

Stability results for abstract evolution equations with intermittent time-delay feedback

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Abstract We consider abstract evolution equations with on–off time delay feedback. Without the time delay term, the model is described by an exponentially stable semigroup. We show that, under appropriate conditions involving the delay term, the system remains asymptotically stable. Under additional assumptions exponential stability results are also obtained. Concrete examples illustrating the abstract results are finally given.

1 Introduction

In this paper we study the stability properties of abstract evolution equations in presence of a time delay term.

In particular, we include into the model an on–off time delay feedback, i.e. the time delay is intermittently present.

Let \mathcal{H} be a Hilbert space, with norm $\|\cdot\|$, and let $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{H}$ be a dissipative operator generating a C_0 – semigroup $(S(t))_{t \geq 0}$ exponentially stable, namely there are two positive constants M and μ such that

$$\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq M e^{-\mu t}, \quad \forall t \geq 0, \quad (1)$$

where $\mathcal{L}(\mathcal{H})$ denotes the space of bounded linear operators from \mathcal{H} into itself.

We consider the following problem

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + \mathcal{B}(t)U(t - \tau) & t > 0, \\ U(0) = U_0, \end{cases} \quad (2)$$

where τ , the time delay, is a fixed positive constant, the initial datum U_0 belongs to \mathcal{H} and, for $t > 0$, $\mathcal{B}(t)$ is a bounded operator from \mathcal{H} to \mathcal{H} .

In particular, we assume that there exists an increasing sequence of positive real numbers $\{t_n\}_n$, with $t_0 = 0$, such that

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- 1) $\mathcal{B}(t) = 0 \quad \forall t \in I_{2n} = [t_{2n}, t_{2n+1}),$
- 2) $\mathcal{B}(t) = \mathcal{B}_{2n+1} \quad \forall t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}).$

We denote $B_{2n+1} = \|\mathcal{B}_{2n+1}\|_{\mathcal{L}(\mathcal{H})}$, $n \in \mathbb{N}$. Moreover, denoted by T_n the length of the interval I_n , that is

$$T_n = t_{n+1} - t_n, \quad n \in \mathbb{N}, \quad (3)$$

we assume

$$T_{2n} \geq \tau, \quad \forall n \in \mathbb{N}. \quad (4)$$

Time delay effects are frequently present in applications and concrete models and it is now well-understood that even an arbitrarily small delay in the feedback may destabilize a system which is uniformly stable in absence of delay (see e.g. [7, 8, 22, 30]).

We want to show that, under appropriate assumptions involving the delay feedback coefficients, the size of the time intervals where the delay appears and the parameters M and μ in (1), the considered model is asymptotically stable or exponentially stable, in spite of the presence of the time delay term.

Stability results for second-order evolution equations with intermittent damping were first studied by Haraux, Martinez and Vancostenoble [14], without any time delay term. They considered a model with intermittent on-off or with positive-negative damping and gave sufficient conditions ensuring that the behavior of the system in the time intervals with the standard dissipative damping, i.e. with positive coefficient, prevails over the *bad* behavior in remaining intervals where the damping is not present or it is present with the negative sign, namely as anti-damping. Therefore, asymptotic/exponential stability results were obtained.

More recently Nicaise and Pignotti [23, 24] considered second-order evolution equations with intermittent delay feedback. These results have been improved and extended to some semilinear equations in [9]. In the studied models, when the delay term (which possess a destabilizing effect) is not present, a not-delayed damping acts. Under appropriate sufficient conditions, stability results are then obtained. Related results for wave equations with intermittent delay feedback have been obtained, in 1-dimension, in [12], [13] and [3] by using a different approach based on the D'Alembert formula. However, this last approach furnishes stability results only for particular choices of the time delay.

In the recent paper [28], the intermittent delay feedback is compensated by a viscoelastic damping with exponentially decaying kernel.

The asymptotic behavior of wave-type equations with infinite memory and time delay feedback has been studied by Guesmia in [11] via a Lyapunov approach and by Alabau-Boussouira, Nicaise and Pignotti [2] by combining multiplier identities and perturbative arguments.

We refer also to Day and Yang [6] for the same kind of problem in the case of finite memory. In these papers the authors prove exponential stability results if the coefficient of the delay damping is sufficiently small. These stability results could be easily extended to a variable coefficient $b(\cdot) \in L^\infty(0, +\infty)$ under a suitable *smallness* assumption on the L^∞ -norm of $b(\cdot)$.

In [28], instead, asymptotic stability results are obtained without smallness conditions related to the L^∞ -norm of the delay coefficient. On the other hand, the analysis is restricted to intermittent delay feedback. Asymptotic stability is proved when the coefficient of the delay feedback belongs to $L^1(0, +\infty)$ and the length of the time intervals where the delay is not present is sufficiently large. The same paper considers also problems with on–off anti–damping instead of a time delay feedback. Stability results are obtained even in this case under analogous assumptions.

The idea is here to generalize the results of [28] by considering abstract evolution equations for which, without considering the intermittent delay term, the associated operator generates an exponentially stable C_0 -semigroup.

For such a class of evolution equations we already know that, under a suitable smallness condition on the delay feedback coefficient, an exponential stability result holds true (see [25]). We want to show that stability results are available also under a condition on the L^1 -norm of the delay coefficient, without restriction on the pointwise L^∞ -norm.

The paper is organized as follows. In section 2 we give a well-posedness result. In sections 3 and 4 we prove asymptotic and exponential stability results, respectively, for the abstract model under appropriate conditions. Stability results are established also for a problem with intermittent anti–damping instead of delay feedback in section 5. Finally, in section 6, we give some concrete applications of the abstract results.

2 Well-posedness

In this section we illustrate a well-posedness results for problem (2).

Theorem 1. *For any initial datum $U_0 \in \mathcal{H}$ there exists a unique (mild) solution $U \in C([0, \infty); \mathcal{H})$ of problem (2). Moreover,*

$$U(t) = S(t)U_0 + \int_0^t S(t-s)\mathcal{B}(s)U(s-\tau)ds. \quad (5)$$

Proof. We prove the existence and uniqueness result on the interval $[0, t_2]$; then the global result follows by translation (cfr. [23]). In the time interval $[0, t_1]$, since $\mathcal{B}(t) = 0 \forall t \in [0, t_1]$, then there exists a unique solution $U \in C([0, \tau], \mathcal{H})$ satisfying (5). The situation is different in the time interval $[t_1, t_2]$ where the delay feedback is present. In this case, we decompose the interval $[t_1, t_2]$ into the successive intervals $[t_1 + j\tau, t_1 + (j+1)\tau]$, for $j = 0, \dots, N-1$, where N is such that $t_1 + (N+1)\tau \geq t_2$. The last interval is then $[t_1 + N\tau, t_2]$. Now, first we look at the problem on the interval $[t_1, t_1 + \tau]$. Here $U(t-\tau)$ can be considered as a known function. Indeed, for $t \in [t_1, t_1 + \tau]$, then $t-\tau \in [0, t_1]$, and we know the solution U on $[0, t_1]$ by the first step. Thus, problem (2) may be reformulated on $[t_1, t_1 + \tau]$ as

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + g_0(t) & t \in (\tau, 2\tau), \\ U(\tau) = U(\tau_-), \end{cases} \quad (6)$$

where $g_0(t) = \mathcal{B}(t)U(t - \tau)$. This problem has a unique solution $U \in C([\tau, 2\tau], \mathcal{H})$ (see e.g. Th. 1.2, Ch. 6 of [27]) given by

$$U(t) = S(t - \tau)U(\tau_-) + \int_{\tau}^t S(t - s)g_0(s) ds, \quad \forall t \in [\tau, 2\tau].$$

Proceedings analogously in the successive time intervals $[t_1 + j\tau, t_1 + (j + 1)\tau)$, we obtain a solution on $[0, t_2]$. ■

3 Asymptotic stability results

Let T^* be defined as

$$T^* := \frac{1}{\mu} \ln M, \quad (7)$$

where M and μ are the constants in (1), that is T^* is the time for which $Me^{-\mu T^*} = 1$.

We can state a first estimate on the intervals I_{2n} where the delay feedback is not present.

Proposition 1. *Assume $T_{2n} > T^*$. Then, there exists a constant $c_n \in (0, 1)$ such that*

$$\|U(t_{2n+1})\|^2 \leq c_n \|U(t_{2n})\|^2, \quad (8)$$

for every solution of problem (2).

Proof. Observe that in the time interval $I_{2n} = [t_{2n}, t_{2n+1}]$ the delay feedback is not present since $\mathcal{B}(t) \equiv 0$. Thus, (8) easily follows from (1) with $\sqrt{c_n} = Me^{-\mu T_{2n}} < Me^{-\mu T^*} = 1$. ■

Let us now introduce the Lyapunov functional

$$F(t) = F(U, t) := \frac{1}{2} \|U(t)\|^2 + \frac{1}{2} \int_{t-\tau}^t \|\mathcal{B}(s + \tau)\|_{\mathcal{L}(\mathcal{H})} \|U(s)\|^2 ds. \quad (9)$$

Proposition 2. *Assume 1), 2). Moreover, assume $T_{2n} \geq \tau$, $\forall n \in \mathbb{N}$. Then,*

$$F'(t) \leq B_{2n+1} \|U(t)\|^2, \quad t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}], \quad \forall n \in \mathbb{N}. \quad (10)$$

for any solution of problem (2).

Proof. By differentiating the energy $F(\cdot)$, we have

$$F'(t) = \langle U(t), \mathcal{A}U(t) \rangle + \langle U(t), \mathcal{B}(t)U(t-\tau) \rangle + \frac{1}{2} \|\mathcal{B}(t+\tau)\|_{\mathcal{L}(\mathcal{H})} \|U(t)\|^2 - \frac{1}{2} \|\mathcal{B}(t)\|_{\mathcal{L}(\mathcal{H})} \|U(t-\tau)\|^2.$$

Then, since the operator \mathcal{A} is dissipative, one can estimate

$$F'(t) \leq \|\mathcal{B}(t)\|_{\mathcal{L}(\mathcal{H})} \|U(t)\| \|U(t-\tau)\| + \frac{1}{2} \|\mathcal{B}(t+\tau)\|_{\mathcal{L}(\mathcal{H})} \|U(t)\|^2 - \frac{1}{2} \|\mathcal{B}(t)\|_{\mathcal{L}(\mathcal{H})} \|U(t-\tau)\|^2. \quad (11)$$

Therefore, from Cauchy–Schwarz inequality,

$$F'(s) \leq \frac{1}{2} \|\mathcal{B}(t)\|_{\mathcal{L}(\mathcal{H})} \|U(t)\|^2 + \frac{1}{2} \|\mathcal{B}(t+\tau)\|_{\mathcal{L}(\mathcal{H})} \|U(t)\|^2.$$

Now, observe that, since $T_{2n} \geq \tau$, for every $n \in \mathbb{N}$, if t belongs to I_{2n+1} then $t + \tau$ belongs to I_{2n+1} or to I_{2n+2} . In the first case $\|\mathcal{B}(t)\|_{\mathcal{L}(\mathcal{H})} = B_{2n+1}$ while, in the second case $\|\mathcal{B}(t)\|_{\mathcal{L}(\mathcal{H})} = 0$. Thus (10) is proved. ■

Theorem 2. Assume 1), 2) and $T_{2n} \geq \tau$ for all $n \in \mathbb{N}$. Moreover assume $T_{2n} > T^*$, for all $n \in \mathbb{N}$, where T^* is the time defined in (7). Then, if

$$\sum_{n=0}^{\infty} \ln [e^{2B_{2n+1}T_{2n+1}} (c_n + T_{2n+1}B_{2n+1})] = -\infty, \quad (12)$$

the equation (2) is asymptotically stable, namely any solution U of (2) satisfies $\|U(t)\| \rightarrow 0$ for $t \rightarrow +\infty$.

Proof. Note that from (10) we obtain

$$F'(t) \leq 2B_{2n+1}F(t), \quad t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}), \quad n \in \mathbb{N}.$$

Then, by integrating on the time interval I_{2n+1} ,

$$F(t_{2n+2}) \leq e^{2B_{2n+1}T_{2n+1}} F(t_{2n+1}), \quad \forall n \in \mathbb{N}. \quad (13)$$

From the definition of the Lyapunov functional F ,

$$F(t_{2n+1}) = \frac{1}{2} \|U(t_{2n+1})\|^2 + \frac{1}{2} \int_{t_{2n+1}-\tau}^{t_{2n+1}} \|\mathcal{B}(s+\tau)\|_{\mathcal{L}(\mathcal{H})} \|U(s)\|^2 ds. \quad (14)$$

Note that, for $t \in [t_{2n+1}-\tau, t_{2n+1})$, then $t + \tau \in [t_{2n+1}, t_{2n+1} + \tau)$ and therefore, since $|I_{2n+2}| \geq \tau$ it results $t + \tau \in I_{2n+1} \cup I_{2n+2}$. Now, if $t + \tau \in I_{2n+2}$, then $\mathcal{B}(t + \tau) = 0$. Otherwise, if $t + \tau \in I_{2n+1}$, then $\|\mathcal{B}(t + \tau)\| = B_{2n+1}$. Then, from (14) we deduce

$$F(t_{2n+1}) = \frac{1}{2} \|U(t_{2n+1})\|^2 + \frac{1}{2} B_{2n+1} \int_{t_{2n+1}-\tau}^{\min(t_{2n+2}-\tau, t_{2n+1})} \|U(s)\|^2 ds, \quad (15)$$

since if $t_{2n+1} > t_{2n+2} - \tau$, then $\mathcal{B}(t) = 0$ for all $t \in [t_{2n+2}, t_{2n+1} + \tau) \subset [t_{2n+2}, t_{2n+3})$.

Then, since $\|U(\cdot)\|$ is decreasing in the intervals I_{2n} (the operator \mathcal{A} is dissipative and $\mathcal{B}(t) \equiv 0$), we deduce

$$\begin{aligned} F(t_{2n+1}) &\leq \frac{1}{2} \|U(t_{2n+1})\|^2 + \frac{1}{2} T_{2n+1} B_{2n+1} \|U(t_{2n+1} - \tau)\|^2 \\ &\leq \frac{1}{2} \|U(t_{2n+1})\|^2 + \frac{1}{2} T_{2n+1} B_{2n+1} \|U(t_{2n})\|^2. \end{aligned} \quad (16)$$

Using this last estimate in (13), we obtain

$$\|U(t_{2n+2})\|^2 \leq 2F(t_{2n+2}) \leq e^{2B_{2n+1}T_{2n+1}} (c_n + T_{2n+1}B_{2n+1}) \|U(t_{2n})\|^2, \quad \forall n \in \mathbb{N}, \quad (17)$$

where we have used also the estimate (8). By iterating this argument we arrive at

$$\|U(t_{2n+2})\|^2 \leq \prod_{k=0}^n e^{2B_{2k+1}T_{2k+1}} (c_k + T_{2k+1}B_{2k+1}) \|U_0\|^2, \quad \forall n \in \mathbb{N}. \quad (18)$$

Now observe that $\|U(t)\|$ is not decreasing in the whole $(0, +\infty)$. However, it is decreasing for $t \in [t_{2n}, t_{2n+1})$, $n \in \mathbb{N}$, where the destabilizing delay feedback does not act and so

$$\|U(t)\| \leq \|U(t_{2n})\|, \quad \forall t \in [t_{2n}, t_{2n+1}). \quad (19)$$

Moreover, from (16), for $t \in [t_{2n+1}, t_{2n+2})$ we have

$$\|U(t)\|^2 \leq 2F(t) \leq e^{2B_{2n+1}T_{2n+1}} (c_n + B_{2n+1}T_{2n+1}) \|U(t_{2n})\|^2, \quad (20)$$

where in the second inequality we have used (8).

Then, we have asymptotic stability if

$$\prod_{k=0}^n e^{2B_{2k+1}T_{2k+1}} (c_k + T_{2k+1}B_{2k+1}) \longrightarrow 0, \quad \text{for } n \rightarrow \infty,$$

or equivalently

$$\ln \left[\prod_{k=0}^n e^{2B_{2k+1}T_{2k+1}} (c_k + T_{2k+1}B_{2k+1}) \right] \longrightarrow -\infty, \quad \text{for } n \rightarrow \infty,$$

namely under the assumption (12). This concludes the proof. \blacksquare

Remark 1. In particular, (12) is verified if the following conditions are satisfied:

$$\sum_{n=0}^{\infty} B_{2n+1}T_{2n+1} < +\infty \quad \text{and} \quad \sum_{n=0}^{\infty} \ln c_n = -\infty. \quad (21)$$

Indeed, it is easy to see that (21) is equivalent to

$$\sum_{n=0}^{\infty} B_{2n+1} T_{2n+1} < +\infty \quad \text{and} \quad \sum_{n=0}^{\infty} \ln(c_n + B_{2n+1} T_{2n+1}) = -\infty \quad (22)$$

and that (22) implies (12).

Therefore, from (21), we have stability if $\|\mathcal{B}(t)\| \in L^1(0, +\infty)$ and, for instance, the length of the *good* intervals I_{2n} is greater than a fixed time \bar{T} , $\bar{T} > T^*$ and $\bar{T} \geq \tau$, namely

$$T_{2n} \geq \bar{T}, \quad \forall n \in \mathbb{N}.$$

Indeed, in this case there exists $\bar{c} \in (0, 1)$ such that $0 < c_n < \bar{c}$.

If we assume that the length of the delay intervals, namely the time intervals where the delay feedback is present, is lower than the time delay τ , that is

$$T_{2n+1} \leq \tau, \quad \forall n \in \mathbb{N}. \quad (23)$$

we can prove another asymptotic stability result which is, in some sense, complementary to the previous one.

In this case we can directly work with $\|U(t)\|$ instead of passing through the function $F(\cdot)$. We can give the following preliminary estimates on the time intervals I_{2n+1} , $n \in \mathbb{N}$.

Proposition 3. *Assume 1), 2). Moreover assume $T_{2n+1} \leq \tau$ and $T_{2n} \geq \tau$, $\forall n \in \mathbb{N}$. Then, for $t \in I_{2n+1}$,*

$$\frac{d}{dt} \|U(t)\|^2 \leq B_{2n+1} \|U(t)\|^2 + B_{2n+1} \|U(t_{2n})\|^2. \quad (24)$$

Proof: By differentiating $\|U(t)\|^2$ we get

$$\frac{d}{dt} \|U(t)\|^2 = 2\langle U(t), \mathcal{A}U(t) \rangle + 2\langle U(t), \mathcal{B}(t)U(t-\tau) \rangle.$$

Then, by using the dissipativeness of the operator \mathcal{A} ,

$$\frac{d}{dt} \|U(t)\|^2 \leq 2\langle U(t), \mathcal{B}(t)U(t-\tau) \rangle.$$

Hence, from 2),

$$\frac{d}{dt} \|U(t)\|^2 \leq B_{2n+1} \|U(t)\|^2 + B_{2n+1} \|U(t-\tau)\|^2.$$

We can now conclude observing that since $T_{2n+1} \leq \tau$ and $T_{2n} \geq \tau$, then for $t \in I_{2n+1}$ it is $t - \tau \in I_{2n}$. Then, since $\|U(t)\|$ is decreasing in I_{2n} , the estimate in the statement is proved. ■

The stability result follows.

Theorem 3. Assume 1), 2), $T_{2n+1} \leq \tau$ and $T_{2n} \geq \tau$, $\forall n \in \mathbb{N}$. Moreover assume $T_{2n} > T^*$, for all $n \in \mathbb{N}$, where T^* is the time defined in (7). If

$$\sum_{n=0}^{\infty} \ln [e^{B_{2n+1}T_{2n+1}}(c_n + 1 - e^{-B_{2n+1}T_{2n+1}})] = -\infty, \quad (25)$$

then every solution U of (2) satisfies $\|U(t)\| \rightarrow 0$ for $t \rightarrow +\infty$.

Proof. For $t \in I_{2n+1} = [t_{2n+1}, t_{2n+2})$, from estimate (24) we have

$$\|U(t)\|^2 \leq e^{B_{2n+1}(t-t_{2n+1})} \left\{ \|U(t_{2n+1})\|^2 + B_{2n+1} \int_{t_{2n+1}}^t \|U(t_{2n})\|^2 e^{-B_{2n+1}(s-t_{2n+1})} ds \right\}.$$

Then we deduce

$$\|U(t)\|^2 \leq e^{B_{2n+1}T_{2n+1}} \|U(t_{2n+1})\|^2 + e^{B_{2n+1}(t-t_{2n+1})} \|U(t_{2n})\|^2 [1 - e^{-B_{2n+1}(t-t_{2n+1})}],$$

and therefore

$$\|U(t)\|^2 \leq e^{B_{2n+1}T_{2n+1}} \|U(t_{2n+1})\|^2 + e^{B_{2n+1}T_{2n+1}} \|U(t_{2n})\|^2 - \|U(t_{2n})\|^2,$$

for $t \in I_{2n+1} = [t_{2n+1}, t_{2n+2})$, $n \in \mathbb{N}$.

Now we use the estimate (8) obtaining

$$\|U(t_{2n+2})\|^2 \leq e^{B_{2n+1}T_{2n+1}} (c_n + 1 - e^{-B_{2n+1}T_{2n+1}}) \|U(t_{2n})\|^2, \quad n \in \mathbb{N}.$$

Thus,

$$\|U(t_{2n+2})\| \leq \left[\prod_{k=0}^n e^{B_{2k+1}T_{2k+1}} (c_k + 1 - e^{-B_{2k+1}T_{2k+1}}) \right]^{\frac{1}{2}} \|U_0\|. \quad (26)$$

Then the asymptotic stability result follows if

$$\prod_{k=0}^n e^{B_{2k+1}T_{2k+1}} (c_k + 1 - e^{-B_{2k+1}T_{2k+1}}) \rightarrow 0, \quad \text{for } n \rightarrow \infty,$$

namely if

$$\sum_{n=0}^{\infty} \ln [e^{B_{2n+1}T_{2n+1}} (c_n + 1 - e^{-B_{2n+1}T_{2n+1}})] \rightarrow -\infty, \quad \text{for } n \rightarrow \infty. \quad \blacksquare$$

Remark 2. Observe that, when the odd intervals I_{2n+1} have length lower or equal than the time delay τ , the assumption (25) is a bit less restrictive than (12). Indeed,

$$e^{B_{2n+1}T_{2n+1}} (c_n + 1 - e^{-B_{2n+1}T_{2n+1}}) < e^{2B_{2n+1}T_{2n+1}} (c_n + B_{2n+1}T_{2n+1}), \quad \forall n \in \mathbb{N}.$$

Remark 3. Arguing as in Remark 1 we can show that condition (25) is verified, in particular, if (21) holds true.

4 Exponential stability

Under additional assumptions on the coefficients T_n, B_{2n+1}, c_n , exponential stability results hold true.

Theorem 4. *Assume 1), 2). Moreover, assume*

$$T_{2n} = T^0 \quad \forall n \in \mathbb{N}, \quad (27)$$

with $T^0 \geq \tau$ and $T^0 > T^*$, where T^* is the constant defined in (7),

$$T_{2n+1} = \tilde{T} \quad \forall n \in \mathbb{N} \quad (28)$$

and

$$\sup_{n \in \mathbb{N}} e^{2B_{2n+1}\tilde{T}} (c + B_{2n+1}\tilde{T}) = d < 1, \quad (29)$$

where $c = Me^{-\mu T^0}$. Then, there exist two positive constants C, α such that

$$\|U(t)\| \leq Ce^{-\alpha t} \|U_0\|, \quad t > 0, \quad (30)$$

for any solution of problem (2).

Proof. Note that, from the definition of the constant c , estimate (8) holds with $c_n = c$, $\forall n \in \mathbb{N}$. Thus, from (29) and (17) we obtain

$$\|U(T^0 + \tilde{T})\| \leq d^{\frac{1}{2}} \|U_0\|,$$

and then,

$$\|U(n(T^0 + \tilde{T}))\| \leq d^{\frac{n}{2}} \|U_0\|, \quad \forall n \in \mathbb{N}.$$

Therefore, $\|U(t)\|$ satisfies an exponential estimate like (30) (see Lemma 1 of [12]). ■

Concerning the case of *small* delay intervals, namely $|I_{2n+1}| \leq \tau$, $\forall n \in \mathbb{N}$, one can state the following asymptotic stability result.

Theorem 5. *Assume 1), 2). Moreover assume*

$$T_{2n} = T^0 \quad \forall n \in \mathbb{N},$$

with $T^0 \geq \tau$ and $T^0 > T^*$, where the time T^* is defined in (7),

$$T_{2n+1} = \tilde{T}, \quad \text{with } \tilde{T} \leq \tau \quad \forall n \in \mathbb{N} \quad (31)$$

and

$$\sup_{n \in \mathbb{N}} e^{B_{2n+1}\tilde{T}} (c + 1 - e^{-B_{2n+1}\tilde{T}}) = d < 1, \quad (32)$$

where $c = Me^{-\mu T^0}$. Then, there exist two positive constants C, α such that

$$\|U(t)\| \leq Ce^{-\alpha t} \|U_0\|, \quad t > 0, \quad (33)$$

for any solution of (2).

Proof. The proof is analogous to the one of Theorem 4. ■

5 Intermittent anti-damping

With analogous technics we can also deal with an intermittent anti-damping term. More precisely, let us consider the model

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + \mathcal{B}(t)U(t) & t > 0, \\ U(0) = U_0, \end{cases} \quad (34)$$

where τ is the time delay, the initial datum U_0 belongs to \mathcal{H} and, for $t > 0$, $\mathcal{B}(t)$ is a bounded operator from \mathcal{H} such that

$$\langle \mathcal{B}(t)U, U \rangle \geq 0, \quad \forall U \in \mathcal{H}.$$

Thus $\mathcal{B}(t)U(t)$ is an anti-damping term (cfr. [14]). In particular we consider an intermittent feedback, that is we assume that there exists an increasing sequence of positive real numbers $\{t_n\}_n$, with $t_0 = 0$, such that

- 3) $\mathcal{B}(t) = 0 \quad \forall t \in I_{2n} = [t_{2n}, t_{2n+1})$,
- 4) $\mathcal{B}(t) = \mathcal{D}_{2n+1} \quad \forall t \in I_{2n+1} = [t_{2n+1}, t_{2n+2})$.

We denote $D_{2n+1} = \|\mathcal{D}_{2n+1}\|_{\mathcal{L}(\mathcal{H})}$, $n \in \mathbb{N}$.

As before, denote by T_n the length of the interval I_n , that is

$$T_n = t_{n+1} - t_n, \quad n \in \mathbb{N}.$$

Note that proposition 1, which gives an observability estimate on the intervals I_{2n} where the anti-damping is not present, still holds true. Concerning the time intervals I_{2n+1} where the anti-damping acts one can obtain the following estimate.

Proposition 4. Assume 3) and 4). For every solution of problem (34),

$$\frac{d}{dt} \|U(t)\|^2 \leq 2D_{2n+1} \|U(t)\|^2, \quad t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}], \quad \forall n \in \mathbb{N}.$$

Proof. Being \mathcal{A} dissipative, the estimate follows immediately from 3). ■

From Proposition 4 we deduce an asymptotic stability result.

Theorem 6. *Assume 3), 4). Moreover assume $T_{2n} > T^*$, for all $n \in \mathbb{N}$, where T^* is the time defined in (7). If*

$$\sum_{n=0}^{\infty} \ln(e^{2D_{2n+1}T_{2n+1}} c_n) = -\infty, \quad (35)$$

then the problem (34) is asymptotically stable, that is any solution U of (34) satisfies $\|U(t)\| \rightarrow 0$ for $t \rightarrow +\infty$.

Proof. From Proposition 4 we have the estimate

$$\frac{d}{dt} \|U(t)\|^2 \leq 2D_{2n+1} \|U(t)\|^2, \quad t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}], \quad \forall n \in \mathbb{N}.$$

This implies

$$\|U(t_{2n+2})\|^2 \leq e^{2D_{2n+1}T_{2n+1}} \|U(t_{2n+1})\|^2, \quad \forall n \in \mathbb{N}. \quad (36)$$

Then, from estimate (8) which is always valid of course in the time intervals without damping,

$$\|U(t_{2n+2})\|^2 \leq e^{2D_{2n+1}T_{2n+1}} c_n \|U(t_{2n})\|^2, \quad \forall n \in \mathbb{N}. \quad (37)$$

By repeating this argument we obtain

$$\|U(t_{2n+2})\|^2 \leq \prod_{k=0}^n e^{2D_{2k+1}T_{2k+1}} c_k \|U_0\|^2, \quad \forall n \in \mathbb{N}. \quad (38)$$

Therefore, asymptotic stability is ensured if

$$\prod_{k=0}^n e^{2D_{2k+1}T_{2k+1}} c_k \longrightarrow 0, \quad \text{for } n \rightarrow \infty,$$

or equivalently

$$\ln\left(\prod_{k=0}^n e^{2D_{2k+1}T_{2k+1}} c_k\right) \longrightarrow -\infty, \quad \text{for } n \rightarrow \infty.$$

This concludes. ■

Remark 4. In particular (35) is verified under the following assumptions:

$$\sum_{n=0}^{\infty} D_{2n+1}T_{2n+1} < +\infty \quad \text{and} \quad \sum_{n=0}^{\infty} \ln c_n = -\infty. \quad (39)$$

Under additional assumptions on the problem coefficients T_n, D_{2n+1}, c_n , an exponential stability result holds.

Theorem 7. *Assume 3), 4) and*

$$T_{2n} = T^0 \quad \forall n \in \mathbb{N}, \quad (40)$$

with $T^0 > T^*$, where the time T^* is defined in (7). Assume also that

$$T_{2n+1} = \tilde{T} \quad \forall n \in \mathbb{N} \quad (41)$$

and

$$\sup_{n \in \mathbb{N}} e^{2D_{2n+1}\tilde{T}} c = d < 1, \quad (42)$$

where, $c = Me^{-\mu T^0}$. Then, there exist two positive constants C, α such that

$$\|U(t)\| \leq Ce^{-\alpha t} \|U_0\|, \quad t > 0, \quad (43)$$

for any solution of problem (34).

6 Concrete examples

In this section we illustrate some examples falling into the previous abstract setting.

6.1 Viscoelastic wave type equation

Let H be a real Hilbert space and let $A : \mathcal{D}(A) \rightarrow H$ be a positive self-adjoint operator with a compact inverse in H . Denote by $V := \mathcal{D}(A^{\frac{1}{2}})$ the domain of $A^{\frac{1}{2}}$.

Let us consider the problem

$$u_{tt}(x, t) + Au(x, t) - \int_0^\infty \mu(s) Au(x, t-s) ds + b(t)u_t(x, t-\tau) = 0 \quad t > 0, \quad (44)$$

$$u(x, t) = 0 \quad \text{on} \quad \partial\Omega \times (0, +\infty), \quad (45)$$

$$u(x, t) = u_0(x, t) \quad \text{in} \quad \Omega \times (-\infty, 0]; \quad (46)$$

where the initial datum u_0 belongs to a suitable space, the constant $\tau > 0$ is the time delay, and the memory kernel $\mu : [0, +\infty) \rightarrow [0, +\infty)$ satisfies

i) $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$;

ii) $\mu(0) = \mu_0 > 0$;

- iii) $\int_0^{+\infty} \mu(t) dt = \tilde{\mu} < 1$;
- iv) $\mu'(t) \leq -\delta \mu(t)$, for some $\delta > 0$.

Moreover, the function $b(\cdot) \in L_{loc}^\infty(0, +\infty)$ is a function which is zero intermittently. That is, we assume that for all $n \in \mathbb{N}$ there exists $t_n > 0$, with $t_0 = 0$ and $t_n < t_{n+1}$, such that

$$\begin{aligned} 1_w) \quad & b(t) = 0 \quad \forall t \in I_{2n} = [t_{2n}, t_{2n+1}), \\ 2_w) \quad & |b(t)| \leq b_{2n+1} \neq 0 \quad \forall t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}). \end{aligned}$$

Stability result for the above problem were firstly obtained in [28]. We want to show that they can also be obtained as particular case of previous abstract setting.

To this aim, following Dafermos [5], we can introduce the new variable

$$\eta^t(x, s) := u(x, t) - u(x, t - s). \quad (47)$$

Then, problem (44)–(46) may be rewritten as

$$u_{tt}(x, t) = -(1 - \tilde{\mu})Au(x, t) - \int_0^\infty \mu(s)A\eta^t(x, s)ds - b(t)u_t(x, t - \tau) \quad \text{in } \Omega \times (0, +\infty), \quad (48)$$

$$\eta_t^t(x, s) = -\eta_s^t(x, s) + u_t(x, t) \quad \text{in } \Omega \times (0, +\infty) \times (0, +\infty), \quad (49)$$

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, +\infty), \quad (50)$$

$$\eta^t(x, s) = 0 \quad \text{in } \partial\Omega \times (0, +\infty), t \geq 0, \quad (51)$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega, \quad (52)$$

$$\eta^0(x, s) = \eta_0(x, s) \quad \text{in } \Omega \times (0, +\infty), \quad (53)$$

where

$$\begin{aligned} u_0(x) &= u_0(x, 0), \quad x \in \Omega, \\ u_1(x) &= \frac{\partial u_0}{\partial t}(x, t)|_{t=0}, \quad x \in \Omega, \\ \eta_0(x, s) &= u_0(x, 0) - u_0(x, -s), \quad x \in \Omega, s \in (0, +\infty). \end{aligned} \quad (54)$$

Set $L_\mu^2((0, \infty); V)$ the Hilbert space of V -valued functions on $(0, +\infty)$, endowed with the inner product

$$\langle \varphi, \psi \rangle_{L_\mu^2((0, \infty); V)} = \int_0^\infty \mu(s) \langle A^{1/2} \varphi(s), A^{1/2} \psi(s) \rangle_H ds.$$

Let \mathcal{H} be the Hilbert space

$$\mathcal{H} = V \times H \times L_\mu^2((0, \infty); V),$$

equipped with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} \right\rangle_{\mathcal{H}} := (1 - \tilde{\mu}) \langle A^{1/2}u, A^{1/2}\tilde{u} \rangle_H + \langle v, \tilde{v} \rangle_H + \int_0^\infty \mu(s) \langle A^{1/2}w, A^{1/2}\tilde{w} \rangle_H ds. \quad (55)$$

Denoting by U the vector $U = (u, u_t, \eta)$, the above problem can be rewritten in the form (2), where $\mathcal{B}U = B(u, v, \eta) = (0, bv, 0)$ and \mathcal{A} is defined by

$$\mathcal{A} \begin{pmatrix} u \\ v \\ w \end{pmatrix} := \begin{pmatrix} v \\ (1 - \tilde{\mu})Au + \int_0^\infty \mu(s)Aw(s)ds \\ -w_s + v \end{pmatrix}, \quad (56)$$

with domain (cfr. [26])

$$\begin{aligned} \mathcal{D}(\mathcal{A}) := \{ (u, v, \eta)^T \in H_0^1(\Omega) \times H_0^1(\Omega) \times L_\mu^2((0, +\infty); H_0^1(\Omega)) : \\ (1 - \tilde{\mu})u + \int_0^\infty \mu(s)\eta(s)ds \in H^2(\Omega) \cap H_0^1(\Omega), \\ \eta_s \in L_\mu^2((0, +\infty); H_0^1(\Omega)) \}. \end{aligned} \quad (57)$$

It has been proved in [10] that the above system is exponentially stable, namely that the operator \mathcal{A} generates a strongly continuous semigroup satisfying the estimate (1), for suitable constants. Moreover, it is well-known that, the operator \mathcal{A} is dissipative. Therefore, our previous results apply to this model.

As a concrete example we can consider the wave equation with memory. More precisely, let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a smooth boundary $\partial\Omega$. Let us consider the initial boundary value problem

$$\begin{aligned} u_{tt}(x, t) - \Delta u(x, t) + \int_0^\infty \mu(s)\Delta u(x, t-s)ds \\ + b(t)u_t(x, t-\tau) = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (58) \\ u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, +\infty), \quad (59) \\ u(x, t) = u_0(x, t) \quad \text{in } \Omega \times (-\infty, 0]. \quad (60) \end{aligned}$$

This problem enters in previous form (44) – (46), if we take $H = L^2(\Omega)$ and the operator A defined by

$$A : \mathcal{D}(A) \rightarrow H : u \rightarrow -\Delta u,$$

where $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$.

The operator A is a self-adjoint and positive operator with a compact inverse in H and is such that $V = \mathcal{D}(A^{1/2}) = H_0^1(\Omega)$.

Under the same conditions that before on the memory kernel $\mu(\cdot)$ and on the function $b(\cdot)$, previous asymptotic/exponential stability results are valid. The case b constant has been studied in [2]. In particular, we have proved that the exponential stability is preserved, in presence of the delay feedback, if the coefficient b of this

one is sufficiently small. The choice b constant was made only for the sake of clearness. The result in [2] remains true if instead of b constant we consider $b = b(t)$, under a suitable smallness condition on the L^∞ -norm of $b(\cdot)$. On the contrary here we give stability results without restrictions on the L^∞ -norm of $b(\cdot)$, even if only for on-off $b(\cdot)$.

Our results also apply to Petrovsky system with viscoelastic damping with Dirichlet and Neumann boundary conditions:

$$u_{tt}(x,t) + \Delta^2 u(x,t) - \int_0^\infty \mu(s) \Delta^2 u(x,t-s) ds + b(t) u_t(x,t-\tau) = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (61)$$

$$u(x,t) = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, +\infty), \quad (62)$$

$$u(x,t) = u_0(x,t) \quad \text{in } \Omega \times (-\infty, 0]. \quad (63)$$

This problem enters into the previous abstract framework, if we take $H = L^2(\Omega)$ and the operator A defined by

$$A : \mathcal{D}(A) \rightarrow H : u \rightarrow \Delta^2 u,$$

where $\mathcal{D}(A) = H_0^2(\Omega) \cap H^4(\Omega)$, with

$$H_0^2(\Omega) = \left\{ v \in H^2(\Omega) : v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \right\}.$$

The operator A is a self-adjoint and positive operator with a compact inverse in H and is such that $V = \mathcal{D}(A^{1/2}) = H_0^2(\Omega)$.

Therefore, under the same conditions that before on the memory kernel $\mu(\cdot)$ and on the function $b(\cdot)$, previous asymptotic/exponential stability results are valid.

6.2 Locally damped wave equation equation

Here we consider the wave equation with local internal damping and intermittent delay feedback. More precisely, let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a boundary $\partial\Omega$ of class C^2 . Denoting by m the standard multiplier $m(x) = x - x_0$, $x_0 \in \mathbb{R}^n$, let ω_1 be the intersection of Ω with an open neighborhood of the subset of $\partial\Omega$

$$\Gamma_0 = \{x \in \partial\Omega : m(x) \cdot \nu(x) > 0\}. \quad (64)$$

Fixed any subset $\omega_2 \subseteq \Omega$, let us consider the initial boundary value problem

$$u_{tt}(x,t) - \Delta u(x,t) + a\chi_{\omega_1}u_t(x,t) + b(t)\chi_{\omega_2}u_t(x,t - \tau) = 0$$

$$\text{in } \Omega \times (0, +\infty), \quad (65)$$

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0, +\infty), \quad (66)$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \quad (67)$$

where χ_{ω_i} denotes the characteristic function of ω_i , $i = 1, 2$, a is a positive number and b in $L^\infty(0, +\infty)$ is an on–off function satisfying (1_w) and (2_w) of subsection 6.1. The initial datum (u_0, u_1) belongs to $H_0^1(\Omega) \times L^2(\Omega)$.

This problem enters into our previous framework, if we take $H = L^2(\Omega)$ and the operator A defined by

$$A : \mathcal{D}(A) \rightarrow H : u \rightarrow -\Delta u,$$

where $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$.

Now, denoting $U = (u, u_t)$, the problem can be restated in the abstract form (2) where $\mathcal{B}U = B(u, v) = (0, b(t)\chi_{\omega_2}v)$ and \mathcal{A} is defined by

$$\mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} v \\ -Au - a\chi_{\omega_1}v \end{pmatrix}, \quad (68)$$

with domain $\mathcal{D}(A) \times L^2(\Omega)$ in the Hilbert space $\mathcal{H} = H \times H$.

Concerning the part without delay feedback, namely the locally damped wave equation

$$u_{tt}(x,t) - \Delta u(x,t) + a\chi_{\omega_1}u_t(x,t) = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (69)$$

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0, +\infty), \quad (70)$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \quad (71)$$

it is well–known that, under the previous Lions geometric condition on the set ω_1 (or under the more general assumption of control geometric property [4]) where the frictional damping is localized, an exponential stability result holds (see e.g. [4, 16, 17, 18, 19, 20, 21, 31]). This is equivalent to say that the strongly continuous semigroup generated by the operator \mathcal{A} associated to (69)–(71), namely the one defined in (68), satisfies (1). As well–known, the operator \mathcal{A} is dissipative. Thus previous abstract stability results are valid also for this model. We emphasize the fact that the set ω_2 may be any subset of Ω , not necessarily a subset of ω_1 . On the contrary, in previous stability results for damped wave equation and intermittent delay feedback (see e.g. [24, 9]) the set ω_2 has to be a subset of ω_1 . On the other hand, now the standard (not delayed) frictional damping is always present in time while in the quoted papers it is on–off like the delay feedback and it acts only on the complementary time intervals with respect to this one.

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