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A remark on the Laplacian flow and the modified Laplacian co-flow in G_2-geometry

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Abstract
We give a shorter proof of the well-posedness of the Laplacian flow in G_2-geometry. This is based on the observation that the DeTurck–Laplacian flow of G_2-structures introduced by Bryant and Xu as a gauge fixing of the Laplacian flow can be regarded as a flow of (not necessarily closed) G_2-structures, which fits in the general framework introduced by Hamilton in J Differ Geom 17(2):255–306, 1982. A similar application is given for the modified Laplacian co-flow.

Keywords Laplacian flow · G2-geometry · Short-time existence

1 Introduction

In[1] Bryant introduced a geometric flow in G_2-geometry which evolves an initial closed G_2-structure φ_0 in the direction of its Laplacian.

Given a compact seven-dimensional manifold with a closed G_2-structure (M, φ_0), a *Laplacian flow* is a solution to the evolution equation

$$\frac{\partial}{\partial t}\varphi_t = \Delta_{\varphi_t}\varphi_t, \quad d\varphi_t = 0, \quad \varphi_{|t=0} = \varphi_0\,. \tag{1}$$

The well-posedness of Eq. (1) is proved in[2] by applying the Nash–Moser theorem to the gauge fixing

$$\frac{\partial}{\partial t}\varphi_t = \Delta_{\varphi_t}\varphi_t + \mathcal{L}_{V(\varphi_t)}\varphi_t, \quad d\varphi_t = 0, \quad \varphi_{|t=0} = \varphi_0, \tag{2}$$

where $\mathcal{L}$ is the Lie derivative and $V : C^\infty(M, \Lambda^3_+) \to C^\infty(M, TM)$ is a first-order differential operator which depends on the choice of a connection on M. Here, Λ^3_+ denotes the open

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subbundle of Λ^3 of G_2-structures on M. A solution to (2) is usually called a *DeTurck–Laplacian flow*.

A DeTurck–Laplacian flow φ_t is also a solution to

$$\frac{\partial}{\partial t}\varphi_t = dd^*_{\varphi_t}\varphi_t + d\iota_{V(\varphi_t)}\varphi_t, \quad \varphi_{|t=0} = \varphi_0 \,. \tag{3}$$

In the present note, we observe that Eq. (3) fits in the general framework introduced by Hamilton in[4]. As a direct consequence, we have the following theorem which in particular implies the well-posedness of (2)

Theorem 1.1 *Let* (M, φ_0) *be a compact* seven-*dimensional manifold with a* G_2-*structure. Then, Eq.* (3) *has a unique short-time solution.*

In[5] Karigiannis, McKay and Tsui introduced the *Laplacian co-flow* as the solution to the evolution equation

$$\frac{\partial}{\partial t}(*_{\varphi_t}\varphi_t) = -\Delta_{\varphi_t} *_{\varphi_t}\varphi_t, \quad d *_{\varphi_t}\varphi_t = 0, \quad \varphi_{|t=0} = \varphi_0, \tag{4}$$

where in this case φ_0 is supposed to be co-closed with respect to the metric induced by itself. The well-posedness of this last equation is still an open problem and Grigorian introduced in[3] the following modification

$$\frac{\partial}{\partial t}(*_{\varphi_t}\varphi_t) = \Delta_{\varphi_t} *_{\varphi_t}\varphi_t + 2d((A - \mathrm{Tr}(T(\varphi_t))\varphi_t), \quad d *_{\varphi_t}\varphi_t = 0, \quad \varphi_{|t=0} = \varphi_0, \tag{5}$$

where A is a constant and $T(\varphi_t)$ is the torsion of φ_t. In[3], the well-posedness of (5) is proved following the same approach of Bryant in[1] by applying the Nash–Moser theorem to the gauge fixing

$$\frac{\partial}{\partial t}(*_{\varphi_t}\varphi_t) = \Delta_{\varphi_t} *_{\varphi_t}\varphi_t + 2d((A - \mathrm{Tr}(T(\varphi_t))\varphi_t) + \mathcal{L}_{V(\varphi_t)}\varphi_t, \quad d *_{\varphi_t}\varphi_t = 0, \quad \varphi_{|t=0} = \varphi_0, \tag{6}$$

Any solution to Eq. (6) satisfies

$$\frac{\partial}{\partial t}(*_{\varphi_t}\varphi_t) = dd^*_{\varphi_t} *_{\varphi_t}\varphi_t + 2d((A - \mathrm{Tr}(T(\varphi_t))\varphi_t) + d\iota_{V(\varphi_t)}\varphi_t, \quad \varphi_{|t=0} = \varphi_0, \tag{7}$$

Analogously to Theorem 1.1, we have

Theorem 1.2 *Let* (M, φ_0) *be a compact* seven-*dimensional manifold with a* G_2-*structure. Then, Eq.* (6) *has a unique short-time solution.*

2 Proof of the results

Both Theorems 1.1 and 1.2 can be proved by using the following setup introduced by Hamilton in [4].

Let M be an oriented compact manifold, F a vector bundle over M, U an open subbundle of F and

$$E : C^\infty(M, U) \to C^\infty(M, F)$$

a second-order differential operator. For $f \in C^\infty(M,U)$, we denote by $DE(f) : C^\infty(M,F) \to C^\infty(M,F)$ the linearization of E at f and by $\sigma DE(f)$ the principal symbol of $DE(f)$.

Definition 2.1 *An integrability condition for E is a first-order linear differential operator*

$$L : C^\infty(M,F) \to C^\infty(M,G),$$

*where G is another vector bundle over M, such that $L(E(f)) = 0$ for all $f \in C^\infty(M,U)$, and for every (x, ξ) in T*M all the eigenvalues of $\sigma DE(f)(x,\xi)$ restricted to $\ker \sigma L(x,\xi)$ have strictly positive real part.*

Theorem 2.1 (Hamilton [4, Theorem 5.1]) *Assume that E admits an integrability condition. Then, for every $f_0 \in C^\infty(M,U)$ the geometric flow*

$$\frac{\partial f}{\partial t} = E(f), \quad f(0) = f_0, \tag{8}$$

has a unique short-time solution.

Now we can focus on the setup of Theorem 1.1. Here, we consider

$$F = \Lambda^3, \quad U = \Lambda^3_+, \quad G = \Lambda^4, \quad E(\varphi) = dd^*_\varphi\varphi + d\iota_{V(\varphi)}\varphi, \quad L = d : C^\infty(M,\Lambda^3) \to C^\infty(M,\Lambda^4)\,.$$

From [2], it follows that for every $\varphi \in C^\infty(M,U)$ and every *closed* $\psi \in C^\infty(M,\Lambda^3)$, we have

$$DE(\varphi)(\psi) = -\Delta_\varphi\psi + \text{l.o.t.}$$

Hence, all the assumptions of Hamilton's Theorem 2.1 are satisfied and Theorem 1.1 follows.

Notice that if the starting form φ_0 is closed, then the solution to (3) is closed for every t since

$$d\tfrac{\partial}{\partial t}\varphi = 0\,.$$

Therefore, if φ_0 is closed, the unique solution φ_t to (3) solves also the DeTurck–Laplacian flow (2) and the short-time existence of the DeTurck–Laplacian flow (2) can be deduced from Theorem 1.1.

About the proof of Theorem 1.2, we set

$$\begin{aligned} &F = \Lambda^4, \quad U = \Lambda^4_+, \quad G = \Lambda^5, \quad E(*_\varphi\,\varphi) = dd^*_\varphi\varphi + d\iota_{V(\varphi)} + 2d((A - \mathrm{Tr}(T(\varphi))\varphi), \\ &L = d : C^\infty(M,\Lambda^4) \to C^\infty(M,\Lambda^5)\,. \end{aligned}$$

From [3], it follows

$$DE(*_\varphi\,\varphi)(\psi) = -\Delta_\varphi\psi + \text{l.o.t.}$$

for every closed $\psi \in C^\infty(M,\Lambda^4)$ and the proof of Theorem 1.2 follows.

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