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p-GROUPS OF FROBENIUS TYPE

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Abstract

A. Camina defined the groups of Frobenius Type as follows:

Let G be a finite group with a proper normal subgroup $N \neq 1$. G is a group of Frobenius Type if it satisfies the following two equivalent conditions:

F1 : There is a set of irreducible non-trivial characters of G, $\{\chi_1, \ldots, \chi_n\}$ with $n \in \mathbb{N}$, such that χ_i vanishes on $G - N$ and there exist natural numbers $a_1, \ldots, a_n > 0$ such that $\sum_{i \in \{1,\ldots,n\}} a_i \chi_i$ is constant on N^*.

F2 : For every $x \in G - N$ and for every $z \in N$, x is conjugate to xz.

A Camina Pair (C-pair for short) (G, N) is a pair (G, N), with $N \lhd G$, that satisfies $F1$ and $F2$. The term *"Camina group"* is also used frequently in the literature, for the particular case $N = G'$.

A number of other group theoretical proprierties that are equivalent to $F1$ and $F2$ can be found in the literature:

Let G be a finite group, and $N \lhd G$. Then G is of Frobenius type with respect to N, (i.e. (G, N) is a *C-pair*) if the following equivalent conditions hold for every element $x \in G - N$:

1. $xN \subseteq x^G$;
2. $N \subseteq [x, G]$;
3. $|C_{G/N}(xN)| = |C_G(x)|$;
4. $\chi(x) = 0$ for every $\chi \in \mathrm{Irr}(G|N)$.

The main result of Camina's paper is a crucial case distinction in the classification of the C-pairs:

Let (G, N) be a C-pair. Then either G is a Frobenius group with Frobenius kernel N, or there is a prime p for which one of the following conditions holds:

1. N is a p-group;
2. $\frac{G}{N}$is a p-group.

The case in which both N and G/N are p-groups, for the same prime p, was first studied by Macdonald and in this thesis I continue the study of Macdonald of p-groups of Frobenius Type, focusing on the set $\mathrm{Irr}(G|N)$ when $N = Z(G)$, where I use the notation $\mathrm{Irr}(G|N)$ for the set of the irreducible characters of G whose kernel does not contain N, i.e. $\mathrm{Irr}(G|N) := \mathrm{Irr}(G) - \mathrm{Irr}(G/N)$.

In his work Camina lists some conjectures:

1. Let G be a finite p-group and $1 < i < c$, if $(G, \gamma_i(G))$ has $F2$ then $(G, \gamma_{i+1}(G))$ has $F2$.
2. If G is a finite p-group of class c and (G, H) has $F2$ then $H = \gamma_{c-1}(G)$ or $H = \gamma_c(G)$.
3. Assume that (G, K) is a Camina Pair and $\frac{G}{K}$ is a p-group for some $p \in \mathbb{P}$, then one of the following holds:
   (a) G is a Frobenius group with Frobenius kernel K and complement of Frobenius that is a p-group;
   (b) G is a Camina p-group with $K = G'$ or $K = G_3$;
   (c) G is a Frobenius group with complement of Frobenius that is quaternionic and $K = G'$.

It's reasonable to think that the previous conjecture is true because if $K = G'$ the conjecture is verified by R. Dark and C. M. Scoppola in their work "On Camina Groups of Prime Power Order".

So in my thesis I discuss this conjecture and I continue my guided tour through finite groups of Frobenius type as follows:

First of all, with the support of professor Norberto Gavioli, we have found examples of p-groups G such that $(G, Z(G))$ is a Camina pair, but G is a not a Camina group.

In my thesis there is not only the code, thanks to which it is possible to find such groups in the "SmallGroups" library of GAP:

```
p:= 5,7,11, ...;
   iscaminawrtcentre := function (g)
   local x, el, z, elz, genz;
      if IsAbelian(g) then return(false);
      fi;
         z :=Centre(g);
         genz :=GeneratorsOfGroup(z);
         cls :=ConjugacyClasses(g);
         cls :=Filtered(cls, x-> (Size(x) > 1));
            for x in cls do
               tmplist=[ ];
                  for el in x do
                     for elz in genz do;
                        if not(elz*el in x) then return(false);
                        fi;
                     od;
                  od;
               od;
            return(true);
         end;
   lst :=AllSmallGroups(Size, p^n, iscaminawrtcentre, true);
```

But also a detailed analysis of the found groups collected in the following simple table:

Size of the group:	*Number of examples found:*	*Description:*
5^5	14	p-groups of order 3125 with 5 generators
5^6	58	p-groups of order 15625 with 6 generators
7^5	14	p-groups of order 16807 with 5 generators
7^6	100	p-groups of order 117649 with 6 generators

Later, together with my tutor Carlo Maria Scoppola, we focused on the class mentioned above and then we have exploited the theory of characters of such groups, showing the following result:

Let G be a p-group with $p > 2$, $|Z(G)| = p^z$ and z a non-negative integer. Then $(G, Z(G))$ is a Camina pair if and only if $|G| = p^{z+2n}$ and there are $p^z - 1$ characters of degree p^n.

As a consequence of previous theorem, I obtain:

Let G be a p-group and (G, K) be a C-pair. Then $|G : K|$ is an even power of p.

As a special case of Conjecture 2 proposed by Camina, I mention:

Let G be a finite p-group of class c, if $(G, Z_2(G))$ has $F2$ then $(G, Z(G))$ has $F2$.

So, by the previous result and assuming that the previous conjecture is true, I have:

Suppose that $|Z_2(G) : Z(G)| = p^n$, $|Z(G)| = p^z$ and $|G : Z_2(G)| = p^m$, if $(G, Z_2(G))$ is a Camina Pair, then n is an even number.

Note that in this case, the existence of $p^n - 1$ characters of degree $p^{\frac{m}{2}}$ would imply the existence of $p^z - 1$ characters of degree $p^{(\frac{m+n}{2})}$.

Contents

Chapter 1

Introduction

About forty years ago B. Beisiegel, in his work [7], introduced the following definition:

Definition 1.0.1. *A p-group $G \neq 1$ is a semi-extraspecial p-group if for all $M <\cdot Z(G)$, G/M is an extraspecial p-group.*

A crucial example of semi-extraspecial p-group is the Heisenberg group $H(p^n)$, consisting of the upper unitriangular matrices of order 3 over the finite field with p^n elements, which is a Sylow p-subgroup of $SL(3, p^n)$.

In 1978, indipendently, A. Camina defined the groups of Frobenius Type as follows:

Definition 1.0.2. *Let G be a finite group with a proper normal subgroup $N \neq 1$. G is a group of Frobenius Type if it satisfies the following two equivalent conditions:*

F1 *: There is a set of irreducible non-trivial characters of G, $\{\chi_1, \ldots, \chi_n\}$ with $n \in \mathbb{N}$, such that χ_i vanishes on $G - N$ and there exist natural numbers $a_1, \ldots, a_n > 0$ such that $\sum_{i \in \{1,\ldots,n\}} a_i\chi_i$ is constant on N^*.*

F2 *: For every $x \in G - N$ and for every $z \in N$, x is conjugate to xz.*

Remark 1. It's easy to see that semi-extraspecial p-groups are groups of Frobenius Type.

A Camina Pair (C-pair for short) (G, N) is a pair (G, N), with $N \lhd G$, that satisfies $F1$ and $F2$.

A number of other group theoretical proprierties that are equivalent to $F1$ and $F2$ can be found in the literature:

Lemma 1.0.3. *Let G be a finite group, and $N \lhd G$. Then G is of Frobenius type with respect to N, (i.e. (G, N) is a C-pair) if the following equivalent conditions hold for every element $x \in G - N$:*

1. $xN \subseteq x^G$;
2. $N \subseteq [x, G]$;
3. $|C_{G/N}(xN)| = |C_G(x)|$;

4. $\chi(x) = 0$ *for every* $\chi \in \mathrm{Irr}(G|N)$ *(Mattarei, [61]).*

It is rather clear that these properties are somehow inherited by quotients over normal subgroup contained in N ([58], Lemma 1.1).

The main result of Camina's paper is a crucial case distinction in the classification of the C-pairs:

Theorem 1.0.4. *Let* (G, N) *be a C-pair. Then either* G *is a Frobenius group with Frobenius kernel* N*, or there is a prime* p *for which one of the following conditions holds:*

1. N *is a* p*-group;*
2. $\frac{G}{N}$ *is a* p*-group.*

Therefore from now on we will implicitly exclude the Frobenius groups from the class of C-pairs, and we will assume that the prime p that appears in the statement of Theorem 1.0.4 divides $gcd(|N|, |G : N|)$.

Extraspecial p-groups and semi extraspecial p-groups mentioned above are an obvious example of groups of Frobenius Type.

The case in which both N and G/N are p-groups, for the same prime p, was first studied by Macdonald, who was probabily unaware of [7].

Our notation is mainly standard, see [9]. It should be noted, though, that for every element $x \in G$ we use the symbol $[x, G]$ to denote the *set* $\{[x, g] | g \in G\}$, rather than the subgroup generated by it. We also use the notation $\mathrm{Irr}(G|N)$ for the set of the irreducible characters of G whose kernel does not contain N, i.e. $\mathrm{Irr}(G|N) := \mathrm{Irr}(G) - \mathrm{Irr}(G/N)$.

The term *"Camina group"* is also used frequently in the literature, for the particular case $N = G'$.

In this thesis we continue the study of Macdonald of p-groups of Frobenius Type, focusing on the set $\mathrm{Irr}(G|N)$ when $N = Z(G)$.

A good and recent survey on Camina p-groups is found in [49], to which the reader is referred for all the results concerning Camina p-groups of class 2, i.e. semiextraspecial p-groups. A brief summary about these groups can be found in the second chapter of this thesis.

In his work he lists some conjectures proposed in [56]:

Conjecture 1.0.5. *Let* G *be a finite* p*-group and* $1 < i < c$*, if* $(G, \gamma_i(G))$ *has* $F2$ *then* $(G, \gamma_{i+1}(G))$ *has* $F2$*.*

Conjecture 1.0.6. *If* G *is a finite* p*-group of class* c *and* (G, H) *has* $F2$ *then* $H = \gamma_{c-1}(G)$ *or* $H = \gamma_c(G)$*.*

As a special case of Conjecture 1.0.5 we mention:

Conjecture 1.0.7. *Let* G *be a finite* p*-group of class* c*, if* $(G, Z_2(G))$ *has* $F2$ *then* $(G, Z(G))$ *has* $F2$*.*

We mention here also a conjecture about C-pairs that are not p-groups:

Conjecture 1.0.8. *Assume that (G, K) is a Camina Pair and $\frac{G}{K}$ is a p-group for some $p \in \mathbb{P}$, then one of the following holds:*

1. *G is a Frobenius group with Frobenius kernel K and complement of Frobenius that is a p-group;*
2. *G is a Camina p-group with $K = G'$ or $K = G_3$;*
3. *G is a Frobenius group with complement of Frobenius that is quaternionic and $K = G'$.*

It's reasonable to think that the previous conjecture is true because if $K = G'$ the conjecture is verified in [19]. So in the third chapter of this thesis I will discuss this conjecture.

In the same chapter of my thesis I continue my guided tour through finite groups of Frobenius type as follows:

1. First of all, with the support of professor Norberto Gavioli, we have found examples of p-groups G such that $(G, Z(G))$ is a Camina pair, but G is a not a Camina group.

   In my thesis there is not only the code, thanks to which it is possible to find such groups in the "SmallGroups" library of GAP; but also a detailed analysis of the found groups.

2. Later, together with my tutor Carlo Maria Scoppola, we focused on the class mentioned above and then we have exploited the theory of characters of such groups, showing the following result:

   Let G be a p-group with $p > 2$, $|Z(G)| = p^z$ and z a non-negative integer. Then $(G, Z(G))$ is a Camina pair if and only if $|G| = p^{z+2n}$ and there are $p^z - 1$ characters of degree p^n.

   As a consequence of previous theorem, we obtain:

   Let G be a p-group and (G, K) be a C-pair. Then $|G : K|$ is an even power of p.

   By the previous result and assuming that the Conjecture 1.0.7 is true, then we have:

   Suppose that $|Z_2(G) : Z(G)| = p^n$, $|Z(G)| = p^z$ and $|G : Z_2(G)| = p^m$, if $(G, Z_2(G))$ is a Camina Pair, then n is an even number.

   Note that in this case, the existence of $p^n - 1$ characters of degree $p^{\frac{m}{2}}$ would imply the existence of $p^z - 1$ characters of degree $p^{(\frac{m+n}{2})}$.

In the fourth chapter of this thesis I survey the groups of Frobenius type that are not p-groups, and then I will proceed to list some generalizations and possible open problems.

Chapter 2

Background Results

2.1 Frobenius Groups

Definition 2.1.1. *Let G be a finite group. G is said to be a Frobenius group if it contains a proper non-identity subgroup H such that $H \cap gHg^{-1}$=1 for all $g \in G - H$. The subgroup H is called a Frobenius complement for G.*

This hypothesis is equivalent to the permutation-group-theoretic definition of Frobenius groups:

Definition 2.1.2. *G is a Frobenius group if it acts transitively on a set X so that no non-identity element fixes more than one point.*

Let G be a Frobenius group, Frobenius proved that the set $N = \{G - \cup_{x \in G} x^{-1}Hx\} \cup \{1\}$ is a normal subgroup of G.

Definition 2.1.3. *The normal subgroup $N = \{G - \cup_{x \in G} x^{-1}Hx\} \cup \{1\}$ is called Frobenius kernel of G.*

2.2 Character Theory

We list a few basic results that we assume, see [29] for proofs. Let G be a finite group, let F be a field and suppose Ψ is a representation of the group algebra $F[G]$ of degree n. Since Ψ is an algebra homomorfism, we have:

1. $\Psi(1) = I$, the identity matrix.
2. $\Psi(g)$ is nonsingular and $\Psi(g)^{-1} = \Psi(g^{-1})$, for every $g \in G$.
3. If we restrict the function Ψ to $G \subseteq F[G]$, we obtain a group homomorphism from G into the general linear group $GL(n, F)$ that is the multiplicative group of nonsingular $n \times n$ matrices over F.

Definition 2.2.1. *Let Ψ be an F-representation of G. Then the F-character χ of G afforded by Ψ is the function given by $\chi(g) = tr\Psi(g)$.*

We restrict our attention to the case that the field $F = \mathbb{C}$ and we fix a finite group G and a representative set $\mathcal{M}(\mathbb{C}[G])$ of irreducible $\mathbb{C}[G]-$modules denoted by $\{M_1, \ldots, M_k\}$.

The following facts are well known:

Theorem 2.2.2. *See [29]:*

1. *For every $h \in G$ the following holds: $(1/|G|)\sum_{g\in G}\chi_i(g)\chi_j(g^{-1}) = \delta_{ij}\frac{\chi_i(h)}{\chi_j(1)}$.*
   (Generalized Orthogonality Relation, [29], Theorem 2.13)
2. *$\sum_{g\in G}\chi(g)\overline{\chi(g)} = |G|$.*
   (First Ortogonality Relation, [29], Corollary 2.14)
3. *If $g \in G$ is not conjugate to $h \in G$ in G, then $\sum_{\chi\in \mathrm{Irr}\, G}\chi(g)\overline{\chi(h)} = 0$.*
   Otherwise $\sum_{\chi\in \mathrm{Irr}\, G}\chi(g)\overline{\chi(h)} = |C_G(g)|$.
   (Second Orthogonality Relation, [29], Theorem 2.18)
4. *$\sum_{\chi\in \mathrm{Irr}\, G}\chi(1)^2 = |G|$ ([29], Corollary 2.7).*
5. *$\chi(1)^2 \leq |G : Z(G)|$ ([29], Corollary 3.12).*
   So if G is a p-group we have $\chi(1)^2$ divides $|G : Z(G)|$.

Theorem 2.2.3. *Let G be a p-group. For every $\chi \in \mathrm{Irr}\, G$ the following facts are well known [29]:*

1. *$\chi(1)$ divides $|G|$ and $|\chi(1)| = p^k$, for some non negative integer k.*
2. *$\chi \in \mathrm{Irr}\, G$ is faithful if and only if $\chi(z) \neq \chi(1)$, for all $1 \neq z \in Z(G)$. In this case, $Z(G)$ is cyclic.*

Theorem 2.2.4. *For every $\chi \in \mathrm{Irr}\, G$ the following facts are well known [29]:*

1. *If G has a faithful irreducible character, then $Z(G)$ is cyclic.*
2. *If G is a p-group and $Z(G)$ is cyclic, then G has a faithful character.*

Theorem 2.2.5 ([29], Lemma 2.25)**.** *Let ρ be a irreducible $\mathbb{C}$-representation of G of degree n. Suppose A is an $n \times n$ matrix over $\mathbb{C}$ which commutes with $\rho(g)$ for all $g \in G$.*

Then $A = \alpha I$ for some $\alpha \in \mathbb{C}$.

2.3 Special and extra-special p-groups

Let G be a finite p-group. The following known results are in [76].

Definition 2.3.1. *A p-group G is a special group if $Z(G) = G' = \mathrm{Frat}(G)$.*

Definition 2.3.2. *Special groups whose derived subgroup has order p are extraspecial groups.*

Therefore an equivalent condition for a p-group G to be an extraspecial group is that $Z(G)$ is cyclic of order p and the quotient $\frac{G}{Z(G)}$ is a non-trivial elementary abelian p-group.

Theorem 2.3.3. *Every extraspecial p-group has order p^{1+2n} for some positive integer n, and conversely for each such number there are exactly two extraspecial groups up to isomorphism.*

Theorem 2.3.4. *A central product of two extraspecial p-groups is extraspecial, and every extraspecial group can be written as a central product of extraspecial groups of order p^3.*

This reduces the classification of extraspecial groups to that of extraspecial groups of order p^3.

For all $p \in \mathbb{P}$, a uniform presentation of the extraspecial groups of order p^{1+2n} can be given as follows. Define the two groups:

$$M(p) = \langle a, b, c \mid a^p = b^p = 1, c^p = 1, ba = abc, ca = ac, cb = bc \rangle$$

$$N(p) = \langle a, b, c \mid a^p = b^p = c, c^p = 1, ba = abc, ca = ac, cb = bc \rangle$$

$M(p)$ and $N(p)$ are non-isomorphic extraspecial groups of order p^3 with center of order p generated by c. The two non-isomorphic extraspecial groups of order p^{1+2n} are the central products of either n copies of $M(p)$ or $n-1$ copies of $M(p)$ and 1 copy of $N(p)$.

This is a special case of a classification of p-groups with cyclic center and simple derived subgroups given by Newman.

If p is an odd prime, there are two extraspecial groups of order p^3, which are given by:

- The group of triangular 3×3 matrices over the field with p elements, with 1's on the diagonal. This group has exponent p.
- The semidirect product of a cyclic group of order p^2 by a cyclic group of order p acting non-trivially on it. This group has exponent p^2.

If p is equal to 2, there are two extraspecial groups of order $8 = 2^3$, which are given by:

- The dihedral group D_8 of order 8, which can also be given by either of the two constructions in the section above for $p = 2$. This group has 2 elements of order 4.
- The quaternion group Q_8 of order 8, which has 6 elements of order 4.

2.3.1 Character theory of extraspecial groups

If G is an extraspecial group of order p^{1+2n}, then its irreducible complex representations are given as follows:

1. There are exactly p^{2n} irreducible representations of dimension 1.

   The center $Z(G)$ acts trivially, and the representations just correspond to the representations of the abelian group $\frac{G}{Z(G)}$.

2. There are exactly $p - 1$ irreducible representations of dimension p^n.

   There is one of these for each non-trivial character χ of the center, on which the center acts as multiplication by χ. The character values are given by $p^n\chi$ on $Z(G)$, and 0 for elements not in $Z(G)$.

Chapter 3

Main Results

3.1 The case of Camina $p-$groups

We recall the following:

Theorem 3.1.1. *Let G be a finite p-group (so G is nilpotent of nilpotence class equal to c), $1 \neq N \lhd G$, and assume that (G, N) is a C-pair. Then:*

1. *$N = Z_{c-r+1}(G) = \gamma_r(G)$ for some r.*

2. *Let $N = Z_{c-r+1}(G)$. Then $Z_i(G)/Z_{i-1}(G)$ is elementary abelian for all $c - r + 1 < i \leq c$.*

3. *Let $N = \gamma_r(G)$. Then $\gamma_i(G)/\gamma_{i+1}(G)$ is elementary abelian for all $r - 1 \leq i \leq c$.*

From now we assume that (G, G') is a C-pair.

Theorem 3.1.2. *Assume that $|G : G'| = p^m$, then:*

1. *[7] , [55] $m = 2n$ is even, and $|\gamma_2(G) : \gamma_3(G)| \leq p^n$.*

2. *[55] if $\gamma_3(G) \neq 1$ then $|\gamma_2(G) : \gamma_3(G)| = p^n$.*

3. *[19] $|\gamma_3(G) : \gamma_4(G)| < p^{3n/2}$.*

4. *[58], if $\gamma_3(G) \geq p^n$ then $G/\gamma_3(G) \simeq H(p^n)$*

5. *[19] $\gamma_4(G) = 1$.*

6. *[57] $G^p \leq \gamma_3(G)$.*

7. *G' is elementary abelian.*

The reader is invited to look at the original papers for the proofs. Here we include only some comments.

On [56], Lemma 2.1 states the inequality

$$(3') \quad |\gamma_3(G) : \gamma_4(G)| \le p^n,$$

which is stronger than the one stated here in item (3) above, but the sketch of proof offered on [56] does not seem to be conclusive.

Some of the results of [58] are based on assumption (3'), but most of the results of [58] are about Camina p-groups of nilpotency class at least 4, and they are therefore superseded by item (5) above, while the results is section 1 of [58] can be easily mended: namely, items (v) and (vi) of Theorem 1.3 in [58] can be stated and proved easily under the hypothesis $\gamma_3(G) \ge p^n$, rather than $\gamma_3(G) = p^n$ as in [58], in view of item 1 of Theorem 3.1.2; similarly, in the proof of Theorem 1.5 of [58], the conclusion of the proof depends only on the inequality $|\gamma_3(G) : \gamma_4(G)| \ge p^n$, which is obtained correctly, i.e. independently of the inequality (3').

This is relevant, because the proof of the main result of [19], which is item (5) of 3.1.2 above, is a reduction (based on the consideration of the Lie algebra associated to the lower central series of G) of the general case to the metabelian case, i.e. to the hypothesis of Theorem 1.5 of [58].

Many authors have tried to prove the inequality (3'), that would certainly be best possible, as shown by the examples 2.12 of [58]. So far, to our best knowledge, the problem is open.

In the direction of Conjecture 1.0.7 we are able to prove the following result:

Theorem 3.1.3. *Let G be a finite p-group and $|Z(G)| = p$, if $(G, Z_2(G))$ has $F2$ then $(G, Z(G))$ has $F2$.*

Proof. Choose an element $a \in Z_2(G) - Z(G)$ and observe that the conjugates of a form the coset $aZ(G)$ because the commutators of a belong to $Z(G)$ and the number of distinct commutators cannot be less than p. For this reason $(G, Z(G))$ has $(F2)$.

□

3.2 Examples of group of Frobenius Type with $N = Z(G)$

Our first aim is to prove the existence of p-group G such that:

1. G isn't a Camina group (that is equivalent to say that (G, G') isn't a Camina pair);
2. $(G, Z(G))$ is a Camina pair;
3. G has class equal to 3.

Recall from ([58], Lemma 1.1)

Lemma 3.2.1. *We have:*

- *If (G,N) is a C-pair and $K \leq N$ is a normal subgroup of G, then (G/K,N/K) is a C-pair;*
- *Assume that K and N are proper normal subgroups of G, $K \leq N$, and that $(G/K,N/K)$ is a C-pair. Then a necessary and sufficient condition for (G,N) to be a C-pair is that if $g \in G-N$ and $k \in K$, then there is $x \in G$ such that $[g,x]=k$.*

According to Lemma 3.2.1, a group G such that $(G,Z(G))$ is a Camina pair while G isn't a Camina group has a quotient of class 2 which is not a Camina group, so we will work in class equal to 3.

MacDonald in his work [55] lists two examples of these groups without proving a detailed proof. We expand the first one here:

$G =< x > \rtimes < y >$ is metacyclic with $o(x)=p^n-1$, $o(y)=p^n-1$ and $x^y=x^{1+p}$. For every $i \leq 1,\dots,p^n-1$ and $i \notin \{1,p^n,2p^{n+1},\dots\}$, we consider $(x^iy^j)^y=(x^i)^yy^j$.

Obviously $(x^iy^j)^y=(x^i)^yy^j=(x^y)^iy^j=x^{(1+p)i}y^j=x^{ip}x^iy^j$, so applying the conjugation an appropriate number of times, we obtain $(x^iy^j)^y=x^{ip^{n-1}}(x^iy^j)^y$.

With this argument we have shown that the elements x^iy^j cover $< x^{p^{n-1}} >$. Using the same argument we obtain the same result also for $i \in \{1,p^{n-1},2p^{n-1},\dots,pp^{n-1}\}$. Since $x^{ip}=x^{sp^{n-1}p}=x^sp^n=1$, for every element we obtain $(x^{sp^{n-1}}y)^x=(x^{(1+p)i}y)^x=x^{(1+p)i}y^j=x^{ip}x^iy^j=1x^iy^j=x^iy^j$.

3.3 Examples constructed by GAP

More examples of p-groups G such that $(G,Z(G))$ is a Camina pair, but G is a not a Camina group, can be found in the SmallGroups library of GAP, using the following code written by Noberto Gavioli:

```
p:= 5,7,11, ...;
   iscaminawrtcentre := function (g)
   local x,el,z,elz,genz;
      if IsAbelian(g) then return(false);
      fi;
         z :=Centre(g);
         genz :=GeneratorsOfGroup(z);
         cls :=ConjugacyClasses(g);
         cls :=Filtered(cls,x-> (Size(x) > 1));
            for x in cls do
               tmplist=[ ];
                  for el in x do
```

```
                      for elz in genz do;
                        if not(elz*el in x) then return(false);
                        fi;
                      od;
                    od;
                  od;
                return(true);
              end;
          lst :=AllSmallGroups(Size, p^n, iscaminawrtcentre, true);
```

It is worthwhile to collect in a simple table all the information described in the following sections:

Size of the group:	*Number of examples found:*	*Description:*
5^5	14	p-groups of order 3125 with 5 generators
5^6	58	p-groups of order 15625 with 6 generators
7^5	14	p-groups of order 16807 with 5 generators
7^6	100	p-groups of order 117649 with 6 generators

3.3.1 Groups of order 5^5

For the prime $p = 5$ and the natural number $n = 5$ we obtain the following 14 p-groups of order 3125 with 5 generators f_1, f_2, f_3, f_4, f_5:

$G_1 = \langle f_1, f_2, f_3, f_4, f_5 \mid f_1^5 * f_4^{-1}, f_1^{-1} * f_2 * f_1 * f_3^{-1} * f_2^{-1}, f_2^5 * f_3^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_3^{-1}, f_3^5 * f_5^{-1}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-4} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5 \rangle$

$G_2 = \langle f_1, f_2, f_3, f_4, f_5 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_3^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_4^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_5^{-1} * f_3^{-1}, f_3^5, f_1^{-1} * f_4 * f_1 * f_5^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5 \rangle$

$G_3 = \langle f_1, f_2, f_3, f_4, f_5 \mid f_1^5 * f_5^{-1}, f_1^{-1} * f_2 * f_1 * f_3^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_4^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_5^{-1} * f_3^{-1}, f_3^5, f_1^{-1} * f_4 * f_1 * f_5^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5 \rangle$

$G_4 = \langle f_1, f_2, f_3, f_4, f_5 \mid f_1^5 * f_5^{-2}, f_1^{-1} * f_2 * f_1 * f_3^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_4^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_5^{-1} * f_3^{-1}, f_3^5, f_1^{-1} * f_4 * f_1 * f_5^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5 \rangle$

$G_5 = \langle f_1, f_2, f_3, f_4, f_5 \mid f_1^5 * f_5^{-3}, f_1^{-1} * f_2 * f_1 * f_3^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_4^{-1} * f_3^{-1}, f_2^{-1} * f_3 *$

$f_2*f_5^{-1}*f_3^{-1}, f_3^5, f_1^{-1}*f_4*f_1*f_5^{-1}*f_4^{-1}, f_2^{-1}*f_4*f_2*f_4^{-1}, f_3^{-1}*f_4*f_3*f_4^{-1}, f_4^5, f_1^{-1}*f_5*f_1*f_5^{-1}, f_2^{-1}*f_5*f_2*f_5^{-1}, f_3^{-1}*f_5*f_3*f_5^{-1}, f_4^{-1}*f_5*f_4*f_5^{-1}, f_5^5\rangle$

$G_6 = \langle f_1, f_2, f_3, f_4, f_5 \mid f_1^5*f_5^{-4}, f_1^{-1}*f_2*f_1*f_3^{-1}*f_2^{-1}, f_2^5, f_1^{-1}*f_3*f_1*f_4^{-1}*f_3^{-1}, f_2^{-1}*f_3*f_2*f_5^{-1}*f_3^{-1}, f_3^5, f_1^{-1}*f_4*f_1*f_5^{-1}*f_4^{-1}, f_2^{-1}*f_4*f_2*f_4^{-1}, f_3^{-1}*f_4*f_3*f_4^{-1}, f_4^5, f_1^{-1}*f_5*f_1*f_5^{-1}, f_2^{-1}*f_5*f_2*f_5^{-1}, f_3^{-1}*f_5*f_3*f_5^{-1}, f_4^{-1}*f_5*f_4*f_5^{-1}, f_5^5\rangle$

$G_7 = \langle f_1, f_2, f_3, f_4, f_5 \mid f_1^5, f_1^{-1}*f_2*f_1*f_3^{-1}*f_2^{-1}, f_2^5*f_5^{-1}, f_1^{-1}*f_3*f_1*f_4^{-1}*f_3^{-1}, f_2^{-1}*f_3*f_2*f_5^{-1}*f_3^{-1}, f_3^5, f_1^{-1}*f_4*f_1*f_5^{-1}*f_4^{-1}, f_2^{-1}*f_4*f_2*f_4^{-1}, f_3^{-1}*f_4*f_3*f_4^{-1}, f_4^5, f_1^{-1}*f_5*f_1*f_5^{-1}, f_2^{-1}*f_5*f_2*f_5^{-1}, f_3^{-1}*f_5*f_3*f_5^{-1}, f_4^{-1}*f_5*f_4*f_5^{-1}, f_5^5\rangle$

$G_8 = \langle f_1, f_2, f_3, f_4, f_5 \mid f_1^5, f_1^{-1}*f_2*f_1*f_4^{-1}*f_2^{-1}, f_2^5, f_1^{-1}*f_3*f_1*f_3^{-1}, f_2^{-1}*f_3*f_2*f_5^{-1}*f_3^{-1}, f_3^5, f_1^{-1}*f_4*f_1*f_5^{-1}*f_4^{-1}, f_2^{-1}*f_4*f_2*f_4^{-1}, f_3^{-1}*f_4*f_3*f_4^{-1}, f_4^5, f_1^{-1}*f_5*f_1*f_5^{-1}, f_2^{-1}*f_5*f_2*f_5^{-1}, f_3^{-1}*f_5*f_3*f_5^{-1}, f_4^{-1}*f_5*f_4*f_5^{-1}, f_5^5\rangle$

$G_9 = \langle f_1, f_2, f_3, f_4, f_5 \mid f_1^5*f_5^{-1}, f_1^{-1}*f_2*f_1*f_4^{-1}*f_2^{-1}, f_2^5, f_1^{-1}*f_3*f_1*f_3^{-1}, f_2^{-1}*f_3*f_2*f_5^{-1}*f_3^{-1}, f_3^5, f_1^{-1}*f_4*f_1*f_5^{-1}*f_4^{-1}, f_2^{-1}*f_4*f_2*f_4^{-1}, f_3^{-1}*f_4*f_3*f_4^{-1}, f_4^5, f_1^{-1}*f_5*f_1*f_5^{-1}, f_2^{-1}*f_5*f_2*f_5^{-1}, f_3^{-1}*f_5*f_3*f_5^{-1}, f_4^{-1}*f_5*f_4*f_5^{-1}, f_5^5\rangle$

$G_{10} = \langle f_1, f_2, f_3, f_4, f_5 \mid f_1^5, f_1^{-1}*f_2*f_1*f_4^{-1}*f_2^{-1}, f_2^5*f_5^{-1}, f_1^{-1}*f_3*f_1*f_3^{-1}, f_2^{-1}*f_3*f_2*f_5^{-1}*f_3^{-1}, f_3^5, f_1^{-1}*f_4*f_1*f_5^{-1}*f_4^{-1}, f_2^{-1}*f_4*f_2*f_4^{-1}, f_3^{-1}*f_4*f_3*f_4^{-1}, f_4^5, f_1^{-1}*f_5*f_1*f_5^{-1}, f_2^{-1}*f_5*f_2*f_5^{-1}, f_3^{-1}*f_5*f_3*f_5^{-1}, f_4^{-1}*f_5*f_4*f_5^{-1}, f_5^5\rangle$

$G_{11} = \langle f_1, f_2, f_3, f_4, f_5 \mid f_1^5, f_1^{-1}*f_2*f_1*f_4^{-1}*f_2^{-1}, f_2^5*f_5^{-2}, f_1^{-1}*f_3*f_1*f_3^{-1}, f_2^{-1}*f_3*f_2*f_5^{-1}*f_3^{-1}, f_3^5, f_1^{-1}*f_4*f_1*f_5^{-1}*f_4^{-1}, f_2^{-1}*f_4*f_2*f_4^{-1}, f_3^{-1}*f_4*f_3*f_4^{-1}, f_4^5, f_1^{-1}*f_5*f_1*f_5^{-1}, f_2^{-1}*f_5*f_2*f_5^{-1}, f_3^{-1}*f_5*f_3*f_5^{-1}, f_4^{-1}*f_5*f_4*f_5^{-1}, f_5^5\rangle$

$G_{12} = \langle f_1, f_2, f_3, f_4, f_5 \mid f_1^5, f_1^{-1}*f_2*f_1*f_4^{-1}*f_2^{-1}, f_2^5, f_1^{-1}*f_3*f_1*f_3^{-1}, f_2^{-1}*f_3*f_2*f_5^{-1}*f_3^{-1}, f_3^5*f_5^{-1}, f_1^{-1}*f_4*f_1*f_5^{-1}*f_4^{-1}, f_2^{-1}*f_4*f_2*f_4^{-1}, f_3^{-1}*f_4*f_3*f_4^{-1}, f_4^5, f_1^{-1}*f_5*f_1*f_5^{-1}, f_2^{-1}*f_5*f_2*f_5^{-1}, f_3^{-1}*f_5*f_3*f_5^{-1}, f_4^{-1}*f_5*f_4*f_5^{-1}, f_5^5\rangle$

$G_{13} = \langle f_1, f_2, f_3, f_4, f_5 \mid f_1^5, f_1^{-1}*f_2*f_1*f_5^{-1}*f_2^{-1}, f_2^5, f_1^{-1}*f_3*f_1*f_3^{-1}, f_2^{-1}*f_3*f_2*f_5^{-1}*f_3^{-1}, f_3^5, f_1^{-1}*f_4*f_1*f_5^{-1}*f_4^{-1}, f_2^{-1}*f_4*f_2*f_4^{-1}, f_3^{-1}*f_4*f_3*f_4^{-1}, f_4^5, f_1^{-1}*f_5*f_1*f_5^{-1}, f_2^{-1}*f_5*f_2*f_5^{-1}, f_3^{-1}*f_5*f_3*f_5^{-1}, f_4^{-1}*f_5*f_4*f_5^{-1}, f_5^5\rangle$

$G_{14} = \langle f_1, f_2, f_3, f_4, f_5 \mid f_1^5*f_5^{-1}, f_1^{-1}*f_2*f_1*f_5^{-1}*f_2^{-1}, f_2^5, f_1^{-1}*f_3*f_1*f_3^{-1}, f_2^{-1}*f_3*f_2*f_5^{-1}*f_3^{-1}, f_3^5, f_1^{-1}*f_4*f_1*f_5^{-1}*f_4^{-1}, f_2^{-1}*f_4*f_2*f_4^{-1}, f_3^{-1}*f_4*f_3*f_4^{-1}, f_4^5, f_1^{-1}*f_5*f_1*f_5^{-1}, f_2^{-1}*f_5*f_2*f_5^{-1}, f_3^{-1}*f_5*f_3*f_5^{-1}, f_4^{-1}*f_5*f_4*f_5^{-1}, f_5^5]\rangle$

3.3.2 Groups of order 5^6

For the prime $p = 5$ and the natural number $n = 6$ we obtain the following 58 p-groups of order 15625 with 6 generators $f_1, f_2, f_3, f_4, f_5, f_6$:

$G_1 = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1}*f_2*f_1*f_2^{-1}, f_2^5, f_1^{-1}*f_3*f_1*f_5^{-1}*f_3^{-1}, f_2^{-1}*f_3*f_2*f_6^{-1}*f_3^{-1}, f_3^5, f_1^{-1}*f_4*f_1*f_6^{-1}*f_4^{-1}, f_2^{-1}*f_4*f_2*f_5^{-2}*f_4^{-1}, f_3^{-1}*f_4*f_3*f_4^{-1}, f_4^5, f_1^{-1}*f_5*f_1*f_5^{-1}, f_2^{-1}*f_5*f_2*f_5^{-1}, f_3^{-1}*f_5*f_3*f_5^{-1}, f_4^{-1}*f_5*f_4*f_5^{-1}, f_5^5, f_1^{-1}*f_6*f_1*f_6^{-1}, f_2^{-1}*f_6*f_2*f_6^{-1}, f_3^{-1}*f_6*f_3*f_6^{-1}, f_4^{-1}*f_6*f_4*f_6^{-1}, f_5^{-1}*f_6*f_5*f_6^{-1}, f_6^5\rangle$

$G_2 = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5*f_5^{-4}, f_1^{-1}*f_2*f_1*f_2^{-1}, f_2^5, f_1^{-1}*f_3*f_1*f_5^{-1}*f_3^{-1}, f_2^{-1}*f_3*f_2*f_6^{-1}*f_3^{-1}, f_3^5, f_1^{-1}*f_4*f_1*f_6^{-1}*f_4^{-1}, f_2^{-1}*f_4*f_2*f_5^{-2}*f_4^{-1}, f_3^{-1}*f_4*f_3*f_4^{-1}, f_4^5, f_1^{-1}*$

$f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_3 = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_5^{-4}, f_1^{-1} * f_2 * f_1 * f_2^{-1}, f_2^5 * f_6^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_6^{-1} * f_3^{-1}, f_3^5, f_1^{-1} * f_4 * f_1 * f_6^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-2} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_4 = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_5^{-4}, f_1^{-1} * f_2 * f_1 * f_2^{-1}, f_2^5 * f_6^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_6^{-1} * f_3^{-1}, f_3^5, f_1^{-1} * f_4 * f_1 * f_6^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-2} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_5 = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_6^{-1} * f_5^{-1}, f_1^{-1} * f_2 * f_1 * f_2^{-1}, f_2^5 * f_6^{-2} * f_5^{-3}, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_6^{-1} * f_3^{-1}, f_3^5, f_1^{-1} * f_4 * f_1 * f_6^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-2} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_6 = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_6^{-3}, f_1^{-1} * f_2 * f_1 * f_2^{-1}, f_2^5 * f_6^{-3} * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_6^{-1} * f_3^{-1}, f_3^5, f_1^{-1} * f_4 * f_1 * f_6^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-2} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_7 = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_6^{-1}, f_1^{-1} * f_2 * f_1 * f_2^{-1}, f_2^5 * f_6^{-3} * f_5^{-3}, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_6^{-1} * f_3^{-1}, f_3^5, f_1^{-1} * f_4 * f_1 * f_6^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-2} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_8 = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_2^{-1}, f_2^5 * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_6^{-1} * f_3^{-1}, f_3^5, f_1^{-1} * f_4 * f_1 * f_6^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-2} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^5 * f_6^{-4}, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_9 = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{10} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_5^{-4}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{11} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{12} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_5^{-3}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} *$

$f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5\rangle$

$G_{13} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5\rangle$

$G_{14} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-3}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5\rangle$

$G_{15} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-4} * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5\rangle$

$G_{16} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-4} * f_5^{-3}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5\rangle$

$G_{17} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-3} * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5\rangle$

$G_{18} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-3} * f_5^{-3}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5\rangle$

$G_{19} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_6^{-4}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5\rangle$

$G_{20} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_6^{-4}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5\rangle$

$G_{21} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_6^{-4}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_5^{-2}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5\rangle$

$G_{22} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_6^{-4}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_5^{-1}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5\rangle$

$G_{23} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_5^{-4}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{24} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_5^{-4}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{25} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_6^{-4} * f_5^{-4}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{26} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_6^{-4} * f_5^{-4}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{27} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_6^{-3} * f_5^{-4}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{28} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_6^{-3} * f_5^{-4}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{29} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_6^{-2} * f_5^{-4}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{30} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_6^{-2} * f_5^{-4}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{31} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_6^{-1} * f_5^{-4}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{32} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5 * f_6^{-1} * f_5^{-4}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{33} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 *$

$f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{34} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-2}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{35} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-2} * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-4} * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{36} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-4} * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-4} * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{37} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_5^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{38} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-3} * f_5^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-4} * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{39} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-2}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{40} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_5^{-3}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-1}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{41} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-2} * f_5^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-4} * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{42} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-4} * f_5^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-4} * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{43} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-2}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-3} * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} *$

$f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{44} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-3} * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{45} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-4} * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-2} * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5])$

$G_{46} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-4} * f_5^{-3}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-1} * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{47} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-2} * f_5^{-2}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{48} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-4} * f_5^{-2}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{49} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-3}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-3} * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{50} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-3} * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-2} * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{51} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_5^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-4} * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{52} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-1} * f_5^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5 \rangle$

$G_{53} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-2}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-3} * f_5^{-2}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 *$

$f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5\rangle$

$G_{54} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-3} * f_5^{-1}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5\rangle$

$G_{55} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-2} * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5\rangle$

$G_{56} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-2} * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-2} * f_5^{-2}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5\rangle$

$G_{57} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-4} * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-2} * f_5^{-1}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5\rangle$

$G_{58} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^5, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^5 * f_6^{-1} * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-2} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^5 * f_6^{-2}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^5, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^5, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^5\rangle$

3.3.3 Groups of order 7^5

For the prime $p = 7$ and the natural number $n = 5$ we obtain the following 14 p-groups of order 16807 with 5 generators f_1, f_2, f_3, f_4, f_5:

$G_1 = \langle\ f_1, f_2, f_3, f_4, f_5 \mid f_1^7 * f_4^{-1}, f_1^{-1} * f_2 * f_1 * f_3^{-1} * f_2^{-1}, f_2^7 * f_3^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_3^{-1}, f_3^7 * f_5^{-1}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-6} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7\rangle$

$G_2 = \langle\ f_1, f_2, f_3, f_4, f_5 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_3^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_4^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_5^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_5^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7\rangle$

$G_3 = \langle\ f_1, f_2, f_3, f_4, f_5 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_3^{-1} * f_2^{-1}, f_2^7 * f_5^{-1}, f_1^{-1} * f_3 * f_1 * f_4^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_5^{-3} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_5^{-3} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7\rangle$

$G_4 = \langle\ f_1, f_2, f_3, f_4, f_5 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_3^{-1} * f_2^{-1}, f_2^7 * f_5^{-1}, f_1^{-1} * f_3 * f_1 * f_4^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_5^{-2} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_5^{-2} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7\rangle$

$G_5 = \langle\ f_1, f_2, f_3, f_4, f_5 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_3^{-1} * f_2^{-1}, f_2^7 * f_5^{-1}, f_1^{-1} * f_3 * f_1 * f_4^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_5^{-6} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_5^{-6} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7\rangle$

$G_6 = \langle\ f_1, f_2, f_3, f_4, f_5 \mid f_1^7 * f_5^{-1}, f_1^{-1} * f_2 * f_1 * f_3^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_4^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_5^{-3} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_5^{-3} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7 \rangle$

$G_7 = \langle\ f_1, f_2, f_3, f_4, f_5 \mid f_1^7 * f_5^{-1}, f_1^{-1} * f_2 * f_1 * f_3^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_4^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_5^{-2} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_5^{-2} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7 \rangle$

$G_8 = \langle\ f_1, f_2, f_3, f_4, f_5 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_4^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7 \rangle$

$G_9 = \langle\ f_1, f_2, f_3, f_4, f_5 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_4^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_3^{-1}, f_3^7 * f_5^{-1}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7 \rangle$

$G_{10} = \langle\ f_1, f_2, f_3, f_4, f_5 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_4^{-1} * f_2^{-1}, f_2^7 * f_5^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7 \rangle$

$G_{11} = \langle\ f_1, f_2, f_3, f_4, f_5 \mid f_1^7 * f_5^{-1}, f_1^{-1} * f_2 * f_1 * f_4^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7 \rangle$

$G_{12} = f_1, f_2, f_3, f_4, f_5 \mid f_1^7 * f_5^{-1}, f_1^{-1} * f_2 * f_1 * f_4^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-3} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7 \rangle$

$G_{13} = \langle\ f_1, f_2, f_3, f_4, f_5 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_5^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_5^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7 \rangle$

$G_{14} = \langle\ f_1, f_2, f_3, f_4, f_5 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_5^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_5^{-1} * f_4^{-1}, f_4^7 * f_5^{-1}, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7 \rangle$

3.3.4 Groups of order 7^6

For the prime $p = 7$ and the natural number $n = 6$ we obtain the following 100 p-groups of order 117649 with 6 generators $f_1, f_2, f_3, f_4, f_5, f_6$:

$G_1 = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_6^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_6^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-3} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_2 = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_6^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_6^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-3} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_3 = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_2^{-1}, f_2^7 * f_6^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_6^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_6^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-3} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_4 = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_2^{-1}, f_2^7 * f_6^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_6^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_6^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-3} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_5 = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-3}, f_1^{-1} * f_2 * f_1 * f_2^{-1}, f_2^7 * f_6^{-5} * f_5^{-5}, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_6^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_6^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-3} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_6 = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-5}, f_1^{-1} * f_2 * f_1 * f_2^{-1}, f_2^7 * f_6^{-5} * f_5^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_6^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_6^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-3} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_7 = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-5} * f_5^{-2}, f_1^{-1} * f_2 * f_1 * f_2^{-1}, f_2^7 * f_6^{-3} * f_5^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_6^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_6^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-3} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_8 = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-1}, f_1^{-1} * f_2 * f_1 * f_2^{-1}, f_2^7 * f_6^{-5} * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_6^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_6^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-3} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_9 = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-5} * f_5^{-1}, f_1^{-1} * f_2 * f_1 * f_2^{-1}, f_2^7 * f_6^{-4} * f_5^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_6^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_6^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-3} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{10} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_2^{-1}, f_2^7 * f_5^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-1} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_6^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_6^{-1} * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-3} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_4^{-1}, f_4^7 * f_6^{-6}, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{11} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{12} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{13} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_5^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} *$

$f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{14} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{15} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{16} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-6} * f_5^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{17} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-6} * f_5^{-5}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{18} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-6} * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{19} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-4} * f_5^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{20} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-4} * f_5^{-5}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{21} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-4} * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{22} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_5^{-6}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{23} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_5^{-5}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} *$

$f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{24} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{25} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{26} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_5^{-2}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{27} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_5^{-1}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{28} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{29} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-5}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{30} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{31} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-6} * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{32} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-6} * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-5}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{33} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-6} * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} *$

$f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{34} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-5} * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{35} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-5} * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-5}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{36} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-5} * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{37} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-4} * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{38} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-4} * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-5}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{39} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-4} * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{40} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-3} * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{41} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-3} * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-5}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{42} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-3} * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{43} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-2} * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{44} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-2} * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-5}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{45} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-2} * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{46} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-1} * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{47} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-1} * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-5}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{48} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7 * f_6^{-1} * f_5^{-6}, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{49} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_5^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{50} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-2}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_5^{-6}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{51} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-2} * f_5^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6} * f_5^{-6}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{52} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-4} * f_5^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6} * f_5^{-5}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{53} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-6} * f_5^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6} * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{54} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_5^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 *$

$f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{55} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-1} * f_5^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6} * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{56} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_5^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{57} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_5^{-5}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{58} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-2} * f_5^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6} * f_5^{-6}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{59} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-6} * f_5^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6} * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{60} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-1} * f_5^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6} * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{61} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-5} * f_5^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6} * f_5^{-1}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{62} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-2}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-5} * f_5^{-6}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{63} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-5} * f_5^{-5}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7\rangle$

$G_{64} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-5} * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} *$

$f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{65} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-5} * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{66} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-3}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-5} * f_5^{-2}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{67} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-5}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-5} * f_5^{-1}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{68} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-2} * f_5^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-4} * f_5^{-6}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{69} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-4} * f_5^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-4} * f_5^{-5}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{70} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-3} * f_5^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-4} * f_5^{-2}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{71} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-5} * f_5^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-4} * f_5^{-1}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{72} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-2} * f_5^{-5}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-3} * f_5^{-6}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{73} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-4} * f_5^{-5}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-3} * f_5^{-5}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{74} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-6} * f_5^{-5}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-3} * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} *$

$f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{75} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-1} * f_5^{-5}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-3} * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{76} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-3} * f_5^{-5}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-3} * f_5^{-2}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{77} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-5} * f_5^{-5}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-3} * f_5^{-1}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{78} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-5} * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{79} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-1} * f_5^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-4} * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{80} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-1} * f_5^{-5}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-3} * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{81} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-1} * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-2} * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{82} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-1} * f_5^{-3}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-1} * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{83} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-1} * f_5^{-2}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_5^{-4}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{84} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-3} * f_5^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6} * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{85} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-5} * f_5^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6} * f_5^{-2}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{86} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_5^{-1}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-6} * f_5^{-1}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{87} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-3}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-5} * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{88} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-5}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-5} * f_5^{-2}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{89} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-3} * f_5^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-4} * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{90} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_5^{-6}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-4} * f_5^{-1}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{91} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-3} * f_5^{-5}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-3} * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{92} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-5} * f_5^{-5}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-3} * f_5^{-2}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{93} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_5^{-5}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-3} * f_5^{-1}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{94} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-3} * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-2} * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{95} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-5} * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-3} *$

$f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-2} * f_5^{-2}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{96} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_5^{-4}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-2} * f_5^{-1}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{97} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-3} * f_5^{-3}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-1} * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{98} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_5^{-3}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_6^{-1} * f_5^{-1}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{99} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-3} * f_5^{-2}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_5^{-3}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

$G_{100} = \langle f_1, f_2, f_3, f_4, f_5, f_6 \mid f_1^7, f_1^{-1} * f_2 * f_1 * f_6^{-1} * f_2^{-1}, f_2^7 * f_6^{-5} * f_5^{-2}, f_1^{-1} * f_3 * f_1 * f_5^{-3} * f_3^{-1}, f_2^{-1} * f_3 * f_2 * f_4^{-1} * f_3^{-1}, f_3^7 * f_5^{-2}, f_1^{-1} * f_4 * f_1 * f_4^{-1}, f_2^{-1} * f_4 * f_2 * f_5^{-1} * f_4^{-1}, f_3^{-1} * f_4 * f_3 * f_6^{-1} * f_4^{-1}, f_4^7, f_1^{-1} * f_5 * f_1 * f_5^{-1}, f_2^{-1} * f_5 * f_2 * f_5^{-1}, f_3^{-1} * f_5 * f_3 * f_5^{-1}, f_4^{-1} * f_5 * f_4 * f_5^{-1}, f_5^7, f_1^{-1} * f_6 * f_1 * f_6^{-1}, f_2^{-1} * f_6 * f_2 * f_6^{-1}, f_3^{-1} * f_6 * f_3 * f_6^{-1}, f_4^{-1} * f_6 * f_4 * f_6^{-1}, f_5^{-1} * f_6 * f_5 * f_6^{-1}, f_6^7 \rangle$

3.4 Group G such that $(G, Z(G))$ is a Camina Pair

Let G be a p-group with $p > 2$ and suppose that $Z(G)$ is the only normal subgroup of order p of G. If the character $\chi \in IrrG$ isn't faithful, then $Z(G)$ is contained in its kernel. For such groups, according to Theorem 2.2.4, we have at least one faithful character $\chi \in IrrG$.

We have:

Theorem 3.4.1. *Let G be a $p-$group with $p > 2$ and $|Z(G)| = p$. Then $(G, Z(G))$ is a Camina pair if and only if $|G| = p^{2n+1}$ for a positive integer n, and G has $p-1$ irreducible characters of degree p^n.*

Proof. Let $\chi \in Irr(G)$, $\chi(1) = p^n$ and let ρ be an irreducible representation affording χ. Since the degree of χ is p^n, χ and ρ are faithful by (5) of Theorem 2.2.2. By [29], Theorem 2.2.5, $\rho(z)$ is a diagonal scalar matrix for $z \in Z(G)$, so it is a multiple of the identical matrix according to λ (the only eigenvalue). For this reason we have $\chi(z) = \lambda\chi(1)$.

Note that:

$$\sum_{z\in Z(G)}\chi(z)\overline{\chi(z)} = \sum_{z\in Z(G)}\lambda\chi(1)\overline{\lambda\chi(1)} = \sum_{z\in Z(G)}\overline{\lambda}\lambda\chi(1)\chi(1) = \sum_{z\in Z(G)}\chi(1)^2 = p(p^{2n}) = p^{2n+1} = |G| = \sum_{g\in G}\chi(g)\overline{\chi(g)}.$$

The last equality comes from (2) of Theorem 2.2.2 and this implies that $\chi(g) = 0$ for every $g \in G - Z(G)$.

This argument applies to the $p - 1$ characters of degree p^n. Let's denote these characters by $\chi_1, \ldots, \chi_{p-1}$, so for (4) of Theorem 2.2.2 we have:

$$\sum_{\chi\in IrrG}\chi(1)^2 = \sum_{\chi\in\{\chi_1,\ldots,\chi_{p-1}\}}\chi(1)^2 + \sum_{\chi\in IrrG-\{\chi_1,\ldots,\chi_{p-1}\}}\chi(1)^2 = (p-1)p^{2n} + \sum_{\chi\in IrrG-\{\chi_1,\ldots,\chi_{p-1}\}}\chi(1)^2 = |G| = p^{2n+1}.$$

Since $(p-1)p^{2n} + \sum_{\chi\in IrrG-\{\chi_1,\ldots,\chi_{p-1}\}}\chi(1)^2 = p^{2n+1}$ then $\sum_{\chi\in IrrG-\{\chi_1,\ldots,\chi_{p-1}\}}\chi(1)^2 = p^{2n}$, thus the characters $\chi \in IrrG - \{\chi_1, \ldots, \chi_{p-1}\}$ are the irreducible characters of $\frac{G}{Z(G)}$.

Finally, irreducible characters that don't contain $Z(G)$ in their kernel, vanish off $Z(G)$, so $(G, Z(G))$ is a C-pair by Lemma 1.0.3 .

Conversely, let $(G, Z(G))$ be a C-pair and let χ be an irreducible faithful character of G. According to Lemma 1.0.3, $\chi(g) = 0\ \forall g \in G - Z(G)$. Now we apply (1) of Theorem 2.2.4 and we have:

$$|G| = \sum_{g\in G}\chi(g)\overline{\chi(g)} = \sum_{z\in Z(G)}\chi(z)\overline{\chi(z)} + \sum_{g\in G-Z(G)}\chi(g)\overline{\chi(g)} = \sum_{z\in Z(G)}\chi(z)\overline{\chi(z)} + 0 = \sum_{z\in Z(G)}\lambda\chi(1)\overline{\lambda\chi(1)} = \sum_{z\in Z(G)}\chi(1)^2 = p\chi(1)^2.$$

Set $\chi(1) = p^n$. Let $\{\chi_1, \ldots, \chi_k\}$ be the set of faithful irreducible characters of G. Since the non-faithful characters of G are characters of $\frac{G}{Z(G)}$, by (4) of Theorem 2.2.2 we have:

$$p^{2n+1} = |G| = \sum_{\chi\in\{\chi_1,\ldots,\chi_k\}}\chi(1)^2 + \sum_{\chi\notin\{\chi_1,\ldots,\chi_k\}}\chi(1)^2 = p^{2n}k + p^{2n}.$$

So there are exactly $k = p - 1$ faithful characters.

□

We can generalise the previous theorem as follows:

Theorem 3.4.2. *Let G be a $p-$group with $p > 2$, $|Z(G)| = p^z$ with z a positive integer, then $(G, Z(G))$ is a Camina pair if and only if $|G| = p^{z+2n}$ and G has $p^z - 1$ irreducible characters of degree p^n.*

Proof. Let $(G, Z(G))$ be a Camina pair. Then, by Theorem 2.2 in [55], $Z(G)$ is elementary abelian and $Z(G) = \gamma_c(G)$, where c is the nilpotency class of G.

Let M be a maximal subgroup of $Z(G)$, i.e. $|Z(G) : M| = p$. Since $(G, Z(G))$ is a C-pair then $(\frac{G}{M}, \frac{Z(G)}{M})$ is C-pair too by Lemma 3.2.1. Also $\gamma_c(\frac{G}{M}) = \frac{\gamma_c(G)}{M}$. Hence Theorem 3.4.1 applies to $(\frac{G}{M}, \gamma_c(\frac{G}{M}))$ and $\frac{G}{M}$ has $p-1$ irreducible characters of degree equal to p^n, where $|G : Z(G)| = p^{2n}$.

This is true for all maximal subgroups M of $Z(G)$. Since $|Z(G)| = p^z$, we have $p^{z-1} + p^{z-2} + p^{z-3} + \cdots + p + 1$ such maximal subgroups, and thus there are $(p-1)(p^{z-1} + p^{z-2} + \cdots + p + 1) = p^z - 1$ irreducible characters of degree p^n.

Conversely, with the same argument, let $|G| = p^{z+2n}$ and let G have $p^z - 1$ characters of degree p^n, say $\{\chi_1, \ldots, \chi_{p^z-1}\}$. Let χ be one of them, by (2) of Theorem 2.2.2 we obtain:

$$|G| = \sum_{g\in G} \chi(g)\overline{\chi(g)} = \sum_{z\in Z(G)} \chi(z)\overline{\chi(z)} + \sum_{g\in G-Z(G)} \chi(g)\overline{\chi(g)} = p^z p^{2n} + \sum_{g\in G-Z(G)} \chi(g)\overline{\chi(g)}$$
$$= |G| + \sum_{g\in G-Z(G)} \chi(g)\overline{\chi(g)}.$$

Therefore $\sum_{g\in G-Z(G)} \chi(g)\overline{\chi(g)} = 0$. The other characters by (4) of Theorem 2.2.2 are exactly those of $\frac{G}{Z(G)}$:

$$\sum_{\chi\in IrrG} \chi(1)^2 = \sum_{\chi\in\{\chi_1,\ldots,\chi_{p^z-1}\}} \chi(1)^2 + \sum_{\chi\in IrrG-\{\chi_1,\ldots,\chi_{p^z-1}\}} \chi(1)^2 =$$
$$(p^z-1)p^{2n} + \sum_{\chi\in IrrG-\{\chi_1,\ldots,\chi_{p^z-1}\}} \chi(1)^2 = |G| = p^{2n+z}.$$

Since $(p^z-1)p^{2n} + \sum_{\chi\in IrrG-\{\chi_1,\ldots,\chi_{p^z-1}\}} \chi(1)^2 = p^{2n+z}$ then $\sum_{\chi\in IrrG-\{\chi_1,\ldots,\chi_{p^z-1}\}} \chi(1)^2 = p^{2n}$, thus the characters $\chi \in IrrG - \{\chi_1, \ldots, \chi_{p^z-1}\}$ are the irreducible characters of $\frac{G}{Z(G)}$.

So, those that aren't characters of $\frac{G}{Z(G)}$ vanish off $Z(G)$ and $(G, Z(G))$ is a C-pair by Lemma 1.0.3.

□

An important consequence of the Theorem 3.4.2 is:

Corollary 3.4.3. *Let G be a p-group and (G, K) be a C-pair. Then $|G : K|$ is an even power of p.*

Proof. Since (G, K) is a C-pair, K is a Camina kernel i.e. $K = Z_r(G)$, for some r, by Lemma 2.1 in [55]. So $(\frac{G}{Z_{r-1}(G)}, \frac{K}{Z_{r-1}(G)})$ is a C-pair too. But $\frac{K}{Z_{r-1}(G)} = Z(\frac{G}{Z_{r-1}(G)})$ by definition.

Finally, $|G : K| = |G : Z_r(G)|$ is an even power of p by Theorem 3.4.2.

□

If Conjecture 1.0.7 were true, then the following corollary would hold:

Corollary 3.4.4. *Suppose that $|Z_2(G) : Z(G)| = p^n$, $|Z(G)| = p^z$ and $|G : Z_2(G)| = p^m$. If $(G, Z_2(G))$ is a C-pair, then n is an even number.*

Proof. According to Theorem 3.4.2, $\frac{G}{Z(G)}$ has $p^n - 1$ characters of degree $p^{\frac{m}{2}}$ and these characters are characters of G too. If $(G, Z(G))$ is also a C-pair, then G has also $p^z - 1$ characters of degree $p^{(\frac{m+n}{2})}$.

Since m is an even number by Corollary 3.4.3 then n is also an even number.

Finally, the existence of $p^n - 1$ characters of degree $p^{\frac{m}{2}}$ would imply the existence of $p^z - 1$ characters of degree $p^{(\frac{m+n}{2})}$.

□

Chapter 4

Generalizations and Applications

Upon suggestion of my advisor, in the early stage of this work I started to collect known results about groups of Frobenius type, generalizations and applications. We had in mind the idea of a survey. After the publication of [50] we abandoned the idea of a survey, but I think it reasonable to include here those parts of the material I collected that somehow complement or illustrate the content of [50]. Therefore the different sections of this chapter are independent, and certainly not self-contained, the only purpose of them being that of providing some references to the large existing literature.

4.1 The non-solvable case

In 1988 D. Chillag, A. Mann and C.M. Scoppola proved that a Camina group G is solvable if G/N is a p-group (see [15]), consequently if (G, N) is a non-solvable C-pair, the group G is a Frobenius group or the subgroup N is a $p-$group. H. Zassenhaus classified all non-solvable Frobenius groups (see [71]):

Theorem 4.1.1 (Zassenhaus)**.** *Let G be a non-solvable Frobenius complement. Then G has a normal subgroup G_0 with $|G : G_0| = 1$ or $|G : G_0| = 2$, such that $G_0 = SL(2, 5) \times M$ with M which is a Z-group of order prime to 2,3 and 5.*

Like in [3] we assume that X^∞ denotes the solvable residual of the group X and $S(G)$ denotes the solvable radical of G. For groups of Frobenius type, or C-pairs, the situation is similar to the one described in the above theorem, as shown in [3]:

Theorem 4.1.2. *Let (G,N) be a C-pair such that G is non-solvable. Then N is a p-group and one of the following holds:*

- $(G/O_p(G))^\infty \simeq SL(2, p^e), p^e > 3$;
- $(G/O_p(G))^\infty \simeq SL(2, 5), p = 3$;

- $(G/O_p(G))^\infty \simeq SL(2,13), p = 3$;
- $(G/O_p(G))^\infty \simeq SL(2,5), p \geq 7$ *and* $(S(G), N)$ *is a C-pair;* $G/S(G) \simeq A_5$ *or* $G/S(G) \simeq S_5$.

It's important to observe that a similar result is mentioned in [12], where Camina mentions a result of W.B. Stewart in which the possible modules and groups which can occur as $G/O_p(G)$ are classified:

Theorem 4.1.3. *Let G be a finite group with a proper normal subgroup* $H \neq 1$ *such that if* $x \in G/H$, x *is conjugate to* xy *for every* $y \in H$. *And assume G is not a soluble group nor is G a Frobenius group. If H is a p-group then* $G/O_p(G) = \overline{G}$ *has a normal subgroup* $\overline{K}$ *such that* $\overline{G}/\overline{K}$ *is soluble and:*

1. *if* $p = 2$ *then* $\overline{K} \simeq SL(2, 2^n), Sz(2^{2m+1})$ *or* $SL(2, 2^n) \times Sz(2^{2mb})$;
2. *if* $p = 3$ *then* $\overline{K} \simeq SL(2, 3^n), SL(2, 5), SL(2, 7)$ *or* $SL(2, 17)$;
3. *if* $p \geq 5$ *then* $\overline{K} \simeq SL(2, p^n)$.

4.2 The infinite case

In Theorem 3.1.2 finite Camina groups are completly described. Here we want to extend the definition of Camina group to the infinite case:

Definition 4.2.1. *A group G is said to be a Camina group if* $G \neq G'$ *and every nontrivial coset of* G' *is a single conjugacy class of G.*

There are many examples of infinite non-abelian Camina groups:

1. The infinite extraspecial p-group G, with $G' = Z(G)$ of order $p \in \mathbb{P}$ is a Camina group and, of course, nilpotent of class 2.
2. The semidirect products $G = A < x >$, where $A < G$, $x \in G/A$, $A^2 = A$, $|x| = 2$ and $a^x = a^{-1}$ for each $a \in A$ is a Camina group and can be neither nilpotent nor Frobenius.
3. The last example is a family of non-solvable Camina groups constructed by Ol'shanskii modifying the construction given by him in [70] of infinite groups of finite exponent p (where p is a big enough prime), in which all subgroups of order p are conjugate. These groups are infinite non-solvable groups G of exponent p, such that $G' \neq G$ and all subgroups of order p in G/G' are conjugate. In fact:

   Theorem 4.2.2. *For each connected algebraic group H over an algebraically closed field of characteristic p with Frobenius map* σ *on H, there exist countably many non-isomorphic infinite Camina groups with* $G' \simeq H$. *In particular, if H is semisimple then G is not locally solvable.*

In [27], Herzog, Longobardi and Maj characterized the infinite Camina groups with finite commutator subgroup:

Theorem 4.2.3. *Let G be a infinite group with $\{1\} \neq G'$ finite. Then G is a Camina group if and only if G is a nilpotent p-group ($p \in \mathbb{P}$) of class 2 and of exponent dividing p^2, with $Z(G) = G'$ and for any maximal subgroup H of G', G/H is an extraspecial group (i.e. G is an infinite semiextraspecial group).*

They also describe locally finite Camina groups, residually finite Camina groups, finitely generated Camina groups and some periodic solvable Camina groups. For **locally finite** Camina groups they have:

Theorem 4.2.4. *If G is a locally finite Camina group, then either G/G' is a p-group for a suitable $p \in \mathbb{P}$, or G' is nilpotent and either G' is a p-group for some $p \in \mathbb{P}$ or G/G' is locally cyclic.*

By Theorem 2.1, finite non-abelian Camina groups are either p-groups of nilpotency class 2 or 3, or Frobenius groups with complements which are either cyclic or quaternion. In the following theorem, Herzog, Longobardi and Maj, extend this result to **residually finite** Camina groups:

Theorem 4.2.5. *If G is a non abelian residually finite Camina group, one of the following holds :*

1. *G is a infinite p-group of nilpotency class 2 and exponent dividing p^2, with $G' = Z(G)$;*
2. *G is a Frobenius group with Frobenius kernel equal to G' nilpotent of class depending on the order $|G/G'|$ and with finite cyclic complements;*
3. *G is a Frobenius group with abelian kernel and complements isomorphic to Q_8.*

Then they showed that an infinite non-abelian **finitely generated** Camina groups must be non-solvable:

Theorem 4.2.6. *Let G be an infinite non-abelian finitely generated Camina group, then G is nonsolvable. Equivalently let G be a non-abelian finitely generated solvable Camina group, then G is finite.*

Theorem 4.2.7. *A infinite non-abelian Camina group with finite commutator subgroup is a nilpotent p-group of class 2, and exponent dividing p^2 with $G'=Z(G)$.*

Theorem 4.2.8. *A finitely generated solvable Camina group is either abelian or finite.*

The following theorems classify the **periodic solvable** Camina groups with G' infinite and satisfying one of the following conditions:

1. $G' \in min$;
2. G' is of finite Prüfer rank.

Theorem 4.2.9. *Let G be a periodic solvable Camina group and assume that G' is an infinite group, then G/G' is finite and one of the following holds:*

1. *If G is a group of finite Prüfer rank then G is a Camina p-group in min;*
2. *If $G' \in min$ is a p-group (p a prime) and there exists a normal abelian subgroup A of finite index in G such that G/A is a finite Camina p-group, A is a direct product of finitely many Prüfer p-groups and $C_A(y)$ is finite for every $y \in G/G'$;*
3. *G is a Frobenius group with complements which are either cyclic or isomorphic to the quaternion group ;*
4. *$G = A \rtimes K$, where K is a finite abelian group, A is a direct product of finitely many Prüfer p_i-groups (p_i primes, not necessarily different), and $C_A(k)$ is finite for any $k \in K/\{1\}$;*
5. *$G = D \rtimes K$, where K is a finite abelian group, D is a nilpotent group, $D = B \times C$, where $C \rtimes K$ is a Frobenius group with the kernel C, B is a direct product of finitely many Prüfer p_i- groups (p_i primes, not necessarily different), and $C_B(k)$ is finite for any $k \in K/\{1\}$;*
6. *$G = (B \times C)K$, where $B \times C$ is normal in G, K is a finite p-group (p a prime), C is a p'-group, B is a direct product of finitely many Prüfer p-groups, G/B is a Frobenius group with the kernel $(B \times C)/B$ and with complements which are either cyclic or isomorphic to the quaternion group, and $C_B(y)$ is finite for any $y \in K/B$.*

Conversely, if one of (1), (2), (3), (4) and (5) holds, then G is a Camina group.

In his work, [23], Ersoy presents some results on periodic linear and finitary groups containing an anticentral element:

Definition 4.2.10. *Let G be a non-perfect group. An element a is an anticentral element if $aG' = a^G$.*

Remark 2. A non-perfect group G is called a Camina group if every element outside its commutator subgroup is anticentral.

Ladisch proved that every finite group with an anticentral element is solvable, however, there are examples of infinite groups with an anticentral element which are not solvable. So Ersoy studied infinite groups containing an anticentral element and constructed some examples of infinite Camina groups which are not locally solvable.

Ersoy constructed some examples of infinite Camina groups which are not locally solvable using the following:

Theorem 4.2.11 (Lang-Steinberg Theorem)**.** *Let G be a connected algebraic group over an algebraically closed field of characteristic p and let σ be a Frobenius map on G. Then the map L defined as $L : x \in G \to x^{-1}x^{\sigma} \in G$ is surjective.*

Theorem 4.2.12. *Let G be any connected (linear) algebraic group over an algebraically closed field of characteristic p and σ be a Frobenius map on G. Then the group $H = G \rtimes < \sigma >$ is an infinite Camina group with a periodic linear commutator subgroup. Moreover, if G is semisimple, then H is not locally solvable.*

Corollary 4.2.13. *For any connected linear algebraic group G defined over an algebraically closed field $\mathbb{F}$ of characteristic p, there exist countably many non-isomorphic copies of infinite Camina groups whose commutator subgroups are isomorphic to G.*

Now, we can construct some explicit examples of infinite Camina groups:

1. Let T be a torus in $GL_n(\mathbb{F})$ where $\mathbb{F}$ is an algebraically closed field of characteristic p and let σ be a standard Frobenius map. Then $G = T \rtimes < \sigma >$ is an infinite solvable Camina group by Theorem 4.3. Here, G' is abelian and $C'_G(\sigma)$ is finite. A finitely generated solvable Camina group is either abelian or finite. Now, G is infinite and non-abelian, so, G is not finitely generated.

2. $H = SL_n(\mathbb{F})$, where $\mathbb{F}$ is an algebraically closed field of characteristic p. Then H is a simple linear algebraic group. Let $\sigma_k : (x_{ij}) \in H \leftarrow (x_{ij})^{p^k} \in H$. Then for each k $\in \mathbb{N}$, the group $G_k = H \rtimes < \sigma_k >$ is an infinite Camina group which is not locally solvable. Here, if $k_1 \neq k_2$ the group G_{k_1} is not isomorphic to G_{k_2} . To see this, assume without loss of generality that $k_1 < k_2$. We have $C_{G_{k_2}}((g, \sigma_{k_2})) \simeq SL_n(p^{k_2})$ and $C_{G_{k_2}}((g, \sigma_{k_2}^s)) \simeq SL_n(p^{sk_2})$ for each $s \geq 0$. Hence, the minimal size of a centralizer in G_{k_2} is the order of $SL_n(p^{k_2})$. But, in G_1, the order of the centralizer of $(1, \sigma_{k_1})$ is smaller. Therefore, G_{k_1} is not isomorphic to G_{k_2} .

3. Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$. Let $\sigma_n : A \in SL_n(\mathbb{F}) \rightarrow \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in SL_{n+1}(\mathbb{F})$ be the embedding of $SL_n(\mathbb{F})$ into $SL_{n+1}(\mathbb{F})$. Consider the direct limit G of $SL_n(\mathbb{F})$'s via these embeddings ϕ_n's. Here, G is a non-linear simple locally finite group. Let σ be the automorphism of G defined as $\alpha|_{SL_n(\mathbb{F})} : (x_{ij}) \rightarrow (x^{q_{ij}})$ where $q = p^k$. Hence, $\alpha|_{SL_n(\mathbb{F})}$ is a standard Frobenius map $\sigma_{n,q}$ for $SL_n(\mathbb{F})$. Consider the map $L : x \in G \rightarrow x^{-1}x^{\alpha} \in G$. Since for each $x \in G$ the map α sends $x \in SL_n(\mathbb{F})$ to an element in $SL_n(\mathbb{F})$, the map $L|_{SL_n(\mathbb{F})}$ is equal to the Lang map $x \rightarrow x^{-1}x^{\sigma_{n,q}}$, so it is surjective. Therefore, by the same argument as in Theorem 5.21, $G \rtimes < \alpha >$ is a Camina group, whose commutator subgroup is isomorphic to G, a non-linear simple locally finite group. Moreover, for each k where $q = p^k$ we obtain a non-isomorphic infinite non-linear Camina group whose commutator subgroup is isomorphic to G. Also, since every finitely generated subgroup of $G \rtimes < \alpha >$ is contained in an extension of $SL_n\mathbb{F}$ by $< \alpha|_{SL_n(\mathbb{F})} >$, the group $G \rtimes < \alpha >$ is not finitely generated.

4.3 Extended Camina Groups, FC-Pairs And Extended Camina Pairs

In [3] there is an extension of the concept of "Camina group" or "CF-pair", which appears in [12], infact:

Definition 4.3.1. *Let (A, B) be a table algebra and let C be a nonempty subset of $B - \{1_A\}$ such that $\overline{b_i} \in C$ whenever $b_i \in C$. Assume that $Irr(b_i, b_j) \in C$ with $b_j \neq b_i$ or $\overline{b_i}$. Then we say that C is a quasi-closed subset of B. Futhermore, C is called notrivial quasi-closed if C is quasi-closed and in not of the form $\{b_i, \overline{b_i}\}$ for some $b_i \in B$. Finally C is called minimal quasi-closed if C, but not proper subset of C, is nontrivial quasi-closed.*

Definition 4.3.2. *An extended Camina group (ECG) is a pair of groups (G, H) such that $\{1\} \neq H \lhd G$ and $hg \in cl_G(g) \cup cl_G(g^{-1})$ for every $h \in H$ and $g \in G - H$ (where $cl_G(g)$ denotes the G-conjugacy class of g).*

Definition 4.3.3. *A CF-pair is an ECG which satisfies the stronger condition that $hg \in cl_G(g) \forall h \in H$ and $g \in G - H$.*

The most important results of [3] are:

Theorem 4.3.4. *Let (G, H) be an ECG such that H is a maximal normal subgroup of G, then one of the following holds:*

1. *G/H is cyclic of prime order and G is a Frobenius group with Frobenius kernel H (hence (G, H) is a FC-pair);*
2. *G has a normal subgroup K of odd order and an element g of order 4 such that g acts fixed-point-freely on K by conjugation, $G = K < g >$ and $H = K < g^2 >$.*

Corollary 4.3.5. *Any pair of groups $H \lhd G$ for which (1) and (2) of Theorem 4.3.4 holds is an ECG and that if (2) of Theorem 4.3.4 holds then (G, H) is not a FC-pair.*

Corollary 4.3.6. *Let (G, H) be an ECG such that G/H is cyclic and $g \in G$ be such that $< gH >= G/H$. Then either G is a Frobenius group with Frobenius kernel H or $|G/H| = 2$, order(g)=4, G has a normal subgroup K of odd order such that H=K $< g^2 >$ and g acts fixed-point-freely on K by conjugation.*

Chillag and Herzog presented a description of finite groups G with C-pairs of type (G, G'), which generalize the Frobenius groups with abelian complements.

Definition 4.3.7. *A pair (G, K), where G is a finite group and $1 < K < G$ is a normal subgroup of G, is called an extended Camina pair if $xK \subseteq class_G(x) \cup class_G(x^{-1})$ for each $x \in G - K$.*

C-pairs are clearly also extended Camina pairs. In [17] they proved:

Theorem 4.3.8. *If (G, G') is an extended Camina pair, then one of the following three statements holds:*

1. *G is a Frobenius group with kernel G';*
2. *G is a p-group for some prime p;*
3. *G/G' is a 2-group.*

4.4 Generalized Gagola Characters

Let G be a finite group, if G has a character $\chi \in \text{Irr(G)}$ so that χ vanishes on all but two conjugacy classes of G, Gagola proved that such a character χ is unique when $|G| > 2$ and that G has a unique minimal normal subgroup N which is necessarily an elementary abelian p-group for some prime p. In this situation we call χ a Gagola character of G and call (G, N) a Gagola pair or a p-Gagola pair to emphasize the prime p when necessary.

Remark 3. If G is a finite group of order m, $Gal(\chi)$ for the set of all Galois conjugates of χ and $n(\chi)$ for the number of conjugacy classes of G on which χ takes nonzero values. Now assume that $G > 1$, and from the first orthogonality relation of characters we obtain $|Gal(\chi)| + 1 \leq n(\chi)$.

In [13] X. Chang, H. Wang and P. Jin introduced some important results:

Definition 4.4.1. *An irreducible character χ of a finite group G is a generalized Gagola character if $|Gal(\chi)| + 1 = n(\chi)$. Similarly, if the group G has a generalized Gagola character, then G also has a unique minimal normal subgroup N which is an elementary abelian p-group for some prime p, and in this case, we call (G, N) a generalized Gagola pair or a generalized p-Gagola pair to indicate the prime p.*

Definition 4.4.2. *(G, N) is a Galois pair if $Irr(G|N)$ is a single Galois orbit of Irr(G), that is, $Irr(G|N) = Gal(\chi)$ for some $\chi \in Irr(G)$.*

In general, a C-pair needs not to be a Galois pair and vice versa. Actually, they proved that a pair (G, N) is a generalized Gagola pair if and only if it is both a C-pair and a Galois pair. Now the main results can be stated as follows:

Theorem 4.4.3. *Assume that G is a finite non-abelian group. Then G has a generalized Gagola character if and only if there exists a normal subgroup N of G with $1 < N < G$, satisfying the following three conditions:*

1. *(G, N) is a C-pair;*
2. *N is an elementary abelian p-group for some prime p;*
3. *G permutes transitively all subgroups of order p of N. In that case, N is a unique minimal normal subgroup of G, and the set of non-vanishing elements of any generalized Gagola character of G, and hence (G, N) is a generalized p-Gagola pair.*

In particular, for any $M \lhd G$ with $1 < M < G$, then (G, M) is a generalized Gagola pair if and only if it is both a C-pair and a Galois pair.

In [12] Camina proved that if (G, N) is a C-pair, then either G is a Frobenius group, or at least one of the groups N and G/N is a p-group for some prime p. Furthermore, if G is not a Frobenius group and N is a p-group, then p must divide the order of G/N. From Theorem 4.4.3 we have:

Corollary 4.4.4. *As an application of Theorem 4.4.3, holds:*

1. *If G is a nonabelian p-group, then G has a generalized Gagola character if and only if $(G, Z(G))$ is a C-pair and $|Z(G)| = p$.*
2. *If G is a Frobenius group with the Frobenius kernel N, then (G, N) is a generalized Gagola pair if and only if N is an elementary abelian p-group for some prime p and G is transitive on all subgroups of order p of N.*

4.5 Generalized Camina Groups And Generalized Camina Pairs

In [41] Lewis generalizes the definition of Camina groups:

Definition 4.5.1. *A group G is said to be a Generalized Camina group (or GCG) if* $Cl_G(g) = gG_2$ *for every* $g \in G/G_2Z(G)$.

Obviously, every Camina group is a generalized Camina group.

Definition 4.5.2. *A pair* (G, N) *is said to be a generalized Camina pair (abbreviated GCP) if* $N \lhd G$ *and, all non linear irreducible characters of G vanish outside N (see [42]).*

The main result of his work is the following:

Theorem 4.5.3. *Let G be a group, then G is a generalized Camina group if and only if G is isoclinic to a Camina group.*

Many results for Camina groups will translate immediately to GCG.

Lewis first studied the nipotent groups with nilpotent class 2 and then he focused on nilpotent groups with nilpotence class at most 3:

Theorem 4.5.4. *Let G be a nilpotent, generalized Camina group with associated prime p, then:*

1. *If* $p = 2$, *then G has nilpotence class 2.*
2. *If p is odd, then G has nilpotence class at most 3.*

Prajapati and Sury, in [72], continued to study this class of groups.

4.6 Finite Generalizations Of Camina Groups

In [78] and [80] Yadav presented a finite generalizations of Camina p-groups:

Definition 4.6.1. *A finite p-group G is called an almost Camina p-group if* $\gamma_2(G) = \Phi(G)$ *and the following condition* (C) *holds for every minimal generating set of G.* (C) *If* $\{x_1, x_2, \dots, x_d\}$ *is a minimal generating set for G, then* $[x_i, G] = \gamma_2(G)$ *for all but at most one* $i = 1, 2, \dots, d$.

Notice that every Camina p-group is almost Camina. But converse is not true. For example we take a group G of order p^4 and nilpotency class 3. It is easy to check that G is an almost Camina p-group but it is not a Camina group.

Definition 4.6.2. *A non-abelian finite group G will be called Camina-type if* $[x, G] = \gamma_2(G)$ *(or equivalently* $x^G = x\gamma_2(G)$*) for all* $x \in G - \Phi(G)$.

Notice that a Camina-type group G is a Camina group if and only if $\gamma_2(G) = \Phi(G)$. In this work he also gave a classification of the almost Camina p-groups and Camina-type groups.

In [28] Isaacs and Lewis presented some Camina p-groups that are generalized Frobenius complements, where the definition of generalized Frobenius complements is the same used in [53]:

Definition 4.6.3. *Let G be a linear group and $\theta(G)$ the normal subgroup of G generated by those elements which have a nonzero fixed point. We call $G/\theta(G)$ a generalized Frobenius complement.*

4.7 Infinite Generalizations Of Camina Groups

In [64] infinite generalizations of Camina groups are discussed and some infinite Camina groups are constructed. Since Ersoy observed in [23] that if G' is finite, then G is solvable, he proved:

Theorem 4.7.1. *Let G be a group with an anticentral element a of order m such that G' is a periodic $\mathbb{F}$-linear group where $\mathbb{F}$ has characteristic p. Then one of the following cases occurs:*

1. *$C_{G'}(a)$ is finite and G is solvable;*
2. *$C_{G'}(a)$ has an infinite abelian subgroup of exponent p^k where p^k divides m.*

Corollary 4.7.2. *If G is a group with an anticentral element a of finite order such that G' is periodic $\mathbb{F}$-linear group and $(|a|, p) = 1$, where char$\mathbb{F}$=p, then G is solvable. In particular, periodic linear groups over fields of characteristic p with semisimple anticentral elements are solvable.*

The following observation shows that if a locally finite group with an anticentral element is residually finite, then it is locally solvable:

Proposition 4.7.3. *Let G be a residually finite and locally finite group with an anticentral element a, then G is locally solvable.*

Then he will show that if a locally finite group containing an anticentral element is residually finite, then it is locally solvable.

4.8 Camina Triples

Previously Mattarei ([59], [61]) and later Mlaiki ([62]) introduced a generalization of C-pairs:

Definition 4.8.1. *Let $1 < M \leq N$ be two nontrivial normal subgroups of a finite group G, we say that (G, M, N) is a Camina Triple if for every $g \in G/N$, g is a conjugate to all of gM.*

The Camina triple condition clearly reduces to the condition that (G, N) is a C-pair when N=M.

In particular, Mlaiki [62], proved the following important results:

Theorem 4.8.2. *If (G, N, M) is a Camina triple then the following are true:*

1. *M is solvable;*
2. *M has a normal $\pi-$complement Q with M/Q is nilpotent, where π is the set of primes that divide $|G:N|$;*
3. *If $x \in M$, then there exists $\chi \in Irr(G|M)$ such that $\chi(x) \neq 0$;*
4. *If $x \in G\ N$, then $\pi(x) = 0$ for all $\chi \in Irr(G|M)$.*

Theorem 4.8.3. *If (G, N, M) is a Camina triple, then at least one of the following hold:*

1. *G/N is a p-group for some prime p;*
2. *M is nilpotent;*
3. *M has a non-trivial $\pi-$complement say Q, where π is the set of primes that divides $|G/N|$. Moreover, M/Q is a non-trivial nilpotent group.*

Theorem 4.8.4. *If (G, M, N) is a Camina triple, then either G/N is a $p-$group, M is nilpotent or M has a non-trivial nilpotent quotient.*

4.9 Applications Of Camina Groups

In [59], [60] and [61], S. Mattarei constructed pairs of p-groups (G, H) with identical character tables and different derived lengths and he also showed that *Brauer pairs* are generalizations of Camina groups, where:

Definition 4.9.1. *[22] Two finite groups G and H form a Brauer pair if they are non-isomorphic, but their character tables including power maps are equivalent; that is, if there exists a bijection $\tau : Cl(G) \to Cl(H)$ on the conjugacy classes of these groups which induces a bijection on the complex characters $\sigma : \chi \in Irr(H) \to \chi \circ \tau \in Irr(G)$ and satisfies $\tau \circ \pi_n^G = \pi_n^H \circ \tau$ for every $n \in \mathbb{Z}$, where $\pi_n^G : Cl(G) \to Cl(G)$ is the n-th powermap induced by $g \in G \to g^n \in G$.*

In 2009 Lewis [39] found a condition that characterizes when two Camina p-groups of nilpotence class 2 form a Brauer pair. Later, in [44], M. Lewis and J. Wilson showed that Camina groups provide simple examples of Brauer pairs.

They derived some isomorphism invariants for quotients of *generalized Heisenberg groups* by observing that these groups are special instances of Camina groups, infact the Camina property is used to show that the complex character tables of quotients of a Heisenberg group are determined solely by their order. Recall that:

Definition 4.9.2. *A group H is a generalized Heisenberg group if there is a field K and an integer m such that H is isomorphic to $H_m(K) = \{ \begin{bmatrix} 1 & u & s \\ 0 & I_m & v^t \\ 0 & 0 & 1 \end{bmatrix} | s \in K; u, v \in K^m \}$. When m = 1 we call H a Heisenberg group. The family of groups in which we are interested are the non-abelian quotients of H.*

Starting from the results obtained in [40], Lewis used the informations collected about generalized Camina groups with nilpotence class 3 to characterize the character tables of these groups. With this results regarding nilpotent GCG he is able to classify the character tables of nilpotent GCG of nilpotence class 3. Surprisingly, he has the same characterization as in the nilpotence class 2 case:

Theorem 4.9.3. *Let G and H be generalized Camina groups of nilpotence class 3. Suppose that there exist isomorphisms* $a : Irr(G|G') \to Irr(H|H')$ *and* $b : Irr(Z(G)) \to Irr(Z(H))$ *so that* $a(\alpha_{Z(H)}) = b(\beta_{Z(G)})$ *for all* $\alpha \in Irr(G|G')$*. Then G and H have identical character tables.*

In particular, in [40], Lewis proved:

Theorem 4.9.4. *If G and J are finite Camina p-groups of nilpotence class 2, then G and J have isomorphic character tables if and only if* $|G : G'| = |H : H'|$ *and* $|G'| = |H'|$*.*

Now we recall that:

Definition 4.9.5. *A group G has the condition no divisibility among degrees (NDAD) if for every* $a, b \in cd(G)$ *with* $1 < a < b$*, a does not divide b.*

In [37] Lewis, Moretó and Wolf showed how to use the Camina p-groups of nilpotency class 3 to construct solvable finite groups that satisfy the condition NDAD for which exixts an absolute constant C such that $|cd(G)| \leq C$.

4.10 Supercharacter Theories Of Camina Pairs

Now we want to recall some important definitions in the Supercharacter Theories:

Definition 4.10.1. *A supercharacter theory C of G is a pair* (X, K) *for X a partition of Irr(G) and K a partition of G which satisfies three criteria:*

1. $|X| = |K|$;
2. *For each set* $X \in \chi$ *there is a character* χ_X *whose irreducible constituents lie in X such that* χ_X *is constant on each member of K;*
3. $\{1\} \in K$.

Definition 4.10.2. *The sets* $k \in K$ *is called superclasses and* $\{1_G\} \in X$ *and* χ_X *is called a supercharacter.*

Definition 4.10.3. *A subgroup N of G is C-normal if we can write N as a union of superclasses of C. Further, we denote the set of all supercharacter theories of G by Sup(G). We write M(G) for the maximal supercharacter theory for G.*

C. W. Wynn [84] explored supercharacter theories of extraspecial p-groups, general C-pairs and Frobenius groups:

Corollary 4.10.4. *If G is an extraspecial p-group or a Frobenius group and C is a supercharacter theory for G where the only C-normal subgroups are* $\{1\}$ *and G, then* $C = M(G)$*.*

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