &= \tilde{R}_1\,\left(\tilde{N}_1\,\tilde{X}_1^2(T_B)'\,\tilde{X}_1^2(T_B)\,\tilde{N}_1\right)^{-1}\left(\tilde{N}_1\tilde{X}_1^2(T_B)'e\right)\\ &\Rightarrow \tilde{R}_1\,\tilde{V}_1(\lambda)^{-1}\,\tilde{E}_1(\lambda).\end{aligned} \tag{A.10}$$

Therefore,

$$F_1(T_B) = \frac{\left(\tilde{R}_1\,\hat{\mu}_1^2 - \tilde{r}_1\right)'\,\tilde{N}_1^{*-1}\,\tilde{N}_1^*\left[\tilde{R}_1\,\left(\tilde{X}_1^{2'}\,\tilde{X}_1^2\right)^{-1}\,\tilde{R}_1'\right]^{-1}\tilde{N}_1^*\,\tilde{N}_1^{*-1}\left(\tilde{R}_1\,\hat{\mu}_1^2 - \tilde{r}_1\right)}{3\,\tilde{\sigma}_1^2(T_B)},$$

implies that:

$$F_1(T_B) = \frac{\left[\tilde{R}_1\tilde{N}_1^{-1}\left(\hat{\mu}_1^2(T_B) - \mu_1^2\right)\right]'\left[\tilde{R}_1\left(\tilde{N}_1\tilde{X}_1^{2'}\tilde{X}_1^2\tilde{N}_1\right)^{-1}\tilde{R}_1'\right]^{-1}\left[\tilde{R}_1\tilde{N}_1^{-1}\left(\hat{\mu}_1^2(T_B) - \mu_1^2\right)\right]}{3\,\tilde{\sigma}_1^2(T_B)}.$$

Using the results in (A.9) and (A.10), we have:

$$F_1(T_B) \Rightarrow \frac{\left(\tilde{R}_1\tilde{V}_1(\lambda)^{-1}\tilde{E}_1(\lambda)\right)'\left[\tilde{R}_1\tilde{V}_1(\lambda)^{-1}\tilde{R}_1'\right]^{-1}\left(\tilde{R}_1\tilde{V}_1(\lambda)^{-1}\tilde{E}_1(\lambda)\right)}{3\,\sigma^2}. \tag{A.11}$$

Therefore, the distribution of $F_1(\hat{T}_B)$ follows from the limiting distribution of $F_1(T_B)$ given in (A.11) and from the property that the break-fraction estimator, $\hat{T}_B/T$, is a T-consistent estimator of the true break-fraction, λ^0, see Harvey and Mills (2004, page 869 and 876). □

Proof of Theorem 3. Consider the data generating process given by Eq. (19). Without loss of generality, we assume that β and γ are each equal to zero given that, under the unit root null hypothesis, $F_2(T_B)$ is invariant to the value of β and γ. For a given break-date, T_B, the regression model (9) can be written as:

$$Y = X_2(T_B)\,\mu_2 + e = X_2^1(T_B)\,\mu_2^1 + X_2^2(T_B)\,\mu_2^2 + e, \tag{A.12}$$

where $X_2(T_B) = [1\ t\ DU_{t-1}\ D_t\ DT_{t-1}\ Y_{-1}]$, $X_2^1(T_B) = [D_t]$, $X_2^2(T_B) = [1\ t\ DU_{t-1}\ DT_{t-1}\ Y_{-1}]$, $\mu_2 = (\alpha_2^*\ \beta_2^*\ \kappa\ \xi\ \zeta\ \rho)'$, $\mu_2^1 = \xi$, and $\mu_2^2 = (\alpha_2^*\ \beta_2^*\ \kappa\ \zeta\ \rho)'$. Let $\tilde{Y}$ and $\tilde{X}_2^2(T_B)$ denote the projections of Y and $X_2^2(T_B)$ on the spanned by the vector $X_2^1(T_B)$, and consider the regression:

$$\tilde{Y} = \tilde{X}_2^2(T_B)\,\mu_2^2 + e. \tag{A.13}$$

The F-statistic $F_2(T_B)$ defined in Eq. (14) for the null hypothesis $H_0^{M_2}: \beta_2^* = 0, \zeta = 0, \rho = 1$ in regression (9) is identical to the F-statistic for this joint null hypothesis based on regression (A.13). The null hypothesis $H_0^{M_2}$ based on regression (A.13) can be expressed as $\tilde{R}_2\, \mu_2^2(T_B) = \tilde{r}_2$ where $\tilde{r}_2 = (0\ 0\ 1)'$ and $\tilde{R}_2$ is a 3x5 matrix given by:

$$\tilde{R}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore, the F-statistic for $H_0^{M_2}$ based on regression (A.13) is given by:

$$F_2(T_B) = \frac{\left(\tilde{R}_2\, \hat{\mu}_2^2(T_B) - \tilde{r}_2\right)' \left[\tilde{R}_2 \left(\tilde{X}_2^2(T_B)'\tilde{X}_2^2(T_B)\right)^{-1} \tilde{R}_2'\right]^{-1} \left(\tilde{R}_2\, \hat{\mu}_2^2(T_B) - \tilde{r}_2\right)}{3\tilde{\sigma}_2^2(T_B)}, \tag{A.14}$$

where $\hat{\mu}_2^2(T_B)$ is the OLS estimator of μ_2^2 based on regression (A.13), and $\tilde{\sigma}_2^2(T_B)$ is the mean square error from regression (A.13). Define $\tilde{N}_2 = diag[T^{-1/2}, T^{-3/2}, T^{-1/2}, T^{-3/2}, T^{-1}]$. It follows that:

$$\tilde{N}_2^{-1}\,(\hat{\mu}_2^2(T_B) - \mu_2^2) = \left(\tilde{N}_2\, \tilde{X}_2^2(T_B)'\, \tilde{X}_2^2(T_B)\, \tilde{N}_2\right)^{-1} \left(\tilde{N}_2 \tilde{X}_2^2(T_B)'e\right).$$

Using the results: $T^{-3/2}\sum_{t=1}^{T} y_{t-1} - y_{T_B} \Rightarrow \sigma\int_0^1 W(r)\,dr$, $T^{-5/2}\sum_{t=1}^{T} t y_{t-1} - (T_B+1)y_{T_B} \Rightarrow \sigma\int_0^1 rW(r)\,dr$, $T^{-3/2}\sum_{t=T_B+2}^{T} y_{t-1} \Rightarrow \sigma\int_\lambda^1 W(r)\,dr$, $T^{-5/2}\sum_{t=T_B+2}^{T}(t-1-T_B)y_{t-1} \Rightarrow \sigma\int_\lambda^1 (r-\lambda)W(r)\,dr$, $T^{-2}\sum_{t=1}^{T} y_{t-1}^2 - y_{T_B}^2 \Rightarrow \sigma^2\int_0^1 W(r)^2\,dr$, $T^{-1/2}\sum_{t=1}^{T} e_t - e_{T_B+1} \Rightarrow \sigma W(1)$, $T^{-3/2}\sum_{t=1}^{T} te_t - (T_B+1)e_{T_B+1} \Rightarrow \sigma\left[W(1) - \int_0^1 W(r)\,dr\right]$, $T^{-1/2}\sum_{t=T_B+2}^{T} e_t \Rightarrow \sigma[W(1) - W(\lambda)]$, $T^{-3/2}\sum_{t=T_B+2}^{T}(t-1-T_B)e_t \Rightarrow \sigma W(1) - \sigma\lambda[W(1)-W(\lambda)] - \sigma\int_0^1 W(r)\,dr$, $T^{-1}\sum_{t=1}^{T} y_{t-1}e_t - y_{T_B}e_{T_B+1} \Rightarrow \frac{1}{2}\sigma^2\left[W(1)^2 - 1\right]$, $T^{-2}\sum_1^T t - (T_B+1) \rightarrow \frac{1}{2}$, $T^{-3}\sum_1^T t^2 - (T_B+1)^2 \rightarrow \frac{1}{3}$, $T^{-2}\sum_{T_B+2}^{T} t \rightarrow \frac{1}{2}(1-\lambda^2)$, $T^{-2}\sum_{T_B+2}^{T}(t-1-T_B) \rightarrow \frac{1}{2}(1-\lambda^2) - \lambda(1-\lambda)$, $T^{-3}\sum_{T_B+2}^{T} t(t-1-T_B) \rightarrow \frac{1}{6}\lambda^3 - \frac{1}{2}\lambda + 1$, and $T^{-3}\sum_{T_B+2}^{T}(t-1-T_B)^2 \rightarrow \frac{1}{3}(1-\lambda)^3$, it can be easily shown that:

$$\tilde{N}_2^{-1}\,(\hat{\mu}_2^2(T_B) - \mu_2^2) \Rightarrow \tilde{V}_2(\lambda)^{-1}\,\tilde{E}_2(\lambda), \tag{A.15}$$

where $\tilde{V}_2$ and $\tilde{E}_2$ are defined in Theorem 3. Since $\tilde{R}_2\,\tilde{N}_2^{-1}\left(\hat{\mu}_2^2(T_B) - \mu_2^2\right) = \tilde{N}_2^{*-1}\left[\tilde{R}_2\hat{\mu}_2^2 - \tilde{r}_2\right]$, where $\tilde{N}_2^* = diag[T^{-3/2}, T^{-3/2}, T^{-1}]$, it follows that:

$$\begin{aligned} \tilde{N}_2^{*-1}\left[\tilde{R}_2\hat{\mu}_2^2 - \tilde{r}_2\right] &= \tilde{R}_2\left(\tilde{N}_2\, \tilde{X}_2^2(T_B)'\, \tilde{X}_2^2(T_B)\, \tilde{N}_2\right)^{-1}\left(\tilde{N}_2\tilde{X}_2^2(T_B)'e\right) \\ &\Rightarrow \tilde{R}_2\, \tilde{V}_2(\lambda)^{-1}\, \tilde{E}_2(\lambda). \end{aligned} \tag{A.16}$$

Therefore,

$$F_2(T_B) = \frac{\left(\tilde{R}_2\, \hat{\mu}_2^2 - \tilde{r}_2\right)'\, \tilde{N}_2^{*-1}\, \tilde{N}_2^* \left[\tilde{R}_2\left(\tilde{X}_2^{2'}\, \tilde{X}_2^2\right)^{-1}\tilde{R}_2'\right]^{-1} \tilde{N}_2^*\, \tilde{N}_2^{*-1}\left(\tilde{R}_2\, \hat{\mu}_2^2 - \tilde{r}_2\right)}{3\,\tilde{\sigma}_2^2(T_B)},$$

implies that:

$$F_2(T_B) = \frac{\left[\tilde{R}_2\tilde{N}_2^{-1}\left(\hat{\mu}_2^2(T_B) - \mu_2^2\right)\right]'\left[\tilde{R}_2\left(\tilde{N}_2\tilde{X}_2^{2'}\tilde{X}_2^2\tilde{N}_2\right)^{-1}\tilde{R}_2'\right]^{-1}\left[\tilde{R}_2\tilde{N}_2^{-1}\left(\hat{\mu}_2^2(T_B) - \mu_2^2\right)\right]}{3\,\tilde{\sigma}_2^2(T_B)}.$$

Using the results in (A.15) and (A.16), we have:

$$F_2(T_B) \Rightarrow \frac{\left(\tilde{R}_2\tilde{V}_2(\lambda)^{-1}\tilde{E}_2(\lambda)\right)'\left[\tilde{R}_2\tilde{V}_2(\lambda)^{-1}\tilde{R}_2'\right]^{-1}\left(\tilde{R}_2\tilde{V}_2(\lambda)^{-1}\tilde{E}_2(\lambda)\right)}{3\,\sigma^2}. \tag{A.17}$$

Therefore, the distribution of $F_2(\hat{T}_B)$ follows from the limiting distribution of $F_2(T_B)$ given in (A.17) and from the property that the break-fraction estimator, $\hat{T}_B/T$, is a T-consistent estimator of the true break-fraction, λ^0, see Harvey and Mills (2004, page 869 and 876). □

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