

Resilience of long modes in cosmological observables

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Abstract. By a careful implementation of gauge transformations involving long-wavelength modes, we show that a variety of effects involving squeezed bispectrum configurations, for which one Fourier mode is much shorter than the other two, *cannot* be gauged away, except for the unphysical *exactly* infinite-wavelength ($k = 0$) limit. Our result applies, in particular, to the Maldacena consistency relation for single-field inflation, yielding a *local* non-Gaussianity strength $f_{\text{NL}}^{\text{local}} = -(5/12)(n_S - 1)$ (with n_S the primordial spectral index of scalar perturbations), and to the $f_{\text{NL}}^{\text{GR}} = -5/3$ term, appearing in the dark matter bispectrum and in the halo bias, as a consequence of the general relativistic non-linear evolution of matter perturbations. Such effects are therefore physical and observable in principle by future high-sensitivity experiments.

Keywords: Inflation, Primordial Non-Gaussianity, Bias, Quantum Field Theory, Cosmological Perturbation Theory

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1 Introduction

The fundamental role of long-wavelength perturbations in cosmology has been recognised in various contexts: the study of primordial non-Gaussianities, the analysis of non-linearities in the dark matter and halo clustering properties, the study of specific secondary anisotropies in the Cosmic Microwave Background (CMB). In all these cases, the role of long-wavelength modes can be accounted for the calculation of specific observables (such as the dark matter bispectrum, the halo bias or the CMB bispectrum) by either adopting traditional higher-order perturbation theory techniques, or by considering - as a sort of useful short-cut tool - long modes as a local background (a “Separate Universe”) on top of which the contribution by shorter-wavelength perturbations can be accounted for, by means of a linear analysis, though the two techniques are completely equivalent. The strength of non-Gaussianity f_{NL} , which in the simplest single-field “standard inflation” case is of the order of the first slow-roll parameters ϵ and η [1–6], contains a *local* contribution, whose dominant signal resides in *squeezed* bispectrum triangles (i.e. such that one triangle side in Fourier space is much shorter than the other two), given by [5] $f_{\text{NL}}^{\text{local}} = -(5/12)(n_S - 1)$, where n_S is the primordial spectral index of scalar perturbations. The leading contribution to the bispectrum in the squeezed limit, for single-field standard inflation models, can be predicted without actually calculating the 3-point function but relying only on the 2-point one [7–10]; the so-called “Maldacena consistency relation” connects the bispectrum and the power-spectrum. The consistency condition in single-field inflation can also be derived by exploiting the residual symmetries of the gauge-fixed action for the comoving curvature perturbation ζ [11–14]. This prediction has to be compared with current upper bounds derived by the analysis of *Planck* satellite CMB data ($f_{\text{NL}}^{\text{local}} = -0.9 \pm 5.1$, [15]), while for the spectral tilt one has $n_S = 0.9652 \pm 0.0042$ [16], thus making this theoretical prediction still far from detectability. Observational prospects on CMB temperature and polarisation data and new large-scale structure surveys will have the capability to tighten the primordial non-Gaussianity bound to $f_{\text{NL}}^{\text{local}} = \mathcal{O}(1)$ [17–20]. More futuristic projects on CMB spectral distortions [21] and measurements of 21cm background fluctuations from the dark ages (see, e.g. [22] and refs. therein) may eventually reach the required sensitivity $f_{\text{NL}}^{\text{local}} \sim \mathcal{O}(10^{-2})$.

The long-short splitting technique was later adopted to study the second-order Sachs-Wolfe contribution to CMB anisotropies [23, 24], whose calculation confirms previous findings obtained by standard second-order perturbative techniques [25, 26]. Similarly, the CMB

non-Gaussianity arising from non-linear effects at recombination was successfully obtained by both techniques [27, 28].

As soon as general relativistic calculations of the dark matter density perturbations were performed up to second-order (see, e.g. [29]), a term mimicking local non-Gaussianity was found, as a manifestation of long-wavelength gravitational potential modes, which, once inside the horizon, modulate short-wavelength density perturbations. The amplitude of this ‘‘General Relativistic’’ non-linear effect (dubbed GR-non-Gaussianity, because of its genuinely non-Newtonian origin) is equivalent to a local non-Gaussianity strength $f_{\text{NL}}^{\text{GR}} = -5/3$ in the matter bispectrum. Later calculations confirmed that result and extended it to all perturbative orders [30]. The same term was then studied in connection with local non-Gaussianity corrections to the dark matter halo bias, where it manifests itself as a scale-dependent bias contribution, thereby adding to any primordial $f_{\text{NL}}^{\text{local}}$ -like term [31, 32]. This would then make such a GR effect observable in the galaxy power-spectrum [33] obtained from ongoing galaxy survey data.

An important issue however arises in this framework and has been long debated in the literature [9, 21, 28–30, 33–47], especially in connection with the observability of the consistency relation and of the GR contribution to halo bias: do these terms correspond to a physical observable effect, or can they be cancelled by a suitable coordinates transformation (spatial dilatation)? This cancellation has been often claimed as a manifestation of the Equivalence Principle [48], according to which the acceleration caused by a *uniform* gravitational field cannot be locally distinguished from that caused by a non-inertial reference frame.

The aim of this paper is to provide a critical discussion of the claim that the squeezed limit of ‘‘single-clock’’ primordial non-Gaussianity is not observable [36, 40, 40, 44, 49]. We find that long perturbation modes, containing any wavenumber $k \neq 0$, *cannot be gauged away*, neither by a spatial dilatation [37–39], nor by resorting to more sophisticated techniques, such as the use of Conformal Fermi Coordinates (CFC) [40–42, 44, 45]; the claimed cancellation indeed only takes place in the unphysical *exactly* infinite wavelength (i.e. $k = 0$) limit. The above mentioned effects, involving long but finite-wavelength perturbation modes, are therefore physical and observable in principle by future high-sensitivity experiments. In particular, primordial non-Gaussianity in single field inflation is observable even in the squeezed limit.

This paper is organised as follows. In Section 2 we discuss the transformation of the metric components under a gauge transformations involving long-wavelength modes. In Section 3 we show that under deformed space dilatations, the comoving curvature ζ behaves like a 3-scalar, and no shift is present for any finite value of the wave-number k . In Section 4 the transformation properties of the bispectrum under a deformed dilatation are discussed. A summary of our main results is given in the Section 5. Appendix A is devoted to a discussion of the CFC approach.

2 Long perturbation modes and local rescaling

Let us consider the perturbed metric around a Friedmann-Lemaître-Robertson-Walker (FLRW) space-time, which takes the following general form¹

$$ds^2 = (\bar{g}_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu = -dt^2 + a^2 \delta_{ij} dx^i dx^j + h_{\mu\nu} dx^\mu dx^\nu. \quad (2.1)$$

The perturbation of a generic quantity \mathcal{F} will be written as follows

$$\mathcal{F} = \bar{\mathcal{F}} + \mathcal{F}^{(1)} + \mathcal{F}^{(2)} + \dots \quad (2.2)$$

¹For simplicity we neglect here the spatial curvature \mathcal{K} , however, the analysis can be easily extended to cover also the case $\mathcal{K} \neq 0$.

Thus, at linear order, the perturbed metric $h_{\mu\nu}$ represents a small deviation from homogeneity and isotropy. Under an infinitesimal coordinate transformation $x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu$ the metric components transform as

$$\begin{aligned}\Delta h_{00} &= 2 \partial_t \epsilon^0, \\ \Delta h_{0i} &= \partial_i \epsilon^0 - a^2 \partial_t \epsilon^i, \\ \Delta h_{ij} &= -2 a \dot{a} \delta_{ij} \epsilon^0 - a^2 (\partial_j \epsilon^i + \partial_i \epsilon^j).\end{aligned}\tag{2.3}$$

As well known, cosmological perturbations can be decomposed into scalar, vector and tensor sectors, according to [50]

$$\begin{aligned}h_{00}^{(1)} &= -2 \phi^{(1)}, \\ h_{0i}^{(1)} &= a \left(\partial_i F^{(1)} + G_i^{(1)} \right), \\ h_{ij}^{(1)} &= a^2 \left(-2 \psi^{(1)} \delta_{ij} + \partial_{ij} B^{(1)} + \partial_j C_i^{(1)} + \partial_i C_j^{(1)} + D_{ij}^{(1)} \right);\end{aligned}\tag{2.4}$$

In particular, focusing on the scalar sector only and setting $\epsilon^i = \partial_i \epsilon$ we get

$$\begin{aligned}\Delta \phi^{(1)} &= -H \dot{\epsilon}^0, & \Delta F^{(1)} &= a^{-1} \epsilon^0 - a \dot{\epsilon}, & \Delta \psi^{(1)} &= H \epsilon^0, & \Delta B^{(1)} &= -2 \epsilon, \\ \Delta \mathcal{F}^{(1)} &= -\dot{\mathcal{F}} \epsilon^0;\end{aligned}\tag{2.5}$$

where \dot{f} denotes the time partial derivative of f . It is worth to notice that the decomposition (2.4) makes sense only if the metric perturbations have a non-trivial space dependence; the case of infinite-wavelength perturbation is special and will be discussed separately below. We will show that the standard transformation rules (2.5) are the only ones that can be continuously deformed into the infinite-wavelength limiting case, or, in Fourier space, to the $k \rightarrow 0$ case.

In cosmological applications, it is often useful to define the curvature of an hyper-surface defined by the condition $S = \text{const.}$, where S is a four-dimensional scalar. Such a curvature \mathcal{R}_S is defined in terms of the 3D Ricci scalar associated to the induced metric in S

$$\chi_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \quad n_\mu = \left(-g^{\alpha\beta} \partial_\alpha S \partial_\beta S \right)^{-1/2} \partial_\mu S,\tag{2.6}$$

as

$$\mathcal{R}_S = R - K_{\mu\nu} K^{\mu\nu} + K^2 + 2 \nabla_\nu (n^\mu \nabla_\mu n^\nu) - 2 \nabla_\mu (n^\mu \nabla_\nu n^\nu);\tag{2.7}$$

where ² $K_{\mu\nu} = \chi_\mu^\alpha \nabla_\alpha n_\nu$ is the extrinsic curvature of S , $K = K_{\mu\nu} \chi^{\mu\nu}$ its trace and R is the four-dimensional Ricci scalar. At the first order in perturbation theory we have

$$\mathcal{R}_S = \frac{4}{a^2} \partial^2 \left(\frac{H}{\dot{S}} S^{(1)} - \psi \right) + \dots \equiv \mathcal{R}_S^{(1)} + \dots \quad \partial^2 = \delta_{ij} \partial_i \partial_j.\tag{2.8}$$

The geometric nature of \mathcal{R}_S leads to the so-called gauge invariance of $\mathcal{R}_S^{(1)}$; namely from eq. (2.5) one gets $\Delta \mathcal{R}_S^{(1)} = 0$. As expected, $\mathcal{R}_S^{(1)}$ is insensitive to any purely spatial transformation, with $\epsilon^0 = 0$, under which neither $S^{(1)}$ nor $\psi^{(1)}$ change. We have assumed that at the background level S coincides with the hypersurface of homogeneity of the unperturbed FLRW spacetime.

²We denote by ∇ the covariant derivative associated to the Levi-Civita connection.

Notice that in generic coordinates, by choosing local coordinates y^a , $a = 1, 2, 3$ on S we have that

$$\frac{\partial t}{\partial y^a} = -\frac{\partial_i S}{\partial_t S} \frac{\partial x^i}{\partial y^a}. \quad (2.9)$$

For instance, in single-field inflation models, taking $S = \text{const.}$ as the hyper-surface whose normal is proportional to $\partial_\mu \varphi$, where φ is the inflaton field, we get

$$\mathcal{R}_S^{(1)} \propto \partial^2 \left(H v - \psi^{(1)} \right) \equiv \partial^2 \zeta, \quad v = \frac{\varphi^{(1)}}{\dot{\varphi}}. \quad (2.10)$$

The quantity ζ is the comoving curvature perturbation and it is well known that ζ is constant in time on super-horizon scales in the case of single-field inflation.

The case of very large-wavelength perturbations needs to be analyzed carefully. Indeed, consider the following transformation

$$x^i \rightarrow x'^i = e^\lambda x^i = (1 + \lambda + \dots) x^i, \quad (2.11)$$

corresponding to spatial diffeomorphisms, where λ can be space and time dependent. In this case, if we take $\epsilon^i = \lambda x^i$ and $\epsilon^0 = 0$, we have

$$\Delta h_{00} = 0 \quad \Delta h_{0i} = -a^2 x^i \dot{\lambda}; \quad (2.12)$$

while

$$\Delta h_{ij} = -a^2 (2 \lambda \delta_{ij} + x^i \partial_j \lambda + x^j \partial_i \lambda). \quad (2.13)$$

The 3-vector ϵ^i can be written as the gradient of a scalar $\epsilon^i = \partial_i \epsilon$, only if

$$\epsilon^i = \partial_i \epsilon \quad \Rightarrow \quad x^i \partial_j \lambda - x^j \partial_i \lambda = 0. \quad (2.14)$$

Of course, this is the case when $\lambda = \lambda_0 = \text{const.}$, where only a δ_{ij} term is generated in (2.13); however there is a one-parameter degeneracy between the variations of $\psi^{(1)}$ and $B^{(1)}$; indeed (2.13) is reproduced by taking

$$\begin{aligned} \Delta_{(\alpha)} \phi^{(1)} &= 0, & \Delta_{(\alpha)} F^{(1)} &= 0, & \Delta_{(\alpha)} S^{(1)} &= 0, \\ \Delta_{(\alpha)} B^{(1)} &= \lambda_0 (\alpha - 1) x^2, & \Delta_{(\alpha)} \psi^{(1)} &= \alpha \lambda_0. \end{aligned} \quad (2.15)$$

Furthermore, transformation rules (2.15) are often used by setting arbitrarily $\alpha = 1$. This 1-parameter degeneracy can be lifted by introducing a linear dependence on the coordinates, by taking $\lambda = \lambda_0 + \lambda_1 x^i n_i$, with n_i a constant 3-vector; we then get

$$\begin{aligned} \Delta \phi^{(1)} &= 0, & \Delta F^{(1)} &= 0, & \Delta S^{(1)} &= 0, \\ \Delta B^{(1)} &= -\frac{\lambda}{2} x^2, & \Delta \psi^{(1)} &= \frac{\lambda}{2}; & \lambda &= \lambda_0 + \lambda_1 x^i n_i. \end{aligned} \quad (2.16)$$

As a result, a linear dependence on \mathbf{x} lifts the degeneracy, by setting $\alpha = 1/2$; however with this choice the spatial metric is no longer diagonal. The situation, in general, is even worse when λ is a quadratic function of x^i ; namely $\lambda = \lambda_0 + x^i n_i + D_{ij} x^i x^j$ where \mathbf{D} is a constant symmetric matrix. In this case Δh_{ij} cannot be reproduced by any variation of ψ and B alone. Such ambiguity is solved by imposing that the variation of the spatial metric h_{ij} can be consistently reproduced by a change of the scalar modes in h_{ij} , i.e. if and only if

$\lambda = \lambda(t, |\mathbf{x}|)$, satisfying automatically (2.14). In general, any coordinates transformation of the form $\delta x^i = \partial_i \epsilon(t, |\mathbf{x}|)$ can be written as a *deformed dilatation* such that

$$\lambda x^i = \partial_i \epsilon \Rightarrow \lambda = \frac{\partial_r \epsilon}{r} \quad r = |\mathbf{x}|. \quad (2.17)$$

Thus, except for the special case $\lambda = \lambda_0 = \text{constant}$, the correct transformation rule is (2.5) and thus, being $\epsilon^0 = 0$, there is no shift of ψ . Such a shift can be present only in the very special case $\lambda = \lambda_0 = \text{constant}$ with the choice $\alpha = 1$. In other words, there is no spatial dilatation with *almost constant* $\lambda(|\mathbf{x}|)$ able to reduce with continuity to the $\lambda = \lambda_0$ case, simply because the large-scale limit $k \rightarrow 0$ and the smallness of the gradient term $x^i \partial_i \lambda$ do not commute, thereby giving rise to different transformation rules. As we will demonstrate in Section 3, this implies that terms of order $x^i \partial_i \lambda$ cannot be neglected or confused in a gradient expansion.

Indeed, the choice to set $\alpha = 1$ in the λ_0 case was first introduced by Weinberg [51, 52] to prove the constancy of ζ in the large-scale limit, and later in [12][13] to get the single-field consistency relations. However, here the *gauge redundancy* approach is applied in the large-scale limit; this consists in setting the scalar variation ΔB to zero by hand, a procedure which cannot be reproduced by a standard coordinate transformation. A typical use of such a “would-be-shift” is the attempt to gauge away any very long-wavelength component of a perturbation in ψ . Consider, for instance, the case of single-field inflation and fix the so-called ζ -gauge, obtained by setting [5]

$$\varphi^{(1)} = B^{(1)} = 0, \quad (2.18)$$

Thus, in this gauge, $\mathcal{R}_S^{(1)} = -\frac{4}{a^2} \partial^2 \psi^{(1)}$ and $\zeta = -\psi^{(1)}$; primordial non-Gaussianity is given in terms of the 3-point function of $\psi^{(1)}$ in this gauge. Suppose that we split $\psi^{(1)}$ in two parts $\psi^{(1)} = \psi_S + \psi_L$, with ψ_S containing only short (high-frequency) modes and ψ_L describing a perturbation made of very long wavelengths; can we get rid of ψ_L by using the shift of $\psi^{(1)}$ induced by a spatial rescaling of coordinates, and, at the same time, remain in the ζ -gauge (2.18)? The answer is negative, unless ψ_L is genuinely constant; otherwise, instead of shifting $\psi^{(1)}$ one is going to produce a non-vanishing $B^{(1)}$, moving away from this gauge (2.18). This is perfectly consistent with the dynamics of single-field inflation; indeed, in the gauge (2.18), the field $\psi^{(1)}$ satisfies the equation

$$\ddot{\psi} - \left(\frac{2\dot{H}}{H} + \frac{\ddot{H} + 6H\dot{H}}{2M_{\text{Pl}}^2 \dot{H}^2} + 3H \right) \dot{\psi} + \frac{1}{a^2} \partial^2 \psi = 0. \quad (2.19)$$

In such a gauge, therefore, a rescaling of the coordinates should send a solution of (2.19) into a new solution. Namely, considering an arbitrary shift $\Delta \psi^{(1)} = \lambda(\mathbf{x})$, λ should solve (2.19). Actually this is the case if $\partial^2 \lambda = 0$; thus $\lambda = \lambda_0 + \lambda_1 x^i n_i$, thereby confirming the above conclusion. In the Fourier basis, this means that λ_k is a delta-function with support in $k = 0$ that this is precisely what is needed in the proof of the generalised consistency relations [13, 14].

3 Deformed dilatations and gradient expansion

In this Section we show that, whatever the selected range of scales is, a deformed dilatation can be described via standard gauge transformation rules, avoiding the ψ shift in the large-scale limit. Consider a deformed dilatation of the form

$$x^i \rightarrow x'^i = x^i + \lambda(x) x^i, \quad \lambda = \frac{1}{(2\pi)^{3/2}} \int d^3 k e^{i\mathbf{k}\mathbf{x}} \lambda_k, \quad (3.1)$$

where usually λ_k is taken to be the filtered Fourier transform of some comoving curvature mode ζ_k . Often the window function W is supposed to select the “long mode” part of ζ , namely

$$\lambda_k = W_k \zeta_k, \quad (3.2)$$

that should be gauged away. A widely used choice ³ for W_k is the Heaviside step function centred on a particular scale k_c ,

$$W_k = \theta \left[\frac{1}{H} (k_c - k) \right], \quad (3.3)$$

where $k_c/H \ll 1$ at the time of interest. Naively, one could argue that performing such a “dilatation”, the off-diagonal terms of the transformed spatial metric $\sim x^j \partial_i \lambda$ in (2.15) are negligible, at leading order in a gradient expansion. However, integrating by parts we get

$$x^i \partial_j \lambda = -\frac{1}{(2\pi)^{3/2}} \int d^3k e^{i\mathbf{k}\cdot\mathbf{x}} \partial_{k^i} (k^j \lambda_k) + \text{BT}. \quad (3.4)$$

The window function is such that the boundary term BT can be set to zero, given that W selects long modes only. We have taken both λ_k and W_k functions of $k = |\mathbf{k}|$; as a result (2.14) is satisfied. Here differentiation with respect to k will be denoted by a prime. Thus, in Fourier space

$$\begin{aligned} (\Delta h_k)_{ij} &= -a^2 [2 \lambda_k \delta_{ij} - \partial_{k^j} (k^i \lambda_k) - \partial_{k^i} (k^j \lambda_k)] \\ &= 2 a^2 \frac{k^i k^j}{k} (\lambda_k)'. \end{aligned} \quad (3.5)$$

Unless $W_k \sim \delta(k)$, there is no term proportional to δ_{ij} , while $(\Delta h_k)_{ij}$ is reproduced by

$$\Delta B_k = -\frac{\Delta h_{ij}}{a^2 k^i k^j} = -\frac{2}{k} \lambda_k'; \quad (3.6)$$

where $\lambda_k' = \partial_k \lambda_k$. No shift of ψ is present and, once again, we recover the general transformation rule (2.5). Indeed, by using (3.4) we obtain ϵ such that $\partial_i \epsilon = \epsilon^i$ is given in Fourier space by

$$\epsilon_k = \frac{\lambda_k'}{k}; \quad (3.7)$$

thus (3.6) is exactly the Fourier transform of (2.5), as expected. Despite the filtering procedure which drops short modes, the transformation rule (2.5) is not altered. Our result is in contrast with the transformation rule of the curvature perturbation and definition advocated in [44], where, instead of the comoving curvature perturbation, a quantity more similar to the local number of e-fold is defined⁴

$$\zeta^{(\text{alt})} = \delta N = \frac{1}{6} \frac{h_{ii}}{a^2} = -\psi + \frac{1}{6} \partial^2 B. \quad (3.8)$$

While the quantity δN , which coincides with ζ in the comoving gauge[54–57], is often assumed to be still approximately equal to ζ in CFC and other gauges where the term $\partial^2 B$ is assumed

³See for instance [40], where a transformation to Conformal-Fermi-Coordinates is approximated by a deformed spatial dilatation with W taken as an Heaviside step function.

⁴Actually, the local number of e-folds contains terms of order $\partial^2 F$, coming from a proper time integration along the fluid world-line, with relative terms lying on the initial hyper-surface [53].

to be negligible in the gradient expansion. The $\partial^2 B$ in CFC is very peculiar, indeed it compensates the diagonal metric term $\psi \delta_{ij}$ leading to $g_{ij} \sim x^2 \partial^2 \zeta$. However, we have just shown that such gradient terms cannot be neglected and in Fourier space it does not matter how close to zero the x^i value is. Let us also stress that $\zeta^{(\text{alt})}$ is rather different from the comoving curvature perturbation ζ ; most crucially it is not gauge invariant! Indeed, in Fourier space, we have that, under the transformation (3.1),

$$\zeta_k^{(\text{alt})} \rightarrow \zeta_k^{(\text{alt})} + \frac{1}{3} k \lambda'_k \quad (3.9)$$

where again there is no shift term, and in addition the gauge invariance is simply lost. In Appendix (A) it is shown that the coordinate transformation that connects CFC to comoving coordinates can be written as a deformed dilatation with a suitable λ . The shift of ψ is crucial in arguing that the long mode can be gauged away by transforming the primordial bispectrum, or any N-point function obtained by a *local measurement* [42], from comoving to CFC coordinates and then canceling the leading term of the Maldacena consistency relation [40, 44, 49]. Contrary to what we have found, it is often claimed that under a spatial “long” deformed dilatation transformation where $\zeta^{(\text{alt})}$ ⁵ is split into a long and a short part, while the “long” $\zeta_L^{(\text{alt})}$ shifts by λ_L , the short part $\zeta_S^{(\text{alt})}$ is a genuine scalar quantity. In our analysis we do not find such a transformation property.

The first application of a constant dilatation was given in [51][52] as a tool to show the constancy of ζ under mild assumptions. A constant dilatation represents the residual gauge ambiguity in the Newtonian gauge of an unperturbed FLRW solution. Such a pure-gauge mode with $k = 0$ ⁶ can be promoted to a physical (adiabatic mode) perturbation by enforcing that it solves the subset of Einstein’s equations that are trivial at $k = 0$ (for instance in the Newtonian gauge this set corresponds to the ij Einstein’s equations.). In this sense a non-physical gauge mode that corresponds to a constant spatial dilatation is promoted to the physical adiabatic mode

$$\zeta|_{k \rightarrow 0} = -\lambda_0 - \frac{H}{a} C; \quad (3.10)$$

where, typically, the constant C corresponds to a decaying mode. Similarly, the Maldacena consistency relation and their extensions can be derived [9, 12–14] by promoting the redundancy of the ζ -gauge to a full-fledged adiabatic mode, extending the transformation including 3-special conformal transformations in addition to constant dilatations.

The Maldacena consistency relation relates the 3-point function of the comoving curvature perturbation to the 2-point function of the same quantity. It is important that such correlation functions are scalars under a change of the coordinates of the hyper-surface, see (2.7) and (2.8), namely

$$\mathcal{B}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \langle \zeta(\vec{x}_1) \zeta(\vec{x}_2) \zeta(\vec{x}_3) \rangle \rightarrow \mathcal{B}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \mathcal{B}(\vec{x}_1, \vec{x}_2, \vec{x}_3); \quad (3.11)$$

a similar relation holds true for the 2-point function⁷. This relation is a consequence of the geometric nature of ζ and of the general covariance of the action describing gravity and the

⁵We stress once again that such a quantity is not the gauge-invariant ζ unambiguously defined in any coordinates as being proportional to the Ricci scalar of the hyper-surface orthogonal to the inflaton velocity in single field inflation.

⁶The crucial difference from standard transformations is to impose a gauge redundancy in the $k = 0$ case

⁷We are interested on super-horizon scales where, at least when the Weinberg theorem applies, the time dependence in \mathcal{B} is negligible (for simplicity of notation, only spatial coordinates will be shown).

inflationary sector. According to [5], in the squeezed limit, where one of the three momenta is much smaller than the others, the primordial bispectrum assumes the following form

$$\langle \zeta(k_1)\zeta(k_2)\zeta(k_3) \rangle |_{k_3 \rightarrow 0} = -(2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) (n_s - 1) P(k_3) P(k_1). \quad (3.12)$$

If one accepts the transformation property (3.11), it seems of little physical interest the choice of coordinates used in the computation of \mathcal{B} . Indeed, once \mathcal{B} is found to be non-vanishing in a set of coordinates, the same will be true in any other set of coordinates. In the non-linear case, taking the expectation value on the Bunch-Davis vacuum state does not commute with the transformation of ζ .

By using CFC, in [40, 44] it was argued that actually *only* the short-wavelength part ζ_s of ζ can be defined as a scalar and, at the leading order, one has

$$\tilde{\zeta}_s(x) = \zeta_s(x) - \lambda x^i \partial_i \zeta_s(x); \quad (3.13)$$

where $\tilde{\zeta}_s$ is the short part of ζ in CFC and λ is the deformed dilatation which relates CFC and comoving coordinates, see A and [40, 42]. Their claim can be summarised by the following relation valid *in the squeezed limit*

$$\Delta\mathcal{B} = \tilde{\mathcal{B}}(x_1, x_2, x_3) - \mathcal{B}(x_1, x_2, x_3) \equiv -\mathcal{B}(x_1, x_2, x_3); \quad (3.14)$$

thus $\tilde{\mathcal{B}}_{\text{squeezed}} = 0$. According to our result, actually

$$\Delta\mathcal{B} \equiv 0, \quad (3.15)$$

in agreement with (2.5). Two independent arguments in favour of (3.15) will be given in Section 4. By using the *in-in formalism*, one can show that the effect of the transformation to CFC changes the Lagrangian by a total derivative. Alternatively, by direct computation of the leading order term in perturbation theory, $\Delta\mathcal{B}$ in Fourier space is found to be just a vanishing boundary term.

In summary, we have argued that single-field primordial non-Gaussianity is observable even in the squeezed limit, and thus the Maldacena consistency relation is physical and observable in principle. Such a result is in agreement with the results of [57, 58], according to which the local f_{NL} is invariant under a non-linear field redefinition, and frame-independent [59]. A similar result was achieved in [60] where it is argued that semi-classical relations are physical away from the exact $k_L = 0$ limit, including the consistency one, as it induces statistical anisotropy in the powerspectrum of short modes. Note that, the Maldacena consistency relation can be derived by using only the residual dilatation symmetry in the ζ -gauge. The sole effect of a change of time or a more complicated special conformal transformation is to give rise to higher order correction in the slow roll-parameters or a gradient correction in the consistency relation [13][61]⁸.

We conclude this section by taking into account another example where space-dependent dilatations were widely used in the literature [37, 45] to study the scale-dependence of the dark matter halo bias, in connection with possible signatures of primordial non-Gaussianity. In particular, disagreement persists in the literature about the possible total or partial cancellation of the effective $f_{\text{NL}}^{\text{GR}} = -5/3$ contribution. Looking at works where CFC or simpler dilatation transformations were used, the following two ingredients are always present

⁸In [61], the change of time is important to find the gauge redundancy in the Newtonian gauge.

1. the off-diagonal terms of the metric are neglected, invoking a gradient expansion; however, as we have just shown, in Fourier space they are of the same order as ζ itself and fundamental for the covariance of the theory, hence they cannot be neglected;
2. an extensive use of the separate universe approach is always present. According to this approach the metric is diagonal on very large scales, ensuring equivalence (in particular for the halo-bias) between comoving and synchronous gauges and between the δN local number of e-folds and the comoving curvature ζ . Finally, the comoving curvature is related to the energy-density field.

It seems that the presence of these two ingredients is sufficient to claim the cancellation of any single-clock primordial f_{NL} contribution in local measurements [42, 45]. However, in the light of our results some doubts are in order. The definition of ζ in CFC coordinates as the trace of the spatial metric g_{ij} and relating it to the energy-density can be dangerous. Such a quantity, which is of second order in the CFC expansion, is not a genuine 3-scalar invariant; see eq. (3.9). On the other hand, the energy density is well known to be related to the gravitational potential ψ via the Poisson equation, which in a generic gauge reads

$$8\pi G\rho = \frac{2}{a^2}\nabla^2\psi - \frac{6}{a^2}H\dot{\psi} + \frac{H}{a^3}\left(a\nabla^2\dot{B} - 2H\nabla^2F\right). \quad (3.16)$$

eq. (3.16) is perfectly invariant under the deformed dilatation because the B gauge change is compensated by the scalar F transformation. This is sufficient to conclude that the long mode with finite momentum k cannot be gauged away by a spatial dilatation, exactly as shown for the consistency relation. Sometimes the cancellation is motivated by resorting to the Equivalence Principle. Now ζ plays the role of a gravitational potential and indeed, according to our result it can be shifted by a constant, as a consequence of the ambiguity present at $k=0$. Such an ambiguity is present in the case of $\lambda(x) = \lambda_0 + \vec{x}\cdot\vec{b}$, see (3.1) and the end of the previous section. However, when $\vec{b}\neq 0$, (2.14) is not satisfied and thus such a λ is not a coordinates transformation that affects the scalar sector. As soon as λ is at least a quadratic function of \vec{x} , tidal effects kick in and no shift of ζ is present. In this sense, our result is in full agreement with the Equivalence Principle.

4 Bispectrum gauge transformation under deformed dilatations

In this Section, we show that under a deformed dilatation, the bispectrum of a generic three-scalar is unchanged, in the sense that $\Delta\mathcal{B} = \tilde{\mathcal{B}}(x_1, x_2, x_3) - \mathcal{B}(x_1, x_2, x_3) \equiv 0$. Tree-level correlation functions can be computed in two ways: by using the in-in formalism in the interacting picture or equivalently by solving the classical equation of motions at the required order in perturbation theory with free-field initial conditions, see for instance [62]. In the in-in formalism, correlations of fields are computed by relating the fields in the interaction picture (free fields) with Heisenberg picture fields perturbatively; thus non-linearities are encoded in such a relation. In the second approach fields are decomposed with creation and annihilation operators with non-linear modes.

Let us start with in-in formalism. It is sufficient to analyse the extra terms in the action, arising from the non-linear coordinate transformation (deformed dilatation) which connects comoving gauge, where v and B are set to zero, with CFC-like reference frame

$$\tilde{x}^i = e^\lambda x^i, \quad g_{ij} = a^2 e^{2\zeta} \delta_{ij}. \quad (4.1)$$

The action is invariant and can be written in ADM form as [5]

$$S = \int d^4x \sqrt{h} N \left[R^{(3)} + K_{ij} K^{ij} - K^2 + \mathcal{L}_m \right] \equiv \int d^4x \sqrt{h(x)} \mathcal{S}(x). \quad (4.2)$$

where h is the spatial metric determinant, N is the shift and K_{ij} is the extrinsic curvature tensor of $t = \text{constant}$, while \mathcal{L}_m is the Lagrangian for the inflaton ϕ . In the comoving gauge, the $t = \text{constant}$ hyper-surface coincides with $\phi = \text{const.}$ hyper-surface. The 3-scalar in (4.2) can be written as \mathcal{S}

$$\tilde{\mathcal{S}}(\tilde{x}) \equiv \mathcal{S}(x) = \bar{\mathcal{S}}(t) + \mathcal{S}^{(1)}(x) + \mathcal{S}^{(2)}(x) + \dots \quad (4.3)$$

Defining the quantity

$$\Delta_S = \sqrt{\tilde{h}(x)} \tilde{\mathcal{S}}(x) - \sqrt{h(x)} \mathcal{S}(x), \quad (4.4)$$

the bispectrum variation in-in formalism is given by

$$\begin{aligned} \Delta \mathcal{B} &= \langle \tilde{\zeta}(t, x)^3 \rangle - \langle \zeta(t, x)^3 \rangle \\ &= i \int_{t_0}^t dt' \left\langle \left[\zeta^3(t, x), \int d^3x \Delta_S^{(3)} \right] \right\rangle, \end{aligned} \quad (4.5)$$

the second equality can be obtained by using the first-order gauge invariance⁹ of ζ : $\tilde{\zeta}^{(1)}(x) = \zeta^{(1)}(x)$ and

$$L_{int} = -H_{int} = \int d^3x \sqrt{h} \mathcal{S}. \quad (4.6)$$

Now, under a generic redefinition of spatial coordinates $\delta x^i = \tilde{x}^i - x^i$, we get the following transformation

$$\left\{ \begin{array}{l} \tilde{\mathcal{S}}(x) = \mathcal{S}(x) - \delta x^i \partial_i (\mathcal{S}^{(1)}(x) + \mathcal{S}^{(2)}(x)) - \frac{1}{2} \delta x^i \delta x^j \partial_{ij} \mathcal{S}^{(1)}(x) + \delta x^j \partial_j (\delta x^i \partial_i \mathcal{S}^{(1)}) ; \\ \sqrt{\tilde{h}(x)} = a^3 + a^3 [3\zeta - \partial_i (x^i \lambda)] + \frac{1}{2} a^3 [9\zeta^2 - 6\zeta \partial_j (x^j \lambda) - 6\lambda x^j \partial_j \zeta] \\ \quad + \frac{1}{2} a^3 [9\zeta^3 - 9\zeta^2 \partial_j (x^j \lambda) - 18\zeta \lambda x^j \partial_j \zeta] + O(\lambda^2, \lambda^3). \end{array} \right. \quad (4.7)$$

For simplicity, we have omitted terms quadratic and cubic in λ , coming from the transformation of the determinant of the spatial metric h . All the λ^2 terms in the quadratic action can be canceled, by integrating by parts, and the same reasoning applies to $\lambda^2 - \lambda^3$ vertices in the cubic action¹⁰. Thus, the change of the spatial coordinates induces the following variation Δ_S up to the third order

$$\left\{ \begin{array}{l} \bar{\Delta}_S = 0, \\ \Delta_S^{(1)} = a^3 \bar{\mathcal{S}}(t) \partial_i (\lambda x^i), \\ \Delta_S^{(2)} = -\frac{1}{2} a^3 \partial_i [2\lambda \mathcal{S}^{(1)} x^i + 6\bar{\mathcal{S}}(t) \lambda \zeta x^i], \\ \Delta_S^{(3)} = -\frac{9}{2} a^3 \bar{\mathcal{S}}(t) \partial_i (\zeta^2 \lambda x^i) - 3a^3 \partial_i (\zeta \lambda \mathcal{S}^{(1)}) - a^3 \partial_i (\lambda \bar{\mathcal{S}}^{(2)} x^i), \end{array} \right. \quad (4.8)$$

we have omitted all the quadratic and cubic terms in λ . The final result is that, for all the relevant vertices, we get that the variation of the cubic Lagrangian is just a boundary term and then, substituting in eq. (4.5), we get

$$\Delta \mathcal{B} \equiv 0. \quad (4.9)$$

⁹As we have previously shown, excluding the very special case of a constant λ , this is the case.

¹⁰Note that being λ defined by long modes only, $\lambda^2 - \lambda^3$ vertices should imply triangles with two and three squeezed momenta that are not relevant in the squeezed limit.

The above analysis can be also extended to the case where a change of time is considered; the result will be again (4.9).

Let us now reproduce the same result in a different way. The first step consists in considering that the ζ curvature is a 3-scalar, i.e.

$$\tilde{\zeta}(\tilde{x}) = \zeta(x) \Rightarrow \tilde{\zeta}(x) = \zeta(x) - \lambda x^i \partial_i \zeta(x) + \dots \quad (4.10)$$

The variation of the spectrum due to the above non-linear transformation can be computed at the leading order as a correction proportional to a 4-point function, which is already non-vanishing for free fields. From (4.10), we get in Fourier space

$$\Delta\mathcal{B} = -\langle \zeta_{k_1} \zeta_{k_2} (\lambda x \cdot \partial \zeta) |_{k_3} \rangle - \langle \zeta_{k_1} (\lambda x \cdot \partial \zeta) |_{k_2} \zeta_{k_3} \rangle - \langle (\lambda x \cdot \partial \zeta) |_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle \equiv \sum_{i=1}^3 \mathcal{B}_i; \quad (4.11)$$

where by definition

$$\begin{aligned} (\lambda x \cdot \partial \zeta) |_k &= \int \frac{d^3x}{(2\pi)^9} e^{-i\mathbf{k}\cdot\mathbf{x}} \left(\prod_{i=1}^2 \int d^3p_i e^{i\mathbf{p}_i \cdot \mathbf{x}} x^i \partial_i [e^{i\mathbf{p}_2 \cdot \mathbf{x}} \zeta_{p_2} \lambda_{p_1}] \right), \\ \zeta_k &= v_k a_{\mathbf{k}} + v_k^* a_{\mathbf{k}}^\dagger, \quad \lambda_k = v_k W_k a_{\mathbf{k}} + \bar{v}_k^* W_k a_{\mathbf{k}}^\dagger. \end{aligned} \quad (4.12)$$

Focusing on the first term in relation (4.11); we get, at the leading order in perturbation theory,

$$\begin{aligned} \Delta\mathcal{B}_3 &= \int \frac{d^3x}{(2\pi)^9} e^{-i\mathbf{k}_3 \cdot \mathbf{x}} \prod_{i=1}^2 \int d^3p_i e^{i\mathbf{p}_i \cdot \mathbf{x}} x^i \partial_i [e^{i\mathbf{p}_2 \cdot \mathbf{x}} \langle \zeta_{k_1} \zeta_{k_2} \lambda_{p_1} \zeta_{p_2} \rangle] \\ &= - \int \frac{d^3p_2}{(2\pi)^6} \partial_{p_2^j} \left[p_2^j \langle \zeta_{k_1} \zeta_{k_2} \lambda_{(k_3-p_2)} \zeta_{p_2} \rangle \right], \end{aligned} \quad (4.13)$$

where the second line was obtained integrating by parts. Integrating $\Delta\mathcal{B}_3$ and applying the Wick's theorem, we get a vanishing boundary term considering both the presence of the window function W_k and the Dirac delta term which centers p_2 on one of the finite momentum k_i . The same procedure applies for the other $\Delta\mathcal{B}_i$ terms, obtaining that the total variation $\Delta\mathcal{B}$ vanishes as expected.

We conclude this Section by stressing how crucial the first-order transformation properties of the field whose bispectrum is analyzed, are. The only way to cancel the f_{NL} term in the squeezed limit is to allow the long modes to shift. Indeed, consider a general field χ such that at linear order

$$x^i \rightarrow \tilde{x}^i = (1 + \lambda) x^i \Rightarrow \tilde{\chi}_k = \chi_k (1 - W_k). \quad (4.14)$$

where W is the above mentioned window function which isolates long modes, i.e. simply

$$\tilde{\chi}_{k_L} \equiv 0, \quad \tilde{\chi}_{k_S} = \chi_{k_S}, \quad (4.15)$$

Being the action corrected thanks to total derivative terms, we trivially get

$$\langle \tilde{\chi}_{k_L} \tilde{\chi}_{k_{S2}} \tilde{\chi}_{k_{S3}} \rangle \sim \left\langle \left[\tilde{\chi}_{k_L} \chi_{k_{S2}} \chi_{k_{S3}}, \sqrt{\hbar} \mathcal{S} \Big|_{k k' k''} \right] \right\rangle. \quad (4.16)$$

The previous relation can be zero if and only if χ shifts, i.e. $\chi_{k_L} \equiv 0$, and in this sense the importance we gave to this point in the previous Section can be understood. Note that here the χ field can be replaced by ζ , ψ or the energy-density ρ , or any other field to which in the

literature such a property of transformation was applied. In this sense, the f_{NL} term can be removed only in the *strictly* $k_L \equiv 0$ limit, where a redundant gauge transformation can be implemented, however this limit has *no physical meaning* in cosmological observables.

Furthermore, if one chooses as χ field the local number of e-folds δN , instead of ζ , as done in CFC coordinates, using eq. (3.9) one should get that the result is window-function-dependent near the reference scale $k_L \sim k_c$ (usually, patch-dependent), it is even singular if W_k is taken to be a Heaviside step function as in eq. (3.3). This makes the claimed f_{NL} cancellation, even less robust in these coordinates.

5 Conclusions

The detection of primordial non-Gaussianity is one of the most important avenues of modern cosmology and forthcoming probes will be able to significantly improve our knowledge of inflation in the Early Universe. On one hand, in the squeezed limit, the amount of non-Gaussianity is completely fixed in a model-independent way in the case of standard single-field inflation; on the other hand, concerns about the physical observability of such a limit have been advocated. In the debate, it is crucial to determine how a very long perturbation affects the quantities of physical interest. We have reanalysed the transformation properties of cosmological observables and, in particular, of the curvature perturbation ζ and its related correlation functions. Our results imply that, excluding the case of infinitely long-wavelength (hence non-physical) perturbations, ζ is a genuine geometrical quantity and in particular it is gauge invariant at first order in perturbation theory. By using a deformed dilatation of spatial coordinates, no shift in ζ is found, no matter what window function is used to filter out short modes. A gauge ambiguity exists only in the strictly $k \rightarrow 0$ limit.

Let us recap briefly the argument here. The key property used in the cancellation is that, under a coordinate transformation, the long-wavelength part of $\psi(x) = \psi_S(x) + \psi_L(x)$ shifts, or equivalently ψ_L is absorbed as a local contribution to the scale-factor. From our analysis in section 2, such a shift does not exist for a physical perturbation. There is an ambiguity when one considers an infinitesimal spatial coordinate transformation of the kind $\delta x^i = \lambda x^i$, used to gauge away the long-wavelength part ψ_L .

Imagine now to expand (in a gradient expansion) λ in powers of x^i , $\lambda = \lambda_0 + n_i x^i + D_{ij} x^i x^j + \dots$. We have shown that at zeroth order, there is a 1-parameter ambiguity in the transformation properties of the scalar part of the metric, see (2.15). One can choose the parameter such that ψ shifts, although this is not the only possibility, and then in the new coordinate system $\tilde{\psi}_L = 0$. When higher-order terms in the expansion are considered, once a scalar-tensor decomposition is set, the ambiguity disappears and the only possibility is to leave ψ unchanged, while the off-diagonal part of g_{ij} is modified, recovering the standard transformation property (2.5). As a result, only unphysical, genuinely x-independent, constant modes can be gauged away. More explicitly, this means that a perfectly constant dilatation can be used to gauge away a truly constant perturbation ψ ; in Fourier space this is equivalent to take the λ_k proportional to a delta function with support in $k = 0$. Consider now the extension of a constant dilatation to long physical modes by introducing a suitable window function, as implicitly done in the literature. Take for instance a Heaviside step window function $W_k = \theta[H^{-1}(k_c - k)]$, as done in [40]; where k_c will be an unspecified cutoff scale, on the edge of the short-long region. If one performs the following coordinate transformation

$$\tilde{x}^i = x^i + \zeta_L(x) x^i, \quad \zeta_L(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int dk^3 e^{i\mathbf{k}\mathbf{x}} W_k \zeta_k, \quad (5.1)$$

in order to gauge away the long part of ζ defined as $\zeta = -\psi + H v$. As commonly stated, one should obtain the transformation

$$\zeta_L \rightarrow \tilde{\zeta}_L = 0, \quad (5.2)$$

at *zero* order in a gradient expansion. However, when one considers the $\partial_i \zeta_L$ terms, the correct gauge transformations are

$$\tilde{\psi} = \psi, \quad \tilde{B} = B - \frac{2}{k} \partial_k (W_k \zeta_k). \quad (5.3)$$

Indeed, the metric tensor term $\partial_i \partial_j B$ is of order ζ_k . Thus, once again, working with a physical mode, no shift is found. The crucial point necessary to get (5.2) is imposing that $\tilde{B} - B = 0$. The subtlety here is that, for more standard coordinates transformations, one can always neglect $\partial_i \partial_j B$ on large scales (doing so $\partial_i \partial_j (\tilde{B} - B) \rightarrow 0$), but this is simply not the case here. Thus, $\tilde{\psi} - \psi = 0$ and, given that at the relevant order in perturbation theory the velocity potential does not change going from comoving coordinates to CFC, namely $\tilde{v} - v = 0$, we get that the curvature perturbation cannot change

$$\tilde{\zeta} - \zeta = 0, \quad (5.4)$$

contrary to (5.2). As one can see from eq. (2.15), one can always choose $\alpha = 0$ and fixing the ambiguity present at $k = 0$ such that B actually transforms, making the gradient expansion continuous. Thus, we believe that any attempt to separate “local” from “global” effects should not introduce a discontinuity in the transformation properties of the metric. The bottom line is that no shift exists for ψ . Thus, a long (but not infinitely long!) physical mode *cannot be gauged away, even locally*. It is worth to notice the choice of what is called ζ in CFC coordinates, $\zeta^{(\text{alt})}$ in (3.8), is not gauge invariant, see (3.9), and does not coincide with ζ at large scale. Of course, the same discussion can be extended to any quantity directly related to ζ , such as the energy density $\rho_k \sim k^2 \zeta_k$. Last but not least, the transformation property (3.8) of ζ_{CFC} depends strongly on the window function W_k used; such a term of order $\partial_k W_k$ would then affect any n-point correlator, in the region close to $k = k_c$.

The impact on the bispectrum of ζ is relevant: we have found that, even in the squeezed limit, no cancellation of primordial non-Gaussianity takes place. Thus, the Maldacena consistency relation still represents an important feature to tame the zoo of inflationary models, according to the pattern of symmetry breaking during inflation. Interestingly, similar techniques, based on deformed spatial dilatations, were used to study the general relativistic scale-dependent contribution to the bias of dark matter halos, claiming the cancellation of the so-called $f_{\text{NL}}^{\text{GR}} = -5/3$ term in local measurements. Our analysis implies that also such a term is unaffected by deformed spatial dilatations.¹¹ Hence, these effects are physical and observable in principle by future high-sensitivity experiments.

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¹¹We will discuss this issue in more detail in a future publication.

A CFC vs. space-dependent dilatation

The coordinates transformation that relates CFC and comoving coordinates is a special case of deformed dilatation of Section 3. The perturbed conformal metric $\tilde{h}_{\mu\nu}$ is related to $h_{\mu\nu}$ by

$$\tilde{h}_{\mu\nu} = \frac{1}{a^2} h_{\mu\nu}. \quad (\text{A.1})$$

Conformal Fermi Coordinates can be described by the following tetrads in comoving coordinates ¹²:

$$\begin{aligned} e_0^\mu &= \frac{1}{a} \left(1 + \frac{1}{2} \tilde{h}_{00}, V^i \right) \\ e_i^\mu &= \frac{1}{a} \left(0, \delta_j^i - \frac{1}{2} \tilde{h}_j^i \right) \end{aligned} \quad (\text{A.2})$$

Neglecting second-order perturbations, the cubic transformation from the comoving to the CFC frame has the following form:

$$\begin{aligned} \Delta x_F^k &= \Delta x^k - \bar{\Delta} x^k \psi|_p - \frac{1}{2} \bar{\Delta} x^i \bar{\Delta} x^j \left(\delta_k^i \partial_j \psi + \delta_k^j \partial_i \psi - \delta_j^i \partial_k \psi \right) |_p \\ &\quad - \frac{1}{6} \bar{\Delta} x^i \bar{\Delta} x^j \bar{\Delta} x^l \left(\delta_k^i \partial_{jl} \psi + \delta_k^j \partial_{il} \psi - \delta_j^i \partial_{ki} \psi \right) |_p \end{aligned} \quad (\text{A.3})$$

where $\Delta x^\mu = x^\mu(\tau) - p^\mu(\tau)$, is the deviation from a central worldline p^μ , and $\bar{\Delta} x^\mu$ its background value.

We can take $\psi(x) = \psi(|\bar{\Delta}x|)$ and $\partial_j \tilde{\psi}|_p = \partial_j \psi|_{\bar{\Delta}x=0}$ and, thus, without loss of generality,

$$\begin{aligned} \Delta x_F^k &\equiv x_F^k = x^k - x^k \mathcal{V}(x) - x^k \psi|_0 - \frac{1}{2} x^i x^j \left(\delta_k^i \partial_j \psi + \delta_k^j \partial_i \psi - \delta_j^i \partial_k \psi \right) |_0 \\ &\quad - \frac{1}{6} x^i x^j x^l \left(\delta_k^i \partial_{jl} \psi + \delta_k^j \partial_{il} \psi - \delta_j^i \partial_{ki} \psi \right) |_0 \end{aligned} \quad (\text{A.4})$$

There is still the freedom to choose the value of the coordinates of the central world-line at the initial proper-time τ_i ; one can set

$$p^i(\tau_i) = 0, \quad p^i(\tau) = \int_{\tau_i}^{\tau} v^i(\tau', 0) d\tau'. \quad (\text{A.5})$$

Focusing on the scalar sector we can always write the 3-velocity as a gradient

$$\int_{\tau_i}^{\tau} v^k|_p d\tau = \partial_k V(|\mathbf{x}|)|_{x^i=p^i} \equiv x^k \mathcal{V}(x). \quad (\text{A.6})$$

It is worth stressing that it is necessary to consider the relation between CFC and comoving coordinates at least at third order to get the correct off-diagonal spatial metric corrections at second order in x^i , which have the form $x^i x^j \partial_{ij} h$.

In order to find the function λ which defines the deformed dilatation one has to express $\psi|_0$, $\partial_i \psi|_0$ and $\partial_{ij} \psi|_0$ as a Taylor series centred on \mathbf{x} , at least at third order in x^i . After some tedious computation one gets, from eq. (A.4):

$$x_F^k = \left[1 + \zeta - \frac{1}{2} |\mathbf{x}| \partial_x \zeta(|\mathbf{x}|) + \frac{1}{6} |\mathbf{x}|^2 \partial_x^2 \zeta(|\mathbf{x}|) - \frac{1}{12} |\mathbf{x}|^3 \partial_x^3 \zeta(|\mathbf{x}|) - \mathcal{V}(|\mathbf{x}|) \right] x^k, \quad (\text{A.7})$$

¹²For details see [42]

where we have imposed $\psi(x) = \psi(|\mathbf{x}|)$. Thus, the transformation between spatial CFC and spatial comoving coordinates perfectly matches a space-time dependent dilatation with

$$\lambda = \zeta - \frac{1}{2}|\mathbf{x}| \partial_x \zeta(|\mathbf{x}|) + \frac{1}{6}|\mathbf{x}|^2 \partial_x^2 \zeta(|\mathbf{x}|) - \frac{1}{12}|\mathbf{x}|^3 \partial_x^3 \zeta(|\mathbf{x}|) - \mathcal{V}(|\mathbf{x}|) + O(|\mathbf{x}|^4). \quad (\text{A.8})$$

This simple procedure can be generalized at any order

$$\lambda^{(n)} = \sum_{l=0}^n \alpha^{(l)} |\mathbf{x}|^l \partial_x^l \zeta(|\mathbf{x}|) - \mathcal{V}(|\mathbf{x}|) + O(|\mathbf{x}|^{n+1}), \quad \alpha^{(0)} = 1. \quad (\text{A.9})$$

Of course, as discussed in the main text, ζ is the gauge-invariant ζ . Once a central world-line is chosen as the origin of the new *Lagrangian* coordinate system, one can take x to be arbitrarily (not necessarily small) by truncating the series to a sufficiently large n -th order. Finally, even if x is taken to be a small displacement from the central value (set to zero), in Fourier space any information about the smallness of a term $x^i \partial_i$ is lost for any value assumed by x , as shown in Section 3. If we neglect the \mathcal{V} presence, the new \tilde{g}_{ij} take the following form

$$\tilde{g}_{ij} = \frac{1}{3} x^k x^l \left(\delta_j^l \partial_{ik} + \delta_i^l \partial_{jk} - \delta_j^i \partial_{lk} - \delta_l^k \partial_{ij} \right) \psi + O(x^3), \quad (\text{A.10})$$

which is the same result obtained in [41]. Still in the series of works [40][41][42][44][45] it is assumed that the CFC transformation works in a region on an *unspecified* scale, depending on the dimension of the patch of Universe under investigation. Thus, all the quantities which appear in the previous relations are coarse-grained. In particular, this holds for eq. (A.9), where an Heaviside theta window function, defined as in (3.3), is understood. Finally, we stress once again that even if in eq. (A.10) the diagonal part of the spatial metric proportional to $\psi \delta_{ij}$ disappears, this does not mean that the ψ function is shifted.

References

- [1] Alejandro Gangui, Francesco Lucchin, Sabino Matarrese, and Silvia Mollerach. The Three point correlation function of the cosmic microwave background in inflationary models. *Astrophys. J.*, 430:447–457, 1994.
- [2] Alejandro Gangui and Jerome Martin. Cosmic microwave background bispectrum and slow roll inflation. *Mon. Not. Roy. Astron. Soc.*, 313:323, 2000.
- [3] Li-Min Wang and Marc Kamionkowski. The Cosmic microwave background bispectrum and inflation. *Phys. Rev. D*, 61:063504, 2000.
- [4] Viviana Acquaviva, Nicola Bartolo, Sabino Matarrese, and Antonio Riotto. Second order cosmological perturbations from inflation. *Nucl. Phys. B*, 667:119–148, 2003.
- [5] Juan Martin Maldacena. Non-Gaussian features of primordial fluctuations in single field inflationary models. *JHEP*, 05:013, 2003.
- [6] David H. Lyth and Yeinzon Rodriguez. Non-Gaussianity from the second-order cosmological perturbation. *Phys. Rev. D*, 71:123508, 2005.
- [7] Paolo Creminelli and Matias Zaldarriaga. CMB 3-point functions generated by non-linearities at recombination. *Phys. Rev. D*, 70:083532, 2004.
- [8] Clifford Cheung, A.Liam Fitzpatrick, Jared Kaplan, and Leonardo Senatore. On the consistency relation of the 3-point function in single field inflation. *JCAP*, 02:021, 2008.

- [9] Paolo Creminelli, Guido D’Amico, Marcello Musso, and Jorge Norena. The (not so) squeezed limit of the primordial 3-point function. *JCAP*, 1111:038, 2011.
- [10] Leonardo Senatore and Matias Zaldarriaga. A Note on the Consistency Condition of Primordial Fluctuations. *JCAP*, 08:001, 2012.
- [11] Paolo Creminelli, Jorge Norena, and Marko Simonovic. Conformal consistency relations for single-field inflation. *JCAP*, 07:052, 2012.
- [12] Kurt Hinterbichler, Lam Hui, and Justin Khoury. Conformal Symmetries of Adiabatic Modes in Cosmology. *JCAP*, 08:017, 2012.
- [13] Kurt Hinterbichler, Lam Hui, and Justin Khoury. An Infinite Set of Ward Identities for Adiabatic Modes in Cosmology. *JCAP*, 01:039, 2014.
- [14] Lam Hui, Austin Joyce, and Sam S.C. Wong. Inflationary soft theorems revisited: A generalized consistency relation. *JCAP*, 02:060, 2019.
- [15] Y. Akrami et al. Planck 2018 results. IX. Constraints on primordial non-Gaussianity. 5 2019.
- [16] N. Aghanim et al. Planck 2018 results. VI. Cosmological parameters. 7 2018.
- [17] Fabio Finelli et al. Exploring cosmic origins with CORE: Inflation. *JCAP*, 04:016, 2018.
- [18] Carmelita Carbone, Licia Verde, and Sabino Matarrese. Non-Gaussian halo bias and future galaxy surveys. *Astrophys. J. Lett.*, 684:L1–L4, 2008.
- [19] Olivier Doré et al. Cosmology with the SPHEREX All-Sky Spectral Survey. 12 2014.
- [20] Dionysios Karagiannis, Andrei Lazanu, Michele Liguori, Alvise Raccanelli, Nicola Bartolo, and Licia Verde. Constraining primordial non-Gaussianity with bispectrum and power spectrum from upcoming optical and radio surveys. *Mon. Not. Roy. Astron. Soc.*, 478(1):1341–1376, 2018.
- [21] Enrico Pajer and Matias Zaldarriaga. A New Window on Primordial non-Gaussianity. *Phys. Rev. Lett.*, 109:021302, 2012.
- [22] Julian B. Muñoz, Yacine Ali-Hamoud, and Marc Kamionkowski. Primordial non-gaussianity from the bispectrum of 21-cm fluctuations in the dark ages. *Phys. Rev. D*, 92(8):083508, 2015.
- [23] Nicola Bartolo, Sabino Matarrese, and Antonio Riotto. Non-Gaussianity of Large-Scale Cosmic Microwave Background Anisotropies beyond Perturbation Theory. *JCAP*, 08:010, 2005.
- [24] Lotfi Boubekur, Paolo Creminelli, Guido D’Amico, Jorge Norena, and Filippo Vernizzi. Sachs-Wolfe at second order: the CMB bispectrum on large angular scales. *JCAP*, 08:029, 2009.
- [25] Silvia Mollerach and Sabino Matarrese. Cosmic microwave background anisotropies from second order gravitational perturbations. *Phys. Rev. D*, 56:4494–4502, 1997.
- [26] Nicola Bartolo, Sabino Matarrese, and Antonio Riotto. Gauge-invariant temperature anisotropies and primordial non-Gaussianity. *Phys. Rev. Lett.*, 93:231301, 2004.
- [27] N. Bartolo, S. Matarrese, and A. Riotto. Non-Gaussianity in the Cosmic Microwave Background Anisotropies at Recombination in the Squeezed limit. *JCAP*, 02:017, 2012.
- [28] Paolo Creminelli, Cyril Pitrou, and Filippo Vernizzi. The CMB bispectrum in the squeezed limit. *JCAP*, 1111:025, 2011.
- [29] Nicola Bartolo, Sabino Matarrese, and Antonio Riotto. Signatures of primordial non-Gaussianity in the large-scale structure of the Universe. *JCAP*, 10:010, 2005.
- [30] Marco Bruni, Juan Carlos Hidalgo, and David Wands. Einstein’s signature in cosmological large-scale structure. *Astrophys. J. Lett.*, 794(1):L11, 2014.
- [31] Neal Dalal, Olivier Doré, Dragan Huterer, and Alexander Shirokov. The imprints of primordial non-gaussianities on large-scale structure: scale dependent bias and abundance of virialized objects. *Phys. Rev. D*, 77:123514, 2008.

- [32] Sabino Matarrese and Licia Verde. The effect of primordial non-Gaussianity on halo bias. *Astrophys. J. Lett.*, 677:L77–L80, 2008.
- [33] Licia Verde and Sabino Matarrese. Detectability of the effect of Inflationary non-Gaussianity on halo bias. *Astrophys. J. Lett.*, 706:L91–L95, 2009.
- [34] Eleonora Villa, Licia Verde, and Sabino Matarrese. General relativistic corrections and non-gaussianity in large-scale structure. *Classical and Quantum Gravity*, 31(23):234005, Nov 2014.
- [35] Stefano Camera, Mario G. Santos, and Roy Maartens. Probing primordial non-Gaussianity with SKA galaxy redshift surveys: a fully relativistic analysis. *Mon. Not. Roy. Astron. Soc.*, 448(2):1035–1043, 2015. [Erratum: *Mon. Not. Roy. Astron. Soc.* 467, 1505–1506 (2017)].
- [36] Takahiro Tanaka and Yuko Urakawa. Dominance of gauge artifact in the consistency relation for the primordial bispectrum. *Journal of Cosmology and Astroparticle Physics*, 2011(05):014?014, May 2011.
- [37] Roland de Putter, Olivier Doré, and Daniel Green. Is There Scale-Dependent Bias in Single-Field Inflation? *JCAP*, 1510(10):024, 2015.
- [38] Tobias Baldauf, Uros Seljak, Leonardo Senatore, and Matias Zaldarriaga. Galaxy bias and non-linear structure formation in general relativity. *Journal of Cosmology and Astroparticle Physics*, 2011(10):031?031, Oct 2011.
- [39] Bart Horn, Lam Hui, and Xiao Xiao. Lagrangian space consistency relation for large scale structure. *Journal of Cosmology and Astroparticle Physics*, 2015(09):068?068, Sep 2015.
- [40] Enrico Pajer, Fabian Schmidt, and Matias Zaldarriaga. The Observed Squeezed Limit of Cosmological Three-Point Functions. *Phys. Rev.*, D88(8):083502, 2013.
- [41] Liang Dai, Enrico Pajer, and Fabian Schmidt. On Separate Universes. *JCAP*, 10:059, 2015.
- [42] Liang Dai, Enrico Pajer, and Fabian Schmidt. Conformal Fermi Coordinates. *JCAP*, 11:043, 2015.
- [43] Nicola Bartolo, Daniele Bertacca, Marco Bruni, Kazuya Koyama, Roy Maartens, Sabino Matarrese, Misao Sasaki, Licia Verde, and David Wands. A relativistic signature in large-scale structure. *Phys. Dark Univ.*, 13:30–34, 2016.
- [44] Giovanni Cabass, Enrico Pajer, and Fabian Schmidt. How Gaussian can our Universe be? *JCAP*, 1701(01):003, 2017.
- [45] Giovanni Cabass, Enrico Pajer, and Fabian Schmidt. Imprints of Oscillatory Bispectra on Galaxy Clustering. *JCAP*, 09:003, 2018.
- [46] Kazuya Koyama, Obinna Umeh, Roy Maartens, and Daniele Bertacca. The observed galaxy bispectrum from single-field inflation in the squeezed limit. *JCAP*, 07:050, 2018.
- [47] Obinna Umeh, Kazuya Koyama, Roy Maartens, Fabian Schmidt, and Chris Clarkson. General relativistic effects in the galaxy bias at second order. *JCAP*, 05:020, 2019.
- [48] Nastassia Grimm, Fulvio Scaccabarozzi, Jaiyul Yoo, Sang Gyu Biern, and Jinn-Ouk Gong. Galaxy Power Spectrum in General Relativity. 5 2020.
- [49] Rafael Bravo, Sander Mooij, Gonzalo A. Palma, and Bastiàn Pradenas. Vanishing of local non-Gaussianity in canonical single field inflation. *JCAP*, 05:025, 2018.
- [50] J.M. Stewart. Perturbations of Friedmann-Robertson-Walker cosmological models. *Class. Quant. Grav.*, 7:1169–1180, 1990.
- [51] S. Weinberg. Adiabatic modes in cosmology. *Phys. Rev.*, D67:123504, 2003.
- [52] S. Weinberg. *Cosmology*. Oxford Univ. Press, 2008.

- [53] Sabino Matarrese, Luigi Pilo, and Rocco Rollo. ΔN formalism and conserved currents in cosmology. *JCAP*, 04:017, 2019.
- [54] Misao Sasaki and Ewan D. Stewart. A General analytic formula for the spectral index of the density perturbations produced during inflation. *Prog. Theor. Phys.*, 95:71–78, 1996.
- [55] David Wands, Karim A. Malik, David H. Lyth, and Andrew R. Liddle. A New approach to the evolution of cosmological perturbations on large scales. *Phys. Rev.*, D62:043527, 2000.
- [56] David H. Lyth, Karim A. Malik, and Misao Sasaki. A General proof of the conservation of the curvature perturbation. *JCAP*, 0505:004, 2005.
- [57] Ali Akbar Abolhasani and Misao Sasaki. Single-field consistency relation and δN -formalism. *JCAP*, 1808(08):025, 2018.
- [58] Guillem Domenech, Jinn-Ouk Gong, and Misao Sasaki. Consistency relation and inflaton field redefinition in the δN formalism. *Phys. Lett. B*, 769:413–417, 2017.
- [59] Cesar Gomez and Raul Jimenez. How Gaussian can the Sky be? Primordial Non-Gaussianity from Quantum Information. 5 2020.
- [60] Steven B. Giddings and Martin S. Sloth. Cosmological observables, infrared growth of fluctuations, and scale-dependent anisotropies. *Physical Review D*, 84(6), Sep 2011.
- [61] Paolo Creminelli, Jorge Noreña, Marko Simonović, and Filippo Vernizzi. Single-Field Consistency Relations of Large Scale Structure. *JCAP*, 1312:025, 2013.
- [62] Steven Weinberg. Quantum contributions to cosmological correlations. *Phys. Rev. D*, 72:043514, 2005.