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An existence result for quasiequilibrium problems in separable Banach spaces

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ABSTRACT

We establish the existence of solutions for a quasiequilibrium problem in the context of infinite-dimensional separable Banach spaces. To this purpose, we use a fixed point technique which is based on the notion of inside point of a convex set that appeared in 1956 in a paper by E. Michael [12]. As particular case, we provide the existence of a generalized Nash equilibrium requiring the lower semicontinuity of the strategy map and the closedness of the fixed point set.

1. Introduction

The well-known Ky Fan inequality [10] is a very general mathematical model, which embraces the formats of several disciplines. It is worthy of notice that Blum and Oettli [4] have considered this problem as a general model of equilibrium and for this reason it was named *equilibrium problem*. A quasiequilibrium problem is an equilibrium problem in which the constraint set is subject to modifications depending on the considered point. More precisely, given a subset C of a topological space $\mathbb{X}$, a set-valued map $K : C \rightrightarrows C$, and a function $f : C \times C \to \mathbb{R}$, the *quasiequilibrium problem* (QEP in short) consists in finding $\bar{x} \in C$ such that

$$\bar{x} \in K(\bar{x}) \quad \text{and} \quad f(\bar{x}, y) \geq 0, \quad \forall y \in K(\bar{x}).$$

This problem setting encompasses many relevant problems as special cases, among which variational and quasivariational inequalities, generalized Nash equilibrium problems, mixed quasivariational-like inequalities and so on.

Unlike the equilibrium problems which have an extensive literature on existence results, stability of the solutions, and solution methods (see [3] for a recent survey), the study of the QEPs to date is at the

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beginning even if they were introduced, to the best of our knowledge, in 1976 in a seminal paper of Mosco where an existence result was given in a real Hausdorff locally convex space [14, Theorem 7.1]. The author required the compactness of C, and both lower and upper semicontinuity on the constraint set-valued map K together with the convexity and the closedness of its values. Moreover, f was assumed to be monotone and upper semicontinuous together with the convexity and the lower semicontinuity with respect to the second variable. An outline of the proof is as follows: first, the fixed points of the map $S : C \rightrightarrows C$ defined by

$$S(x) = \{z \in K(x) : f(z, y) \geq 0,\ \forall y \in K(x)\}$$

coincide with the solutions of the QEP. A generalization of the Minty's lemma [13] allows to define a suitable equilibrium problem whose set of solutions is convex and coincides with $S(x)$. The Ky Fan's lemma guarantees that $S(x)$ is nonempty, and finally the Kakutani's fixed point theorem applied to the selection map S gives the solution.

By the application of this technique and exploiting some generalized monotonicity assumptions, Aussel, Cotrina and Iusem [2] succeeded in weakening the continuity assumptions on f exploiting the concept of upper sign property introduced in [7]; however, they required that the space $\mathbb{X}$ is finite dimensional and the values of the constraint map K have nonempty interior.

Conversely, other authors provided several existence results which do not involve any monotonicity assumption on f (see for instance the book [1]). In [8], a result was proved in the setting of finite-dimensional spaces, where the upper semicontinuity of K and the monotonicity on f were avoided. To be more precise, the upper semicontinuity of K was replaced by the weaker assumption of closedness of the fixed point set of K. We notice that under the assumption of compactness of C, the upper semicontinuity and closed valuedness of K is equivalent to the closedness of the graph of K which implies the closedness of the fixed point set, while the converse is not necessarily true. Nevertheless the finite-dimensionality assumption on $\mathbb{X}$ is a key tool for the proof of the existence result which relies on a selection theorem due to Michael. More recently, Cubiotti and Yao proposed [9] an extension of the results in [8] to the setting of infinite-dimensional normed space. They considered a particular case of QEP, the generalized Nash equilibrium problem, and an interesting existence result was established using the stronger assumptions of Hausdorff lower semicontinuity on K and nonemptiness of the interior of its values. The compactness of C was relaxed with a classical form of coercivity.

The aim of our paper is to extend the result in [8] to the setting of infinite-dimensional separable Banach space without requiring the Hausdorff lower semicontinuity of K. Moreover, we assume that K is constrained to have values on a suitable family of sets, denoted by $\mathcal{D}(\mathbb{X})$, and this assumption is weaker than the assumption of nonemptiness of the topological interior of $K(x)$. A similar result is obtained for generalized Nash equilibrium problems.

Instrumental to our results is the analysis of a generalized concept of relative interior introduced by Michael in [12] which, to the best of our knowledge, fell into oblivion despite its intrinsic interest. Section 2 is devoted to the study of this notion which will be compared with the notion of quasi relative interior more recently introduced in [6]. Section 3 is provided to establish existence results for a large class of QEPs and generalized Nash equilibria.

We conclude this introduction giving some notations. All along the paper, $\mathbb{X}$ stands for a real Banach space, $\mathbb{X}^*$ for its topological dual and $\langle \cdot, \cdot \rangle$ for the associated duality product. For a set $A \subseteq \mathbb{X}$, we denote by $\operatorname{cl} A$ the closure of A and by $\operatorname{cone} A$ the conic hull of A.

Let $\mathbb{Y}$ be a Banach space and $C \subseteq \mathbb{X}$ be a fixed convex set. A set-valued map F from C into $\mathbb{Y}$ assigns to each $x \in C$ a subset (eventually empty) $F(x)$ of $\mathbb{Y}$ and we write $F : C \rightrightarrows \mathbb{Y}$. The graph of F is just the subset of $C \times \mathbb{Y}$ defined by

$$\operatorname{gph} F = \{(x, y) \in C \times \mathbb{Y} : y \in F(x)\}.$$

A *selection* of F is a single-valued mapping $\varphi : C \to \mathbb{Y}$ such that $\varphi(x) \in F(x)$ for each $x \in C$. A fixed point of a set-valued map F from C to C itself is a point $x \in C$ satisfying $x \in F(x)$. We denote the set of fixed points of F by $\operatorname{fix} F$.

A set-valued map $F : C \rightrightarrows \mathbb{Y}$ is said *lower semicontinuous* if for any open subset Ω of $\mathbb{Y}$ the lower inverse set

$$F^{-1}(\Omega) = \{x \in C : F(x) \cap \Omega \neq \emptyset\}$$

is open. This is equivalent to say that if $x \in C$ and $\Omega \subset \mathbb{Y}$ is an open set with $F(x) \cap \Omega \neq \emptyset$, then there exists a neighborhood U_x of x such that $F(x') \cap \Omega \neq \emptyset$ for every $x' \in U_x \cap C$.

2. Notions of relative interior in Banach spaces

We shall begin by surveying some of the properties of faces and generalized interior that we require in our analysis. Suppose that $C \subseteq \mathbb{X}$ is convex; a convex set $S \subseteq C$ is a *face* of C if $x_1, x_2 \in C$, $t \in (0,1)$ and $tx_1 + (1-t)x_2 \in S$ implies $x_1, x_2 \in S$. The element of a singleton face is called an *extreme point* of C.

It is immediate that the intersection of any collection of faces is a face. Equally immediate are the following two properties which will be recalled in the sequel.

Lemma 2.1. *Let $C \subseteq \mathbb{X}$ be a convex set and $S \subseteq C$ be a face.*

(i) *If $D \subseteq C$ is a convex set, then $S \cap D$ is a face of D.*
(ii) *If H is a hyperplane which supports S, then $S \cap H$ is a face of C.*

Proof. Assertion (i) descends from the definition. For the latter assertion, fix a linear functional φ and $\alpha \in \mathbb{R}$ such that $H = \{x \in \mathbb{X} : \varphi(x) = \alpha\}$. Assume that $x_1, x_2 \in C$ and $t \in (0,1)$ such that $tx_1 + (1-t)x_2 \in S \cap H$. Since S is a face of C, we deduce that $x_1, x_2 \in S$; moreover, since H supports S,

$$\alpha = \varphi(tx_1 + (1-t)x_2) = t\varphi(x_1) + (1-t)\varphi(x_2) \geq \alpha.$$

Therefore $\varphi(x_1) = \varphi(x_2) = \alpha$ and x_1 and x_2 belong to H. □

In general the closed faces of a closed convex set have more tractable structure than arbitrary faces and they play a key role in the definition of a suitable algebraic interior of closed convex sets used by Michael in [12, Theorem 3.1‴]. We extend this definition to a generic convex set.

Definition 2.1. Let C be a convex subset of $\mathbb{X}$ and $\mathcal{F}$ the (possibly empty) collection of all proper closed faces of $\operatorname{cl} C$; then

$$I(C) = C \setminus \bigcup_{S \in \mathcal{F}} S$$

is the set of the *inside points* of C.

It is noteworthy that every nonempty, closed, convex, subset C of a separable Banach space $\mathbb{X}$ has $I(C)$ not empty [12, Lemma 5.1].

Let us compare in the following the set of the inside points with other three important generalized interiority notions for convex subsets of Banach spaces: the relative interior, the relative algebraic interior and the quasi relative interior. Assume that $C \subseteq \mathbb{X}$ is a nonempty and convex set, then

- the *relative interior* of C, denoted $\operatorname{ri} C$, is the interior of C within the closed affine hull of C;
- the *relative algebraic interior* of C is the set

$$\operatorname{icr}(C) = \{x \in C : \operatorname{cone}(C - x) \text{ is a linear subspace of } \mathbb{X}\};$$

- the *quasi relative interior* of C is the set

$$\operatorname{qri}(C) = \{x \in C : \operatorname{cl} \operatorname{cone}(C - x) \text{ is a linear subspace of } \mathbb{X}\}.$$

The quasi relative interior of a convex set is characterized by means of the normal cone as follows [6, Proposition 2.8].

Theorem 2.1. *Let C be a nonempty convex subset of $\mathbb{X}$ and $x \in C$. Then $x \in \operatorname{qri}(C)$ if and only if the normal cone of C at x*

$$N_C(x) = \{x^* \in \mathbb{X}^* : \langle x^*, y - x\rangle \le 0,\ \forall y \in C\}$$

is a linear subspace of $\mathbb{X}^*$.

Geometrically, $x \in \operatorname{qri}(C)$ if there is no proper closed supporting hyperplane to C at x, instead $x \in I(C)$ if there is no proper closed face of $\operatorname{cl} C$ containing x. More exactly we have the following relations between these notions.

Theorem 2.2. *For every nonempty convex set $C \subseteq \mathbb{X}$ the following inclusions holds*

$$\operatorname{ri}(C) \subseteq \operatorname{icr}(C) \subseteq I(C) \subseteq \operatorname{qri}(C).$$

If $\operatorname{ri}(C)$ is nonempty (in particular when $\mathbb{X}$ is a finite-dimensional space) then all the inclusions collapse to equality.

Proof. Of course $\operatorname{ri}(C) \subseteq \operatorname{icr}(C)$. Given $x \in \operatorname{icr}(C)$, assume by contradiction that $x \notin I(C)$ that means there exists a proper closed face S of $\operatorname{cl} C$ such that $x \in S$. Fix $y \in C$ different from x. By the definition of relative algebraic interior $x - y \in \operatorname{cone}(C - x)$ and therefore there exists $\varepsilon > 0$ such that $x + \varepsilon(x - y) \in C$. Hence x belongs to the interior of the segment $(y, x + \varepsilon(x - y))$ (it is enough to choose $t = (1 + \varepsilon)^{-1}$). Then $y \in S$ and, from the arbitrariness of y, we have $C \subseteq S \subseteq \operatorname{cl} C$. Taking the closure of all sets, we obtain the contradiction $S = \operatorname{cl} C$.

For the last inclusion, assume that $x \in C \setminus \operatorname{qri}(C)$. From Theorem 2.1 there exists $x^* \in N_C(x)$ such that $-x^* \notin N_C(x)$. Therefore

$$\langle x^*, y\rangle \le \alpha = \langle x^*, x\rangle, \quad \forall y \in C \tag{1}$$

and there exists $y_0 \in C$ such that

$$\langle x^*, y_0\rangle < \alpha = \langle x^*, x\rangle. \tag{2}$$

From (1) we deduce that $H = \{z \in \mathbb{X} : \langle x^*, z\rangle = \alpha\}$ is a hyperplane which supports C and, from the continuity of x^*, it also supports $\operatorname{cl} C$. Hence, from (2) and (ii) of Lemma 2.1, $H \cap \operatorname{cl} C$ is a closed proper face of $\operatorname{cl} C$ which contains x and therefore $x \notin I(C)$.

The reverse inclusions descend from [5, Proposition 2.12]. □

We conclude this short description showing that all the inclusions of Theorem 2.2 are in general strict.

Example 2.1. Consider $\mathbb{X} = \mathcal{C}([0,1])$; from the Weierstrass approximation theorem the convex subset $\mathbb{R}[x]$ of the polynomials has empty relative interior but $\operatorname{icr}(\mathbb{R}[x]) = \mathbb{R}[x]$.

Example 2.2. Consider $\mathbb{X} = \ell_2$ and

$$C = \{x \in \ell_2 : \|x\|_2 \le 1 \text{ and } x_k \ge 0 \text{ for all } k\}.$$

Since C is weakly compact and convex, it is a closed set and therefore $I(C) \neq \emptyset$ as observed above while $\operatorname{icr}(C)$ is empty. Indeed if $x \in \operatorname{icr}(C)$, then for all $y \in C$ there exists $t > 0$ such that $(1+t)x - ty \in C$. Clearly, $x_k > 0$ for all k (as otherwise choosing y with $y_k > 0$ the point $(1+t)x - ty$ doesn't belong in C), moreover there exists $\varepsilon > 0$ such that $\|x\|_2 < 1 - \varepsilon$ (as otherwise choosing $y = 0$ the point $(1+t)x - ty$ doesn't belong in C). Let $\{x_{k_n}\}$ be a subsequence of x such that $0 < x_{k_n} < \varepsilon/4^n$ and define $y \in C$ by $y_{k_n} = \varepsilon/2^n$ and $y_k = 0$ otherwise. Then the k_n-th component of $(1+t)x - ty$ is

$$(1+t)x_{k_n} - ty_{k_n} < (1+t)\frac{\varepsilon}{4^n} - t\frac{\varepsilon}{2^n} = \left(1 + t - 2^n t\right)\frac{\varepsilon}{4^n} < 0$$

for large enough n.

Example 2.3. Consider $\mathbb{X} = \ell_3$ and

$$C = \{x \in \ell_3 : \|x\|_2 \le 1\}.$$

Since the identity map embedding ℓ_2 in ℓ_3 is continuous and the unit ball in ℓ_2 is weakly compact, then C is weakly compact in ℓ_3. Moreover, $\|\cdot\|_2$ is a strictly convex norm, hence every point $x \in C$ with $\|x\|_2 = 1$ is an extreme point of C. So $x \notin I(C)$.

On the other hand, fix $x \in C$ with $\|x\|_2 = 1$ but $x \notin \ell_{3/2}$ which is the dual space of ℓ_3. We show that $N_C(x) = \{0\}$ which implies that $x \in \operatorname{qri}(C)$. Suppose $x^* \in N_C(x) \subseteq \ell_{3/2}$; then $\langle x^*, y - x\rangle \le 0$ for all $y \in C$ which implies

$$\|x^*\|_2 = \sup\{\langle x^*, y\rangle : y \in C\} \le \langle x^*, x\rangle \le \|x^*\|_2 \cdot \|x\|_2 = \|x^*\|_2.$$

Therefore we have equality throughout and, by Cauchy–Schwarz, x^* and x are linearly dependent. But $x^* \in \ell_{3/2}$ whereas $x \notin \ell_{3/2}$ and therefore $x^* = 0$.

The last part of this section deals with the family of convex sets $\mathcal{D}(\mathbb{X})$ defined as follows:

$$\mathcal{D}(\mathbb{X}) = \{C \subseteq \mathbb{X} : C \text{ is convex and } I(\operatorname{cl} C) \subseteq C\}.$$

This family was introduced in [12] where it was proved that it contains all the convex sets which are either closed or with nonempty interior. Furthermore, when $\mathbb{X}$ is finite dimensional, from Theorem 2.2 we have $I(\operatorname{cl} C) = \operatorname{ri}(\operatorname{cl} C) = \operatorname{ri}(C) \subseteq C$ for every convex set C and then $\mathcal{D}(\mathbb{X})$ coincides with the family of all convex sets.

For establishing our existence result, it is crucial to know when the intersection of two convex sets belongs to $\mathcal{D}(\mathbb{X})$. Unfortunately this class is not closed under the intersection as the following example shows.

Example 2.4. Fixed $x^* \in \mathbb{X}^*$ with $x^* \neq 0$, define the closed hyperplane $H = \{x \in \mathbb{X} : \langle x^*, x\rangle = 0\}$ which is a Banach space. Thanks to the axiom of choice, there exists a discontinuous linear functional φ from H to $\mathbb{R}$ which in turn defines the hyperplane $H_\varphi = \{x \in H : \varphi(x) = 0\}$ that is dense in H. Consider the convex sets

$$C_1 = \{x \in \mathbb{X} : \langle x^*, x\rangle < 0\} \cup H_\varphi \quad \text{and} \quad C_2 = \{x \in \mathbb{X} : \langle x^*, x\rangle > 0\} \cup H_\varphi.$$

Since C_1 and C_2 have nonempty interior, they belong to $\mathcal{D}(\mathbb{X})$. On the converse $C_1 \cap C_2 = H_\varphi \notin \mathcal{D}(\mathbb{X})$ since clearly $\operatorname{cl}(C_1 \cap C_2) = H$ and $I(H) = H$.

However, if at least one of the two sets is open, then the following result is obtained.

Lemma 2.2. *Given $C \in \mathcal{D}(\mathbb{X})$ and $\Omega \subseteq \mathbb{X}$ open and convex, then $C \cap \Omega \in \mathcal{D}(\mathbb{X})$.*

Proof. Clearly $\Theta = C \cap \Omega$ is convex and hence we have only to prove that $I(\operatorname{cl}\Theta) \subseteq \Theta$ that means that every $x \in \operatorname{cl}\Theta \setminus \Theta$ doesn't belong to $I(\operatorname{cl}\Theta)$. Since $x \notin \Theta$, then either $x \notin \Omega$ or $x \in \Omega \setminus C$. In the first case, since Ω is open from the separation theorem, there exist $x^* \in \mathbb{X}^*$ and $\alpha \in \mathbb{R}$ such that

$$\langle x^*, x\rangle = \alpha < \langle x^*, y\rangle, \quad \forall y \in \Omega.$$

Hence, from the continuity of x^* we deduce

$$\langle x^*, y\rangle \geq \alpha, \quad \forall y \in \operatorname{cl}\Theta.$$

Therefore $H = \{x \in \mathbb{X} : \langle x^*, x\rangle = \alpha\}$ is a closed hyperplane which supports $\operatorname{cl}\Theta$ at x but does not contain Θ. So, by item (ii) of Lemma 2.1, $H \cap \operatorname{cl}\Theta$ is a proper closed face of $\operatorname{cl}\Theta$. Since x belongs to this face it can't belong to $I(\operatorname{cl}\Theta)$.

In the second case, since C belongs to $\mathcal{D}(\mathbb{X})$ and $x \in \operatorname{cl}C \setminus C$, then there exists a proper closed face S of $\operatorname{cl}C$ such that $x \in S$. From item (i) of Lemma 2.1, the closed and convex set $S \cap \operatorname{cl}\Theta$ is a face of $\operatorname{cl}\Theta$. It will be enough to prove that $S \cap \operatorname{cl}\Theta \neq \operatorname{cl}\Theta$. By contradiction, assume that $\operatorname{cl}\Theta \subseteq S$. Pick $y \in \operatorname{cl}C$, then $(x, y) \cap \operatorname{cl}C \cap \Omega \neq \emptyset$ and $\operatorname{cl}C \cap \Omega \subseteq \operatorname{cl}\Theta \subseteq S$. This implies that $y \in S$ and $\operatorname{cl}C = S$ which is absurd. □

The following result dealing with the Cartesian product of two sets is easier to prove.

Lemma 2.3. *Given $C_1 \in \mathcal{D}(\mathbb{X}_1)$ and $C_2 \in \mathcal{D}(\mathbb{X}_2)$ where $\mathbb{X}_1$ and $\mathbb{X}_2$ are two Banach spaces, then $C_1 \times C_2 \in \mathcal{D}(\mathbb{X}_1 \times \mathbb{X}_2)$.*

Proof. It is enough to verify the following chain of inclusions

$$I(\operatorname{cl}(C_1 \times C_2)) = I(\operatorname{cl}C_1 \times \operatorname{cl}C_2) \subseteq I(\operatorname{cl}C_1) \times I(\operatorname{cl}C_2) \subseteq C_1 \times C_2.$$

The first equivalence is clear and the last inclusion descends from the assumption. Now take $(x_1, x_2) \in \operatorname{cl}C_1 \times \operatorname{cl}C_2$ such that $(x_1, x_2) \notin I(\operatorname{cl}C_1) \times I(\operatorname{cl}C_2)$. We may assume without loss of generality that $x_1 \notin I(\operatorname{cl}C_1)$ and therefore there is a proper and closed face S_1 of $\operatorname{cl}C_1$ such that $x_1 \in S_1$. But $S_1 \times \operatorname{cl}C_2$ is a proper and closed face of $\operatorname{cl}C_1 \times \operatorname{cl}C_2$ containing (x_1, x_2). Hence $(x_1, x_2) \notin I(\operatorname{cl}C_1 \times \operatorname{cl}C_2)$ which ends the proof. □

3. The existence result

We are in position to describe the main two results we need to prove our existence theorem. The former is the Schauder's fixed point theorem whose history dates back almost one hundred years [16].

Theorem 3.1 *(Schauder's fixed point). Every continuous function φ from a convex compact subset C of a Banach space to C itself has a fixed point.*

The latter is one of a series of subtle theorems proved by Michael [12] and concerning the existence of continuous selections. Let $\mathbb{X}$ and $\mathbb{Y}$ be topological spaces and $\mathcal{S}$ be a collections of subsets of $\mathbb{Y}$. The selection problem handled by Michael can be rephrased as follows. Under what conditions on $\mathbb{X}$, $\mathbb{Y}$ and $\mathcal{S}$, every lower semicontinuous set-valued map $F : \mathbb{X} \rightrightarrows \mathbb{Y}$ with values in $\mathcal{S}$ admits a continuous selection? A possible solution of this problem is indicated in the following result [12, Theorem 3.1‴].

Theorem 3.2 *(Michael selection). Every lower semicontinuous set-valued map F from a perfectly normal T_1 space to a separable Banach space $\mathbb{Y}$ with nonempty values in the class $\mathcal{D}(\mathbb{Y})$ admits a continuous selection.*

We recall that a perfectly normal space is a topological space in which every two disjoint nonempty closed sets can be precisely separated by a continuous function from $\mathbb{X}$ to the real line $\mathbb{R}$. All metric spaces are perfectly normal Hausdorff. Collecting these two results, we deduce the following fixed point theorem for lower semicontinuous set-valued maps.

Corollary 3.1. *Every lower semicontinuous set-valued map F from a convex compact subset C of a separable Banach space $\mathbb{X}$ to C itself with nonempty values in the class $\mathcal{D}(\mathbb{X})$ has a fixed point.*

Notice that, unlike the famous Kakutani fixed point Theorem [11] in which the closedness of $\operatorname{gph} f$ is required, we need the lower semicontinuity of the set-valued map.

Theorem 3.3. *Let $\mathbb{X}$ be a separable Banach space and C be a convex and compact set. Assume that K is lower semicontinuous with $K(x) \in \mathcal{D}(\mathbb{X})$ nonempty for every $x \in C$, and $\operatorname{fix} K$ is closed. Moreover, suppose that $f(x,x) \geq 0$ for all $x \in C$, the level set*

$$\{y \in C : f(x,y) < 0\} \tag{3}$$

is convex for all $x \in \operatorname{fix} K$ and the level set

$$\{(x,y) \in \operatorname{fix} K \times C : f(x,y) < 0\} \tag{4}$$

is open on $\operatorname{fix} K \times C$, then the QEP has solution.

Proof. First, Corollary 3.1 ensures the nonemptiness of the set $\operatorname{fix} K$.

The set-valued map $F : \operatorname{fix} K \rightrightarrows C$ defined by

$$F(x) = \{y \in C : f(x,y) < 0\}$$

is convex valued and its graph is open on $\operatorname{fix} K \times C$ by assumption.

Now we show that the set-valued map $G = K \cap F$ from $\operatorname{fix} K$ to C is lower semicontinuous. Take $x \in \operatorname{fix} K$ and $\Omega \subseteq \mathbb{X}$ open set such that $G(x) \cap \Omega \neq \emptyset$. Choose an element y belonging to $G(x) \cap \Omega$. Since $\operatorname{gph} F$ is open, there exist two neighborhoods U'_x and U_y such that

$$U_y \cap C \subseteq F(x'), \quad \forall x' \in U'_x \cap \operatorname{fix} K.$$

Moreover, $y \in K(x) \cap U_y \cap \Omega$ and K is lower semicontinuous, then there exists a neighborhood U''_x such that

$$K(x') \cap U_y \cap \Omega \neq \emptyset, \quad \forall x' \in U''_x \cap C.$$

Hence, for all $x' \in U_x = U'_x \cap U''_x$ we have

$$G(x') \cap \Omega = K(x') \cap F(x') \cap \Omega \supseteq K(x') \cap U_y \cap C \cap \Omega \neq \emptyset$$

as claimed.

In the last part of the proof we show that there exists $x \in \operatorname{fix} K$ such that $G(x) = \emptyset$ which implies that x is a solution of QEP. Assume by contradiction that $G(x) \neq \emptyset$ for all $x \in \operatorname{fix} K$ and consider the set-valued map $H : C \rightrightarrows C$ defined by

$$H(x) = \begin{cases} G(x) & \text{if } x \in \operatorname{fix} K, \\ K(x) & \text{if } x \notin \operatorname{fix} K. \end{cases}$$

Clearly H is nonempty valued and Lemma 2.2 guarantees that $H(x) \in \mathcal{D}(\mathbb{X})$ for each $x \in C$. Moreover, H is lower semicontinuous. Indeed fix $x \in C$ and an open set $\Omega \subseteq \mathbb{X}$ such that $H(x) \cap \Omega \neq \emptyset$. If $x \in \operatorname{fix} K$, from the lower semicontinuity of G there exists a neighborhood U'_x such that

$$G(x') \cap \Omega \neq \emptyset, \quad \forall x' \in U'_x \cap \operatorname{fix} K.$$

Since $G(x) \subseteq K(x)$, from the lower semicontinuity of K there exists a neighborhood U''_x such that

$$K(x') \cap \Omega \neq \emptyset, \quad \forall x' \in U''_x \cap C$$

and then

$$H(x') \cap \Omega \neq \emptyset, \quad \forall x' \in U'_x \cap U''_x \cap C.$$

Instead if $x \notin \operatorname{fix} K$, since K is lower semicontinuous, then there exists a neighborhood U'_x such that

$$K(x') \cap \Omega \neq \emptyset, \quad \forall x' \in U'_x \cap C.$$

Moreover, $C \setminus \operatorname{fix} K$ is open on C, and therefore there exists U''_x such that $U''_x \cap C \subseteq C \setminus \operatorname{fix} K$. Again

$$H(x') \cap \Omega \neq \emptyset, \quad \forall x' \in U'_x \cap U''_x \cap C$$

and the lower semicontinuity of H is achieved.

Hence, all the assumptions of Corollary 3.1 are satisfied and then there exists a point $x \in C$ such that $x \in H(x)$. Clearly $x \in \operatorname{fix} K$ and this implies $f(x, x) < 0$ which contradicts the assumption on f. □

It is useful to give sufficient conditions implying the assumptions of the theorem. Clearly the convexity of the level set (3) can be deduced from the quasiconvexity of $f(x, \cdot)$ for all $x \in \operatorname{fix} K$. Instead the upper semicontinuity of f implies the openness of the level set (4).

As a corollary of Theorem 3.3 one recovers a result by Cubiotti [8, Theorem 2.1], who also considers the case when the set-valued map K involved in QEP is lower semicontinuous. However, his proof depends heavily on the finite dimensionality of the space, while ours is valid in separable Banach spaces.

More recently Cubiotti and Yao proved [9, Theorem 1.1] an existence result for the Nash equilibria of a noncooperative generalized game with infinite-dimensional strategy spaces. The main specificity of their result, that was in the same spirit of [8], was the absence of upper semicontinuity assumptions on the set-valued functions defining the feasible strategies of the players. Nevertheless the authors required the Hausdorff lower semicontinuity instead of the weaker notion of lower semicontinuity. The last part of our paper is devoted to deduce from Theorem 3.3 a new existence result which will be compared with theirs.

The generalized Nash equilibrium problem (GNEP for short) is a noncooperative game in which, in contrast to the standard Nash equilibrium problem, the strategy set of each player depends on the strategies of all the other players. The GNEP is a model which has been used actively in many fields (electricity, telecommunications, transportation and others) in the past fifty years but it is only from nineties that research on this topic picked up speed and strength.

Formally the GNEP consists of N players, each labeled by an integer $i = 1, \dots, N$. Each player i controls the variables $x^i \in C_i$, where C_i is a nonempty convex and closed subset of a Banach space $\mathbb{X}_i$. We denote by $\mathbf{x} = (x^1, \dots, x^N) \in \prod_{i=1}^N C_i = C$ the vector formed by all these decision variables and by $\mathbf{x}^{-i}$ we denote the strategy vector of all the players different from player i. The set of all such vectors will be denoted by C^{-i}. We sometimes write $(x^i, \mathbf{x}^{-i})$ instead of $\mathbf{x}$ in order to emphasize the i-th player's variables within $\mathbf{x}$. Note that this is still the vector $\mathbf{x} = (x^1, \dots, x^i, \dots, x^N)$, and the notation $(x^i, \mathbf{x}^{-i})$ does not mean that the block components of $\mathbf{x}$ are reordered in such a way that x^i becomes the first block. Each player i has an objective function $\theta_i : C \to \mathbb{R}$ that depends on all players' strategies. Furthermore, each player's strategy must belong to a set identified by the set-valued map $K_i : C^{-i} \rightrightarrows C_i$ in the sense that the strategy space of the player i is $K_i(\mathbf{x}^{-i})$ which depends on the rival players' strategies $\mathbf{x}^{-i}$.

The aim of player i, given the other players' strategies $\mathbf{x}^{-i}$, is to choose a strategy x^i that solves the following optimization problem

$$\min\{\theta_i(x^i, \mathbf{x}^{-i}) : x_i \in K_i(\mathbf{x}^{-i})\}. \tag{5}$$

For any $\mathbf{x}^{-i}$, the solution set of the problem (5) is denoted by $S_i(\mathbf{x}^{-i})$. The GNEP consists in finding a vector $\bar{\mathbf{x}}$ such that

$$\bar{x}^i \in S_i(\bar{\mathbf{x}}^{-i}), \quad \forall i = 1, \dots, N.$$

The point $\bar{\mathbf{x}}$ is therefore an equilibrium if no player can decrease his objective function by changing unilaterally $\bar{x}^i$ to any other feasible point.

The standard existence theorems for GNEP usually require both the upper and the lower semicontinuity of the set-valued functions K_i, together with the convexity and the closedness of their values. They also typically assume convexity and compactness of the strategy sets C_i, continuity of the functions θ_i, and convexity of each θ_i with respect to the i-th strategy x^i (see for instance [1,15]).

A known approach for solving GNEP is to consider an equivalent formulation. We introduce the so-called Nikaido–Isoda function that was first introduced in [15] as a tool to improve the original existence result for Nash equilibrium problem:

$$\Psi(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^N \left[\theta_i(x^i, \mathbf{x}^{-i}) - \theta_i(y^i, \mathbf{x}^{-i})\right].$$

The Nikaido–Isoda function has a simple interpretation: suppose that $\mathbf{x}$ and $\mathbf{y}$ are two feasible points for the GNEP, each summand in the definition represents the improvement in the objective function of player i when he changes his action from x^i to y^i while all the other players stick to the choice $\mathbf{x}^{-i}$.

It is worth noting that equilibria of the GNEP are characterized by the impossibility to get any improvement for any feasible choice $\mathbf{y}$. This is stated in the following lemma which will be proved for the sake of completeness, since it is fundamental for our next existence result.

Lemma 3.1. *The vector of strategies $\bar{\mathbf{x}}$ solves the GNEP if and only if*

$$\bar{\mathbf{x}} \in K(\bar{\mathbf{x}}) \quad \textit{and} \quad \Psi(\bar{\mathbf{x}}, \mathbf{y}) \le 0, \quad \forall \mathbf{y} \in K(\bar{\mathbf{x}}), \tag{6}$$

where $K : C \rightrightarrows C$ is defined by $K(\mathbf{x}) = \prod_{i=1}^N K_i(\mathbf{x}^{-i})$.

Proof. Clearly every solution of the GNEP solves (6). On the converse if $\bar{\mathbf{x}}$ solves (6), then for all $y^i \in K_i(\bar{\mathbf{x}}^{-i})$ the vector $(y^i, \bar{\mathbf{x}}^{-i}) \in K(\bar{\mathbf{x}})$ and the inequality in (6) becomes

$$\theta_i\big(\bar{x}^i, \bar{\mathbf{x}}^{-i}\big) \le \theta_i\big(y^i, \bar{\mathbf{x}}^{-i}\big)$$

that is $\bar{x}^i$ solves (5) for all i. □

Thanks to Lemma 3.1, we can deduce from Theorem 3.3 the following general result about existence of solutions of a GNEP.

Theorem 3.4. *Let all the Banach spaces $\mathbb{X}_i$ be separable and the convex sets C_i be compact. Moreover, assume that for every player i the set-valued map K_i is lower semicontinuous with nonempty images belonging to $\mathcal{D}(\mathbb{X}_i)$, the function θ_i is continuous on C and the function $\theta_i(\cdot, \mathbf{x}^{-i})$ is convex on C_i for every $\mathbf{x} \in \operatorname{fix} K$. If the set $\operatorname{fix} K$ is closed, then GNEP has solution.*

Proof. Defining $f(\mathbf{x}, \mathbf{y}) = -\Psi(\mathbf{x}, \mathbf{y})$, Lemma 3.1 affirms that the set of solutions of the GNEP coincides with the set of solutions of the QEP associated to f and K. For this reason it will be enough to verify all the assumptions of Theorem 3.3.

The Banach space $\mathbb{X} = \prod_{i=1}^{N} \mathbb{X}_i$ is separable since product of N separable spaces. Analogously the subset C is convex and compact. The lower semicontinuity of the set-valued map K can be achieved evaluating the lower inverse set of $A = \prod_{i=1}^{N} A_i$, where each $A_i \subseteq \mathbb{X}_i$ is open. Clearly,

$$\begin{aligned} K^{-1}(A) &= \big\{\mathbf{x} \in C : K_i\big(\mathbf{x}^{-i}\big) \cap A_i \neq \emptyset \ \forall i = 1, \dots, N\big\} \\ &= \bigcap_{i=1}^{N} \big\{\mathbf{x} \in C : K_i\big(\mathbf{x}^{-i}\big) \cap A_i \neq \emptyset\big\} \\ &= \bigcap_{i=1}^{N} \big(C_i \times \big\{\mathbf{x}^{-i} \in C^{-i} : K_i\big(\mathbf{x}^{-i}\big) \cap A_i \neq \emptyset\big\}\big) \\ &= \bigcap_{i=1}^{N} \big(C_i \times K_i^{-1}(A_i)\big). \end{aligned}$$

Thus $K^{-1}(A)$ is derived from finite intersection of open sets on C and hence is open on C. Moreover, for every $\mathbf{x} \in C$ the set $K(\mathbf{x})$ belongs to $\mathcal{D}(\mathbb{X})$ from Lemma 2.3. Finally the convexity of the level set descends from the convexity of $\theta_i(\cdot, \mathbf{x}^{-i})$ and the openness is a consequence of the continuity of θ_i. Hence the existence of solutions descends from Theorem 3.3. □

We end with a comparison between our result and Theorem 1.1 in [9]. Cubiotti and Yao don't require the completeness and separability of the spaces $\mathbb{X}_i$ and they weaken the compactness of the strategy sets C_i by means of certain technical conditions which, however, are complex and not easy to verify. But above all, they assume that the set-valued maps K_i are Hausdorff lower semicontinuous with $K_i(\mathbf{x}^{-i})$ closed and with nonempty interior with respect to the affine space generated by C_i. On the converse, in the case of separable Banach spaces, our result requires only the lower semicontinuity of K_i providing a positive answer to a question raised in [9, Section 4].

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