A note on sets of type $(0, mq, 2mq)_2$ in $PG(3, q)$

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Abstract

In this paper we investigate sets of type $(0, mq, 2mq)_2$ in $PG(3, q)$.

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1 Introduction and motivation

Let $q = p^h$ be a prime power and $PG(k-1, q)$ be the $(k-1)$-dimensional desarguesian projective space of order q. It is well-known the correspondence between generalized equivalence classes of linear $[n, k]_q$ codes, and equivalence classes of projective systems with n points in $PG(k-1, q)$. A projective system

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$P = (P_1, ..., P_n)$ in $PG(k-1, q)$ is an enumeration of points P_j in this projective space, such that not all these points lie in a hyperplane. Let P_j be given by the homogeneous coordinates $(p_{1j}, p_{2j}, ..., p_{nj})$ and let G_P be the $k \times n$ matrix with $(p_{1j}, p_{2j}, ..., p_{nj})^t$ as j-th column. Then G_P is the generator matrix of a nondegenerate linear code C_P over F_q of length n and dimension k, since not all points lie in a hyperplane. Conversely, since a generator matrix G of a nondegenerate linear $[n, k]_q$ code C over F_q has no zero columns, take the columns of G as homogeneous coordinates of points in $PG(k-1, q)$. This gives the projective system P_C over F_q of G.

Thus a codeword $\mathbf{c} \in C$ may be written as $\mathbf{c} = \mathbf{x}G$, with $\mathbf{x} \in (F_q)^k$. The i-th coordinate of c is zero if and only if the standard inner product of x and the i-th column of G is zero. So in terms of projective systems, P_i is in the hyperplane perpendicular to x. Therefore, P_C is an *h-character projective system* (the size of the intersection of P_C with any hyperplane might assume just one out of h possible different values called the intersection numbers of P_C) if and only if C_P is a linear h weight code (the number of nonzero coordinates of the words of C_P might assume just one of h possible different values called the weights of C_P). This correspondence motivates us to investigate in $PG(3, q)$ sets of type $(0, mq, 2mq)_2$, i.e. projective systems which intersect any plane either in 0, or in mq, or in $2mq$ points, see [4], [5], [7] and [15].

2 The size of a set of type $(0, mq, 2mq)_2$

Throughout this section K will ever denote a k-set of type $(0, mq, 2mq)_2$ in $PG(3, q)$. It follows that $m \leq [\frac{q+1}{2}]$ where $[\frac{q+1}{2}]$ indicates the largest integer that is less than or equal to $\frac{q+1}{2}$. In [10], to classify k-sets in a projective Galois space with respect to their behaviour with the subspaces, M. Tallini Scafati introduced the notion of character of a k-set, that is the number of subspaces having a given number of points in common with the set itself. She gave fundamental relations among the characters in order to characterize arithmetically and geometrically the k-sets of such a space. The projective characters, as well as other integers attached in a similar way to K, are connected by systems of Diophantine equations. Simple discussion of these systems gives necessary conditions for the existence of K, see [1], [6], [9] and [13]. The question of obtaining sufficient conditions is much more difficult, see [2], [12] and [14], and can be studied only in special cases, see [3], [8] and [11].

Let x, y and z denote the number of the 0-planes, mq-planes and $2mq$-planes, respectively. The definition of x, y and z and double countings give the following equatons:

$$x + y + z = (q + 1)(q^2 + 1) \tag{1}$$

$$mqy + 2mqz = k(q^2 + q + 1) \tag{2}$$

$$mq(mq - 1)y + 2mq(2mq - 1)z = k(k - 1)(q + 1) \tag{3}$$

By (2) we see that $q|k$. Putting $k = aq$ and solving the system we get

$$2m^2x = (q + 1)a^2 - [3m(q^2 + q + 1) - q]a + 2(q + 1)(q^2 + 1)m^2 \tag{4}$$

$$m^2y = a[2m(q^2 + q + 1) - q - a(q + 1)] \tag{5}$$

$$2m^2z = a[a(q + 1) + q - m(q^2 + q + 1)] \tag{6}$$

Now if we denote by Δ the discriminant of the following equation

$$(q + 1)a^2 - [3m(q^2 + q + 1) - q]a + 2(q + 1)(q^2 + 1)m^2 = 0 \tag{7}$$

of second degree in the unknown a then we have that
$\Delta = (q^4+2q^3+11q^2+2q+1)(m-1)^2+2[(q-1)q^3+(8q-1)q+1](m-1)+(q-1)^4$. It is immediate to see that for any $m \geq 1$ and for any $q \geq 2$ it is $\Delta > 0$. Thus there is a positive real number δ such that $\delta^2 = \Delta$.

Now put

- $a_0 := \frac{m(q^2+q+1)-q}{q+1} = mq - \frac{q-m}{q+1}$
- $a_1 := \frac{3m(q^2+q+1)-q-\delta}{2(q+1)} = a_0 + \frac{m(q^2+q+1)+q-\delta}{2(q+1)}$
- $a_2 := \frac{3m(q^2+q+1)-q+\delta}{2(q+1)} = a_1 + \frac{\delta}{q+1}$;
- $a_3 := \frac{2m(q^2+q+1)-q}{q+1} = 2mq - \frac{q-2m}{q+1} = a_2 + \frac{m(q^2+q+1)-q-\delta}{2(q+1)}$

If $1 \leq m \leq [\frac{q+1}{2}]$ and $q \geq 2$, then it is easy to prove that

$$0 < \frac{m(q^2 + q + 1) - q - \delta}{2(q + 1)} < \frac{m(q^2 + q + 1) + q - \delta}{2(q + 1)} < 2 \tag{8}$$

where δ is a positive real number such that $\delta^2 = \Delta$.

Now it is immediate to see that

1. $mq - 1 < a_0 < mq$;
2. if $m < [\frac{q+1}{2}]$, then $(2mq - 1) < a_3 < 2mq$;
3. if q is even and $m = [\frac{q+1}{2}] = \frac{q}{2}$, then $a_3 = 2mq$;
4. if q is odd and $m = [\frac{q+1}{2}] = \frac{q+1}{2}$, then $2mq < a_3 < 2mq + 1$;
5. $a_1 - a_0 < 2$;
6. $a_3 - a_2 < 2$;
7. $a_0 < a_1 < a_2 < a_3$.

By (4) and (7) we have that $x > 0$ if and only if $a < a_1$ or $a_2 < a$. Furthermore by (5) and (6) we have that $y > 0$ and $z > 0$ if and only if $a_0 < a < a_3$. Finally we have that $a_0 < a < a_1$ or $a_2 < a < a_3$. Therefore

- if $m < [\frac{q+1}{2}]$, then $a \in \{mq, mq + 1, 2mq - 2, 2mq - 1\}$;
- if q is even and $m = [\frac{q+1}{2}] = \frac{q}{2}$, then $a \in \{mq, mq + 1, 2mq - 1\}$;
- if q is odd and $m = [\frac{q+1}{2}] = \frac{q+1}{2}$, then $a \in \{mq, mq + 1, 2mq - 1, 2mq\}$.

If $a = mq + 1$, then we get $x \leq 0$, a contradiction. If $a = 2mq - 2$, then we get $x < 0$, a contradiction. If q is odd, $m = [\frac{q+1}{2}] = \frac{q+1}{2}$, and $a = 2mq$, then we get $x < 0$, a contradiction.

If $a = 2mq - 1$, then by (4) we get $2m^2x = 2m^2(1 - 2q) + m(q^2 - q + 3) + 1$. So $m|1$ and hence $m = 1$. Finally K is a $(2q^2 - q)$-set of type $(0, q, 2q)_2$. Furthermore $x = \frac{1}{2}(q - 2)(q - 3)$, $y = 3(2q - 1)$, $z = [\frac{1}{2}q(q + 1) - 1](2q - 1)$. Now let us prove that such a set does not exist. On the contrary, let us suppose that such a set exists. So there is at least a plane π_0 external to K. Let r_0 be a line on π_0. Obviously the line r_0 is external to K. Now we denote by u_i the number of i-planes passing through r_0 with $i \in \{0, q, 2q\}$. By counting the points of K via the planes through r_0 we get $u_q = 3 - 2u_0$. Being $u_q \geq 0$ it is $u_0 \in \{0, 1\}$ necessarily. Therefore $u_0 = 1$ since the external plane π_0 passes through r_0. Finally it is $u_q = 1$. So exactly one q-plane passes through each line r_0 on π_0. Hence the number y of q-planes is $q^2 + q + 1$, a contradiction.

If $a = mq$, then by (6) we get $2mz = q^2 - mq = q(q-m)$. So $m|q^2$ and $m < q$ since $z > 0$. Hence $m|q = p^h$. Therefore there is an integer r such that $0 \leq r < h$ and $m = p^r$. Finally K is a (q^2p^r)-set of type $(0, qp^r, 2qp^r)_2$. Furthermore, $z = \frac{1}{2}q(p^{h-r} - 1)$, $x = z + 1$, $y = q(q^2 + q + 1) - 2z$.

Hence in this section we have proved the following

Theorem 2.1. *If K is a k-set of type $(0, mq, 2mq)_2$ in $PG(3, q)$, $q = p^h$, then there is an integer r such that $0 \leq r < h$, $m = p^r$, and $k = mq^2$, i.e.*

- *K is a (q^2p^r)-set of type $(0, qp^r, 2qp^r)_2$.*

3 On (q^2p^r)-sets of type $(0, qp^r, 2qp^r)_2$.

In this section K is a (q^2p^r)-sets of type $(0, qp^r, 2qp^r)_2$ of $PG(3, q)$, $q = p^h$, $0 \leq r < h$. We recall that there are exactly:

- $z = \frac{1}{2}q(p^{h-r} - 1)$ planes meeting K in $2qp^r$ points;
- $x = z + 1$ planes meeting K in no point;
- $y = q(q^2 + q + 1) - 2z$ planes meeting K in qp^r points.

Lemma 3.1. *In $PG(3,q)$ there is a $(\frac{q^3}{p})$-set of type $(0, \frac{q^2}{p}, q^2)$.*

Proof. Let us consider a line l and denote by $F(l)$ the $(q+1)$-set of the planes through l. Now consider a $\frac{q}{p}$-subset $\mathcal{F}$ of $F(l)$. If we denote by K the pointset of $\bigcup_{\pi \in \mathcal{F}} (\pi \setminus l)$, then K has $k = \frac{q}{p} q^2 = \frac{q^3}{p}$ points. Any plane $\pi \in \mathcal{F}$ meets K in exactly q^2 points. Any plane $\pi \in F(l) \setminus \mathcal{F}$ meets K in exactly 0 points. Any plane not through l meets K in exactly $q \frac{q}{p} = \frac{q^2}{p}$ points. □

Theorem 3.2. *If $q = 2^h$ and $r = h-1$, then in $PG(3,q)$ there is a $(q^2 p^r)$-set of type $(0, qp^r, 2qp^r)_2$.*

Question 1. Are there sets K with $p > 2$?

Let l denote an s-line of K, with $s > 0$ and let u be the number of (qp^r)-planes passing through l. By counting the size of K by the planes through l we have $(qp^r - s)u + (2qp^r - s)(q+1-u) = q^2 p^r - s$ from which we get $s = (q+2-u)p^r$. So $p^r | s$ and therefore K is a set of class $[0, p^r, 2p^r, 3p^r, ..., p^h]_1$, i.e. K meets any line in a number of points congruent zero modulo p^r.

Thus, a necessary condition for the existence such a set K is the existence in $PG(2,q)$ of k-sets of class $[0, p^r, 2p^r, 3p^r, ..., p^h]$ with $k \in \{qp^r, 2qp^r\}$.

Question 2. In $PG(2,q)$ do k-sets of class $[0, p^r, 2p^r, 3p^r, ..., p^h]$ with $k \in \{qp^r, 2qp^r\}$ exist?

Let α be an affine plane contained in $AG(3,q)$ and denote by l_∞ its line at infinity that is contained in π_∞, the plane at infinity of $AG(3,q)$. Let $V_\infty \subset \pi_\infty$ be a point non belonging to l_∞ and let S be a subset of points of α. The cone with vertex V_∞ and base S is the subset of $PG(3,q) = AG(3,q) \cup \pi_\infty$ that is the union of the lines of $PG(3,q)$ spanned by V_∞ and a point $P \in S$. In the sequel it will be denoted by $Cone(V_\infty, S)$. The cylinder with vertex V_∞ and base S is the subset of $AG(3,q)$ given by $AG(3,q) \cap Cone(V_\infty, S) = Cone(V_\infty, S)/V_\infty$. In the sequel it will be denoted by $Cyl(V_\infty, S)$. If S is a (qp^r)-sets of type $(0, p^r, 2p^r)$ in $AG(2,q)$, then $Cyl(V_\infty, S)$ is a $(q^2 p^r)$-sets of type $(0, qp^r, 2qp^r)_2$ in $PG(3,q)$.

Question 3. In $AG(2,q)$, $q = p^h$ and $p > 2$, do (qp^r)-sets of type $(0, p^r, 2p^r)$ esist?

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