GLOBAL SOLUTIONS FOR A CHEMOTAXIS HYPERBOLIC-PARABOLIC SYSTEM ON NETWORKS WITH NONHOMOGENEOUS BOUNDARY CONDITIONS

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Abstract. In this paper we study a semilinear hyperbolic-parabolic system as a model for some chemotaxis phenomena evolving on networks; we consider transmission conditions at the inner nodes which preserve the fluxes and nonhomogeneous boundary conditions having in mind phenomena with inflow of cells and food providing at the network exits. We give some conditions on the boundary data which ensure the existence of stationary solutions and we prove that these ones are asymptotic profiles for a class of global solutions.

1. Introduction

In this paper we consider the one dimensional semilinear hyperbolic-parabolic system

$$
\begin{cases} u_t + \lambda v_x = 0 \ , \\ v_t + \lambda u_x = u\psi_x - \beta v \ , \\ \psi_t = D\psi_{xx} + au - b\psi \ , \end{cases} \tag{1.1}
$$

on a finite network, where $\lambda, \beta, D, b > 0$ and $a \geq 0$.

The system has been proposed as a model for chemosensitive movements of bacteria or cells; the unknown u stands for the cells concentration, λv denotes their average flux and ψ is the chemo-attractant concentration produced by the cells themselves; the individuals move at a constant velocity, whose modulus is λ, towards the right or left along the axis; β is the friction coefficient while D, a, b are respectively the diffusion coefficient, the production rate and the degradation one for the chemoattractant .

Systems like (1.1) are adaptations of the so-called Cattaneo equation to the chemotactic case, introducing the nonlinear term $u\psi_x$ in the equation for the flux [22, 8], and their solutions have been studied in [14, 15, 10]; they are included among hyperbolic models which have been recently introduced in contrast to the parabolic ones considered before, since they give rise to a finite speed of propagation and allow better observation of the phenomena during the initial phase.

In recent years, one dimensional models on networks have been developed in order to describe particular chemotactic phenomena like the process of dermal wound healing and the behavior of the slime mold Physarum polycephalum as a model for amoeboid movements. Actually, during the healing process, the stem cells in charge

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of the reparation of dermal tissue (fibroblasts), create a new extracellular matrix, essentially made by collagen, and move along it to fill the wound driven by chemotaxis and tissue engineers insert artificial scaffolds within the wound to accelerate this process [13, 16, 23]; also, the body of Physarum polycephalum contains a network of tubes which are used by nutrients and chemical signals to circulate throughout the organism [18].

These models are heavily characterize by the transmission conditions set at the internal nodes of the network, which couple the solutions on different arcs.

Here we consider the system (1.1) on a network whose arcs I_i are characterized by the parameters $\lambda_i, \beta_i, D_i, a_i, b_i$. The triples of unkowns (u_i, v_i, ψ_i) corresponding to each arc are coupled by the transmission conditions introduced in [11] set at the inner nodes, which impose that the sum of the incoming fluxes equals the sum of the outgoing ones, rather than the continuity of the densities, since the eventuality of jumps at the nodes for these quantities seems a more appropriate framework to describe movements of individuals.

This model, complemented with homogeneous boundary conditions at the external vertices of the network, was studied in [11], concerning the existence and the uniqueness of global solutions in the case of suitably small initial data; moreover, results about existence of stationary solutions and asymptotic behaviour are given in [9]; finally, in [3] the authors carry out a numerical study of the same system with transmission conditions set for the Riemann invariants of the hyperbolic part, which are equivalent to our ones for some choices of the coefficients.

Results about hyperbolic models on networks can be found in [6, 7, 25, 19, 24], with different kinds of transmission conditions; moreover parabolic chemotaxis models on networks were studied in [1, 5, 17], with continuity conditions at the nodes.

It is worth considering system (1.1) with nonhomogeneous boundary conditions at the outer nodes of the network, having in mind phenomena with inflow of cells and food providing at the network exits, in particular experiments on the behaviour of Physarum [18]. We remark that [2] contains a numerical approach to system (1.1) on networks, with transmission conditions given for the Riemann invariants and nonhomogeneous conditions at the boundaries; the numerical tests show the correspondence with the main features of the real behavior of slime mold examined through the laboratory experiments: dead end cutting and the selection of the solution path among the competitive paths.

So, in the present paper we consider system (1.1) with the dissipative transmission conditions introduced in [11] at the inner nodes, and nonhomogeneous Neumann conditions for the hyperbolic part and nonhomogeneous Robin condition for the parabolic equation at the external ones. The boundary data are assumed to satisfy suitable hypothesis ensuring, in particular, the boundedness of the total mass of cells during the phenomenon evolution; the mass is preserved in case of homogeneous Neumann conditions, since the conservation of the fluxes holds at each inner nodes, due to the transmission conditions [11, 9], but in the present case it depends on the evolution in time of the boundary values for the fluxes $\lambda_i v_i$.

The first result in the paper is the existence of local solutions; it is achieved by linear contraction semigroups theory together with the abstract theory for semilinear problems, and the dissipative transmission conditions at the inner nodes play a fundamental role.

The existence of global solutions is achieved under assumptions of smallness of the data, proceeding in some steps. First we assume the existence of a stationary solution $(U(x), V(x), \Psi(x))$ to the problem and we obtain a priori estimates for solutions corresponding to initial and boundary data which are small perturbations of the possible stationary solution. Here a fondamental role is played by a suitable condition stated for the transmission coefficients, which allows to express the jumps of the densitiy u at each inner node as linear combinations of the values of the fluxes at the same node. This fact and assumptions on the data provide a control of the evolution in time of the L^∞- norm of the density which permits to remove some conditions on the parameters a_i and b_i considered in [11, 9]. When the boundary data for the fluxes $\lambda_i v_i$ are constant functions, the hypothesys necessary to prove the a priori estimates imply that the sum of the fluxes incoming in the network have to equal the sum of the outgoing ones and that the initial mass of cells has to equal the mass of the stationary solution.

If a stationary solution $(U(x), V(x), \Psi(x))$ exists and the quantities $\|U\|_\infty$ and $\|\Psi'\|_\infty$ are small, the a priori estimates provide a bound, uniform in time, for a norm of the solutions having small perturbations of the stationary one as initial and boundary data; in this way, after the proof of real existence of stationary solutions, we would obtain the existence of global solutions for a class of initial and boundary data and would identify the stationary solutions as the asymptotic profiles for such class of solutions.

For this reason we devote part of this paper to study the existence of stationary solutions. In the cases of acyclic networks we prove two results, under different smallness conditions on the boundary data and on the total mass; in particular we give conditions which ensure the existence of a stationary solution with non-negative density U. For general networks we exhibit some stationary solutions in very particular cases for the parameters of the problem.

We stress that, although the techniques used in proving the a priori estimates for solutions of (1.1) are similar to the ones in [11], here we need to control the growth in time of the L^∞- norm of the densities to treat the non homogeneous boundary conditions, removing, at the same time, some restrictions on the parameters a_i and b_i considered in [11, 9]. Moreover, in this paper, the proof of existence of global solutions is strictly connected to the existence of stationary solutions, since we consider initial data which are small perturbations of stationary solutions. The study of these solutions (on acyclic graphs) in presence of non homogeneous boundary conditions is more complex than the one in [9], where homogeneous conditions are considered; in both cases, if a stationary solution $(U(x), V(x), \Psi(x))$ exists, the function $V(x)$ is constant on each arc but in the case of null boundary data it has to be zero, while here, in general, it is not, so that different techniques are necessary in proving existence results.

The paper is organized as follows. In Section 2 we give the statement of the problem and, in particular, we introduce the transmission conditions and the assumption on the data, while in Section 3 we prove the local existence result. Section 4 is devoted to the a priori estimates and to the consequent global existence and asymptotic behaviour results, under the assumption that a *small* stationary solution exists. In Section 5 we prove the results of existence of stationary solutions in the case of acyclic networks. Finally, in Section 6 we present the global existence and asymptotic behaviour results under assumptions which ensure the real existence of stationary solutions.

2. **Statement of the problem**

We consider a finite connected graph $\mathcal{G} = (\mathcal{Z}, \mathcal{A})$ composed by a set $\mathcal{Z}$ of n nodes (or vertexes) and a set $\mathcal{A}$ of m oriented arcs, $\mathcal{A} = \{I_i : i \in \mathcal{M} = \{1, 2, ..., m\}\}$. Each node is a point of the plane and each oriented arc I_i is an oriented segment joining two nodes.

We use e_j, $j \in \mathcal{J}$, to indicate the external vertexes of the graph, i.e. the vertexes belonging to only one arc, and by $I_{i(j)}$ the external arc incident with e_j. Moreover, we denote by N_ν, $\nu \in \mathcal{N}$, the internal nodes; for each of them we consider the set of incoming arcs $\mathcal{A}^\nu_{in} = \{I_i : i \in \mathcal{I}^\nu\}$ and the set of the outgoing ones $\mathcal{A}^\nu_{out} = \{I_i : i \in \mathcal{O}^\nu\}$; let $\mathcal{M}^\nu = \mathcal{I}^\nu \cup \mathcal{O}^\nu$.

In this paper, a *path* in the graph is a sequence of arcs, two by two adjacent, without taking into account orientations. Moreover, we call *acyclic* a graph which does not contains cycles, i.e. for each couple of nodes there exists a unique path connecting them, whose arcs are covered only one time.

Each arc I_i is considered as a one dimensional interval $(0, L_i)$. A function f defined on $\mathcal{A}$ is a m-tuple of functions f_i, $i \in \mathcal{M}$, each one defined on I_i. The expression $f_i(N_\nu)$ means $f_i(0)$ if N_ν is the starting point of the arc I_i and $f_i(L_i)$ if N_ν is the endpoint, and similarly for $f(e_j)$.

We set $L^p(\mathcal{A}) := \{f : f_i \in L^p(I_i)\}$, $H^s(\mathcal{A}) := \{f : f_i \in H^s(I_i)\}$ and

$$\|f\|_p := \sum_{i\in\mathcal{M}} \|f_i\|_p \ , \ \|f\|_\infty := \max_{i\in\mathcal{M}} \|f_i\|_\infty \ , \ \|f\|_{H^s} := \sum_{i\in\mathcal{M}} \|f_i\|_{H^s} .$$

We consider the evolution of the following problem on the graph $\mathcal{G}$

$$\left\{\begin{array}{l} u_{it} + \lambda_i v_{ix} = 0 \ , \\ \\ v_{it} + \lambda_i u_{ix} = u_i \psi_{ix} - \beta_i v_i \ , \qquad t \geq 0, \ x \in I_i, \ i \in \mathcal{M}, \\ \\ \psi_{it} = D_i \psi_{ixx} + a_i u_i - b_i \psi_i \ , \end{array}\right. \tag{2.1}$$

where $a_i \geq 0$, λ_i $b_i, D_i, \beta_i > 0$, complemented with the initial conditions

$$(u_{0i}, v_{0i}) \in (H^1(I_i))^2 \ , \ \psi_{0i} \in H^2(I_i) \ , \quad \text{for } i \in \mathcal{M} \ . \tag{2.2}$$

In order to set boundary and transmission conditions, we introduce the following parameters:

$$\delta^\nu_i = 1 \ \text{ if } i \in \mathcal{I}^\nu \ , \ \delta^\nu_i = -1 \text{ if } i \in \mathcal{O}^\nu \ , \quad \nu \in \mathcal{N},$$

$$\eta_j = \left\{\begin{array}{lll} 1 & \text{if the arc } I_{i(j)} \text{ is incoming in } e_j \ , & j \in \mathcal{J}, \\ -1 & \text{if the arc } I_{i(j)} \text{ is outgoing from } e_j \ , & j \in \mathcal{J}, \end{array}\right.$$

where we used the notation introduced at the beginning of this section for the external arcs. Moreover we write $f(e_j)$ in place of $f_{i(j)}(e_j)$, since no ambiguity arises.

The boundary conditions for v, at each outer point e_j, are

$$\eta_j \lambda_{i(j)} v(e_j, t) = \mathcal{W}_j(t) \in W^{2,1}(0,T) \ , \quad \text{for each } T > 0 \ , \ j \in \mathcal{J} \ , \tag{2.3}$$

while for ψ we set the Robin boundary conditions

$$\eta_j D_{i(j)} \psi_x(e_j, t) + d_j \psi(e_j, t) = \mathcal{P}_j(t) \in H^2(0,T) \ , \ d_j \geq 0 \ , \ \text{for each } T > 0 \ , \ j \in \mathcal{J}. \tag{2.4}$$

In addition, at each internal node N_ν we impose the following transmission conditions for the unknown ψ

$$\left\{\begin{array}{l} \delta^\nu_i D_i \psi_{ix}(N_\nu, t) = \displaystyle\sum_{j\in\mathcal{M}^\nu} \alpha^\nu_{ij}(\psi_j(N_\nu, t) - \psi_i(N_\nu, t)) \ \ , \ i \in \mathcal{M}^\nu \ , \ t > 0 \ , \\ \alpha^\nu_{ij} \geq 0 \ , \ \alpha^\nu_{ij} = \alpha^\nu_{ji} \ \text{ for all } i, j \in \mathcal{M}^\nu \ , \end{array}\right. \tag{2.5}$$

and the following ones for the unknowns v and u

$$\left\{\begin{array}{l} -\delta^\nu_i \lambda_i v_i(N_\nu, t) = \displaystyle\sum_{j\in\mathcal{M}^\nu} \sigma^\nu_{ij} \left(u_j(N_\nu, t) - u_i(N_\nu, t)\right) \ , \ i \in \mathcal{M}^\nu \ , \ t > 0 \ , \\ \sigma^\nu_{ij} \geq 0 \ , \sigma^\nu_{ij} = \sigma^\nu_{ji} \ \text{ for all } i, j \in \mathcal{M}^\nu \ . \end{array}\right. \tag{2.6}$$

Motivations for the above constraints on the coefficients in the transmission conditions can be found in [11] . These kind of transmission conditions, known as Kedem-Katchalsky permeability conditions, were introduced in [12] in a parabolic model for the description of passive transport through biological membranes and are used in models where discontinuities for the solutions at the internal nodes are expected [20],[21].

Finally, we impose the following compatibility conditions

$$(2.7)\quad \begin{cases} \eta_j \lambda_{i(j)} v_0(e_j) = \mathcal{W}_j(0)\ , \quad \eta_j D_{i(j)} \psi_{0x}(e_j) + d_j \psi_0(e_j) = \mathcal{P}_j(0), \quad j \in \mathcal{J}, \\ \delta_i^\nu D_i \psi_{0ix}(N_\nu) = \displaystyle\sum_{j\in\mathcal{M}^\nu} \alpha_{ij}^\nu (\psi_{0j}(N_\nu) - \psi_{0i}(N_\nu)), \qquad i \in \mathcal{M}^\nu,\ \nu \in \mathcal{N}, \\ -\delta_i^\nu \lambda_i v_{0i}(N_\nu) = \displaystyle\sum_{j\in\mathcal{M}^\nu} \sigma_{ij}^\nu \left(u_{0j}(N_\nu) - u_{0i}(N_\nu)\right), \qquad i \in \mathcal{M}^\nu,\ \nu \in \mathcal{N}. \end{cases}$$

First we are going to prove that the problem (2.1)-(2.7) has a unique local solution

$$u, v \in C([0,T]; H^1(\mathcal{A})) \cap C^1([0,T]; L^2(\mathcal{A}))\ ,$$

$$\psi \in C([0,T]; H^2(\mathcal{A})) \cap C^1([0,T]; L^2(\mathcal{A})) \cap H^1((0,T); H^1(\mathcal{A}))\ ,$$

for some $T > 0$.

On the other hand, the proofs of the existence of global solutions and of the existence of stationary solutions on acyclic graphs, carried out in the last sections, require the following further conditions on the transmission coefficients,

(2.8)
$$\text{for all } \nu \in \mathcal{N},\ \text{there exists } k \in \mathcal{M}^\nu \text{ such that } \sigma_{ik}^\nu \neq 0 \text{ for all } i \in \mathcal{M}^\nu, i \neq k\ ,$$

in addition to suitable smallness and smoothness assumptions on the data.

We conclude this section by deriving some identities from the transmission conditions (2.5) and (2.6), which will be useful in the next sections.

First, the assumptions on σ_{ij}^ν in (2.6) and on α_{ij}^ν in (2.5) allow to write

$$(2.9)\quad \sum_{\nu\in\mathcal{N}} \sum_{i\in\mathcal{M}^\nu} \delta_i^\nu \lambda_i u_i(N_\nu,t) v_i(N_\nu,t) = \sum_{\nu\in\mathcal{N}} \sum_{i,j\in\mathcal{M}^\nu} \frac{\sigma_{ij}^\nu}{2} \left(u_j(N_\nu,t) - u_i(N_\nu,t)\right)^2\ ,$$

(2.10)
$$\sum_{\nu\in\mathcal{N}} \sum_{i\in\mathcal{M}^\nu} \delta_i^\nu D_i \psi_{i_x}(N_\nu,t) \psi_i(N_\nu,t) = -\sum_{\nu\in\mathcal{N}} \sum_{i,j\in\mathcal{M}^\nu} \frac{\alpha_{ij}^\nu}{2} (\psi_j(N_\nu,t) - \psi_i(N_\nu,t))^2.$$

Finally we remark that the transmission conditions (2.5) imply the conservation of the flux at each inner node N_ν, for all $t > 0$,

$$(2.11)\qquad \sum_{i\in\mathcal{I}^\nu} D_i \psi_{i_x}(N_\nu,t) = \sum_{i\in\mathcal{O}^\nu} D_i \psi_{i_x}(N_\nu,t),$$

and the conditions (2.6) ensure the conservation of the flux of the density of cells at each inner node N_ν , for $t > 0$,

$$(2.12)\qquad \sum_{i\in\mathcal{I}^\nu} \lambda_i v_i(N_\nu,t) = \sum_{i\in\mathcal{O}^\nu} \lambda_i v_i(N_\nu,t)\ ,$$

which corresponds to the following condition for the evolution in time of the total mass

$$\sum_{i\in\mathcal{M}} \int_{I_i} u_i(x,t)\ dx = \sum_{i\in\mathcal{M}} \int_{I_i} u_{0i}(x)\ dx - \sum_{j\in\mathcal{J}} \int_0^t \mathcal{W}_j(s) ds\ .$$

3. **Local solutions**

In order to prove the existence and the uniqueness of a local solution to problem (2.1)-(2.7) we need to introduce the auxiliary functions $\mathcal{V}(x,t)$ and $\Phi(x,t)$, defined on the network as follows

$$\begin{cases} \mathcal{V}_i(x,t), \Phi_i(x,t) = 0 & \text{if } I_i \text{ is an internal arc,} \\ \eta_j \lambda_{i(j)} \mathcal{V}_{i(j)}(x,t) = \dfrac{\mathcal{W}_j(t)}{L_{i(j)}} \left(\eta_j x + \dfrac{1-\eta_j}{2} L_{i(j)} \right) & \text{for all } j \in \mathcal{J}, \\ \eta_j D_{i(j)} \Phi_{i(j)}(x,t) = \dfrac{\mathcal{P}_j(t)}{L_{i(j)}^2} x (x - L_{i(j)}) \left(x + \dfrac{\eta_j - 1}{2} L_{i(j)} \right) & \text{for all } j \in \mathcal{J} \ , \end{cases}$$

where η_j is defined in the previous section.

Let the triple (u, v, ψ) be a solution to (2.1)-(2.7) and let

$$w := v - \mathcal{V}, \qquad \phi = \psi - \Phi \ ; \tag{3.1}$$

then the triple (u, w, ϕ) satisfies the following system

$$\begin{cases} u_{it} + \lambda_i w_{ix} = -\lambda_i \mathcal{V}_{ix} \\ w_{it} + \lambda_i u_{ix} = u_i(\phi_{ix} + \Phi_{ix}) - \beta_i w_i - \mathcal{V}_{it} - \beta_i \mathcal{V}_i \\ \phi_{it} = D_i \phi_{ixx} + a_i u_i - b_i \phi_i - \Phi_{it} + D_i \Phi_{ixx} - b_i \Phi_i \ , \end{cases} \tag{3.2}$$

for $x \in I_i$, $i \in \mathcal{M}$, $t > 0$, with the initial conditions

$$(u_i(x,0), w_i(x,0), \phi_i(x,0)) = (u_{i0}(x), w_{i0}(x), \phi_{i0}(x)) \ , \tag{3.3}$$

where $w_{i0}(x) := v_{i0}(x) - \mathcal{V}_i(x,0), \phi_{i0}(x) := \psi_{i0}(x) - \Phi_i(x,0)$; moreover, it easy to check that the triple (u, w, ϕ) satifies the homogeneous boundary conditions

$$\eta_j \lambda_{i(j)} w(e_j, t) = 0, \qquad t > 0 \ , \ j \in \mathcal{J} \ , \tag{3.4}$$

$$\eta_j D_{i(j)} \phi_x(e_j, t) + d_j \phi(e_j, t) = 0 \ , \ d_j \geq 0 \ , \ t > 0 \ , \ j \in \mathcal{J} \ , \tag{3.5}$$

and the transmission conditions at each inner node N_ν

$$\delta_i^\nu D_i \phi_{ix}(N_\nu, t) = \sum_{j \in \mathcal{M}^\nu} \alpha_{ij}^\nu (\phi_j(N_\nu, t) - \phi_i(N_\nu, t)) \ \ , \ i \in \mathcal{M}^\nu \ , \ t > 0 \ , \tag{3.6}$$

$$-\delta_i^\nu \lambda_i w_i(N_\nu, t) = \sum_{j \in \mathcal{M}^\nu} \sigma_{ij}^\nu \left(u_j(N_\nu, t) - u_i(N_\nu, t) \right) \ , \ i \in \mathcal{M}^\nu \ , \ t > 0 \ . \tag{3.7}$$

We are going to prove the existence and uniqueness result for local solutions to problem (2.1)-(2.7); it will be a consequence of the proof of the existence and uniqueness result for problem (3.2)-(3.7).

We consider the unbounded operator $A_1 : D(A_1) \to (L^2(\mathcal{A}))^2$:

$$D(A_1) = \left\{ \begin{array}{l} \mathcal{U} = (u, w) \in (H^1(\mathcal{A}))^2 : \ \ \eta_j \lambda_{i(j)} w(e_j) = 0, \ j \in \mathcal{J} \ , \\ -\delta_i^\nu \lambda_i w_i(N_\nu) = \displaystyle\sum_{j \in \mathcal{M}^\nu} \sigma_{ij}^\nu \left(u_j(N_\nu) - u_i(N_\nu) \right) \ , \ i \in \mathcal{M}^\nu \ , \ \nu \in \mathcal{N} \end{array} \right\} ,$$

$$A_1 \mathcal{U} = \{(-\lambda_i w_{ix}, -\lambda_i u_{ix})\}_{i \in \mathcal{M}} \ ;$$

moreover we introduce the unbounded operator $A_2 : D(A_2) \to L^2(\mathcal{A})$,

(3.8)
$$D(A_2) = \left\{ \begin{array}{l} \phi \in H^2(\mathcal{A}) : \eta_j D_{i(j)} \phi_x(e_j) + d_j \phi(e_j) = 0 \ , \ j \in \mathcal{J} \ , \\ \\ \delta_i^\nu D_i \phi_{i_x}(N_\nu) = \displaystyle\sum_{j \in \mathcal{M}^\nu} \alpha_{ij}^\nu (\phi_j(N_\nu) - \phi_i(N_\nu)) \ , \quad i \in \mathcal{M}^\nu, \ \nu \in \mathcal{N} \end{array} \right\},$$

$$A_2 \phi = \{D_i \phi_{ixx} - b_i \phi_i\}_{i \in \mathcal{M}} \ .$$

Proposition 3.1. *A_1 and A_2 are m-dissipative operators.*

Proof. The proof for the operator A_1 can be achieved as in [11] (see the proof of Proposition 4.2), taking into account that, if $(u, w) \in D(A_1)$, then

$$\sum_{\nu \in \mathcal{N}} \sum_{i \in \mathcal{M}^\nu} \delta_i^\nu \lambda_i u_i(N_\nu) w_i(N_\nu) \geq 0 \ , \quad \nu \in \mathcal{N} \ ,$$

since an equality as (2.9) holds (with no dependence on time).

For the operator A_2 we similarly notice that the transmission conditions in the definition of $D(A_2)$ imply that, for $\phi \in D(A_2)$,

$$\sum_{\nu \in \mathcal{N}} \sum_{i \in \mathcal{M}^\nu} \delta_i^\nu D_i \phi_{i_x}(N_\nu) \phi_i(N_\nu) \leq 0 \ , \quad \nu \in \mathcal{N} \ , \tag{3.9}$$

since an equality as (2.10) holds (with no dependence on time); moreover, thanks to the boundary conditions (3.6), it is easy to prove that

$$(A_2 \phi, \phi) = \sum_{i \in \mathcal{M}} \int_{I_i} \left(D_i \phi_i \phi_{ixx} - b_i \phi_i^2 \right) \ dx \leq 0 \ ,$$

so that A_2 reveals to be a dissipative operator [4]. In order to prove that the operator is m-dissipative, we introduce the bilinear form $a(\phi, \chi) : (H^1(\mathcal{A}))^2 \to \mathbb{R}$

$$a(\phi, \chi) = \sum_{i \in \mathcal{M}} \int_{I_i} (D_i \phi_{ix} \chi_{i_x} + (1 + b_i) \phi_i \chi_i) \ dx$$

$$- \sum_{\nu \in \mathcal{N}} \sum_{i,j \in \mathcal{M}^\nu} \alpha_{ij}^\nu \left(\phi_j(N_\nu) - \phi_i(N_\nu) \right) \chi_i(N_\nu) + \sum_{j \in \mathcal{J}} d_j \phi(e_j) \chi(e_j) \ ;$$

the form is continuous and coercive, hence, by the Lax-Milgram theorem, we know that, for each $\varphi \in L^2(\mathcal{A})$, there exists a unique $\phi \in H^1(\mathcal{A})$ such that, for all $\chi \in H^1(\mathcal{A})$ it holds $a(\phi, \chi) = \displaystyle\sum_{i \in \mathcal{M}} \int_{I_i} \varphi_i \chi_i \ dx$; taking $\chi_i \in H_0^1(I_i)$ for all $i \in \mathcal{M}$, we obtain that $\phi_{i_x} \in H^1(I_i)$, then taking $\chi_i \in C_0^\infty(I_i)$, as in [11], we prove the equality

$$-D_i \phi_{ixx} + (1 + b_i) \phi_i = \varphi_i \quad \text{ a.e. } x \in I_i \ , \text{ for all } i \in \mathcal{M};$$

finally, thanks to suitable choices of $\chi_i(N_\nu), \chi(e_j)$, we obtain that ϕ satisfies the right boundary and transmission conditions to belong to $D(A_2)$. □

Thanks to the above proposition we conclude that the operator A_1 is the generator of a contraction semigroup $\mathcal{T}_1$ in $(L^2(\mathcal{A}))^2$ while the operator A_2 is the generator of a contraction semigroup $\mathcal{T}_2$ in $L^2(\mathcal{A})$.

We notice that

$$\max_{i \in \mathcal{M}} \|\Phi_i(t)\|_{\mathcal{C}^2(I_i)} \ \leq C_\Phi \max_{j \in \mathcal{J}} |\mathcal{P}_j(t)| \ , \quad t \geq 0 \ , \tag{3.10}$$

where C_Φ is a constant depending on the parameters D_i and L_i, and

$$\sup_{[0,T]} \|\Phi'_x(t)\|_2 \ \leq \tilde{C}_\Phi \max_{j \in \mathcal{J}} \|\mathcal{P}_j\|_{H^2(0,T)} \ , \tag{3.11}$$

where $\tilde{C}_\Phi$ is a constant depending on D_i, L_i, the number of external nodes and the Sobolev constant of $(0,T)$.

Lemma 3.1. *Let $T>0$ and let $(u,w,\phi),(\overline{u},\overline{w},\overline{\phi})$ be two solutions to the problem (3.2)-(3.7) such that $\phi,\overline{\phi}\in C([0,T];H^2(\mathcal{A}))\cap C^1([0,T];L^2(\mathcal{A}))$ and $(u,w),(\overline{u},\overline{w})\in C([0,T];(H^1(\mathcal{A}))^2)\cap C^1([0,T];(L^2(\mathcal{A}))^2)$. Then $(u,w,\phi)=(\overline{u},\overline{w},\overline{\phi})$.*

Proof. Using the first and the second equation in (3.2), holding for (u,w,ϕ) and $(\overline{u},\overline{w},\overline{\phi})$, we obtain, for $t\in[0,T]$,

(3.12)
$$\begin{aligned}&\sum_{i\in\mathcal{M}}\left(\frac{1}{2}\|u_i(t)-\overline{u}_i(t)\|_2^2+\frac{1}{2}\|w_i(t)-\overline{w}_i(t)\|_2^2+\beta_i\int_0^t\|w_i(s)-\overline{w}_i(s)\|_2^2ds\right)\\&+\sum_{\nu\in\mathcal{N}}\sum_{i\in\mathcal{M}^\nu}\int_0^t\delta_i^\nu\lambda_i(u_i(N_\nu,s)-(\overline{u}_i(N_\nu,s))(w_i(N_\nu,s)-\overline{w}_i(N_\nu,s))ds\\&=\sum_{i\in\mathcal{M}}\int_0^t\int_{I_i}\Phi_{ix}(x,s)(u_i(x,s)-\overline{u}_i(x,s))(w_i(x,s)-\overline{w}_i(x,s))dx\,ds\\&+\sum_{i\in\mathcal{M}}\int_0^t\int_{I_i}(u_i(x,s)\phi_{ix}(x,s)-\overline{u}_i(x,s)\overline{\phi}_{ix}(x,s))(w_i(x,s)-\overline{w}_i(x,s))dxds.\end{aligned}$$

The last term on the left hand side is non negative, since $u-\overline{u}$ and $w-\overline{w}$ satisfy the conditions (3.7) so that an analogous identity to (2.9) holds. Then (3.12) implies the inequality

$$\begin{aligned}&\|u(t)-\overline{u}(t)\|_2^2+\|w(t)-\overline{w}(t)\|_2^2+\int_0^t\|w(s)-\overline{w}(s)\|_2^2ds\\&\le c_1\int_0^t\left(\|\phi_x(s)-\overline{\phi}_x(s)\|_2^2+\|u(s)-\overline{u}(s)\|_2^2\right)ds,\end{aligned}$$

for $t\in[0,T]$, where c_1 depends on $\sup_{[0,T]}\|u(t)\|_{H^1}$, $\sup_{[0,T]}\|\overline{\phi}_x(t)\|_{H^1}$, $\max_{j\in\mathcal{J}}\|\mathcal{P}_j\|_{L^\infty(0,T)}$ and on the parameters L_i,λ_i,β_i (see(3.10)).

Next, since $\phi-\overline{\phi}$ satisfies the boundary conditions (3.5) and the transmission conditions (3.6), then the following equalities hold, for $j\in\mathcal{J}$ and $\nu\in\mathcal{N}$,

$$\begin{aligned}&\sum_{j\in\mathcal{J}}\eta_jD_{i(j)}(\phi_x(e_j,t)-\overline{\phi}_x(e_j,t))(\phi(e_j,t)-\overline{\phi}(e_j,t))\le 0\ ,\\&\sum_{\nu\in\mathcal{N}}\sum_{i\in\mathcal{M}^\nu}\delta_i^\nu D_i(\phi_{i_x}(N_\nu,t)-\overline{\phi}_{i_x}(N_\nu,t))(\phi_i(N_\nu,t)-\overline{\phi}_i(N_\nu,t))\le 0\ ;\end{aligned}$$

then, using the third equation in (3.2), we easily obtain

$$\|\phi(t)-\overline{\phi}(t)\|_2^2+\int_0^t\left(\|\phi(s)-\overline{\phi}(s)\|_2^2+\|\phi_x(s)-\overline{\phi}_x(s)\|_2^2\right)ds\le c_2\int_0^t\|u(s)-\overline{u}(s)\|_2^2ds\ ,$$

where the constant c_2 depends on the parameters a_i,b_i,D_i.

Then the result follows by Gronwall Lemma. □

In order to prove the local existence theorem we need some preliminary results. Let $f\in C([0,T];H^2(\mathcal{A}))\cap H^1((0,T);H^1(\mathcal{A}))$ and $g=(g_1,g_2)\in C([0,T];(H^1(\mathcal{A}))^2)\cap C^1([0,T];(L^2(\mathcal{A}))^2)$, we set

$$F_{fg}(t)=\{(0\ ,\ (f_{ix}(t)+\Phi_{ix}(t))\,g_{1i}(t)-\beta_ig_{2i}(t))\}_{i\in\mathcal{M}}. \tag{3.13}$$

Lemma 3.2. *Let $T, K > 0$. Let $\overline{g}, g \in C([0,T];(H^1(\mathcal{A}))^2) \cap C^1([0,T];(L^2(\mathcal{A}))^2)$, $\overline{f}, f \in C([0,T];H^2(\mathcal{A})) \cap H^1((0,T);H^1(\mathcal{A}))$ and*

$$\sup_{[0,T]} \|f(t)\|_{H^2} + \left(\int_0^T \|f'_x(t)\|_2^2 dt\right)^{\frac{1}{2}} \ , \ \sup_{[0,T]} \|\overline{f}(t)\|_{H^2} + \left(\int_0^T \|\overline{f}'_x(t)\|_2^2 dt\right)^{\frac{1}{2}} \leq K.$$

Then there exist two positive constants L_1^{TK}, L_2^{TK}, depending on K and T , such that

$$\sup_{[0,T]} \|F_{fg}(t) - F_{\overline{f}\overline{g}}(t)\|_{(L^2)^2} \leq L_1^{TK} \sup_{[0,T]} \|g(t) - \overline{g}(t)\|_{(L^2)^2} + \sup_{[0,T]} \|\overline{g}_1(t)\|_\infty \sup_{[0,T]} \|f_x(t) - \overline{f}_x(t)\|_2 \ ,$$

and

(3.14)
$$\int_0^T \|F'_{fg}(t) - F'_{\overline{f}\overline{g}}(t)\|_{(L^2)^2} dt \leq \sqrt{T} L_2^{TK} \left(\sup_{[0,T]} \|g(t) - \overline{g}(t)\|_{(H^1)^2} + \sqrt{T}\|g'(t) - \overline{g}'(t)\|_{(L^2)^2}\right)$$

$$+\sqrt{T} \sup_{[0,T]} \|\overline{g}_1(t)\|_\infty \left(\int_0^T \|f'_x(t) - \overline{f}'_x(t)\|_2^2 dt\right)^{\frac{1}{2}} + T \sup_{[0,T]} \|\overline{g}'_1(t)\|_2 \sup_{[0,T]} \|f_x(t) - \overline{f}_x(t)\|_\infty.$$

Proof. We have

$$\sup_{[0,T]} \|F_{fg}(t) - F_{\overline{f}\overline{g}}(t)\|_{(L^2)^2} \leq \left(\sup_{[0,T]} \|f_x(t)\|_\infty + \sup_{[0,T]} \|\Phi_x(t)\|_\infty\right) \sup_{[0,T]} \|g_1(t) - \overline{g}_1(t)\|_2$$

$$+\overline{\beta} \sup_{[0,T]} \|g_2(t) - \overline{g}_2(t)\|_2 + \sup_{[0,T]} \|\overline{g}_1(t)\|_\infty \sup_{[0,T]} \|f_x(t) - \overline{f}_x(t)\|_2 \ ,$$

where $\overline{\beta} := \max\{\beta_i\}_{i\in\mathcal{M}}$; then the first inequality in the claim follows with

$$L_1^{TK} = c_S K + C_\Phi \sup_{j\in\mathcal{J}} \|\mathcal{P}_j\|_{L^\infty(0,T)} + \overline{\beta}, \tag{3.15}$$

where c_S depends on Sobolev constants and C_Φ is the constant in (3.10).

As regard to the second inequality we have

$$\int_0^T \|F'_{fg}(t) - F'_{\overline{f}\overline{g}}(t)\|_{(L^2)^2} \leq \left(\int_0^T \|g_1(t) - \overline{g}_1(t)\|_\infty^2 dt\right)^{\frac{1}{2}} \left(\int_0^T \|f'_x(t) + \Phi'_x(t)\|_2^2 dt\right)^{\frac{1}{2}}$$

$$+ \left(\int_0^T \|\overline{g}_1(t)\|_\infty^2 dt\right)^{\frac{1}{2}} \left(\int_0^T \|f'_x(t) - \overline{f}'_x(t)\|_2^2 dt\right)^{\frac{1}{2}} + T\overline{\beta} \sup_{[0,T]} \|g'_2(t) - \overline{g}'_2(t)\|_2$$

$$+T\left(\sup_{[0,T]} \|f_x(t)\|_\infty + \sup_{[0,T]} \|\Phi_x(t)\|_\infty\right) \sup_{[0,T]} \|g'_1(t) - \overline{g}'_1(t)\|_2 +$$

$$+T \sup_{[0,T]} \|\overline{g}'_1(t)\|_2 \sup_{[0,T]} \|f_x(t) - \overline{f}_x(t)\|_\infty$$

$$\leq \sqrt{T} c_{1S} \left(K + \tilde{C}_\Phi \max_{j\in\mathcal{J}} \|\mathcal{P}_j\|_{H^2(0,T)}\right) \sup_{[0,T]} \|g(t) - \overline{g}(t)\|_{(H^1)^2}$$

$$+T(c_{2S}(K + C_\Phi \max_{j\in\mathcal{J}} \|\mathcal{P}_j\|_{H^1(0,T)}) + \overline{\beta}) \sup_{[0,T]} \|g'(t) - \overline{g}'(t)\|_{(L^2)^2}$$

$$+\sqrt{T} \sup_{[0,T]} \|\overline{g}_1(t)\|_\infty \left(\int_0^T \|f'_x(t) - \overline{f}'_x(t)\|_2^2\right)^{\frac{1}{2}} + T \sup_{[0,T]} \|\overline{g}'_1(t)\|_2 \sup_{[0,T]} \|f_x(t) - \overline{f}_x(t)\|_\infty \ ,$$

where C_Φ and $\tilde{C}_\Phi$ are the constants in (3.10) and (3.11) and c_{1S}, c_{2S} depend on Sobolev constants; then, setting

$$L_2^{TK} = (c_{1S} + c_{2S})(K + (C_\Phi + \tilde{C}_\phi) \max_{j\in\mathcal{J}} \|\mathcal{P}_j\|_{H^2(0,T)}) + \overline{\beta},$$

we obtain the second inequality. □

Theorem 3.1. *(Local existence) There exists $T > 0$ such that problem (2.1)-(2.7) has a unique local solution (u, v, ψ),*

$$(u, v) \in C([0,T]; (H^1(\mathcal{A}))^2) \cap C^1([0,T], (L^2(\mathcal{A}))^2),$$

$$\psi \in C([0,T]; H^2(\mathcal{A})) \cap C^1([0,T], L^2(\mathcal{A})) \cap H^1((0,T); H^1(\mathcal{A})) \ .$$

Proof. We set $\overline{a} := \max_{i\in\mathcal{M}} a_i$, $\underline{b} := \min_{i\in\mathcal{M}} b_i$, $\overline{b} := \max_{i\in\mathcal{M}} b_i$ and $\underline{D} := \min_{i\in\mathcal{M}} D_i$.

We consider the problem (3.2)-(3.7) and we set $\mathcal{U}_0 := (u_0, w_0)$ and

$$Z_i(t) = (Z_{1i}(t), Z_{2i}(t))) := (-\lambda_i \mathcal{V}_{ix}, -\mathcal{V}_{it} - \beta_i \mathcal{V}_i) \ ,$$

$$Z_{3i}(t) := -\Phi_{it} + D_i \Phi_{ixx} - b_i \Phi_i.$$

We fix $\overline{T} > 0$; it is readily seen that there exists a constant $\overline{C}_\Phi$ depending on D_i, L_i, such that

$$\|Z_3(t)\|_2 \le \overline{C}_\Phi \max_{j\in\mathcal{J}} \left(|\mathcal{P}_j(t)| + |\mathcal{P}_j'(t)|\right) \ , \quad t \in [0, \overline{T}] \ , \tag{3.16}$$

$$\|Z_3\|_{H^1((0,\overline{T}); L^2(\mathcal{A}))} \le \overline{C}_\Phi \max_{j\in\mathcal{J}} \|\mathcal{P}_j\|_{H^2(0,\overline{T})}, \tag{3.17}$$

and there exists a constant $C_\mathcal{V}$, dependind on λ_i, L_i, such that

$$\|Z(t)\|_{(L^2)^2} \le C_\mathcal{V} \ \max_{j\in\mathcal{J}} \left(|\mathcal{W}_j(t)| + |\mathcal{W}_j'(t)|\right) \ , \quad t \in [0, \overline{T}] \ , \tag{3.18}$$

$$\|Z\|_{W^{1,1}((0,\overline{T}); (L^2(\mathcal{A}))^2)} \le C_\mathcal{V} \max_{j\in\mathcal{J}} \|\mathcal{W}_j\|_{W^{2,1}(0,\overline{T})} \ . \tag{3.19}$$

Now we introduce the following quantities, M, K_1, K_2, K, depending on the boundary and initial data and on $\overline{T}$,

$$M \ge 2 \left(\left(1 + \|\phi_{0x}\|_\infty + C_\Phi \max_{j\in\mathcal{J}} \|\mathcal{P}_j\|_{L^\infty(0,\overline{T})} + \overline{\beta} \right) \|\mathcal{U}_0\|_{D(A_1)} \right.$$

$$\left. + C_\mathcal{V} \left(\max_{j\in\mathcal{J}} \left(|\mathcal{W}_j(0)| + |\mathcal{W}_j'(0)|\right) + \max_{j\in\mathcal{J}} \|\mathcal{W}_j\|_{W^{2,1}(0,\overline{T})} \right) \right),$$

where C_Φ is the constant in (3.10),

$$K_1 = \|\phi_0\|_{D(A_2)} + 2\overline{a}\|u_0\|_2 + 3\overline{T}\overline{a}M$$

$$+ 2\overline{C}_\Phi \left(\max_{j\in\mathcal{J}} \left(|\mathcal{P}_j(0)| + |\mathcal{P}_j'(0)|\right) + \sqrt{\overline{T}} \max_{j\in\mathcal{J}} \|\mathcal{P}_j\|_{H^2(0,\overline{T})} \right),$$

$$K_2 = \frac{1}{2\underline{D}} \left(\|\phi_0\|_{D(A_2)} + \overline{a}\|u_0\|_2 + \overline{C}_\Phi \max_{j\in\mathcal{J}} \left(|\mathcal{P}_j(0)| + |\mathcal{P}_j'(0)|\right) \right)^2$$

$$+ \frac{1}{2\underline{Db}} \left(\overline{C}_\Phi^2 \max_{j\in\mathcal{J}} \|\mathcal{P}_j\|^2_{H^2(0,\overline{T}))} + \overline{T}\overline{a}^2 M^2 \right),$$

$$K \ge \left(1 + \frac{1 + \overline{b}}{\underline{D}} \right) K_1 + \sqrt{K_2} \ .$$

Let $L_1^{\overline{T}K}, L_2^{\overline{T}K}$ be the constants in Lemma 3.2 and let $T \leq \overline{T}$. Let consider the set

(3.20)
$$B_{MK} = \left\{ \begin{array}{l} \mathcal{U} = (u,w) \in (C([0,T];(H^1(\mathcal{A}))^2) \cap C^1([0,T];(L^2(\mathcal{A}))^2) \ , \\ \\ \phi \in C([0,T];H^2(\mathcal{A})) \cap C^1([0,T];L^2(\mathcal{A})) \cap H^1((0,T);H^1(\mathcal{A})) \ : \\ \\ \mathcal{U}(0) = (u_0, w_0), \quad \phi(0) = \phi_0 \ , \\ \\ \sup_{[0,T]} \|\mathcal{U}(t)\|_{(L^2)^2} \ , \ \sup_{[0,T]} \|\mathcal{U}'(t)\|_{(L^2)^2} \leq M \ , \\ \\ \sup_{[0,T]} \|A_1\mathcal{U}(t)\|_{(L^2)^2} \leq (1 + L_1^{\overline{T}K})M \\ \\ \qquad\qquad +C_{\mathcal{V}} \max_{j\in\mathcal{J}} \left(\|\mathcal{W}_j\|_{L^\infty(0,\overline{T})} + \|\mathcal{W}_j'\|_{L^\infty(0,\overline{T})} \right) \ , \\ \\ \sup_{[0,T]} \|\phi(t)\|_{H^2} \leq \left(1 + \dfrac{1+\bar{b}}{\underline{D}}\right) K_1, \quad \displaystyle\int_0^T \|\phi_x'(t)\|_2^2 \, dt \leq K_2 \end{array} \right\} ;$$

we equip B_{MK} with the metric generated by the norms of the involved spaces, obtaining a complete metric space .

We define a map G on B_{MK} in the following way: given $(\mathcal{U}^I, \phi^I) = (u^I, w^I, \phi^I) \in B_{MK}$, then $(\mathcal{U}, \phi) = G(\mathcal{U}^I, \phi^I)$ is such that ϕ is the solution to

$$\text{(3.21)} \qquad \left\{ \begin{array}{l} \phi \in C([0,T];D(A_2)) \cap C^1([0,T];L^2(\mathcal{A})) \\ \\ \phi'(t) = A_2\phi(t) + au^I(t) + Z_3(t) \ , \quad t \in [0,T] \ , \\ \\ \phi(0) = \phi_0 \in D(A_2) \ , \end{array} \right.$$

where $au^I(t) := \{a_i u_i^I(t)\}_{i\in\mathcal{M}}$, and $\mathcal{U}$ is the solution to

$$\text{(3.22)} \qquad \left\{ \begin{array}{l} \mathcal{U} \in C([0,T];D(A_1)) \cap C^1([0,T];(L^2(\mathcal{A}))^2) \\ \\ \mathcal{U}'(t) = A_1\mathcal{U}(t) + F_{\phi\mathcal{U}^I}(t) + Z(t) \ , \quad t \in [0,T] \ , \\ \\ \mathcal{U}(0) = (u_0, w_0) \in D(A_1) \ , \end{array} \right.$$

where we used the notation (3.13).

First we prove that G is well defined and $G(B_{MK}) \subseteq B_{MK}$. Since $\mathcal{U}^I \in C([0,T];(H^1(\mathcal{A}))^2) \cap C^1([0,T];(L^2(\mathcal{A}))^2)$ and $Z_3 \in H^1((0,T);L^2(\mathcal{A}))$ we can use the theory for nonhomogeneous problems in [4] and we infer the existence and uniqueness of a solution ϕ to problem (3.21) given by

$$\phi(t) = \mathcal{T}_2(t)\phi_0 + \int_0^t \mathcal{T}_2(t-s)(au^I(s) + Z_3(s))ds \ ,$$

see[4]. If we set

$$\mathcal{F}(t) := \int_0^t \mathcal{T}_2(t-s)(au^I(s) + Z_3(s)) \ ds \ ,$$

the assumption on u^I and Z_3 imply that $\mathcal{F} \in C^1([0,T];L^2(\mathcal{A})) \cap C([0,T];D(A_2))$, $A_2\mathcal{F}(t) = \mathcal{F}'(t) - au^I(t) - Z_3(t)$ for all $t \in [0,T]$ (see [4]) and

$$\mathcal{F}'(t) = \int_0^t \mathcal{T}_2(t-s)(au^{I'}(s) + Z_3'(s)) \ ds + \mathcal{T}_2(t)(au^I(0) + Z_3(0)) \ .$$

Then we have

$$
\begin{aligned}
&\|\phi(t)\|_{D(A_2)} \le \|\phi_0\|_{D(A_2)} + \|\mathcal{F}(t)\|_2 + \|A_2(\mathcal{F}(t)\|_2 \\
&\le \|\phi_0\|_{D(A_2)} + \overline{a}\|u^I(t) - \mathcal{T}_2(t)u^I(0)\|_2 + \|Z_3(t) - \mathcal{T}_2(t)Z_3(0)\|_2 \\
&+ \int_0^t \left(\overline{a}\|u^I(s)\|_2 + \|Z_3(s)\|_2 + \overline{a}\|(u^{I'}(s)\|_2 + \|Z_3'(s)\|_2\right) ds \\
&\le \|\phi_0\|_{D(A_2)} + \overline{a}T\left(2\sup_{[0,T]}\|u^{I'}(s)\|_2 + \sup_{[0,T]}\|u^I(s)\|_2\right) \\
&+2\overline{a}\|u(0)\|_2 + 2\|Z_3(0)\|_2 + 2\|Z_3\|_{W^{11}((0,T);L^2(\mathcal{A}))} \le K_1 \ ,
\end{aligned}
\tag{3.23}
$$

whence, using (3.16) and (3.17), the first inequality for ϕ in B_{MK} follows.

Moreover, let $0 < t_1 < t_2 < T$ and let $\Delta^h f(t) := f(t+h) - f(t)$; using the equation in (3.21) we can write

$$
\int_{t_1}^{t_2}\int_{I_i} \left((\Delta^h\phi_i)_t\Delta^h\phi_i - D_i(\Delta^h\phi_i)_{xx}\Delta^h\phi_i\right) dxdt =
$$

$$
\int_{t_1}^{t_2}\int_{I_i} \left(a_i\Delta^h u_i^I\Delta^h\phi_i - b_i(\Delta^h\phi_i)^2 + \Delta^h Z_{3i}\Delta^h\phi_i\right) dxdt \ ;
$$

then we have

$$
\begin{aligned}
&\sum_{i\in\mathcal{M}}\left(\int_{I_i}\frac{(\Delta^h\phi_i(t_2))^2}{2}dx + \int_{t_1}^{t_2}\int_{I_i} D_i(\Delta^h\phi_{i_x})^2 \ dxdt\right) \\
&\le \sum_{\nu\in\mathcal{N}}\sum_{i\in\mathcal{M}^\nu}\int_{t_1}^{t_2}\delta_\nu^i D_i(\Delta^h\phi_{i_x})(\Delta^h\phi_i)(N_\nu,t)dt \\
&+\sum_{j\in\mathcal{J}}\int_{t_1}^{t_2}\eta_j D_{i(j)}(\Delta^h\phi_x)(\Delta^h\phi)(e_j,t)dt \\
&+\sum_{i\in\mathcal{M}}\int_{I_i}\frac{(\Delta^h\phi_i(t_1))^2}{2}dx + \sum_{i\in\mathcal{M}}\int_{t_1}^{t_2}\int_{I_i}\left(\frac{(a_i\Delta^h u_i^I)^2 + (\Delta^h Z_{3i})^2}{2b_i}\right) dxdt \ .
\end{aligned}
\tag{3.24}
$$

Since the first and the second terms on the right hand side are non positive and $u^I, \phi \in C^1([0,T];L^2(\mathcal{A}))$ and $Z_3 \in H^1(0,T);L^2(\mathcal{A}))$, the above inequality implies that $\phi \in H^1((0,T);H^1(\mathcal{A}))$; moreover, for $h \to 0$ and then $t_1 \to 0$, $t_2 \to T$, we have

$$
\begin{aligned}
\sum_{i\in\mathcal{M}}\int_0^T\int_{I_i} D_i\phi_{ixt}^2 \ dxdt &\le \frac{1}{2}\left(\|\phi_0\|_{D(A_2)} + \overline{a}\|u_0\|_2 + \|Z_3(0)\|_2\right)^2 \\
&+\frac{1}{2\underline{b}}\left(T\overline{a}^2\sup_{[0,T]}\|u^{I'}(t)\|_2^2 + \|Z_3\|^2_{H^1((0,T);L^2(\mathcal{A}))}\right) \ ,
\end{aligned}
$$

so that ϕ satisfies the last condition in (3.20).

Now we consider the problem (3.22) and we set $F(t) := F_{\phi\mathcal{U}^I}(t)$. We know that $Z \in W^{1,1}((0,T);(L^2(\mathcal{A}))^2)$ and, from Lemma 3.2, $F \in W^{1,1}((0,T);(L^2(\mathcal{A}))^2)$, then there exists a unique solution $\mathcal{U} \in$ given by

$$
\mathcal{U}(t) = \mathcal{T}_1(t)\mathcal{U}_0 + \int_0^t \mathcal{T}_1(t-s)(F(s)+Z(s))ds \tag{3.25}
$$

see [4]; moreover, using Lemma 3.2, choosing T small such that $T \le (2L_1^{TK})^{-1}$ (see (3.15)) and using (3.19) and the condition on M, we obtain the following inequality

$$\|\mathcal{U}(t)\|_{(L^2)^2} \le \|\mathcal{U}_0\|_{(L^2)^2} + \int_0^t \|F(s) + Z(s)\|_{(L^2)^2} ds$$
$$\le \|\mathcal{U}_0\|_{(L^2)^2} + TL_1^{TK} \sup_{[0,T]} \|\mathcal{U}^I(t)\|_{(L^2)^2} + \|Z\|_{L^1((0,T);(L^2)^2)} \le M \ ;$$

then we can argue as we did before for ϕ, using [4] and (3.14), to obtain

(3.26)
$$\|\mathcal{U}'(t)\|_{(L^2)^2} \le \|A_1\mathcal{U}_0\|_{(L^2)^2} + \|F(0) + Z(0)\|_{(L^2)^2} + \int_0^t \|F'(s) + Z'(s)\|_{(L^2)^2} ds$$
$$\le \|A_1\mathcal{U}_0\|_{(L^2)^2} + (\|\phi_x(0)\|_\infty + \|\Phi_x(0)\|_\infty + \overline{\beta})\|\mathcal{U}_0\|_{(L^2)^2} + \|Z(0)\|_{(L^2)^2}$$
$$+\sqrt{T}L_2^{TK}\left(\sup_{[0,T]}\|\mathcal{U}^I(t)\|_{(H^1)^2} + \sqrt{T}\sup_{[0,T]}\|\mathcal{U}^{I'}(t)\|_{(L^2)^2}\right) + \|Z\|_{W^{1,1}((0,T);(L^2)^2)} \ ,$$

where L_2^{TK} is given at the end of the proof of Lemma 3.2; setting $\underline{\lambda} := \min_{i\in\mathcal{M}} \lambda_i$, using (3.10), (3.18), (3.19)and the condition on the quantity M, the previous inequality implies

$$\sup_{[0,T]}\|\mathcal{U}'(t)\|_{(L^2)^2} \le \frac{M}{2}$$
$$+\sqrt{T}L_2^{TK}\left(\left(1 + \frac{1 + L_1^{TK}}{\underline{\lambda}} + \sqrt{T}\right)M + \frac{C_{\mathcal{V}} \max_{j\in\mathcal{J}}\left(\|\mathcal{W}_j\|_{L^\infty(0,T)} + \|\mathcal{W}_j'\|_{L^\infty(0,T)}\right)}{\underline{\lambda}}\right) \ ,$$

so that, choosing T sufficiently small, we have

$$\sup_{[0,T]}\|\mathcal{U}'(t)\|_{(L^2)^2} \le M.$$

Finally, using Lemma 3.2 and (3.18)

$$\sup_{[0,T]}\|A_1\mathcal{U}(t)\|_{(L^2)^2} \le \sup_{[0,T]}\|\mathcal{U}'(t)\|_{(L^2)^2} + \sup_{[0,T]}\|F(t) + Z(t)\|_{(L^2)^2}$$
$$\le (1 + L_1^{TK})M + C_{\mathcal{V}} \max_{j\in\mathcal{J}}\left(\|\mathcal{W}_j\|_{L^\infty(0,T)} + \|\mathcal{W}_j'\|_{L^\infty(0,T)}\right).$$

The above computations show that $(\mathcal{U}, \phi) = G(\mathcal{U}^I, \phi^I) \in B_{MK}$ if T is small enough.

Now we are going to prove that G is a contraction mapping on B_{MK}, for small values of T.

Let

$$(\mathcal{U}^I, \phi^I) = (u^I, w^I, \phi^I) \ , \ (\overline{\mathcal{U}}^I, \overline{\phi}^I) = (\overline{u}^I, \overline{w}^I, \overline{\phi}^I) \in B_{MK} \ ,$$
$$(\mathcal{U}, \phi) = G(\mathcal{U}^I, \phi^I) \ , \ (\overline{\mathcal{U}}, \overline{\phi}) = G(\overline{\mathcal{U}}^I, \overline{\phi}^I) \ ,$$
$$\overline{F}(t) = F_{\overline{\phi}\,\overline{\mathcal{U}}^I}(t) \ , \ F(t) = F_{\phi\mathcal{U}^I}(t) \ ;$$

Then, arguing as in (3.23) and in (3.24), we have

$$\sup_{[0,T]}\|\phi(t) - \overline{\phi}(t)\|_{D(A_2)} \le T\overline{a}\left(2\sup_{[0,T]}\|\mathcal{U}^{I'}(t) - \overline{\mathcal{U}}^{I'}(t)\|_{(L^2)^2} + \sup_{[0,T]}\|\mathcal{U}^I(t) - \overline{\mathcal{U}}^I(t)\|_{(L^2)^2}\right),$$

$$\frac{1}{2}\sup_{[0,T]}\|\phi'(t) - \overline{\phi}'(t)\|_2^2 + \underline{D}\int_0^T \|\phi_{xt}(t) - \overline{\phi}_{xt}(t)\|_2^2 dt \le \frac{T\overline{a}^2}{2\underline{b}}\sup_{[0,T]}\|\mathcal{U}^{I'}(t) - \overline{\mathcal{U}}^{I'}(t)\|_{(L^2)^2}^2 \ .$$

Then, using (3.25) and Lemma 3.2, we obtain the inequality

$$\sup_{[0,T]} \|\overline{\mathcal{U}}(t) - \mathcal{U}(t)\|_{(L^2)^2} \leq \sup_{[0,T]} \left\| \int_0^t \mathcal{T}_1(t-s)(\overline{F}(s) - F(s))\, ds \right\|_{(L^2)^2}$$

$$\leq T L_1^{TK} \sup_{[0,T]} \|\overline{\mathcal{U}}^I(t) - \mathcal{U}^I(t)\|_{(L^2)^2} + T c_S \|\overline{\mathcal{U}}^I(t)\|_{(H^1)^2} \sup_{[0,T]} \|\overline{\phi}_x(t) - \phi_x(t)\|_2 \ ,$$

where c_S is a Sobolev constant; moreover, arguing as in (3.26) and using (3.14)

$$\sup_{[0,T]} \|\overline{\mathcal{U}}'(t) - \mathcal{U}'(t)\|_{(L^2)^2} \leq \int_0^T \|F'(t) - \overline{F}'(t)\|_{(L^2)^2} dt$$

$$\leq \ \sqrt{T} L_2^{TK} \|\overline{\mathcal{U}}^I - \mathcal{U}^I\|_{C([0,T];(H^1)^2)} + \sqrt{T} c_S \|\overline{\mathcal{U}}^I(t)\|_{(H^1)^2} \|\phi_x - \overline{\phi}_x\|_{H^1((0,T);L^2)}$$

$$+T \left(L_2^{TK} \|\overline{\mathcal{U}}^I - \mathcal{U}^I\|_{C^1[0,T];(L^2)^2)} + c_S \, \|\overline{\mathcal{U}}^{I'}(t)\|_{(H^1)^2} \, \|\phi - \overline{\phi}\|_{C([0,T];H^2)} \right) \ .$$

Finally, for $t \leq T$,

$$\|A_1(\mathcal{U}(t)) - A_1(\overline{\mathcal{U}}(t))\|_{(L^2)^2} \leq \|\mathcal{U}'(t) - \overline{\mathcal{U}}'(t)\|_{(L^2)^2} + \|F(t) - \overline{F}(t)\|_{(L^2)^2}$$

$$\leq \|\mathcal{U}'(t) - \overline{\mathcal{U}}'(t)\|_{(L^2)^2} + L_1^{TK} T \|\overline{\mathcal{U}}^{I'} - \mathcal{U}^{I'}\|_{C([0,T];(L^2)^2)}$$

$$+ c_S \|\overline{\mathcal{U}}^I(t)\|_{(H^1)^2} \|\overline{\phi}_x(t) - \phi_x(t)\|_2.$$

The function $\overline{\mathcal{U}}^I$ belongs to B_{MK}, so it is possible choosing T sufficiently small, depending on the data and on the parameters of the problem, to make G a contraction mapping in B_{MK} and its unique fixed point in B_{MK} is the local solution to problem (3.2)-(3.7). The existence and uniqueness of local solutions for problem (2.1)-(2.7) follow from Lemma 3.1 and from (3.1). □

4. **A priori estimates**

In this section we assume the condition (2.8) . Moreover, we assume that there exists a stationary solution to problem (2.1),(2.5), (2.6), $(U(x), V(x), \Psi(x)) \in (H^1(\mathcal{A}))^2 \times H^2(\mathcal{A})$, verifying the boundary conditions

$$\eta_j \lambda_{i(j)} V(e_j) = W_j \ , \quad \eta_j D_{i(j)} \Psi'(e_j) + d_j \Psi(e_j) = P_j \ , \quad j \in \mathcal{J} \ ; \tag{4.1}$$

we notice that, integrating the first equation in (2.1) and using the conservation of the flux (2.12) at each inner node, it turns out to be necessary that $\sum_{j\in\mathcal{J}} W_j = 0$.

We set

$$\mu_s = \sum_{i\in\mathcal{M}} \int_{I_i} U_i(x) dx \ . \tag{4.2}$$

Let $(\overline{u}(x,t), \overline{v}(x,t), \overline{\psi}(x,t))$ be the local solution in Theorem 3.1 to problem (2.1)-(2.7) corresponding to initial data $(\overline{u}_0(x), \overline{v}_0(x), \overline{\psi}_0(x))$ and boundary data $\overline{\mathcal{W}}(t)$, $\overline{\mathcal{P}}(t)$; let $\overline{\mu}(0) := \sum_{i\in\mathcal{M}} \int_{I_i} \overline{u}_0(x) dx$ be the initial mass so that, integrating the first equation in (2.1) and using (2.12), we have the following expression for the evolution of the mass

$$\overline{\mu}(t) := \sum_{i\in\mathcal{M}} \int_{I_i} \overline{u}(x) dx = \overline{\mu}(0) - \sum_{j\in\mathcal{J}} \int_0^t \overline{\mathcal{W}}_j(s) ds \ .$$

Due to the assumption of existence of the stationary solution $(U(x), V(x), \Psi(x))$, the triple

$$(u(x,t), v(x,t), \psi(x,t)) = (\overline{u}(x,t) - U(x), \overline{v}(x,t) - V(x), \overline{\psi}(x,t) - \Psi(x))$$

is the local solution of the following problem

$$\begin{cases} u_{it} + \lambda_i v_{ix} = 0 \ , \\ v_{it} + \lambda_i u_{ix} = u_i \psi_{ix} - \beta_i v_i + u_i \Psi_i' + U_i \psi_{ix} \ , \qquad t \in [0,T], \ x \in I_i, \ i \in \mathcal{M}, \\ \psi_{it} = D_i \psi_{ixx} + a_i u_i - b_i \psi_i \ , \\ u_i(x,0) = u_{0i}(x) := \overline{u}_{0i}(x) - U_i(x), \ v_i(x,0) = v_{0i}(x) := \overline{v}_{0i}(x) - V_i(x), \\ \psi_i(x,0) = \psi_{0i}(x) := \overline{\psi}_{0i}(x) - \Psi_i(x) \ , \qquad x \in I_i \ , \ i \in \mathcal{M} \\ \eta_j \lambda_{i(j)} v(e_j, t) = \mathcal{W}_j(t) := \overline{\mathcal{W}}_j(t) - W_j \ , \qquad t \in [0,T] \ , \ j \in \mathcal{J} \ , \\ \eta_j D_{i(j)} \psi_x(e_j, t) + d_j \psi(e_j, t) = \mathcal{P}_j(t) := \overline{\mathcal{P}}_j(t) - P_j \ , \qquad t \in [0,T] \ , \ j \in \mathcal{J} \ , \end{cases} \tag{4.3}$$

complemented with the transmission conditions (2.6) and (2.5). We set

$$\mu(t) := \sum_{i \in \mathcal{M}} \int_{I_i} u_i(x,t) dx = \overline{\mu}(t) - \mu_s \ . \tag{4.4}$$

We are going to prove some a priori estimates for the solution to the above problem, assuming suitable conditions on the data. If the stationary solution and the data in (4.3) are small in some suitable norms, then these estimates provide a global existence result for problem (4.3). In this way, after the proof of the real existence of stationary solutions in the next section, the results in the following propositions will be the tools to prove the existence of global solutions to problem (2.1)-(2.7). To this aim we assume some conditions on the functions $\mathcal{W}_j(t) = \overline{\mathcal{W}}_j(t) - W_j$, $\mathcal{P} = \overline{\mathcal{P}}_j(t) - P_j$ and $\mu(t) = \overline{\mu}(t) - \mu_s$; precisely, we assume that the data $\overline{u}_0$, $\overline{\mathcal{W}}_j$, $\overline{\mathcal{P}}_j$, W_j and P_j, and $U(x)$ are such that

$$\mathcal{W}_j \in W^{2,1}((0,+\infty)) \ , \quad \text{for all } j \in \mathcal{J} \ , \tag{4.5}$$

$$\mathcal{P}_j \in H^1((0,+\infty)) \cap H^2((0,T)) \ \text{ for all } T > 0 \ , \quad \text{for all } j \in \mathcal{J} \ , \tag{4.6}$$

$$\mu \in L^2((0,+\infty)) \ . \tag{4.7}$$

The a priori estimates we are going to obtain are the tools to prove the uniform (in time) boundedness of some norms of (u, v, ψ) when the data are small. Similar results are proved in [11] in the case of homogeneous boundary conditions, when $\frac{a_i}{b_i}$ does not change with i, and some of the proofs have minor differences from the ones in that paper. However, here we prove the result in Proposition 4.1, where $\|u_i(\cdot,t)\|_\infty$ is estimated by a linear combination of $|\mu(t)|$, $\|u_{ix}(\cdot,t)\|_1$ and $\|v_i(\cdot,t)\|_\infty$, $i \in \mathcal{M}$; the use of this result, which is necessary in treating the boundary terms in some steps in the proofs of the following propositions, is based on condition (4.7) and allows to discard the condition that $\frac{a_i}{b_i}$ is constant for $i \in \mathcal{M}$, assumed in [11].

First we remark that the assumption (2.8) implies that the condition (2.6) can be rewritten as follows

$$u_j(N_\nu, t) = u_{k_\nu}(N_\nu, t) + \sum_{i \in \mathcal{M}^\nu} \gamma_{ij}^\nu v_i(N_\nu, t) \qquad \text{for all } j \in \mathcal{M}^\nu, \nu \in \mathcal{N} \ , \tag{4.8}$$

where γ_{ij}^{ν} are suitable coefficients depending on σ_{ij}^{ν} (see the proof of Lemma 5.9 in [11]).

Proposition 4.1. *Let (2.8) hold; let* $u, v \in C([0,T]; L^2(\mathcal{A})) \cap C^1([0,T]; H^1(\mathcal{A}))$ *satisfying the conditions (2.6), let* $\mu(t) := \sum_{i\in\mathcal{M}} \int_{I_i} u_i(x,t)dx$ *and* $\gamma := \max\{|\gamma_{ij}^{\nu}|\}$*; then, for all* $i \in \mathcal{M}$, $0 \le t \le T$,

$$\|u_i(\cdot,t)\|_\infty \le \left(\sum_{j\in\mathcal{M}} L_j\right)^{-1} |\mu(t)| + 2\sum_{j\in\mathcal{M}} \left(2\|u_{jx}(\cdot,t)\|_1 + 3\gamma\|v_j(\cdot,t)\|_\infty\right) .$$

Proof. We consider two consecutive nodes, N_ν and N_h, and let I_l be the arc linking them. For all $x \in I_l$, $t \in [0,T]$

$$u_l(x,t) = u_l(N_\nu,t) + \int_{N_\nu}^{x} u_{l_y}(y,t)dy = u_l(N_h,t) + \int_{N_h}^{x} u_{l_y}(y,t)dy$$

(by N_ν we mean 0 if N_ν is the starting node of I_l and we mean L_l otherwise).

Let k_ν, k_h the indexes corresponding respectively to the nodes N_ν and N_h in condition (2.8); then, using (4.8), we can write for all $t \in [0,T]$,

$$u_{k_\nu}(N_\nu,t) + \sum_{j\in\mathcal{M}^\nu} \gamma_{jl}^{\nu} v_j(N_\nu,t) + \int_{N_\nu}^{x} u_{l_y}(y,t)dy$$

$$= u_{k_h}(N_h,t) + \sum_{j\in\mathcal{M}^h} \gamma_{jl}^{h} v_j(N_h,t) + \int_{N_h}^{x} u_{l_y}(y,t)dy \;;$$

then

$$u_{k_\nu}(N_\nu,t) = u_{k_h}(N_h,t) - \sum_{j\in\mathcal{M}^\nu} \gamma_{jl}^{\nu} v_j(N_\nu,t) + \sum_{j\in\mathcal{M}^h} \gamma_{jl}^{h} v_j(N_h,t) + \int_{N_h}^{N_\nu} u_{l_y}(y,t)dy.$$

Since each node of the network is connected with the node N_1, the above relation implies that, for all $p \in \mathcal{N}$, we can express the value of $u_{k_p}(N_p,t)$ in the following way

$$u_{k_p}(N_p,t) = u_{k_1}(N_1,t) + \Gamma_p(t) \;, \tag{4.9}$$

where k_p and k_1 are the indexes in condition (2.8) corresponding respectively to N_p and N_1, and $\Gamma_p(t)$ is a quantity which can be estimated as follows

$$|\Gamma_p(t)| \le \sum_{j\in\mathcal{M}} \left(2\gamma\|v_j(t)\|_\infty + \|u_{j_x}(t)\|_1\right) \;. \tag{4.10}$$

Arguing as in the previous computations, thanks to conditions (4.8) and (4.9), for each $p \in \mathcal{N}$ and each $i \in \mathcal{M}^p$ we can write, for all $x \in I_i$ and $t \in [0,T]$,

$$u_i(x,t) = u_{k_1}(N_1,t) + \Gamma_p(t) + \sum_{j\in\mathcal{M}^p} \gamma_{ji}^{p} v_j(N_p,t) + \int_{N_p}^{x} u_{iy}(y,t)dy \tag{4.11}$$

which implies, for all $i \in \mathcal{M}^p$,

$$\int_{I_i} u_i(x,t)dx = L_i \left(u_{k_1}(N_1,t) + \Gamma_p(t) + \sum_{j\in\mathcal{M}^p} \gamma_{ij}^{p} v_j(N_p,t) \right) + \int_{I_i}\int_{N_p}^{x} u_{iy}(y,t)dydx \;.$$

Letting p vary in $\mathcal{N}$, we can obtain an expression as the above one for each $i \in \mathcal{M}$, so that, after summing for $i \in \mathcal{M}$ and using (4.10), we get

$$\left(\sum_{i\in\mathcal{M}} L_i\right) |u_{k_1}(N_1,t)| \le |\mu(t)| + \left(\sum_{i\in\mathcal{M}} L_i\right) \sum_{i\in\mathcal{M}} \left(3\gamma\|v_i(t)\|_\infty + 2\|u_{ix}(t)\|_1\right) \;.$$

Using the above inequality and (4.10) in (4.11) we obtain the claim. □

In the following we use c_S to denote positive quantities depending on Sobolev constants.

Proposition 4.2. *Let (2.8) hold and let* (u, v, ψ) *be the local solution to (4.3)-(4.7),(2.5)- (2.7); then*

$$\sum_{i\in\mathcal{M}}\left(\sup_{[0,T]}\|u_i(t)\|_2^2+\sup_{[0,T]}\|v_i(t)\|_2^2+2\beta_i\int_0^T\|v_i(t)\|_2^2dt\right)$$

$$\leq \|u_0\|_2^2+\|v_0\|_2^2+2c_S\sum_{j\in\mathcal{J}}\sup_{[0,T]}\|u_{i(j)}(t)\|_{H^1}\|\mathcal{W}_j\|_{L^1(0,+\infty)}$$

$$+\sum_{i\in\mathcal{M}}\|U_i\|_\infty\int_0^T\left(\|\psi_{ix}(t)\|_2^2+\|v_i(t)\|_2^2\right)dt$$

$$+c_1\left(\sum_{i\in\mathcal{M}}\|\Psi_i'\|_\infty\right)\left(\int_0^{+\infty}|\mu(t)|^2dt+\int_0^T(\|u_x(t)\|_2^2+\|v(t)\|_{H^1}^2)dt\right)$$

$$+c_S\sum_{i\in\mathcal{M}}\sup_{[0,T]}\|u_i(t)\|_{H^1}\int_0^T\left(\|\psi_{ix}(t)\|_2^2+\|v_i(t)\|_2^2\right)dt$$

where c_1 *is a suitable constant. depending on Sobolev constants, on the parameters* L_i $(i\in\mathcal{M})$ *and on the quantity* γ *in Proposition 4.1.*

Proof. We multiply the first equation in (4.3) by u_i, the second one by v_i and we sum them; after summing up for $i \in \mathcal{M}$, we obtain the claim, taking into account that from Proposition 4.1 we have, for each $i \in \mathcal{M}$,

$$\int_0^T\int_{I_i}u_i^2(x,t)dxdt\leq c\int_0^T\left(|\mu(t)|^2+\|u_x(t)\|_2^2+\|v(t)\|_{H^1}^2\right)dt$$

where c is a suitable constant depending on L_j $(j \in \mathcal{M})$, γ and Sobolev constants, and that condition (2.9) holds, so that the sum of the terms at nodes is non positive.

□

Proposition 4.3. *Let* (u, v, ψ) *be the local solution to (4.3)-(4.7),(2.5)-(2.7); then*

$$\sum_{i\in\mathcal{M}}\left(\sup_{[0,T]}\|v_{ix}(t)\|_2^2+\sup_{[0,T]}\|v_{it}(t)\|_2^2+2\beta_i\int_0^T\|v_{it}(t)\|_2^2dt\right)$$

$$\leq\|v_{0x}\|_2^2+\|v_t(0)\|_2^2+2c_S\sum_{j\in\mathcal{J}}\|\mathcal{W}_j''\|_{L^1(0,+\infty)}\sup_{[0,T]}\|u_{i(j)}(t)\|_{H^1}$$

$$+2c_S\sum_{j\in\mathcal{J}}\left(\|\mathcal{W}_j'\|_{L^\infty(0,+\infty)}\sup_{[0,T]}\|u_{i(j)}(t)\|_{H^1}+|\mathcal{W}_j'(0)|\|u_{i(j)}(0)\|_{H^1}\right)$$

$$+\sum_{i\in\mathcal{M}}\left(c_S\sup_{[0,T]}\|u_i(t)\|_{H^1}+\|U_i\|_\infty\right)\int_0^T\left(\|\psi_{ixt}(t)\|_2^2+\|v_{it}(t)\|_2^2\right)dt$$

$$+\sum_{i\in\mathcal{M}}\left(c_S\sup_{[0,T]}\|\psi_{ix}(t)\|_{H^1}+\|\Psi_i'\|_\infty\right)\int_0^T\left(\|v_{it}(t)\|_2^2+\|v_{ix}(t)\|_2^2\right)dt\ .$$

Proof. We set $\Delta^h f(x,t) = f(x,t+h) - f(x,t)$; using the first two equations in (4.3), for $0 < \delta < \tau < T$ and $|h| \le \min\{\delta, T-\tau\}$, we obtain

$$\begin{aligned}(4.12)\quad &\int_\delta^\tau \int_{I_i} \left(\frac{(\Delta^h u_i)^2 + (\Delta^h v_i)^2}{2}\right)_t dx\,dt + \int_\delta^\tau \int_{I_i} \lambda_i \left(\Delta^h v_i \Delta^h u_i\right)_x \, dx\,dt \\ &= \int_\delta^\tau \int_{I_i} \left((\Delta^h(u_i\psi_{ix}) + \Psi_i'\Delta^h u_i + U_i \Delta^h \psi_{ix})\Delta^h v_i - \beta_i(\Delta^h v_i)^2\right) \, dx\,dt \ .\end{aligned}$$

Using condition (2.9) and the boundary conditions we can write

$$\begin{aligned}&\sum_{i\in\mathcal{M}} \int_\delta^\tau \int_{I_i} \lambda_i \left(\Delta^h v_i(x,t)\Delta^h u_i(x,t)\right)_x dxdt \ge \sum_{j\in\mathcal{J}} \int_\delta^\tau \Delta^h u(e_j,t)\Delta^h \mathcal{W}_j(t)dt \\ &= h\sum_{j\in\mathcal{J}} \int_0^1 (\Delta^h \mathcal{W}_j(\tau) u(e_j, \tau+\theta h) - \Delta^h \mathcal{W}_j(\delta)u(e_j,\delta+\theta h))d\theta \\ &-h\sum_{j\in\mathcal{J}} \int_0^1 \int_\delta^\tau \Delta^h \mathcal{W}_j'(t) u(e_j, t+\theta h)dtd\theta \ ,\end{aligned}$$

so that, after dividing the equalities (4.12) by h^2, summming them for $i \in \mathcal{M}$ and letting first h and then δ go to zero, we obtain the claim. □

Proposition 4.4. *Let (u, v, ψ) be the local solution to (4.3)-(4.7),(2.5)-(2.7); then*

$$\begin{aligned}&\sum_{i\in\mathcal{M}} \sup_{[0,T]} \lambda_i \|u_{ix}(t)\|_2^2 \le \sum_{i\in\mathcal{M}} \frac{2}{\lambda_i}\left(\sup_{[0,T]} \|v_{it}(t)\|_2^2 + \beta_i^2 \sup_{[0,T]}\|v_i(t)\|_2^2\right) \\ &+ \sum_{i\in\mathcal{M}} (c_S \sup_{[0,T]} \|u_i(t)\|_{H^1} + \|U_i\|_\infty)\left(\sup_{[0,T]} \|u_{ix}(t)\|_2^2 + \sup_{[0,T]} \|\psi_{i_x}(t)\|_2^2\right) \\ &+ \sum_{i\in\mathcal{M}} \|\Psi_i'\|_\infty \sup_{[0,T]} \|u_i(t)\|_{H^1}^2 \ .\end{aligned}$$

Proof. We multiply the second equation in (4.3) by u_{ix}, we integrate over I_i and we sum for $i \in \mathcal{M}$; using the Cauchy-Schwartz inequality we obtain the claim. □

Proposition 4.5. *Let (2.8) hold and let (u, v, ψ) be the local solution to (4.3)-(4.7),(2.5)-(2.7); then*

$$\begin{aligned}&\sum_{i\in\mathcal{M}} \lambda_i \int_0^T \|u_{ix}(t)\|_2^2 \, dt \le \sum_{i\in\mathcal{M}} \frac{2}{\lambda_i}\int_0^T \left(\|v_{it}(t)\|_2^2 + \beta_i^2 \|v_i(t)\|_2^2\right) \, dt \\ &+ \sum_{i\in\mathcal{M}} (c_S \sup_{[0,T]} \|u_i(t)\|_{H^1} + \|U_i\|_\infty) \int_0^T \left(\|u_{ix}(t)\|_2^2 + \|\psi_{ix}(t)\|_2^2\right) \, dt \\ &+ c_2 \sum_{i\in\mathcal{M}} \|\Psi_i'\|_\infty \left(\int_0^{+\infty} |\mu(t)|^2 dt + \int_0^T \left(\|u_{ix}(t)\|_2^2 + \|v_i(t)\|_{H^1}^2\right) dt\right)\end{aligned}$$

where c_2 is a constant depending on Sobolev constants, on L_i $(i \in \mathcal{M})$ and on γ .

Proof. We multiply the second equation (4.3) by u_{ix}, we integrate over $I_i \times (0, T)$ and we sum for $i \in \mathcal{M}$; using the Cauchy-Schwartz inequality and Proposition 4.1, we obtain the claim. □

Proposition 4.6. *Let (u, v, ψ) be the local solution to (4.3)-(4.7),(2.5)-(2.7); then*

$$\sum_{i\in\mathcal{M}} \lambda_i^2 \int_0^T \|v_{ix}(t)\|_2^2\,dt \le c_3\left(\|v_0\|_2^2 + \|u_0\|_{H^1}^2\left(1+\|\psi_0\|_{H^1}^2 + \|\Psi'\|_2^2\right) + \|U\|_\infty^2\|\psi_{0x}\|_2^2\right)$$

$$+c_S\sum_{j\in\mathcal{J}}\left(|\mathcal{W}_j(0)|\,\|u_{i(j)}(0)\|_{H^1} \; + \|\mathcal{W}_j\|_{L^\infty(0,+\infty)}\sup_{[0,T]}\|u_{i(j)}(t)\|_{H^1}\right)$$

$$+c_S\sum_{j\in\mathcal{J}}\|\mathcal{W}_j'\|_{L^1(0,+\infty)}\sup_{[0,T]}\|u_{i(j)}(t)\|_{H^1} + c_4\sum_{i\in\mathcal{M}}\left(\sup_{[0,T]}\|v_{it}(t)\|_2^2 + \int_0^T\|v_{it}(t)\|_2^2 dt\right)$$

$$+\sum_{i\in\mathcal{M}}\left(c_S\sup_{[0,T]}\|u_i(t)\|_{H^1} + \|U_i\|_\infty\right)\int_0^T\left(\|v_i(t)\|_2^2 + \|\psi_{ixt}(t)\|_2^2\right)\,dt$$

$$+c_5\sum_{i\in\mathcal{M}}\left(\sup_{[0,T]}\|\psi_{ix}(t)\|_{H^1} + \|\Psi_i'\|_\infty\right)\int_0^T\|v_i(t)\|_{H^1}^2\,dt$$

where c_3, c_4, c_5 are positive constants depending on $\lambda_i, \beta_i, \sigma_{ij}^\nu$ $(i, j \in \mathcal{M},\ \nu \in \mathcal{N})$, and on Sobolev constants.

Proof. Using the same notations as in the proof of Proposition 4.3, by the second equation in (4.3) we obtain, for $0 < \delta < \tau < T$, $|h| \le \min\{\delta, T-\tau\}$,

$$\int_\delta^\tau\int_{I_i}\left((v_i\Delta^h v_i)_t - v_{it}\Delta^h v_i - \lambda_i v_{ix}\Delta^h u_i \; + \lambda_i\left(v_i\Delta^h u_i\right)_x\right)dx\ dt$$

$$= \int_\delta^\tau\int_{I_i} v_i\left(\Delta^h(u_i\psi_{i_x}) + \Delta^h u_i\Psi' + U_i\Delta^h\phi_{i_x} - \beta_i\Delta^h v_i\right)\ dx\ dt\ .$$

Using (2.6) (as in (2.9)) and the boundary conditions in (4.3) , we can write

(4.13)
$$\sum_{i\in\mathcal{M}}\frac{1}{h}\int_\delta^\tau\int_{I_i}\left(-\lambda_i v_{ix}\Delta^h u_i + \beta_i v_i\Delta^h v_i\right)\ dx\ dt$$

$$= \frac{1}{h}\sum_{i\in\mathcal{M}}\int_{I_i}\left(-v_i(\tau)\Delta^h v_i(\tau)\ dx + v_i(\delta)\Delta^h v_i(\delta)\right)dx - \frac{1}{h}\int_\delta^\tau\sum_{j\in\mathcal{J}}\mathcal{W}_j\Delta^h u(e_j)dt$$

$$-\frac{1}{h}\int_\delta^\tau\sum_{\nu\in\mathcal{N}}\sum_{i,j\in\mathcal{M}^\nu}\frac{\sigma_{ij}^\nu}{2}\left(u_j(N_\nu) - u_i(N_\nu)\right)\Delta^h\left(u_j(N_\nu) - u_i(N_\nu)\right)dt$$

$$+\frac{1}{h}\sum_{i\in\mathcal{M}}\int_\delta^\tau\int_{I_i}\left(v_{it}\Delta^h v_i + v_i\left(\Delta^h(u_i\psi_{i_x}) + \Psi_i'\Delta^h u_i + U_i\Delta^h\psi_{i_x}\right)\right)dx\ dt\ .$$

In order to treat the terms at the inner nodes, we set $H(t) = u_j(N, t) - u_i(N, t)$ and arguing as in [11] we have

$$\lim_{h\to 0}\frac{1}{h}\int_\delta^\tau H(t)\Delta^h H(t)\ dt = \frac{1}{2}\left(H^2(\tau) - H^2(\delta)\right)\ .$$

As regard to the terms at the boundary nodes, we argue as in the proof of Proposition 4.3. Then we obtain the claim letting h and then δ go to zero and τ go to T in (4.13). □

Proposition 4.7. *Let (u, v, ψ) be the local solution to (4.3)-(4.7),(2.5)-(2.7); then*

$$\sum_{i\in\mathcal{M}} \left(\sup_{[0,T]} \|\psi_{i t}(t)\|_2^2 + \int_0^T \left(\|\psi_{i t}(t)\|_2^2 + \|\psi_{i tx}(t)\|_2^2 \right) \, dt \right)$$

$$\leq c_6 \left(\|\psi_0\|_{H^2}^2 + \|u_0\|_2^2 \right) + c_7 \sum_{i\in\mathcal{M}} \int_0^T \|u_{i t}(t)\|_2^2 \, dt$$

$$+c_8 \sum_{j\in\mathcal{J}} \|\mathcal{P}_j'\|_{L^2(0,+\infty)} \left(\left(\int_0^T \|\psi_{i(j)_t}(t)\|_2^2 dt \right)^{\frac{1}{2}} + \left(\int_0^T \|\psi_{i(j)_{tx}}(t)\|_2^2 dt \right)^{\frac{1}{2}} \right)$$

where $c_6, c_7 > 0$ depend on D_i, b_i, a_i ($i \in \mathcal{M}$) and $c_8 > 0$ depends on the same parameters and Sobolev constants.

Proof. The boundary conditions in (4.3) imply that

$$\sum_{j\in\mathcal{J}} \eta_j D_{i(j)} \psi_x(e_j, t) \psi(e_j, t) \leq \sum_{j\in\mathcal{J}} \mathcal{P}_j(t) \psi(e_j, t) \ , \quad t \in [0, T] \ .$$

Then, from the third equation in (4.3), arguing as in Proposition 4.3 and using (2.10) and the boundary conditions in (4.3), we obtain, for $0 < \delta < \tau < T$,

$$\sum_{i\in\mathcal{M}} \left(\int_{I_i} (\Delta^h \psi_i(\tau))^2 dx + 2 \int_\delta^\tau \int_{I_i} \left(b_i (\Delta^h \psi_i)^2 + D_i (\Delta^h \psi_{ix})^2 \right) dx dt \right)$$

$$\leq \sum_{i\in\mathcal{M}} \left(\int_{I_i} (\Delta^h \psi_i(\delta))^2 dx + 2a_i \int_\delta^\tau \int_{I_i} \Delta^h u_i \Delta^h \psi_i dx dt \right) + 2 \sum_{j\in\mathcal{J}} \int_\delta^\tau \Delta^h \mathcal{P}_j \Delta^h \psi(e_j) dt.$$

Since

$$\sum_{j\in\mathcal{J}} \int_\delta^\tau \Delta^h \mathcal{P}_j(t) \Delta^h \psi(e_j, t) dt$$

$$\leq c_S \sum_{j\in\mathcal{J}} \|\Delta^h \mathcal{P}_j\|_{L^2(0,+\infty)} \left(\int_\delta^\tau \left(\|\Delta^h \psi_{i(j)}(t)\|_2^2 + \|\Delta^h \psi_{i(j)_x}(t)\|_2^2 \right) dt \right)^{\frac{1}{2}} ,$$

we obtain the claim arguing as in the previous proof. □

Proposition 4.8. *Let (u, v, ψ) be the local solution to (4.3)-(4.7),(2.5)-(2.7); then*

$$\sum_{i\in\mathcal{M}} \left(\frac{D_i^2}{b_i} \sup_{[0,T]} \|\psi_{ixx}(t)\|_2^2 + 2D_i \sup_{[0,T]} \|\psi_{ix}(t)\|_2^2 \right)$$

$$\leq \sum_{i\in\mathcal{M}} \left(\frac{2}{b_i} \sup_{[0,T]} \|\psi_{i t}(t)\|_2^2 + \frac{2a_i^2}{b_i} \sup_{[0,T]} \|u_i(t)\|_2^2 \right)$$

$$+2c_S \sum_{j\in\mathcal{J}} \sup_{[0,T]} \|\psi_{i(j)}(t)\|_{H^1} \|\mathcal{P}_j\|_{L^\infty(0,+\infty)};$$

moreover, if (2.8) holds,

$$\sum_{i\in\mathcal{M}}\int_0^T\left(\frac{D_i^2}{b_i}\|\psi_{ixx}(t)\|_2^2+2D_i\|\psi_{ix}(t)\|_2^2\right)dt\le\sum_{i\in\mathcal{M}}\int_0^T\frac{2}{b_i}\sup_{[0,T]}\|\psi_{it}(t)\|_2^2dt$$

$$+c_9\sum_{i\in\mathcal{M}}\int_0^T\left(\mu^2(t)+\|u_{ix}(t)\|_2^2+\|v_i(t)\|_{H^1}^2\right)dt+c_{10}\sum_{j\in\mathcal{J}}\|\mathcal{P}_j\|_{L^2(0,+\infty)}\|\mu(t)\|_{L^2(0,+\infty)}$$

$$+c_{11}\sum_{j\in\mathcal{J}}\|\mathcal{P}_j\|_{L^2(0,+\infty)}\left(\int_0^T\left(\|\psi_x(t)\|_{H^1}^2+\|\psi_t(t)\|_2^2+\|u_x(t)\|_2^2+\|v(t)\|_{H^1}^2\right)dt\right)^{\frac{1}{2}}$$

where c_9, c_{10}, c_{11} are positive constants depending on $\gamma, L_i, a_i, b_i, D_i$ $(i\in\mathcal{M})$ and Sobolev constants.

Proof. The first inequality can be achieved multiplying the third equation in (4.3) by $\frac{D_i}{b_i}\psi_{ixx}$, integrating on I_i, summing for $i\in\mathcal{M}$ and using the Cauchy-Schwartz inequality, (2.10) and the boundary conditions in (4.3).

In order to obtain the second inequality, using Proposition 4.1 we obtain

$$\sum_{i\in\mathcal{M}}\int_0^T\left(\frac{D_i^2}{b_i}\|\psi_{ixx}(t)\|_2^2+2D_i\|\psi_{ix}(t)\|_2^2\right)dt\le\sum_{i\in\mathcal{M}}\int_0^T\frac{2}{b_i}\sup_{[0,T]}\|\psi_{it}(t)\|_2^2dt$$

$$+c_9\sum_{i\in\mathcal{M}}\int_0^T\left(\mu^2(t)+\|u_{ix}(t)\|_2^2+\|v_i(t)\|_{H^1}^2\right)dt+\sum_{j\in\mathcal{J}}\|\psi(e_j)\|_{L^2(0,T)}\|\mathcal{P}_j\|_{L^2(0,T)}$$

where c_8 depends on γ, a_i, b_i, L_i $(i\in\mathcal{M})$. The claim follows using the equation satisfied by ψ and Proposition 4.1 . □

Now we introduce the functional F_T as follows:

$$F_T^2(u,v,\psi):=\sup_{t\in[0,T]}\|u(t)\|_{H^1}^2+\sup_{t\in[0,T]}\|v(t)\|_{H^1}^2+\sup_{t\in[0,T]}\|\psi(t)\|_{H^2}^2$$

$$+\int_0^T\left(\|u(t)\|_{H^1}^2+\|v(t)\|_{H^1}^2+\|v_t(t)\|_2^2+\|\psi(t)\|_{H^2}^2+\|\psi_t(t)\|_2^2+\|\psi_{xt}(t)\|_2^2\right)\,dt\ .$$

The a priori estimates in the previous propositions allow to prove the following theorem.

Theorem 4.1. *Let (2.8) hold. Let (U,V,Ψ) be a stationary solution to problem (2.1), (2.5), (2.6), (4.1), (4.2) and let (u,v,ψ) be the solution to problem (4.3)-(4.7), (2.5)-(2.7). There exists $\epsilon_0>0$ such that, if*

$$\|U\|_\infty+\|\Psi'\|_\infty\le\epsilon_0\ ,$$

then, if the quantities

(4.14)

$$\sum_{j\in\mathcal{J}}\|\mathcal{P}_j\|_{H^1(0,+\infty)},\sum_{j\in\mathcal{J}}\|\mathcal{W}_j\|_{W^{2,1}(0,+\infty)},\|\mu\|_{L^2(0,+\infty)},\|u_0\|_{H^1},\|v_0\|_{H^1},\|\psi_0\|_{H^2}$$

are suitably small, then $F_T(u,v,\phi)$ is bounded uniformly in T ,

$$u,v\in C([0,+\infty);H^1(\mathcal{A}))\cap C^1([0,+\infty);L^2(\mathcal{A}))\ ,$$

$$\psi\in C([0,+\infty);H^2(\mathcal{A}))\cap C^1([0,+\infty);L^2(\mathcal{A}))\cap H^1((0,+\infty);H^1(\mathcal{A}))$$

and, for all $i\in\mathcal{M}$,

$$\lim_{t\to+\infty}\|u_i(\cdot,t)\|_{C(\bar{I}_i)},\quad\lim_{t\to+\infty}\|v_i(\cdot,t)\|_{C(\bar{I}_i)}\ ,\quad\lim_{t\to+\infty}\|\psi_i(\cdot,t)\|_{C^1(\bar{I}_i)}=0\ .$$

Proof. Let (u, v, ψ) be the solution to problem (4.3)-(4.7), (2.5)-(2.7) given in Theorem 3.1; using the estimates proved in Propositions 4.1-4.8, it is easy to prove the following inequality

$$F_T^2(u,v,\phi) \le \overline{C}_0 + \overline{C}_2 F_T^2(u,v,\psi) + \overline{C}_3 F_T^3(u,v,\psi), \tag{4.15}$$

where

$$\overline{C}_0 = c_0 \Big(\|u_0\|_{H^1}^2 \left(1 + \|\psi_0\|_{H^2}^2\right) + \|\psi_0\|_{H^2}^2 + \|v_0\|_{H^1}^2 + (\|\Psi'\|_\infty + 1)\, \|\mu\|_{L^2(0,+\infty)}^2$$

$$+ \sup_{j\in\mathcal{J}} \|\mathcal{P}_j\|_{L^2(0,+\infty)} \|\mu\|_{L^2(0,+\infty)} + \frac{1}{2\delta} \Big(\sup_{j\in\mathcal{J}} \|\mathcal{W}_j\|_{W^{2,1}(0,+\infty)}^2 + \sup_{j\in\mathcal{J}} \|\mathcal{P}_j\|_{H^1(0,+\infty)}^2 \Big)\Big),$$

$$\overline{C}_2 = c_2 \left(\|U\|_\infty + \|\Psi'\|_\infty + \frac{\delta}{2} \right) ,$$

$\overline{C}_3, c_0, c_2 > 0$ depend on Sobolev constants, $m, \beta_i, \lambda_i, b_i, a_i, D_i, L_i, \gamma, \sigma_{ij}^\nu$ $(i, j \in \mathcal{M}, \nu \in \mathcal{N})$, and δ is any positive quantity.

Let $\|U\|_\infty, \|\Psi'\|_\infty, \delta$ be in such a way that $\overline{C}_2 < 1$, and let the quantities in (4.14) be small enough to have $\overline{C}_0 \le \frac{4(1-\overline{C}_2)^3}{27}$; these choices imply that the function $f(y) := \overline{C}_3 y^3 - (1-\overline{C}_2)y^2 + \overline{C}_0$ has a negative minimum in $y = \frac{2(1-\overline{C}_2)}{3\overline{C}_3}$. Finally, if $F_0(u, v, \psi) < \dfrac{2(1-\overline{C}_2)}{3\overline{C}_3}$, then we can conclude that the inequality (4.15) implies that $F_T(u, v, \psi)$ remains uniformly bounded for all $T > 0$; then the solution is globally defined.

Moreover the set $\{u(t), v(t), \psi(t)\}_{t\in[0,+\infty)}$ is uniformly bounded in $(H^1(\mathcal{A}))^2 \times H^2(\mathcal{A})$; thus, if we call E_s the set of accumulation points of $\{u(t), v(t), \psi(t)\}_{t\ge s}$ in $(C(\mathcal{A}))^2 \times C^1(\mathcal{A})$, then E_s is not empty and $E := \cap_{s\ge 0} E_s \ne \emptyset$. Let $\hat{v}(x)$ be such that, for a sequence $t_n \to +\infty$,

$$\lim_{n\to+\infty} \sum_{i\in\mathcal{M}} \|v_i(\cdot, t_n) - \hat{v}_i(\cdot)\|_{C(\overline{I}_i)} = 0 .$$

If we set $\omega_i(t) := \|v_i(t,\cdot)\|_{L^2(I_i)}$ then the estimates obtained for the functions v_i imply that $\omega_i \in H^1((0,+\infty))$ and, as a consequence, $\lim_{t\to+\infty} \omega_i(t) = 0$. As $\lim_{n\to+\infty} \|v_i(\cdot, t_n)\|_2 = \|\hat{v}_i(\cdot)\|_2$, we obtain $\|\hat{v}\|_2 = 0$. The same argument can be applied to the functions u_i and ψ_i. □

5. **Stationary solutions on acyclic networks**

In this section we study the real existence of stationary solutions to problem (2.1)-(2.8). Concerning the uniqueness, we can notice that the results of the previous section imply that two stationary solutions with the same mass and the same boundary data , which are small in $H^1 \times H^1 \times H^2$ norm, have to coincide.

In this section we restrict our attention to acyclic graphs and we approach the study of existence of stationary solutions $(U(x), V(x), \Psi(x))$ with fixed mass μ_s,

$$\mu_s = \sum_{i\in\mathcal{M}} \int_{I_i} U_i(x)\, dx, \tag{5.1}$$

and boundary data

$$\eta_j \lambda_{i(j)} V(e_j) = W_j \ , \qquad \eta_j \Psi'(e_j) + d_j \Psi(e_j) = P_j \ , \quad j \in \mathcal{J} \ , \tag{5.2}$$

assuming conditions (2.8) and some suitable smallness conditions on $|\mu_s|$, $|W_j|$ and $|P_j|$.

Of course, for all $i \in \mathcal{M}$, $V_i(x)$ is a constant function, $V_i(x) = V_i$; moreover, we recall that a set of boundary data $\{W_j\}_{j\in\mathcal{J}}$ is compatible with the transmission conditions only if $\sum_{j\in\mathcal{J}} W_j = 0$ (see previous section). These facts hold true for general networks.

In the case of acyclic network, a set of admissible boundary values $\{W_j\}_{j\in\mathcal{J}}$ determines univokely the costant value of each function V_i on the internal arc I_i. Actually, let consider an internal arc I_i and its starting node N_η and the sets

$$
\begin{aligned}
&\mathcal{Q} = \{\nu \in \mathcal{N} : N_\nu \text{ is linked to } N_\eta \text{ by a path not covering } I_i\} \ , \\
&\mathcal{J}' = \{j \in \mathcal{J} : e_j \text{ is linked to } N_\eta \text{ by a path not covering } I_i \ ;
\end{aligned} \tag{5.3}
$$

at each inner node the conservation of the flux (2.12) holds, then

$$
\sum_{\nu\in\mathcal{Q}\cup\{\eta\}} \left(\sum_{\iota\in I^\nu} \lambda_\iota V_\iota(N_\nu) - \sum_{\iota\in O^\nu} \lambda\iota V_\iota(N_\nu) \right) = 0 \ .
$$

Since $V_\iota(x)$ is constant on I_ι for all $\iota \in \mathcal{M}$, using the first condition in (5.2), the above equality reduces to

$$
\lambda_i V_i(x) = -\sum_{j\in\mathcal{J}'} W_j \qquad \forall x \in I_i. \tag{5.4}
$$

Hence, if $\mathcal{G}$ is an acyclic graph, a stationary solution to problem (2.1),(2.5),(2.6), satisfying (5.2) and (5.1), is a triple $(U(x), V, \Psi(x))$, where $V = \{V_i\}_{i\in\mathcal{M}}$ is determined by the boundary values W_j in (5.4), and the functions U and Ψ solve the following problem.

Find C_i, $i = 1, ...m$, and $\Psi \in H^2(\mathcal{A})$ such that

$$
\begin{cases}
-D_i\Psi_i''(x) + b_i\Psi_i(x) = a_iU_i(x) & x \in I_i \ , \quad i \in \mathcal{M} \ , \\
U_i(x) = \exp\left(\frac{\Psi_i(x)}{\lambda_i}\right)\left(C_i - \dfrac{\beta_i}{\lambda_i}V_i \displaystyle\int_0^x \exp\left(-\dfrac{\Psi_i(s)}{\lambda_i}\right)\, ds\right) \ , & \\
\eta_j D_{i(j)}\Psi'(e_j) + d_j\Psi(e_j) = P_j \ , \qquad j \in \mathcal{J}, & \\
\delta_i^\nu D_i\Psi_i'(N_\nu) = \displaystyle\sum_{j\in\mathcal{M}^\nu} \alpha_{ij}^\nu(\Psi_j(N_\nu) - \Psi_i(N_\nu)) \ , & i \in \mathcal{M}^\nu, \nu \in \mathcal{N}, \\
-\delta_i^\nu\lambda_iV_i = \displaystyle\sum_{j\in\mathcal{M}^\nu} \sigma_{ij}^\nu(U_j(N_\nu) - U_i(N_\nu)) \ , & i \in \mathcal{M}^\nu, \nu \in \mathcal{N}, \\
\displaystyle\sum_{i\in\mathcal{M}} \int_{I_i} U_i(x)dx = \mu_s \ . &
\end{cases} \tag{5.5}
$$

We are going to prove existence of solutions to problem (5.5) using a fixed point technique; we need some preliminary results.

Lemma 5.1. *Let $\mathcal{G}$ an acyclic graph and let (2.8) hold. Given a function $f \in H^2(\mathcal{A})$ and real values μ_s and V_i ($i \in \mathcal{M}$), there exists a unique $C^f = (C_1^f, C_2^f, ..., C_m^f)$ such that the functions*

$$
\mathcal{U}_i^f(x) = \exp\left(\frac{f_i(x)}{\lambda_i}\right)\left(C_i^f - \frac{\beta_i}{\lambda_i}V_i \int_0^x \exp\left(-\frac{f_i(s)}{\lambda_i}\right)\, ds\right) \tag{5.6}
$$

satisfy

$$
-\delta_\nu^i\lambda_iV_i = \sum_{j\in\mathcal{M}^\nu} \sigma_{ij}^\nu(\mathcal{U}_j^f(N_\nu) - \mathcal{U}_i^f(N_\nu)) \ , \qquad \nu \in \mathcal{N} \ , i \in \mathcal{M}^\nu \ , \tag{5.7}
$$

$$\sum_{i\in\mathcal{M}}\int_{I_i}\mathcal{U}_i^f(x)dx=\mu_s\ . \tag{5.8}$$

Proof. Given $f_i\in H^2(I_i)$, for $i\in\mathcal{M}$, we introduce the functions

$$E_i^f(x)=\exp\left(\frac{f_i(x)}{\lambda_i}\right)\ ,\qquad J_i^f(x)=\frac{\beta_i}{\lambda_i}\int_0^x\exp\left(-\frac{f_i(s)}{\lambda_i}\right)ds\ . \tag{5.9}$$

The conditions (5.7) can be rewritten as in (4.8)

$$\mathcal{U}_i^f(N_\nu)=\mathcal{U}_{k_\nu}^f(N_\nu)+\sum_{j\in\mathcal{M}^\nu}\gamma_{ji}^\nu V_j\qquad \text{for all } i\in\mathcal{M}^\nu,\nu\in\mathcal{N}\ , \tag{5.10}$$

where γ_{ij}^ν are suitable coefficients depending on σ_{ij}^ν. In order to satisfy such relations at the node N_1, each coeffcient C_i^f, for $i\in\mathcal{M}^1$, have to be a linear combination of the values $C_{k_1}^f$ and V_i $(i\in\mathcal{M}^1)$, where k_1 is the index in (2.8),

$$C_i^f=(E_i^f(N_1))^{-1}\left(E_{k_1}^f(N_1)\left(C_{k_1}^f-V_{k_1}J_{k_1}^f(N_1)\right)+\sum_{j\in\mathcal{M}^1}\gamma_{ji}^1V_j\right)+V_iJ_i^f(N_1)\ .$$

Setting

$$Q_{i\nu}^f=E_{k_\nu}^f(N_\nu)(E_i^f(N_\nu))^{-1}\ ,$$

$$O_{i\nu}^f=(E_i^f(N_\nu))^{-1}\left(-V_{k_\nu}E_{k_\nu}^f(N_\nu)J_{k_\nu}^f(N_\nu)+\sum_{j\in\mathcal{M}^\nu}\gamma_{ji}^1V_j\right)+V_iJ_i^f(N_\nu)$$

we have

$$C_i^f=Q_{i1}^fC_{k_1}^f+O_{i1}^f\ ,\quad i\in\mathcal{M}^1\ ;$$

now, if N_ν and N_1 are two consecutive nodes, linked by the arc I_l, arguing as before we infer that the coefficients $C_{k_\nu}^f$ and $C_{k_1}^f$ have to satisfy the following relation

$$C_l^f=Q_{l\nu}^fC_{k_\nu}^f+O_{l\nu}^f=Q_{l1}^fC_{k_1}^f+O_{l1}^f\ ,$$

which expresses $C_{k_\nu}^f$ in terms of $C_{k_1}^f$; so, for all $i\in\mathcal{M}^\nu$, we have the expression

$$C_i^f=Q_{i\nu}^fC_{k_\nu}^f+O_{i\nu}^f=\frac{Q_{l1}^fQ_{i\nu}^f}{Q_{l\nu}^f}C_{k_1}^f+Q_{i\nu}^f\frac{O_{l1}^f-O_{l\nu}^f}{Q_{l\nu}^f}+O_{i\nu}^f\ .$$

Since there are no cycles in the network, iterating this procedure we can write univokely all the coeffcients C_i^f, $i\in\mathcal{M}$, in terms of $C_{k_1}^f$,

$$C_i^f=\tilde{Q}_i^fC_{k_1}^f+\tilde{O}_i^f\ ,\ i\in\mathcal{M}\ , \tag{5.11}$$

where $\tilde{Q}_i^f$ and $\tilde{O}_i^f$ are suitable quantities depending on the function f and on the values V_ι $(\iota\in\mathcal{M})$. In other words, system (5.7) has ∞^1 solutions given by (5.11), for $C_{k_1}\in\mathbb{R}$. In order to determine $C_{k_1}^f$ we use condition (5.8):

$$C_{k_1}^f=\left(\sum_{i\in\mathcal{M}}\tilde{Q}_i^f\int_{I_i}E_i^f(x)dx\right)^{-1}\left(\mu_s-\sum_{i\in\mathcal{M}}\int_{I_i}\left(\tilde{O}_i^f-V_iJ_i^f(x)\right)E_i^f(x)dx\right).$$

□

Now, given $f \in H^2(\mathcal{A})$ we consider $\mathcal{U}_i^f$ defined in (5.6)-(5.8) and the problem

$$
(5.12)\qquad \begin{cases} -D_i\Psi_i''(x) + b_i\Psi_i(x) = a_i\mathcal{U}_i^f(x) \;, & i \in \mathcal{M} \;, \\ \eta_j D_{i(j)}\Psi'(e_j) + d_j\Psi(e_j) = P_j \;, & j \in \mathcal{J}, \\ \delta_i^\nu D_i\Psi_i'(N_\nu) = \sum_{j\in\mathcal{M}^\nu} \alpha_{ij}^\nu(\Psi_j(N_\nu) - \Psi_i(N_\nu)) \;, & i \in \mathcal{M}^\nu \;,\; \nu \in \mathcal{N} \;, \end{cases}
$$

which has a unique solution (see the proof of Proposition 3.1). We set

$$
(5.13)\qquad \Theta := \sum_{j\in\mathcal{J}} |P_j| + \mu_s \max\{a_i\}_{i\in\mathcal{M}};
$$

then the following estimates hold .

Lemma 5.2. *Let $\mathcal{G}$ be an acyclic graph. Let $\mathcal{U}_i^f(x) \geq 0$, $i \in \mathcal{M}$, and let $\Psi \in H^2(\mathcal{A})$ be the solution to problem (5.12). Then there exist two positive constants K_1, K_2, depending on the parameters b_i, D_i, L_i, d_j ($i \in \mathcal{M}$, $j \in \mathcal{J}$), such that*

$$
(5.14)\qquad \|\Psi\|_\infty \leq K_1\Theta \;, \quad \|\Psi'\|_\infty \leq K_2\Theta \;.
$$

Proof. Using the first equation in (5.12), and (2.10) to treat the terms evaluated at the internal nodes, we obtain

$$
\sum_{i\in\mathcal{M}} \int_{I_i} \left(D_i\Psi_i'^2 + b_i\Psi_i^2\right) dx \leq \sum_{j\in\mathcal{J}} \left(P_j\Psi(e_j) - d_j\Psi^2(e_j)\right) + \sum_{i\in\mathcal{M}} a_i \int_{I_i} \mathcal{U}_i^f |\Psi_i| dx
$$

$$
\leq \left(\sum_{j\in\mathcal{J}} |P_j| + \mu_s \max\{a_i\}_{i\in\mathcal{M}}\right) \sum_{i\in\mathcal{M}} c_i^S \|\Psi_i\|_{H^1} \;,
$$

where c_i^S are Sobolev constants. This yields the first inequality in the claim.

In order to obtain the second inequality, first we notice that, if e_j is an external node and $I_{i(j)}$ is the corresponding external arc, then the following inequality holds

$$
|D_{i(j)}\Psi_{i(j)}'(x)| \leq \int_{I_{i(j)}} D_{i(j)}|\Psi_{i(j)}''(y)| \; dy + |P_j - d_j\Psi(e_j)| \;, \quad x \in I_{i(j)} \;.
$$

Then we consider an internal arc I_ι and its starting node N_η, the sets $\mathcal{Q}$ and $\mathcal{J}'$ as in (5.3) and $\mathcal{S} = \{\iota \in \mathcal{M} : I_\iota \text{ is incident with } N_\nu \text{ for some } \nu \in \mathcal{Q}\}$; at each node the conservation of the flux (2.11) holds, so

$$
\sum_{\nu\in\mathcal{Q}\cup\{\eta\}} \left(\sum_{i\in I^\nu} D_i\Psi_i'(N_\nu) - \sum_{i\in O^\nu} D_i\Psi_i'(N_\nu)\right) = 0 \;.
$$

Then, for all $x \in I_\iota$, using the above equality and the boundary conditions in (5.12), we have

$$
D_\iota\Psi_\iota'(x) = \sum_{i\in\mathcal{S}} \int_{I_i} D_i\Psi_i''(y) \; dy + \int_{N_\eta}^x D_\iota\Psi_\iota''(y) \; dy - \sum_{j\in\mathcal{J}'} \left(P_j - d_j\Psi(e_j)\right) \;.
$$

Then , for all $l \in \mathcal{M}$,

$$
D_l|\Psi_l'(x)| \leq \sum_{j\in\mathcal{J}} |P_j - d_j\Psi(e_j)| + \sum_{i\in\mathcal{M}} \int_{I_i} |b_i\Psi_i(y) - a_i\mathcal{U}_i^f(y)| dy
$$

and using the first inequality in (5.14) we obtain

$$
\max_{i\in\mathcal{M}} \|\Psi_i'\|_\infty \leq \frac{\Theta}{\min\{D_i\}_{i\in\mathcal{M}}} \left(1 + K_1 \left(\sum_{i\in\mathcal{M}} b_i L_i + \sum_{j\in\mathcal{J}} d_j\right)\right) \;.
$$

□

The previous results provides the tools to prove the following theorem of existence for stationary solutions under smallness conditions for some data; in particular we remark that the condition on $\sum_{i\in\mathcal{M}} |V_i|$ is a condition on W_j, $j \in \mathcal{J}$, thanks to (5.4).

Theorem 5.1. *Let $\mathcal{G}$ be an acyclic graph and let (2.8) hold. Let $\mu_s \geq 0$, $W_j, P_j \in \mathbb{R}$, for $j \in \mathcal{J}$, and Θ be given in (5.13); let $\sum_{j\in\mathcal{J}} W_j = 0$ and $V = \{V_i\}_{i\in\mathcal{M}}$ given by (5.4). There exists $\epsilon > 0$ and $\delta = \delta(\Theta) > 0$, increasing with Θ, such that, if*

$$\delta \sum_{i\in\mathcal{M}} |V_i| \leq \mu_s \qquad \text{and} \qquad \mu_s + \sum_{i\in\mathcal{M}} |V_i| < \epsilon \ ,$$

then problem (2.1), (5.2), (2.5), (2.6) has a stationary solution $(U(x), V, \Psi(x))$ satisfying (5.1), with $U_i, \Psi_i \in C^\infty(\overline{I}_i)$ and $U_i \geq 0$, for $i \in \mathcal{M}$. Moreover, it is the unique stationary solution with non-negative U.

Proof. If a stationary solution $(U(x), V, \Psi(x))$ exists then, for all $i \in \mathcal{M}$, V_i are univokely determined by the boundary data W_j, $j \in \mathcal{J}$, in (5.4); moreover U, Ψ satisfy (5.5) and $U_i(x)$ are univokely determined by $\Psi_i(x)$ and the values V_i, σ_{ij}^ν and μ_s (Lemma 5.1). We remark that, if $U(x) \geq 0$, then the estimates in Lemma 5.2 hold for Ψ.

Let G be the operator defined in $D(A_2)$ (see (3.8)) such that, if $\Psi^I \in D(A_2)$ then $\Psi = G(\Psi^I)$ is the solution of problem (5.12) where $f = \Psi^I$ and $\mathcal{U}^{\Psi^I}$ is the function in Lemma 5.1. We consider G on the set

$$B_\Theta := \{\Psi \in D(A_2) : \|\Psi\|_\infty \leq K_1\Theta \ , \|\Psi'\|_\infty \leq K_2\Theta \ \} \ ,$$

where K_1, K_2 are the constants in Lemma 5.2, equipped with the distance d generated by $H^2(\mathcal{A})-$norm; (B_Θ, d) is a complete metric space.

Using the expression of C_i^f given in the proof of Lemma 5.1 and setting

$$\Lambda_1^{\Psi^I} := \sum_{j\in\mathcal{M}} \int_{I_j} \left(\tilde{O}_j^{\Psi^I} - V_j J_j^{\Psi^I}(x)\right) E_j^{\Psi^I}(x)dx \ , \quad \Lambda_2^{\Psi^I} := \sum_{j\in\mathcal{M}} \tilde{Q}_j^{\Psi^I} \int_{I_j} E_j^{\Psi^I}(x)dx$$

(we are using the notations in (5.9)), we can write

$$\mathcal{U}_i^{\Psi^I}(x) = E_i^{\Psi^I}(x) \, \frac{\tilde{Q}_i^{\Psi^I}(\mu_s - \Lambda_1^{\Psi^I}) + (\tilde{O}_i^{\Psi^I} - V_i J_i^{\Psi^I}(x))\Lambda_2^{\Psi^I}}{\Lambda_2^{\Psi^I}} . \tag{5.15}$$

It is readily seen that there exist some positive quantities $q_i = q_i(\Theta)$, increasing in Θ, and some positive quantities $r_i = r_i(\Theta)$, decreasing in Θ, depending also on the parameters of the problem, such that, for all $f \in B_\Theta$,

$$0 < r_1 \leq E_i^f(x) \leq q_1 \ , \quad 0 \leq J_i^f(x) \leq q_2 \ , \quad \forall x \in I_i \ , \quad \forall i \in \mathcal{M} \ , \tag{5.16}$$

$$0 < r_3 \leq \tilde{Q}_i^f \leq q_3 \ , \quad |\tilde{O}_i^f| \leq q_4 \sum_{j\in\mathcal{M}} |V_j| \ , \quad \forall i \in \mathcal{M} \ , \tag{5.17}$$

$$0 < r_5 \leq \Lambda_2^f \leq q_5 \ , \quad |\Lambda_1^f| \leq q_6 \sum_{j\in\mathcal{M}} |V_j| \ . \tag{5.18}$$

Hence, fixed $\mu_s \geq 0$ and P_j, $j \in \mathcal{J}$, it is possible to find a quantity $\delta = \delta(\Theta)$, increasing with Θ, such that, if $\delta \sum_{i\in\mathcal{M}} |V_i| \leq \mu_s$ then $\mathcal{U}_i^{\Psi^I}(x) \geq 0$ for all $i \in \mathcal{M}$. This fact allows us to use Lemma 5.2 and infer that $\Psi \in B_\Theta$.

Now we are going to prove that, if $\mu_s + \sum_{i\in\mathcal{M}} |V_i|$ is small then G is a contraction mapping in B_Θ. We consider $\Psi^I, \overline{\Psi}^I \in B_\Theta$ and the corresponding $\Psi = G(\Psi^I)$ and $\overline{\Psi} = G(\overline{\Psi}^I)$; using the equation satisfied by Ψ and $\overline{\Psi}$ and (2.10), we infer that

$$\sum_{i\in\mathcal{M}} \|\Psi_i - \overline{\Psi_i}\|^2_{H^2} \le K_3 \sum_{i\in\mathcal{M}} \|\mathcal{U}_i^{\Psi^I} - \mathcal{U}_i^{\overline{\Psi}^I}\|_2^2 \ , \tag{5.19}$$

where K_3 is a suitable constant depending on a_i, b_i, D_i, $(i \in \mathcal{M})$; moreover, for $i \in \mathcal{M}$ and $x \in I_i$, we have

(5.20)

$$\left|\mathcal{U}_i^{\Psi^I}(x) - \mathcal{U}_i^{\overline{\Psi}^I}(x)\right| \le \left|E_i^{\Psi^I}(x) - E_i^{\overline{\Psi}^I}(x)\right| \ \frac{q_3\mu_s + (q_3q_6 + q_4 + q_2q_5)\sum_{j\in\mathcal{M}}|V_j|}{r_5}$$

$$+\frac{q_1}{{r_5}^2}\left(\mu_s + q_6\sum_{j\in\mathcal{M}}|V_i|\right)\left(q_5\left|\tilde{Q}_i^{\Psi^I} - \tilde{Q}_i^{\overline{\Psi}^I}\right| + q_3\left|\Lambda_2^{\Psi^I} - \Lambda_2^{\overline{\Psi}^I}\right|\right)$$

$$+q_1\left(\frac{q_3q_5}{{r_5}^2}\left|\Lambda_1^{\Psi^I} - \Lambda_1^{\overline{\Psi}^I}\right| + \left|\tilde{O}_i^{\Psi^I} - \tilde{O}_i^{\overline{\Psi}^I}\right| + |V_i|\left|J_i^{\Psi^I}(x) - J_i^{\overline{\Psi}^I}(x)\right|\right) \ .$$

It is easily seen that, for suitable positive quantities $q_i = q_i(\Theta)$, depending on Θ and on the parameters of the problem, increasing with Θ, the following inequalities hold, for each $\Psi^I, \overline{\Psi}^I \in B_\Theta$, for each $i \in \mathcal{M}$ and $x \in I_i$,

$$\left|E_i^{\Psi^I}(x) - E_i^{\overline{\Psi}^I}(x)\right| \le q_7|\Psi_i^I(x) - \overline{\Psi}_i^I(x)| \ , \quad \left|J_i^{\Psi^I}(x) - J_i^{\overline{\Psi}^I}(x)\right| \le q_8|\Psi_i^I(x) - \overline{\Psi}_i^I(x)| \ ,$$

$$\left|\tilde{Q}_i^{\Psi^I} - \tilde{Q}_i^{\overline{\Psi}^I}\right| \le q_9\sum_{j\in\mathcal{M}}\|\Psi_j^I - \overline{\Psi}_j^I\|_\infty \ , \quad \left|\tilde{O}_i^{\Psi^I} - \tilde{O}_i^{\overline{\Psi}^I}\right| \le q_{10}\left(\sum_{j\in\mathcal{M}}|V_j|\right)\sum_{j\in\mathcal{M}}\|\Psi_j^I - \overline{\Psi}_j^I\|_\infty ,$$

$$\left|\Lambda_1^{\Psi^I} - \Lambda_1^{\overline{\Psi}^I}\right| \le q_{11}\left(\sum_{j\in\mathcal{M}}|V_j|\right)\sum_{i\in\mathcal{M}}\|\Psi_i^I - \overline{\Psi}_i^I\|_\infty \ ,$$

$$\left|\Lambda_2^{\Psi^I} - \Lambda_2^{\overline{\Psi}^I}\right| \le q_{12}\sum_{i\in\mathcal{M}}\|\Psi_i^I - \overline{\Psi}_i^I\|_\infty ;$$

the above inequalities can be used in (5.20) so that (5.19) implies

$$\sum_{i\in\mathcal{M}}\|\Psi_i - \overline{\Psi}_i\|_{H^2} \le q(\Theta)\left(\mu_s + \sum_{j\in\mathcal{M}}|V_j|\right)\sum_{i\in\mathcal{M}}\|\Psi_i^I - \overline{\Psi}_i^I\|_{H^1} \ , \tag{5.21}$$

where $q(\Theta)$ is a quantity increasing with Θ, depending also on the parameters of the problem; hence, for $\mu_s + \sum_{j\in\mathcal{M}}|V_j|$ small enough, G is a contraction mapping on B_Θ. Let $\hat{\Psi}$ be the unique fixed point of G in B_Θ and let $\hat{U} = \mathcal{U}^{\hat{\Psi}}$; then $(\hat{\Psi}, \hat{U})$ is a solution to Problem (5.5). The last assertion in the claim follows from Lemmas 5.1 and 5.2. Regularity properties follow from the equation in (5.5). □

Remark 5.1. *If $\mathcal{G}$ is an acyclic graph and $W_j = 0$ for $j \in \mathcal{J}$ then $V_i = 0$ for $i \in \mathcal{M}$ and $U_i(N_\nu) = U_j(N_\nu)$ for $i, j \in \mathcal{M}^\nu$, for all $\nu \in \mathcal{N}$; in particular, if $\mu_s \ge 0$ then $C_i \ge 0$ for all $i \in \mathcal{M}$ (see the proof of Lemma 5.1), i.e. $U(x) \ge 0$. In this case the stationary solution of the previous theorem is the unique stationary solution with mass μ_s.*

When the quantity $\sum_{i\in\mathcal{M}}|V_i|$ is not small enough respect to μ_s we do not have information about the sign of $U_i(x)$; however, if the boundary data, μ_s and the parameters of the problem satisfy some relations, a stationary solution with mass μ_s exists.

First, given $f\in H^2(\mathcal{A})$ and $\mathcal{U}_i^f$ defined in (5.6)-(5.8), as in Lemma 5.2 we can prove that if $\Psi\in H^2(\mathcal{A})$ is the solution to problem (5.12), then there exist two positive constants $\overline{K}_1,\overline{K}_2$, depending on the parameters $b_i,D_i,\ L_i,d_j$ $(i\in\mathcal{M},\ j\in\mathcal{J})$, such that

$$\|\Psi\|_\infty\le\overline{K}_1(\overline{a}\|\mathcal{U}^f\|_1+\sum_{j\in\mathcal{J}}|P_j|)\ ,\quad \|\Psi'\|_\infty\le\overline{K}_2(\overline{a}\|\mathcal{U}^f\|_1+\sum_{j\in\mathcal{J}}|P_j|)\ , \tag{5.22}$$

where $\overline{a}:=\max\{a_i\}_{i\in\mathcal{M}}$.

Moreover, let $\overline{\beta}$, $\underline{\lambda}$ be as in Section 3 and γ as in Lemma 4.1, and let

$$|\mathcal{A}|:=\sum_{i\in\mathcal{M}}L_i\ ,\qquad \Omega:=|\mu_s|+2|\mathcal{A}|\left(\frac{2\overline{\beta}}{\underline{\lambda}}|\mathcal{A}|+3\gamma\right)\sum_{i\in\mathcal{M}}|V_i|\ ; \tag{5.23}$$

if $(U(x),V,\Psi(x))$ is a stationary solution, then $\lambda_iU_i'(x)=U_i(x)\Psi'(x)-\beta_iV_i$ for each $i\in\mathcal{M}$, so that using Proposition 4.1 we obtain

$$\|U\|_1\le\Omega+\frac{4|\mathcal{A}|\sup_{i\in\mathcal{M}}\|\Psi_i'\|_\infty}{\underline{\lambda}}\|U\|_1\ ,$$

and then, using (5.22),

$$\frac{4|\mathcal{A}|\overline{K}_2\overline{a}}{\underline{\lambda}}\|U\|_1^2-\left(1-\frac{4|\mathcal{A}|\overline{K}_2}{\underline{\lambda}}\sum_{j\in\mathcal{J}}|P_j|\right)\|U\|_1+\Omega\ge0\ . \tag{5.24}$$

Then, if

$$1-\frac{4|\mathcal{A}|\overline{K}_2}{\underline{\lambda}}\sum_{j\in\mathcal{J}}|P_j|>0\ ,\ \ \Omega<\frac{\underline{\lambda}}{16|\mathcal{A}|\overline{K}_2\overline{a}}\left(1-\frac{4|\mathcal{A}|\overline{K}_2}{\underline{\lambda}}\sum_{j\in\mathcal{J}}|P_j|\right)^2\ , \tag{5.25}$$

setting

$$\mu^\pm:=\frac{\underline{\lambda}}{8|\mathcal{A}|\overline{K}_2\overline{a}}\left(1-\frac{4|\mathcal{A}|\overline{K}_2}{\underline{\lambda}}\sum_{j\in\mathcal{J}}|P_j|\pm\sqrt{\left(1-\frac{4|\mathcal{A}|\overline{K}_2}{\underline{\lambda}}\sum_{j\in\mathcal{J}}|P_j|\right)^2-\frac{16|\mathcal{A}|\Omega\overline{K}_2\overline{a}}{\underline{\lambda}}}\right), \tag{5.26}$$

we can conclude that $\mu^\pm>0$ and , if a stationary solution $(U(x),V,\Psi(x))$ exists, then $\|U\|_1\le\mu^-$ or $\|U\|_1\ge\mu^+$.

So, under suitable smallness conditons for the boundary data and $|\mu_s|$, in the following theorem we are able to prove the existence of a stationary solution verifying $\|U\|_1\le\mu^-$.

Theorem 5.2. *Let $\mathcal{G}$ be an acyclic graph and let (2.8) hold. Let $\mu_s>0$, $W_j,P_j\in\mathbb{R}$, for $j\in\mathcal{J}$, $\sum_{j\in\mathcal{J}}W_j=0$ and $V=\{V_i\}_{i\in\mathcal{M}}$ given by (5.4). Let (5.23), (5.25) , (5.26) hold, where $\overline{K}_1$ and $\overline{K}_2$ are the constants in (5.22). There exists $\epsilon>0$ such that, if $|\mu_s|+\sum_{i\in\mathcal{M}}|V_i|<\epsilon$, then problem (2.1),(5.2),(2.5),(2.6) has a stationary solution $(U(x),V,\Psi(x))$ satisfying (5.1) with $U_i,\Psi_i\in C^\infty(\overline{I}_i)$, for all $i\in\mathcal{M}$. Moreover, it is the unique stationary solution verifying $\|U\|_1\le\mu^-$.*

Proof. We proceed as in the proof of Theorem 5.1: we set $\Theta_1 := \overline{a}\mu^- + \sum_{j\in\mathcal{J}} |P_j|$ and we consider the map G defined in the proof of that theorem, acting on the set

$$B_{\Theta_1} := \{\Psi \in D(A_2) : \|\Psi\|_\infty \le \overline{K}_1\Theta_1 \ , \|\Psi'\|_\infty \le \overline{K}_2\Theta_1 \ \}$$

equipped with the distance d generated by the $H^2(\mathcal{A})-$norm; (B_{Θ_1}, d) is a complete metric space.

Fixed $\Psi^I \in B_{\Theta_1}$, $\mathcal{U}^{\Psi^I}$ is still given by (5.15) and the relations (5.16)-(5.18) hold, where the quantities q_i here depend on Θ_1.

Thanks to (5.22), we can prove that $\Psi = G(\Psi^I) \in B_{\Theta_1}$ if we show that $\|\mathcal{U}^{\Psi^I}\|_1 \le \mu^-$; this inequality can be achieved taking into account that $\Psi^I \in B_{\Theta_1}$ and

$$\lambda_i(\mathcal{U}_i^{\Psi^I})' = \mathcal{U}_i^{\Psi^I}(\Psi_i^I)' - \beta_i V_i \quad \text{for all } i \in \mathcal{M} \ ,$$

and then arguing as for (5.24) to obtain

$$\frac{4|\mathcal{A}|\overline{K}_2\overline{a}}{\underline{\lambda}}\ \mu^-\|U^{\Psi^I}\|_1 - \left(1 - \frac{4|\mathcal{A}|\overline{K}_2}{\underline{\lambda}} \sum_{j\in\mathcal{J}} |P_j|\right) \|U^{\Psi^I}\|_1 + \Omega \ge 0 \ .$$

The last part of the proof is equal to the one of Theorem 5.1 since, for $\Psi^I, \overline{\Psi}^I \in B_{\Theta_1}$ an equality like (5.21) holds, with Θ_1 in place of Θ and $|\mu_s|$ in place of μ_s. □

6. **Global solutions**

Here we use the results of Sections 4 and 5 to prove the existence of global solutions to problem (2.1)-(2.7). First we assume that $\mathcal{G}$ is an acyclic graph, so that the existence of some stationary solutions holds.

Let $\mu_s \ge 0$, let the assumptions of Theorem 5.1 hold and let $(U(x), V, \Psi(x))$ be the stationary solution satisfying (5.1) and (5.2); due to (5.14) we can control the size of the quantity $\|U\|_\infty + \|\Psi'\|_\infty$ by means of the size of μ_s, $|P_j|$ and $|V_i|$ ($i \in \mathcal{M}$, $j \in \mathcal{J}$), in order to satisfy the hypothesis in Theorem 4.1. So, such theorem yields the following one.

Let (u_0, v_0, ψ_0) satisfy (2.2) and let $\mathcal{W}_j \in W^{2,1}(0,T)$, $\mathcal{P}_j \in H^2(0,T)$, for each $T > 0$ and $j \in \mathcal{J}$. We set $\mu(t) := \sum_{i\in\mathcal{M}} \int_{I_i} u_{0i}(x)dx - \sum_{j\in\mathcal{J}} \int_0^t \mathcal{W}_j(s)ds$ and we assume that

$$(6.1) \qquad u_0(\cdot) - U \in H^1(\mathcal{A}) \ , \quad v_0(\cdot) - V \in H^1(\mathcal{A}) \ , \quad \psi_0(\cdot) - \Psi \in H^2(\mathcal{A}) \ ,$$

$$(6.2) \ \mathcal{P}_j(\cdot) - P_j \in H^1(0,+\infty), \ \mathcal{W}_j(\cdot) - W_j \in W^{2,1}(0,+\infty), \ \mu(\cdot) - \mu_s \in L^2(0,+\infty).$$

Theorem 6.1. *Let $\mathcal{G}$ be an acyclic graph and let (2.8) hold. Let the assumptions of Theorem 5.1 hold and let $(U(x), V, \Psi(x))$ be the stationary solution to problem (2.1), (2.5),(2.6), (5.1), (5.2) verifying $U(x) \ge 0$. Then, if (6.1)-(6.2) hold and the quantities*

$$\mu_s \ , \ \|u_0 - U\|_{H^1}, \ \ \|v_0 - V\|_{H^1}, \ \ \|\psi_0 - \Psi\|_{H^2} \ , \sum_{j\in\mathcal{J}} |P_j| \ , \ |V_\iota| \ (\iota \in \mathcal{M}) \ ,$$

$$\sum_{j\in\mathcal{J}} \|\mathcal{P}_j - P_j\|_{H^1(0,+\infty)}, \quad \sum_{j\in\mathcal{J}} \|\mathcal{W}_j - W_j\|_{W^{2,1}(0,+\infty)}, \quad \|\mu - \mu_s\|_{L^2(0,+\infty)}$$

are suitably small, then the problem (2.1)-(2.7) has a global solution (u, v, ψ) such that

$$u, v \in C([0,+\infty); H^1(\mathcal{A})) \cap C^1([0,+\infty); L^2(\mathcal{A}))$$
$$\psi \in C([0,+\infty); H^2(\mathcal{A})) \cap C^1([0,+\infty); L^2(\mathcal{A})) \cap H^1((0,+\infty); H^1(\mathcal{A})) \ .$$

Moreover, for each $i \in \mathcal{M}$,

$$\lim_{t\to+\infty} \|u_i(t)-U_i\|_{C(\bar I_i)} = 0\ ,\ \lim_{t\to+\infty} \|v_i(t)-V_i\|_{C(\bar I_i)} = 0\ ,\ \lim_{t\to+\infty} \|\psi_i(t)-\Psi_i\|_{C^1(\bar I_i)} = 0\ .$$

On the other hand, if the assumptions of Theorem 5.2 hold, then the hypothesis in Theorem 4.1 can be satisfied by controlling the size of $|\mu_s|$, $|P_j|$ and $|V_i|$ ($i \in \mathcal{M}$, $j \in \mathcal{J}$); then, similarly to the above result, we obtain the existence of global solutions corresponding to data which are small perturbations of the stationary solution of Theorem 5.2, assuming that $|\mu_s|$, $|P_j|$ and $|V_i|$ ($i \in \mathcal{M}$, $j \in \mathcal{J}$) are suitably small.

In the cases of general networks, we notice that for each $\{P_j\}_{j\in\mathcal{J}}$, it easy to prove the existence of the stationary solution $(U(x), V(x), \Psi(x)) = (0, 0, \Psi^0(x))$, where Ψ^0 is the unique solution to problem (5.12) with $\mathcal{U}^f = 0$; the solution has null mass and satisfies the boundary conditions

$$V(e_j) = 0\ , \qquad \eta_j \Psi'(e_j) + d_j \Psi(e_j) = P_j\ , \quad j \in \mathcal{J}\ . \tag{6.3}$$

Moreover, if some particular relations among the parameters of the problem hold, then there exist stationary solutions constant on the whole network: if we assume that

$$\begin{cases} \dfrac{a_i}{b_i} = r \quad \text{for all } i \in \mathcal{M}, & \\ \text{if } P_j = 0 \text{ then } d_j = 0\ , & j \in \mathcal{J}\ , \\ \text{if } P_j \neq 0 \text{ then } d_j \neq 0 \ \text{ and } \dfrac{P_j}{d_j} = r\dfrac{\mu_s}{|\mathcal{A}|}, & j \in \mathcal{J}\ , \end{cases} \tag{6.4}$$

then, for any $\mu_s \in \mathbb{R}$ the triple $\left(\dfrac{\mu_s}{|\mathcal{A}|}, 0, r\dfrac{\mu_s}{|\mathcal{A}|}\right)$ is a stationary solution to (2.1),(2.5) -(2.8),(5.1) satisfying the boundary conditions (6.3).

Finally, Theorem 4.1 yields that the results of Theorem 6.1 hold for general networks when $\mu_s = 0$, $(U_i(x), V_i, \Psi_i(x)) = (0, 0, \Psi_i^0(x))$, $W_j = 0$ ($i \in \mathcal{M}$, $j \in \mathcal{J}$), and when $(U_i(x), V_i, \Psi_i(x)) = \left(\dfrac{\mu_s}{|\mathcal{A}|}, 0, r\dfrac{\mu_s}{|\mathcal{A}|}\right)$, $W_j = 0$ ($i \in \mathcal{M}$, $j \in \mathcal{J}$) and conditions (6.4) hold, for each $\mu_s \in \mathbb{R}$.

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