

# THIN SUBALGEBRAS OF LIE ALGEBRAS OF MAXIMAL CLASS

M. AVITABILE, A. CARANTI, N. GAVIOLI, V. MONTI, M. F. NEWMAN,  
AND E. A. O'BRIEN

ABSTRACT. For every field  $\mathbb{F}$  which has a quadratic extension  $\mathbb{E}$  we show there are non-metabelian infinite-dimensional thin graded Lie algebras all of whose homogeneous components, except the second one, have dimension 2. We construct such Lie algebras as  $\mathbb{F}$ -subalgebras of Lie algebras  $M$  of maximal class over  $\mathbb{E}$ . We characterise the thin Lie  $\mathbb{F}$ -subalgebras of  $M$  generated in degree 1. Moreover we show that every thin Lie algebra  $L$  whose ring of graded endomorphisms of degree zero of  $L^3$  is a quadratic extension of  $\mathbb{F}$  can be obtained in this way. We also characterise the 2-generator  $\mathbb{F}$ -subalgebras of a Lie algebra of maximal class over  $\mathbb{E}$  which are ideally  $r$ -constrained for a positive integer  $r$ .

## 1. INTRODUCTION

A *thin* Lie algebra is a Lie algebra

$$L = \bigoplus_{i=1}^{\infty} L_i,$$

over a field  $\mathbb{F}$ , graded over the positive integers, such that  $\dim_{\mathbb{F}} L_1 = 2$  and the following *covering property* holds: for every  $i \geq 1$  and for every nonzero  $u \in L_i$ ,  $[u, L_1] = L_{i+1}$ . Hence each homogeneous component of  $L$  has dimension at most 2. If  $\dim L_i = 1$  for every  $i \geq 2$ , the algebra  $L$  is referred to as a (*graded*) *Lie algebra of maximal class* (see Section 2 for details). We use the convention that, unless otherwise specified, thin algebras are infinite-dimensional and not of maximal class. The smallest integer  $k \geq 2$  such that  $\dim L_k = 2$  is a parameter that has been studied in several papers ([CMNS96, AJ01, AJM10]); in particular,  $k$  can be 3.

In this paper we study the class of thin algebras all of whose homogeneous components, except the second, have dimension 2. The only known examples of such algebras are those considered in [GM01]. There it is proved that all metabelian thin Lie algebras belong to this

---

The first four authors are members of the Italian INdAM-GNSAGA, and thank the Department of Mathematics and its Applications of the University of Milano – Bicocca for the hospitality.

A. Caranti is grateful to the Department of Mathematics of the University of Trento for its support.

E. A. O'Brien has been supported by the Marsden Fund of New Zealand grant UOA 1626.

class, and they are in one-to-one correspondence with the quadratic extensions of the field  $\mathbb{F}$ . In particular, in one of the constructions in [GMY01], a thin metabelian Lie algebra is realised as a subalgebra over  $\mathbb{F}$  of the tensor product of the unique infinite-dimensional metabelian Lie algebra of maximal class by a quadratic extension of  $\mathbb{F}$ .

The main goal of this paper is to generalise the results of [GMY01] to the non-metabelian case. Let  $\mathbb{E}$  be a quadratic extension of  $\mathbb{F}$  and let  $M$  be a Lie algebra of maximal class over  $\mathbb{E}$ . In Theorem 4 we consider the  $\mathbb{F}$ -subalgebras of  $M$  generated by two elements of degree 1. Amongst these we explicitly characterise those which are thin and prove that every homogeneous component, except the second one, has dimension 2. We also prove that, for every such thin algebra  $L$ , the ring of  $L$ -endomorphisms of the  $L$ -module  $L^3$  preserving each homogeneous component is isomorphic to  $\mathbb{E}$ .

We consider more generally the  $\mathbb{F}$ -subalgebras of  $M$  which are 2-generated in degree 1 under the assumption that  $\mathbb{E}$  is an arbitrary extension of  $\mathbb{F}$ . In Proposition 3 we characterise those subalgebras which are of maximal class over  $\mathbb{F}$ . The covering property for a graded Lie algebra  $L = \bigoplus_{i=1}^{\infty} L_i$  can be restated as follows (see [GM02]): every nonzero graded ideal of  $L$  is located between two consecutive terms of the lower central series of  $L$ . A natural generalisation is to require that there is a positive integer  $r$  such that every graded nonzero ideal of  $L$  is located between  $L^i$  and  $L^{i+r}$  for some  $i$ . Lie algebras generated by  $L_1$  satisfying this condition are *ideally  $r$ -constrained*. They were introduced in [GM02]. In Proposition 8 we characterise the 2-generator ideally  $r$ -constrained  $\mathbb{F}$ -subalgebras of  $M$  in positive characteristic under the assumption that  $\mathbb{E}$  is a quadratic extension of  $\mathbb{F}$ .

In Section 4 we consider a *just infinite-dimensional* algebra  $T$  over  $\mathbb{F}$  with  $\dim_{\mathbb{F}} T_i = 2$  in each degree  $i$  different from 2. We prove that the ring  $\mathbb{E}$  of  $T$ -endomorphisms of the  $T$ -module  $T^3$  preserving the homogeneous components is a field extension of  $\mathbb{F}$  of degree at most 2. When  $\mathbb{E} \neq \mathbb{F}$ , we construct a Lie algebra of maximal class over  $\mathbb{E}$  which contains  $T$  as an  $\mathbb{F}$ -subalgebra generated by two elements in degree 1.

Finally, we show that if we apply this construction, starting from a thin  $\mathbb{F}$ -subalgebra of a Lie algebra  $M$  of maximal class over a quadratic extension  $\mathbb{E}$  of  $\mathbb{F}$ , then up to isomorphism we recover  $M$ . We deduce that the number of thin Lie  $\mathbb{F}$ -algebras, all of whose homogeneous components of degree at least 3 have dimension 2, is  $|\mathbb{F}|^{\aleph_0}$ , the maximum possible. The work of [CM99, You01] provided a lower bound  $2^{\aleph_0}$  for the number of thin Lie algebras over an arbitrary field of positive characteristic.

## 2. PRELIMINARIES

Throughout the paper all Lie algebras are infinite-dimensional over the underlying field, unless explicitly mentioned.

We briefly recall the definition and the basic notions of Lie algebras of maximal class, referring the reader to [CMN97, CN00] for details. A (graded) Lie algebra of maximal class (generated in degree 1) is a Lie algebra

$$M = \bigoplus_{i=1}^{\infty} M_i$$

over a field  $\mathbb{E}$ , graded over the positive integers, having  $\dim_{\mathbb{E}} M_1 = 2$ ,  $\dim_{\mathbb{E}} M_i = 1$  otherwise and  $[M_i, M_1] = M_{i+1}$ . The last condition is equivalent to the requirement that  $M_1$  generates  $M$  as a Lie algebra. The name is motivated by the observation that the quotient  $M/M^j$ , where  $M^j = \bigoplus_{i \geq j} M_i$  is the  $j$ -th Lie power, is a Lie algebra of (finite) dimension  $j$  and nilpotency class  $j - 1$ , the maximum possible value.

See [SZ97, CVL00, CVL03] for Lie algebras of maximal class not generated in degree 1.

The 2-step centralisers of  $M$  are the 1-dimensional subspaces of  $M_1$  defined by

$$C_i = C_{M_1}(M_i) = \{a \in M_1 \mid [a, M_i] = 0\},$$

for  $i \geq 2$ . Note that  $C_i = C_{M_1}(u_i)$  for every  $0 \neq u_i \in M_i$ . Homogeneous generators,  $x$  and  $y$ , of  $M$  are chosen as follows. Let  $y \in M_1$  such that  $C_2 = \mathbb{E}y$ . If all other  $C_i$  coincide with  $C_2$ , then  $M$  is the unique metabelian Lie algebra of maximal class (this is the only possibility if the underlying field has characteristic zero), and we choose  $x$  in  $M_1$  such that  $x$  and  $y$  are  $\mathbb{E}$ -linearly independent. Otherwise, if  $i$  is the smallest integer such that  $C_i \neq C_2$ , then we choose  $x$  such that  $C_i = \mathbb{E}x$ . Such  $x$  and  $y$  are *standard generators* of  $M$ ; each is defined up to a nonzero constant in  $\mathbb{E}$ .

The sequence of centralisers of  $M$  consists of consecutive occurrences of  $\mathbb{E}y$  interrupted by isolated occurrences of other centralisers. If  $C$  is a 2-step centraliser and  $m$  is the smallest integer such that  $C_m = C$ , then  $m = 2p^n$  for some non-negative integer  $n$ . If  $C_i$  and  $C_j$  are successive occurrences of  $C$ , then  $j - i \leq m$  (see [CN00, Secs. 3 and 10] and [Jur05, p. 439]). Furthermore, as a consequence of the *specialisation technique* (see [CN00, Proposition 4.1]), every 2-step centraliser occurs infinitely often.

### 3. FROM MAXIMAL CLASS TO THIN

Let  $\mathbb{E}$  be an arbitrary extension of  $\mathbb{F}$  and let  $M$  be a Lie algebra of maximal class over  $\mathbb{E}$ . We start with the following result whose proof is immediate.

**Lemma 1.** *Let  $i \geq 2$  and  $\ell \in M_1 \setminus C_i$ . The map*

$$\begin{aligned} \text{ad } \ell: M_i &\rightarrow M_{i+1} \\ v &\mapsto [v, \ell] \end{aligned}$$

defines an  $\mathbb{E}$ -isomorphism; it is also an  $\mathbb{F}$ -isomorphism. If  $V$  is an  $\mathbb{F}$ -subspace of  $M_i$ , then  $\dim_{\mathbb{F}} \text{ad } \ell(V) = \dim_{\mathbb{F}} [V, \ell] = \dim_{\mathbb{F}} V$ .

We now consider the  $\mathbb{F}$ -subalgebra  $L$  of  $M$  generated by two elements  $X$  and  $Y$  of  $M_1$ . If  $X$  and  $Y$  are  $\mathbb{E}$ -linearly dependent then  $[Y, X] = 0$ ; thus  $L$  has  $\mathbb{F}$ -dimension at most 2 and the Lie product is trivial. So assume that  $X$  and  $Y$  are  $\mathbb{E}$ -linearly independent. Now  $[Y, X] \neq 0$ , so  $\dim_{\mathbb{F}} L_2 = 1$ . Let  $l$  be a nonzero element of  $L_i$  for  $i \geq 2$ . Since  $C_{M_1}(l) = C_i$  it follows that  $C_{L_1}(l) = C_i \cap L_1$ . Define  $d_i = \dim_{\mathbb{F}} C_i \cap L_1$ .

The following lemma is crucial for what follows.

**Lemma 2.** *Let  $L$  be the Lie  $\mathbb{F}$ -subalgebra of  $M$  generated by two  $\mathbb{E}$ -linearly independent elements  $X$  and  $Y$  of  $M_1$ . The following hold:*

- (1)  $\dim_{\mathbb{F}} L_2 = 1$ ;
- (2)  $\dim_{\mathbb{F}} L_{i+1} \geq \dim_{\mathbb{F}} L_i$  for every  $i \geq 2$  (in particular,  $L$  is infinite-dimensional);
- (3)  $d_i \leq 1$  for every  $i \geq 2$ ;
- (4) if  $d_i = 0$  then  $\dim_{\mathbb{F}} [l, L_1] = 2$  for every  $l \in L_i \setminus \{0\}$ ;
- (5) if  $d_i = 1$  then  $\dim_{\mathbb{F}} [l, L_1] = 1$  for every  $l \in L_i \setminus \{0\}$ .

*Proof.* Item (1) is trivial. Since  $X$  and  $Y$  are  $\mathbb{E}$ -linearly independent, there is no integer  $i$  such that both  $X$  and  $Y$  belong to  $C_i$  (whose  $\mathbb{E}$ -dimension is 1). This immediately yields (3), and (2) follows by  $L_{i+1} = [L_i, X] + [L_i, Y]$  and Lemma 1. Items (4) and (5) are obvious.  $\square$

We now characterise the 2-generator  $\mathbb{F}$ -subalgebras of  $M$  of maximal class.

**Proposition 3.** *The algebra  $L$  has maximal class if and only if  $d_i = 1$  for every  $i \geq 2$ .*

*Proof.* Observe  $L$  has maximal class if and only if  $\dim_{\mathbb{F}} L_i = 1$  for every  $i \geq 2$ . Lemma 2, items (1), (5) and (4) imply the claim.  $\square$

We now restrict to quadratic extensions. In this case we show that  $L$  is thin if and only if  $L_1$  intersects every 2-step centraliser of  $M$  trivially.

**Theorem 4.** *If  $|\mathbb{E} : \mathbb{F}| = 2$ , then  $L$  is a thin Lie algebra if and only if  $d_i = 0$  for every  $i \geq 2$ . In this case  $\dim_{\mathbb{F}} L_i = 2$  for every  $i \neq 2$ .*

*Proof.* If  $l$  is a nonzero element of  $L_1$  then  $[l, L_1] = L_2$ . Thus  $L$  is thin if and only if  $[l, L_1] = L_{i+1}$  for every  $i \geq 2$  and every nonzero  $l \in L_i$  (and  $L$  is not of maximal class).

Suppose that  $d_i = 0$  for every  $i \geq 2$ . Let  $0 \neq l \in L_i$  with  $i \geq 2$ : by Lemma 2, item (4),  $\dim_{\mathbb{F}} [l, L_1] = 2$ . Since  $\dim_{\mathbb{F}} L_{i+1} \leq \dim_{\mathbb{F}} M_{i+1} = 2$ , it follows that  $[l, L_1] = L_{i+1}$  and  $L$  is thin. In particular,  $\dim_{\mathbb{F}} L_i = 2$  for every  $i \neq 2$ .

Conversely, suppose that  $L$  is thin. The covering property and items (4) and (5) of Lemma 2 imply that  $\dim_{\mathbb{F}} L_{i+1} = 2 - d_i$ , for every  $i \geq 2$ . Thus, Lemma 2, item (2), yields  $d_i \geq d_{i+1}$ . By [CMN97,

Lemma 3.3],  $d_i = d_2$  for infinitely many values of  $i$ . This implies that the sequence of  $d_i$ s is constant. By Proposition 3,  $d_i = 0$  for every  $i \geq 2$ .  $\square$

**Example 5.** Suppose that  $M$  has exactly two distinct 2-step centralisers  $\mathbb{E}x$  and  $\mathbb{E}y$ . For every  $\gamma, \delta \in \mathbb{E} \setminus \mathbb{F}$  with  $\gamma \neq \delta$ , the  $\mathbb{F}$ -subalgebra of  $M$  generated by  $x+y$  and  $\gamma x + \delta y$  is non-metabelian and thin and every homogeneous component of degree different from 2 has dimension 2.

*Remark 6.* Let  $\mathbb{E} = \mathbb{F}(\lambda)$  be a quadratic extension of  $\mathbb{F}$ . Let  $x$  and  $y$  be standard generators of  $M$  and set  $X = \alpha x + \beta y$ ,  $Y = \gamma x + \delta y$ , where  $\alpha, \beta, \gamma, \delta \in \mathbb{E}$  are such that  $\alpha\delta - \beta\gamma \neq 0$ . But  $\mathbb{E}y$  is a 2-step centraliser of  $M$ , so  $s\alpha + t\gamma \neq 0$  for every  $s, t \in \mathbb{F}$ , not both zero. In particular,  $\alpha, \gamma \neq 0$ . Since the linear map that multiplies each homogeneous component  $M_i$  by  $\alpha^{-i}$  is an automorphism of  $M$ , we can assume that  $X = x + \beta y$  and  $Y = \gamma x + \delta y$  with  $\gamma \notin \mathbb{F}$ . By possibly adding to  $Y$  an  $\mathbb{F}$ -multiple of  $X$  and multiplying by a nonzero element of  $\mathbb{F}$ , we obtain  $\gamma = \lambda$ .

If  $M$  is not metabelian, then  $\mathbb{E}x$  is also a 2-step centraliser of  $M$ , so  $s\beta + t\delta \neq 0$  for every  $s, t \in \mathbb{F}$ , not both zero. It follows that  $\beta$  and  $\delta$  are independent over  $\mathbb{F}$ , so both are nonzero. Since  $y$  is defined up to a nonzero constant in  $\mathbb{E}$ , we can take  $y' = \beta y$  in place of  $y$ . Under this assumption, we can represent the generators of  $L$ , with respect to this new basis of  $M_1$ , as  $X = x + y$  and  $Y = \lambda x + \delta y$ , with  $\delta \in \mathbb{E} \setminus \mathbb{F}$ ,  $\delta \neq \lambda$ . Conversely, these generate a thin subalgebra of  $M$  under the assumptions of Theorem 4. Hence there exist at most  $|\mathbb{E} \setminus \mathbb{F}|$  isomorphism classes of thin subalgebras of  $M$ .

*Remark 7.* Let  $L$  be a thin algebra as in Theorem 4. Then  $L_i = M_i$  as sets, for  $i \geq 3$ , but we regard  $L_i$  as a vector space over  $\mathbb{F}$  and  $M_i$  as a vector space over  $\mathbb{E}$ . Consider the  $L$ -module  $L^3$  via the adjoint representation. Observe  $\mathbb{E}$  acts by multiplication as a ring of  $L$ -endomorphisms of  $L^3$ , preserving each  $L_i = M_i$ . Conversely, let  $f$  be an  $L$ -endomorphism of  $L^3$  preserving the homogeneous components. For each  $i \geq 3$  take a nonzero  $l_i \in L_i$ . Then  $f(l_i) = \sigma_i l_i$  for some  $\sigma_i \in \mathbb{E}$ . The covering property implies that  $l_{i+1} = [l_i, l_1]$  for some  $l_1 \in L_1$ , hence

$$\sigma_{i+1} l_{i+1} = f(l_{i+1}) = f([l_i, l_1]) = [f(l_i), l_1] = \sigma_i [l_i, l_1] = \sigma_i l_{i+1},$$

and so  $\sigma_{i+1} = \sigma_i$ . In particular, this shows that  $\sigma_i$  is independent of the choice of  $l_i$  in  $L_i$ . It follows that  $f$  acts on  $L^3$  as multiplication by  $\sigma_3 \in \mathbb{E}$ .

In characteristic zero, the unique Lie algebra of maximal class has just one 2-step centraliser, so  $d_i = d_2$  for every  $i \geq 2$ . Therefore the previous results settle the case of characteristic zero.

Recall from the introduction that  $L$  is ideally  $r$ -constrained if, for every graded nonzero ideal  $I$ , there is a positive integer  $i$  such that  $L^i \supseteq I \supseteq L^{i+r}$ . It is shown in [GM02, Proposition 2] that a 2-generated

ideally 1-constrained Lie algebra is either thin or of maximal class. By Proposition 3 and Theorem 4 this happens if and only if the sequence  $(d_i)_{i \geq 2}$  is constant.

We now address the case of the subalgebras  $L$  for which this sequence is not constant.

**Proposition 8.** *Let  $|\mathbb{E} : \mathbb{F}| = 2$ . Assume that the sequence  $(d_i)_{i \geq 2}$  is not constant and set  $D_0 = \{i \geq 2 \mid d_i = 0\}$ . Let  $t_1 < t_2 < \dots < t_{j-1} < t_j < \dots$  be the elements of  $D_0$ . Then  $t_j - t_{j-1} \leq t_1$  for every  $j \geq 2$ . Let  $r = \max(t_j - t_{j-1})$ : then  $2 \leq r \leq t_1$  and  $L$  is an ideally  $r$ -constrained Lie algebra but  $L$  is not ideally  $(r-1)$ -constrained. Moreover  $\dim_{\mathbb{F}} L_i = 1$  if  $2 \leq i \leq t_1$  and  $\dim_{\mathbb{F}} L_i = 2$  otherwise.*

*Proof.* As  $|\mathbb{E} : \mathbb{F}| = 2$ , we have  $\dim_{\mathbb{F}} L_i \leq 2$  for every  $i \geq 2$ . Since  $d_{t_1}$  is the first 0, by Lemma 2, items (4) and (5),  $\dim_{\mathbb{F}} L_i = 1$  for every  $2 \leq i \leq t_1$  and  $\dim_{\mathbb{F}} L_{t_1+1} = 2$ . Lemma 2, item (2) then yields that  $\dim_{\mathbb{F}} L_i \geq 2$  for every  $i \geq t_1 + 1$  and equality holds since  $|\mathbb{E} : \mathbb{F}| = 2$  and  $\dim_{\mathbb{E}} L_i \leq \dim_{\mathbb{E}} M_i = 1$ .

As mentioned in Section 2, there are infinitely many occurrences of  $C_{t_1}$  and if  $C_u$  and  $C_v$  are two successive such occurrences then  $v - u \leq t_1$ . Hence  $D_0$  is infinite and  $t_j - t_{j-1} \leq t_1$  for every  $j \geq 2$ . Also  $D_1 = \{i \geq 2 \mid d_i = 1\}$  is infinite, so  $r \geq 2$ .

Let  $I$  be a nonzero graded ideal of  $L$  and let  $i$  be such that  $I \subseteq L^i$  but  $I \not\subseteq L^{i+1}$ . We claim that the ideal generated by a nonzero element  $l$  of  $I$  of degree  $i$  contains  $L^{i+r}$ . In particular,  $I \supseteq L^{i+r}$  and therefore we need only consider the case when  $I$  is generated by  $l$ . If  $i = 1$ , then  $[l, L_1] = L_2$  and we are done. If  $2 \leq i \leq t_1$  then  $l$  generates  $L_i$  as an  $\mathbb{F}$ -vector space and we are done. So assume that  $i > t_1$  and let  $t_j$  be such that  $t_j \geq i > t_{j-1}$ ; then  $t_j < i + r$  by definition of  $r$ . By Lemma 2, items (4) and (5), we have  $\dim_{\mathbb{F}}(I \cap L_h) = 1$  for  $i \leq h \leq t_j$  and  $\dim_{\mathbb{F}}(I \cap L_{t_j+1}) = 2$  thus  $I \supseteq L^{t_j+1} \supseteq L^{i+r}$  and  $I \not\supseteq L^{t_j}$ . If we choose  $j_0$  such that  $t_{j_0} - t_{j_0-1} = r$  and put  $i_0 = t_{j_0-1} + 1$ , then the ideal generated by an element of degree  $i_0$  does not contain  $L^{i_0+r-1}$ . Hence  $L$  is not ideally  $(r-1)$ -constrained.  $\square$

#### 4. FROM THIN TO MAXIMAL CLASS

Let  $L = \bigoplus_{i=1}^{\infty} L_i$  be a graded Lie algebra, over a field  $\mathbb{F}$ , generated by  $L_1$  as a Lie algebra. A *graded* module is an  $L$ -module  $V = \bigoplus_{i=n_0}^{\infty} V_i$  over  $\mathbb{F}$ , for some  $n_0 \in \mathbb{N}$ , such that  $V_i \cdot L_j \leq V_{i+j}$ . The module  $V$  is *just infinite-dimensional* if it is infinite-dimensional and every graded submodule is either trivial or has finite codimension. If the image of the adjoint representation of  $L$  is a just infinite-dimensional  $L$ -module, then we say that  $L$  is a just infinite-dimensional Lie algebra. This is always the case when  $L$  is an ideally  $r$ -constrained Lie algebra. Every non-trivial (possibly non-graded) submodule of a just infinite-dimensional graded module  $V$  has finite codimension (see [GMS04, Lemma 2.3]). In

particular, a finite-dimensional submodule of  $V$  is necessarily trivial. A linear map  $\varphi: V \rightarrow V$  is *graded* of degree  $d$  if  $\varphi(V_i) \leq V_{i+d}$  for a non-negative integer  $d$ . Let  $\text{GrEnd}(V)$  be the ring generated by the graded  $\mathbb{F}$ -linear maps of  $V$ . Thus  $\text{GrEnd}(V) = \bigoplus_{d=0}^{\infty} \text{GrEnd}_d(V)$  is a graded ring, where each homogeneous component  $\text{GrEnd}_d(V)$  consists of the  $\mathbb{F}$ -linear maps of  $V$  having degree  $d$ .

**Lemma 9** (Schur's lemma). *Let  $V$  be a graded just infinite-dimensional  $L$ -module. If  $0 \neq \varphi \in \text{End}(V)$  is an  $L$ -endomorphism of  $V$ , then  $\varphi$  is an injective map.*

*Proof.* Let  $U = \ker \varphi$  and  $W = \text{Im } \varphi$ . By assumption,  $W \neq 0$  so  $\dim U = \text{codim } W$  is finite. Hence  $\ker \varphi = 0$ .  $\square$

**Corollary 10.** *Let  $V$  be a graded just infinite-dimensional  $L$ -module. The ring  $\text{GrEnd}_L(V)$ , consisting of the  $L$ -endomorphisms in  $\text{GrEnd}(V)$ , is a graded domain whose homogeneous component  $\text{GrEnd}_{L,d}(V)$  of degree  $d$  has  $\mathbb{F}$ -dimension at most  $\inf_{i \geq n_0} (\dim V_{d+i})$ . In particular, the subring  $\mathbb{E} = \text{GrEnd}_{L,0}(V)$  of the graded  $L$ -endomorphisms of  $V$  of degree 0 is a skew-field extension of  $\mathbb{F}$  of degree at most  $\inf_{i \geq n_0} (\dim V_i)$  and contains  $\mathbb{F}$  in its centre.*

*Proof.* Let  $v$  be a nonzero element in  $V_i$ . The map  $\psi: \text{GrEnd}_{L,d}(V) \rightarrow V_{d+i}$  defined by  $\varphi \mapsto \varphi(v)$  is an injective  $\mathbb{F}$ -linear map, so

$$\dim_{\mathbb{F}} \text{GrEnd}_{L,d}(V) = \dim_{\mathbb{F}} \text{Im } \psi \leq \dim_{\mathbb{F}} V_{d+i}.$$

$\square$

*Remark 11.* For every  $i$  the homogeneous component  $V_i$  is an  $\mathbb{E}$ -module. In particular,  $|\mathbb{E} : \mathbb{F}| \cdot \dim_{\mathbb{E}} V_i = \dim_{\mathbb{F}} V_i$ . If  $\dim_{\mathbb{F}} V_i$  is a prime  $p$  for some  $i$ , then  $|\mathbb{E} : \mathbb{F}|$  is 1 or  $p$ , so  $\mathbb{E}$  is a simple extension of the field  $\mathbb{F}$ .

*Remark 12.* Let  $I$  be a non-trivial graded ideal in a just infinite-dimensional Lie algebra  $L$ . The centraliser  $C_L(I)$  is also a graded ideal of  $L$ , so either  $C_L(I) = 0$ , or the quotient  $L/C_L(I)$  has finite dimension. In the latter case,  $C_L(I) \cap I$  is abelian and has finite codimension. Hence  $L$  is soluble if and only if  $L$  has a non-trivial graded ideal  $I$  with  $C_L(I) \neq 0$ . Also, a graded ideal  $I$  of  $L$  is abelian if and only if  $C_L(I) \neq 0$  (see [GMS04, Proposition 2.9 and Corollary 2.11]). Therefore  $L$  is insoluble if and only if for every non-trivial graded ideal  $I$  of  $L$  the adjoint representation of  $L$  over  $I$  is faithful.

From now on,  $T$  is a just infinite-dimensional Lie algebra over  $\mathbb{F}$  such that  $\dim_{\mathbb{F}} T_i = 2$  for every  $i \neq 2$ . By Corollary 10 and Remark 11, the ring  $\mathbb{E}$  of  $T$ -endomorphisms of  $T^3$  preserving the homogeneous components is a field extension of  $\mathbb{F}$  of degree at most 2.

**Lemma 13.** *Suppose the field  $\mathbb{E}$  has degree 2 over  $\mathbb{F}$ . If  $I$  is a non-trivial maximal abelian graded ideal of  $T$ , then it equals  $T^k$  for some  $k \geq 2$  and is unique.*

*Proof.* Let  $k$  be the smallest integer such that  $I \cap T_k \neq 0$ . Note  $k \geq 2$  because  $\dim_{\mathbb{F}}(T_3) = 2$ . We claim that  $I = T^k$ : since the ideal generated by  $T_k$  is  $T^k$ , it is enough to show that  $I \cap T_k = T_k$ . When  $k = 2$ , this is obvious as  $\dim_{\mathbb{F}} T_2 = 1$ . When  $I \leq T^3$ , the ideal  $\alpha I$  is abelian for every  $\alpha \in \mathbb{E}$ . As  $[x, \alpha y] = \alpha[x, y] = 0$  for every  $x$  and  $y$  in  $I$ , the ideal  $I + \alpha I$  is abelian and  $\alpha I \subseteq I$  by the maximality of  $I$ . Therefore  $I \cap T_k$  is a nonzero vector space over  $\mathbb{E}$ , whence the claim.  $\square$

When  $|\mathbb{E} : \mathbb{F}| = 2$  we construct a Lie algebra  $N = \bigoplus_{i=1}^{\infty} N_i$  of maximal class over  $\mathbb{E}$  containing  $T$  as an  $\mathbb{F}$ -subalgebra generated by two elements of  $N_1$ .

Assume first that  $T$  is not metabelian. Let  $R = \mathfrak{gl}_{\mathbb{E}}(\mathbb{E} \oplus I)$  be the Lie algebra of  $\mathbb{E}$ -linear maps of the vector space  $\mathbb{E} \oplus I$ , where  $I = T^3$  if  $T$  is insoluble, or  $I = T^k$  is the unique maximal abelian graded ideal of  $T$ , otherwise. Note that  $k \geq 3$  as  $T$  is not metabelian, so  $I$  is an  $\mathbb{E}$ -vector space. Let  $z \in T \setminus I$  be a homogeneous element which is central modulo  $I$ . By Remark 12, the map

$$\rho: t \mapsto \left[ \begin{array}{c|c} 0 & [z, t] \\ \hline 0 & \text{ad } t|_I \end{array} \right],$$

which is the adjoint representation of  $T$  over the ideal  $\langle z \rangle \oplus I$ , is a faithful representation of  $T$  in  $R$ . This is easy in the insoluble case; for soluble  $T$ , see [GMS04, Lemma 2.8], taking  $J = \mathbb{F}z + I$ .

The field  $\mathbb{E}$  and  $\rho(T)$  are  $\mathbb{F}$ -subspaces of  $R$ . We claim that

$$N = \mathbb{E} \cdot \rho(T) = \{ \lambda \cdot \rho(t) \mid \lambda \in \mathbb{E} \text{ and } t \in T \} \subseteq R$$

is a Lie algebra of maximal class over  $\mathbb{E}$ . Let  $N_i = \mathbb{E} \cdot \rho(T_i)$ . If  $i \geq 3$  then  $N_i = \mathbb{E} \cdot \rho(T_i) = \rho(\mathbb{E} \cdot T_i) = \rho(T_i)$ . Hence  $N_i$  is a module over  $\mathbb{E}$  of dimension 1 for every  $i \geq 3$ . Clearly, the homogeneous component  $N_2 = \mathbb{E} \cdot \rho(T_2)$  has dimension 1 over  $\mathbb{E}$ . The dimension of  $N_1$  over  $\mathbb{E}$  is 2, otherwise  $T$  is abelian. Let  $S$  be the  $\mathbb{E}$ -subalgebra of  $N$  generated by  $N_1$ . This algebra contains the  $\mathbb{F}$ -algebra generated by  $\rho(T_1)$ , namely  $\rho(T)$ , so  $S$  contains also  $\mathbb{E} \cdot \rho(T) = N$ . Hence  $N$  is generated over  $\mathbb{E}$  by its first homogeneous component  $N_1$ .

When  $T$  is metabelian, the construction must be modified by considering, instead of  $\rho$ , the adjoint representation  $\rho'$  over the ideal  $\langle Y \rangle \oplus T^2$ :

$$\rho': t \mapsto \left[ \begin{array}{c|c|c} 0 & \alpha & [Y, s] \\ \hline 0 & 0 & [[Y, X], t] \\ \hline 0 & 0 & \text{ad } t|_{T^3} \end{array} \right] \in R' = \mathfrak{gl}(\mathbb{E} \oplus \mathbb{E} \oplus T^3)$$

where  $T_1 = \mathbb{F} \cdot X + \mathbb{F} \cdot Y$  and  $t = \alpha X + \beta Y + s$  with  $s \in T^2$ . As above, the Lie algebra  $N = \mathbb{E} \cdot \rho'(T)$  is of maximal class over  $\mathbb{E}$ .

Now let  $\mathbb{E} \supset \mathbb{F}$  be a quadratic extension of fields and let  $M$  be a Lie algebra of maximal class over  $\mathbb{E}$ . Let  $L$  be the thin  $\mathbb{F}$ -subalgebra of  $M$  generated by two elements of  $M_1$  as in Theorem 4. By Remark 7, we can apply the above construction, starting from the algebra  $L$ , to obtain a



Lie algebra  $N$  of maximal class over  $\mathbb{E}$  containing  $L$  as an  $\mathbb{F}$ -subalgebra. We show that we can recover the algebra  $M$  up to isomorphism.

**Proposition 14.** *The Lie algebras  $M$  and  $N$  of maximal class are isomorphic.*

*Proof.* Observe  $N = \mathbb{E} \cdot \rho(L) = \rho(\mathbb{E} \cdot L) = \rho(M) \cong M$  where the last isomorphism holds because  $M$  is a just infinite-dimensional Lie algebra.  $\square$

*Remark 15.* Let  $M$  be an arbitrary algebra of maximal class over  $\mathbb{E}$  with standard generators  $x$  and  $y$ . Every 2-step centraliser other than  $\mathbb{E}y$  can be written as  $\mathbb{E}(x+\lambda y)$  for unique  $\lambda \in \mathbb{E}$ . Let  $\mathcal{L}$  be the set of all such  $\lambda$ s. Let  $X = \alpha x + \beta y = \alpha(x + \alpha^{-1}\beta y)$  and  $Y = \gamma x + \delta y = \gamma(x + \gamma^{-1}\delta y)$  be two  $\mathbb{E}$ -linearly independent elements in  $M_1$ . By Theorem 4 the  $\mathbb{F}$ -subalgebra of  $M$  generated by  $X$  and  $Y$  is thin if and only if  $\mathbb{F} \cdot X + \mathbb{F} \cdot Y$  intersects every 2-step centraliser trivially. This occurs if and only if the affine line  $\Lambda = \{t\alpha^{-1}\beta + (1-t)\gamma^{-1}\delta \mid t \in \mathbb{F}\}$  intersects  $\mathcal{L}$  trivially, so  $\mathcal{L} \subseteq \mathbb{E} \setminus \Lambda$ . There are at least  $|\mathbb{E} \setminus \mathbb{F}|^{n_0} = |\mathbb{E}|^{n_0} = |\mathbb{F}|^{n_0}$  algebras of maximal class satisfying this condition [CMN97]. By Proposition 14, there are at least as many thin  $\mathbb{F}$ -algebras all of whose homogeneous components of degree at least 3 have dimension 2. Hence the upper bound  $|\mathbb{F}|^{n_0}$  for the number of such thin  $\mathbb{F}$ -algebras is attained.

## REFERENCES

- [AJ01] M. Avitabile and G. Jurman, *Diamonds in thin Lie algebras*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **4** (2001), no. 3, 597–608. MR 1859998 (2003a:17038)
- [AJM10] Marina Avitabile, Giuseppe Jurman, and Sandro Mattarei, *The structure of thin Lie algebras with characteristic two*, Internat. J. Algebra Comput. **20** (2010), no. 6, 731–768. MR 2726572 (2011j:17034)
- [CM99] Andrea Caranti and Sandro Mattarei, *Some thin Lie algebras related to Albert-Frank algebras and algebras of maximal class*, J. Austral. Math. Soc. Ser. A **67** (1999), no. 2, 157–184, Group theory. MR 1717411 (2000j:17036)
- [CMN97] A. Caranti, S. Mattarei, and M. F. Newman, *Graded Lie algebras of maximal class*, Trans. Amer. Math. Soc. **349** (1997), no. 10, 4021–4051. MR 1443190 (98a:17027)
- [CMNS96] A. Caranti, S. Mattarei, M. F. Newman, and C. M. Scoppola, *Thin groups of prime-power order and thin Lie algebras*, Quart. J. Math. Oxford Ser. (2) **47** (1996), no. 187, 279–296. MR 1412556 (97h:20036)
- [CN00] A. Caranti and M. F. Newman, *Graded Lie algebras of maximal class. II*, J. Algebra **229** (2000), no. 2, 750–784. MR 1769297 (2001g:17041)
- [CVL00] A. Caranti and M. R. Vaughan-Lee, *Graded Lie algebras of maximal class. IV*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **29** (2000), no. 2, 269–312. MR 1784176 (2002c:17044)
- [CVL03] ———, *Graded Lie algebras of maximal class. V*, Israel J. Math. **133** (2003), 157–175. MR 1968427 (2004d:17034)
- [GM02] Norberto Gavioli and Valerio Monti, *Ideally constrained Lie algebras*, J. Algebra **253** (2002), no. 1, 31–49. MR 1925007

- [GMS04] Norberto Gavioli, Valerio Monti, and Carlo Maria Scoppola, *Just infinite periodic Lie algebras*, Finite groups 2003, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 73–85. MR 2125067 (2006i:17044)
- [GM01] Norberto Gavioli, Valerio Monti, and David S. Young, *Metabelian thin Lie algebras*, J. Algebra **241** (2001), no. 1, 102–117. MR 1838846 (2002d:17032)
- [Jur05] G. Jurman, *Graded Lie algebras of maximal class. III.*, J. Algebra **284** (2005), 435–461. MR 2114564 (2005k:17041)
- [SZ97] Aner Shalev and Efim I. Zelmanov, *Narrow Lie algebras: a coclass theory and a characterization of the Witt algebra*, J. Algebra **189** (1997), no. 2, 294–331. MR 1438178 (98d:17032)
- [You01] David S. Young, *Thin Lie algebras with long second chains*, Ph.D. thesis, The Australian National University, March 2001.

(M. Avitabile) DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DEGLI STUDI DI MILANO - BICOCCA, VIA COZZI, 55, I-20125 MILANO, ITALY  
*Email address:* marina.avitabile@unimib.it

(A. Caranti) DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI TRENTO, VIA SOMMARIVE, 14, I-38123 TRENTO, ITALY  
*Email address:* andrea.caranti@unitn.it

(N. Gavioli) DIPARTIMENTO DI INGEGNERIA E SCIENZE DELL'INFORMAZIONE E MATEMATICA, UNIVERSITÀ DEGLI STUDI DELL'AQUILA, VIA VETOIO, I-67100 COPPITO (AQ), ITALY  
*Email address:* norberto.gavioli@univaq.it

(V. Monti) DIPARTIMENTO DI SCIENZA E ALTA TECNOLOGIA, UNIVERSITÀ DEGLI STUDI DELL'INSUBRIA, VIA VALLEGGIO, 11, I-22100 COMO, ITALY  
*Email address:* valerio.monti@uninsubria.it

(M. F. Newman) MATHEMATICAL SCIENCES INSTITUTE, AUSTRALIAN NATIONAL UNIVERSITY, CANBERRA, ACT 2600 AUSTRALIA  
*Email address:* mike.newman@anu.edu.au

(E. A. O'Brien) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF AUCKLAND, AUCKLAND, NEW ZEALAND  
*Email address:* e.obrien@auckland.ac.nz