Università degli Studi di L'Aquila

Dipartimento di Scienze fisiche e chimiche
Corso di Dottorato di Ricerca in Scienze fisiche e chimiche

Tesi di Dottorato di Ricerca
Ciclo: XXXII

Post-Newtonian approximation in higher-order metric theories of gravity

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Anno Accademico 2020-2021

To my mother
with my deepest gratitude

Acknowledgements

I still remember with great pleasure the day when I got the letter in which I was informed that I had been admitted to the doctoral course at the Physics Department of L'Aquila University. That was the realization of a long-cherished dream for me.

I immediately shared the news with Prof. L. Pilo, who strongly adviced me to attend the lectures of Gravitation and more generally of Cosmology, both held at the Gran Sasso Science Institute (GSSI) - School of Advanced Study - in l'Aquila, in order to fill my big gap on the subject. In fact, I had already attended with the utmost interest some lectures by Prof. L. Pilo (I had *enlightenment on the way to Damascus* by attending his course on Gauge Theories), concerning the theoretical part of Elementary Particle Physics - with which I also obtained my degree - as well as others dealing with the experimental side held by Prof. S. Petrera, Prof. F. Cavanna and Prof. A. Grillo, since I was convinced to continue my studies in this area, with the aim of dealing with String Theories. I am particularly grateful to Prof. L. Pilo and Prof. S. Petrera for the fruitful discussions we had, both on issues related to their respective courses and my degree thesis as well as for their support in finding the direction to be taken at the beginning of my doctoral course.

By a strange twist of fate I found myself dealing with Gravitation!

During the courses I attended at GSSI, I had the pleasure to meet Prof. S. Capozziello, with whom I decided to undertake the research work, then condensed into this doctoral thesis. Even though unexpected events occurred, I managed to complete the job, thanks to the constant dialogue I had with him during these years of PhD, despite the physical distance between us, due to our respective lives and the difficulties associated with the pandemic. He was a guide, generous in his advice, punctual and thorough in discussions of merit, able to involve me thanks to the enthusiasm he usually puts in whatever he does. Generally speaking, he was a fun interlocutor during this adventure.

The observations made by Prof. L. Pilo were useful too. They concerned some critical aspects related to the theoretical model treated in this thesis and certainly helped me to gain a broader view of the topic. Furthermore, I am also thankful for the suggestions he gave when preparing the slides needed to present the progressive steps of my job during the years of PhD.

A big thank you goes to the two reviewers, Prof. G. Lambiase and Prof. L. Fatibene, of whom I appreciated the care they had in commenting on what I wrote in my thesis; in particular, I am not exaggerating by saying that I am in great debt with Prof. L. Fatibene for the meticulousness (he says he is particularly prone to compulsiveness when he finds himself correcting the writings of others) that transpires both from the observations made on the topics covered and from the advice on the shape of my 'shaky' English.

Finally, I am deeply grateful to my partner, Luisa, for having patiently shared with me all the moments of this experience with me; I well remember the day I decided, with a deep sense of guilt, to force her to take a brief but intense training period for the use of LaTeXwriting software, so that she could help me with the transcription of the large amount of computations to be included in the appendices. Honestly, I took advantage of her abilities, being her technical qualities well known to me. Her work was decisive for the presentation of the thesis to Doctoral College within the prescribed deadlines.

I hope that all the work done so far will be the launching pad for the achievement of further

satisfactions.

Pineto, April 2021 Alessandro Breccione Mattucci

Synopsis

The Post-Newtonian limit of a metric theory of gravity is discussed, based on the most general quadratic Lagrangian in the curvature scalars. This approach involves the use of the perturbative expansion of the metric tensor extended up to the fourth-order and expressed in terms of three gravitational potentials. The perturbative developments of the field equations related to the aforementioned Lagrangian, imply to have to deal with two systems of coupled fourth-order differential equations, corresponding to the second order and fourth respectively. Actually, only the former admits solutions, giving back modified Newtonian potentials, which consist of a purely Newtonian term plus two Yukawa-type contributions. These results are part of a larger work, which consists in making explicit the second-order coupled differential equations system, as the conditions placed on the coupling constants, appearing in the quadratic Lagrangian, vary and in the consequent research and analysis of the relative solutions, where they exist, in order to give a picture as general as possible of the theoretical model under study. The solutions are computed using the Green's function method. Finally, the gravitational potentials, obtained in the weak-field and low-velocity limit, are compared with the Newtonian one derived in General Relativity.

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1. *Post-Newtonian approximation in higher-order metric theories of gravity.*
   A. Breccione Mattucci, S. Capozziello, submitted.

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Chapter 1

Introduction

1.1 The actors on the stage of the Universe

All the phenomena occurring in nature, either at microscopic level or at macroscopic one, are controlled, essentially, by four fundamental interactions: the electromagnetic one, responsible for all the aspects inherent radiation; the weak-nuclear one, which explains the instability of atomic nuclei; the strong-nuclear one, justifying the existence of hadrons, i.e. bound-states of quarks, nucleons *in primis*, the blocks of atomic nuclei and, last but not least, the gravitational one, with which we have to do everyday, involving all the objects having mass, independently of their own characteristics.

Their properties are quite different, as it results from the synoptic Table 1.1 on the following page [1].

Since the fifthies, a lot of efforts have been made, thanks to the hard work of many scientists, in order to put them in a unique, consistent quantum field theory. Until now, it has been possible to unify the electromagnetic, weak and strong interactions only, well described by the symmetry group $SU_C(3) \otimes SU_L(2) \otimes U_Y(1)$.

But urgent issue remains to solve: the gravitational interaction seems to oppose to any attempt to link it to the others, due to its, apparent, incompatibility with Quantum Mechanics.

On the other hand, its relative strength (with respect to the strong one[1]) is very small, that may be interpreted as a signal that its nature is completely different from that of the other fundamental interactions.

Furthermore, many other issues, concerning its behavior either at small scale or at large scale, are waiting for an answer.

This work takes place in the context of the theories which point to extend General Relativity (GR), in order to try to give an answer to urgent issues, such as the dynamics of galaxies, clusters and superclusters of galaxies at astrophysical and higher scales and the current accelerated expansion of the Universe on cosmic scale, avoiding the hypothesis of the existence of exotic ingredients, such as the *Dark Matter* - the missing mass problem - and the *Dark Energy* - the cosmological constant problem.

1.2 A brief historical survey

What I am going to present, here, is only a brief review, without any claim of completeness, about the evolution of the concepts of space and time, strictly connected to the study of the bodies' motion, or more generally to the various phenomena occurring in the Universe, starting from the late of XVII'th century, the *Newtonian Age*, to arrive up to our days, the *Grand Unification Theories* era.

[1] In Table 1.1 it has been normalized to 1.

Table 1.1: Basic properties of fundamental interactions.

Interaction	**Acting on**	**Strength**	**Range** (m)	**Propagator(s)**	**Spin**
strong nuclear	hadrons	1	10^{-15}	gluons (8)	1
electromagnetic	electric charge	10^{-2}	∞	photon	1
weak nuclear	leptons,hadrons	10^{-5}	10^{-18}	$W^{\pm}, Z^{0}$	1
gravitational	mass	10^{-39}	∞	graviton (?)	2

Just to cite the most important, in the ambit of Particles Physics, multi-dimensional manifolds, in which some dimensions are *warped* or *enlarged*, concerning *Superstring Theory*, or its extreme version, known as *M-Theory*; in the context of Gravity, metric theories pointing to extend the GR, or non-metric theories, such as the Teleparallel Gravity, based on the concept of torsion, in turn strictly connected to the spin of particles. Finally, the *Supergravity Theory*, which is a Field Theory collecting the *SuperSymmetry* (SUSY) with the GR.

All of them aim to give, although from different points of view, a plausible explanation to two urgent issues, which arose as a result of observations made from Earth and via satellite, over the past ninety years: the first, the problem of missing mass, as an excess of gravitation is observed at astrophysical and higher scales, impossible to justify through the ordinary (barionic) mass and energy, detected through brightness measurements; the second, the current accelerated expansion observed on a cosmic scale, based on measurements made on the most distant type Ia *Supernovae*.

1.2.1 First steps

Any phenomenon, occuring in the Universe, to be described in a consistent way, needs a context. This fundamental issue has stressed, since the ancient times, many brilliant minds; to cite a few, from the *pre-Hellenistic Age*, Pythagoras, Democritus, Plato, Aristotle (VI-IV a.C.), to the time of *Renaissance*, R. Descartes (1596-1650) in particular, who described, first, in *Principia Philosophiae*, the inertia of bodies, in terms of uniform rectilinear motion, without specifying with respect to what.

On the other side, it was known that velocity makes sense in a relational point of view only. Moreover, G. Galilei (1564-1642), the father of *Scientific Method*, pointed out that an observer which is moving along a straightline, at constant velocity and one at rest, are indistinguishable, in the sense that the same experiment, in both cases, has the same evolution. In other words, he established that it is impossible *to feel* the uniform rectilinear motion.

And what about accelerated motion?

An observer, whose velocity is changing in value or direction, or both, *feels* the acceleration, without the necessity of any external reference; in other words, he *feels* it intrinsically.

It seemed that velocity needed a reference, but acceleration did not. The situation, in some sense, was confused.

I. Newton (1642-1727), in his treatise *Philosophiae Naturalis Principia Mathematica*, clarified this question (and many other, gravitation included, so laying the foundations of *Classical Mechanics*), by using the well-known bucket experiment (1689) [2]. It consists of a rotating, hanging bucket, filled with water; in the beginning, the water is at rest with respect to the rotating bucket and its surface is flat, but, at last, it rotates along with the bucket and its surface is concave.The situation remains the same also soon after the inversion. He asked himself, with respect to what the water was rotating and he arrived to the conclusion, that the *absolute space* was the entity, to which to relate the motion of the water, in order to justify the behaviour of its surface.

He states that «*the absolute space, by its own nature, without relation with anything external, remain always similar and immobile*»[2], namely, the space is a universal cosmic stage in which phenomena occur, but these ones cannot modify it anyway. Moreover, he does not clarify this concept further, limiting himself to define it *Sensorium Dei*, influenced by the theories of the theologian and philosopher H. More (1614-1687) [3].

And also, «*the absolute time, real, mathematic by itself and by its own nature, without relation with anything external, flows uniformly*»[2], that is, time runs in the same way for all observers.

G.W. Leibniz (1646-1716), a coeval mathematician and philosopher, did not agree about Newton's statements, asserting that space and time are terms useful to define relations between the positions of the objects and the moment during which they occur.

1.2.2 Einstein's revolution

In the middle of XIX'th century, the austrian physicist and philosopher E. Mach (1838-1916), the inspirer, in some sense, of Einstein's theory concerning the GR, intervened in this debate, asserting that, in vacuum space, rotation and any other motion do not make sense; they make sense, only if they are related to *the mean distribution of the remaining matter in the Universe*, without specifying how.

But clouds were thickening at the horizon.

First of all, Maxwell's equations, describing the behaviour of electric and magnetic fields, involved the constancy of the velocity of light (c), without needing any reference frame; it seemed that it was independent of the motion of the observer with respect to the source, as the Michelson-Morley's experiment (Cleveland, 1887) had established clearly.

This fact induced A. Einstein (1879-1955) to raise it to a postulate, that involved a review of the concepts of space and time, no more absolute and separated, but closely connected to each other, to form the *space-time*, continuous and flat. Any observer has its time and space, in other words its own reference frame, from which to see the light travelling with velocity c; this is due to the possibility to define two kinds of velocity, a space-like velocity and a time-like one, in which the total velocity splits. On these foundations, it was simply to justify, for instance, strange effects such as the *contraction-lengths* (an observer at rest and a second one, moving with velocity v, find different values about the distance separating two fixed points) or the *time-dilation* (as extreme example: on a light-wave, time is at rest, since velocity is completely space-like) and many other situations, involving observers in uniform rectilinear relative motion.

The Theory of Special Relativity (SR) was born, published in 1905. The situation is summarized in Table 1.2 on the next page [4].

Another issue concerned the Newton's law, the well-known law of the inverse square of the distance,

$$F = G_N \frac{Mm}{r^2} \tag{1.1}$$

where does not appear any dependence on time, namely, the interaction between two objects is transmitted *instantaneously*. That was, evidently, in contrast with SR.

Furthermore, it remained to explain the way according to which gravity acts in SR.

The starting point, in this case, was to understand that acceleration and gravity are equivalent, since they produce the same effects. This intuition, constituting the *Equivalence Principle* (1907) (EP), the foundation of GR (1915) [5], allowed A. Einstein to establish a strict connection between the curved texture of space-time and gravity due to the distribution of the mass/energy (the conceptual experiment of the *Rotor* was probably his first idea about it) [6]. Indeed, any accelerated observer describes, step by step, a curved world-line in the space-time, due to the presence of gravity, in turn generated by the local distribution of mass/energy. In this way, space and time become dynamical variables, as a consequence of the dynamics of the mass/energy present in the universe. In other words, Einstein's brilliant intuition consists in using the concept of acceleration, which is a measurable physical quantity, to establish the identity between gravity and the curvature of space-time. The situation is summarized in the following scheme:

$$\begin{array}{c} \textit{gravity} \longleftrightarrow \textit{acceleration} \longleftrightarrow \textit{space-time curvature} \\ \Downarrow \\ \textit{gravity} \longleftrightarrow \textit{space-time curvature} \end{array} \tag{1.2}$$

On the other hand, the GR theory did not get rid of the previous acquisitions, since, in presence of weak gravitational fields, where small masses give negligible curvature, SR was restored; moreover, in case of objects with low velocities, the Newtonian physics was recovered.

Table 1.2: Summary of the different positions concerning the nature of space and time

Newton	The space is an entity; the accelerated motion is not relative; absolutist position.
Leibniz	The space is not an entity; every aspect of motion is relative; relationist position.
Mach	The space is not an entity; the accelerated motion is relative to the mean distribution of matter in the universe; relationist position.
Einstein (SR)	Space and time, separately, are relative; the space-time is an absolute entity.

The first confirmations were not long in coming, since A. Einstein himself, on the basis of the equivalence between gravitation and inertia, in 1907, predicted the gravitational red-shift. Furthermore, the theory matched exactly the experimental results concerning the precession of the apsidal line of the Mercury's orbit, the extent of which was calculated by U. Le Verrier as early as 1855, realizing that $34\,arcsec$ were missing, subsequently improved by S. Newcomb in 1882, evaluating the gap in $43\,arcsec$, exactly those predicted by the GR, as well as the deflection of the light in the gravitational field of the Sun, as measured in 1919 by A. Eddington [15], during a solar eclipse [7, 8, 9, 10, 11].

Subsequently, other effects of GR were predicted and measured, which definitively sanctioned its validity with respect to other competing metric theories, primarily the Brans-Dicke scalar-tensor theory: the delay of the echo of an electromagnetic wave in radar-ranging measurements on the inner planets made from 1959 onwards, as predicted by I. Shapiro in 1955; the verification of the validity of the *Weak Equivalence Principle* also for large bodies, for which the ratio E_{grav}/E_{rest} is not negligible, thanks to the theoretical test developed by K. Nordtvedt in 1965, better known as the *Nordtvedt effect*, applicable to any gravitational theory that predicted a curved space-time; the *geodetic effect* concerning the precession of the axis of a gyroscope moving in a gravitational field (a parallel transport-like effect needed to define the covariant derivative); finally, the *Lense-Thirring effect*, i.e. the frame-dragging effect, which describes the precession of the axis of a gyroscope placed in the gravitational field of a rotating source [12, 13, 14].

1.3 Gravity at different scales

More than a century has passed from time during which have seen the light the two pillars of modern physics, the GR and the Quantum Mechanics (QM). These theories concern ambits completely different, since the first describes the dynamics of the macroscopic world, while the second, born in response to burning problems related to the microscopic world, with respect to which nineteenth-century physics failed, concerns objects at atomic and subatomic scales.

In the beginning, it seemed that they were irreconcilable; in fact, whenever the QM has to be used along with the GR, that is, when dealing with extremely dense and compact objects, practically point-like, as in the case of black holes or, on the other extreme, in the case of the universe itself when it was born, their equations, when put together, produce results that make no sense.

But, during the eighties, some physicists realized, after many efforts, that it was possible to merge the QM with the GR, in what is known as *String Theory*, through the basic idea according to which each particle can be described in terms of the vibrations of a string, a non point-like elementary object, having an intrinsic extension along a single dimension, comparable to the Planck's scale, $L_P \sim 10^{-33}\,cm$, that is to say a sort of vibrating ring. However, although this appears to be a very promising theory, it still requires a lot of work (and probably a lot of time) to be complete and consistent, due to the complexity of the mathematics it requires. I do not intend

to go further, as this is not the subject of the thesis and not having the necessary skills, although I consider it very intriguing.

In what follows, without any claim to completeness, I move on to deal with some issues concerning gravity on a small and large scale, in order to highlight the strengths and weaknesses of the GR when applied to such different contexts.

1.3.1 Gravity at small scale

Many experiments have been done (and others are in preparation), by using the most modern devices, such as torsion balances, torsion pendulums, torsion oscillators at low and high frequencies, microcantilevers [16], to test the Newton's law at small scales (but not at the smallest to require QM), searching for possible deviations from the inverse square law of distance.

To date, no one deviation has been yet observed, at least up to $10^{-2}\,cm$, the lowest distance reached by current experiments.

On the other hand, GR provides for corrections at distances comparable with the Schwarzschild radius, $R_S \sim 10^{-25}\,cm$ for a mass of $1\,kg$, that one used in current experiments, while Quantum Gravity at distances comparable with Planck's scale. Therefore, there exists a wide range of distance scales to investigate.

Supergravity and *Superstring Theory*, involving small deviations from EP, predict a gravitational interaction given by two contributions: the first one, universal, with infinite range and coupling constant $G = G_N$; the second one, the so-called *fifth force*, acting directly on the atomic components of bodies, with short range and coupling constant $G \sim G_N$.

There are various candidates capable of propagating the *fifth force*. In *Extended SuperSymmetry* (SUSY), we have a vector field, connected with the *graviphoton* [17], and a scalar field related to the *dilaton*, the scalar partner of graviton. Outside of SUSY, I want to remind, incidentally, the well known Brans-Dicke scalar-tensor theory (1961), containing a scalar field non-minimally coupled with gravity.

Here, I am going to point out some aspects of the graviphoton and the dilaton, which involve corrections to the Newtonian force at small scale.

Graviphoton, a boson with spin 1, unlike photon, has mass; so, it transmits a short-range interaction, having strength similar to the gravitational one and travelling with velocity $v < c$. Most importantly, it couples with hypercharge, or better with baryon charge, what involves the dependence on the chemical composition of bodies and it results to be attractive or repulsive depending on the sign of the interacting baryon charges, like the electromagnetic interaction, being related to a vector field. Consequently, as far as the baryon matter is concerned, the graviphoton gives a repulsive contribution to the total gravitational force, in this way weakening it.

There is no evidence for it at the energy scales reached by LHC collider at CERN, around $14\ TeV$, corresponding to a range of $10^{-16}\,cm$ ($\ll 10^{-2}\,cm$ reached in the current experiments), which means that, if it exists, it has to be even more massive. I remind, here, that range and mass of a particle are linked by the relation $R = 1/m$, when expressed in natural units.

Another aspect, important in astrophysical and cosmological ambit, is given by the graviphoton-photon coupling, altought very weak, in vacuum. In fact, an e.m. wave, having constant momentum, contains both components, graviphotonic and photonic, respectively with mass different from zero and zero. This implies the existence of two frequencies, which interfere, giving oscillations in the total intensity of the wave, because of a mutual conversion between the two components; that could underlie the process generating the strong magnetic fields observed at galactic and extragalactic scales [37].

Dilaton owes his name to 'conformal' symmetry, characterizing the dynamics of unidimensional elementary objects, such as strings; in other words, the dynamics of a string has not to be modified by a change of scale of length, time or energy. In order to prevent dilaton from violating this symmetry, certain conditions must be imposed and the latter set its role in the context of gravitation. In the specific, it can interacts with all particles, but this interaction is not universal, depending on

the nature of them. According to some computations [19], its coupling constant with hadrons is $G_{had} \sim 40 - 50\, G_N$, whereas that one with leptons is $G_{lep} \sim G_N$. Consequently, the dilatonic force results to be stronger than the gravitational one, depending on the chemical composition of bodies. But the most important thing is that it is attractive, so reinforcing the total gravitational force.

As in the case of graviphoton, its mass is unknown; the current limits of distance ($10^{-2}\, cm$) impose for it a smaller range, that is to say a mass $m_D \gtrsim 10^{-3}\, eV$ (m_D dilaton mass). It is interesting to observe, incidentally, that a mass $m_D \lesssim 1/10\, m_p$ (m_p being the proton mass), allows dilatons coming from remote ages to survive up to now, giving a contribution to the fossil radiation, eventually observable.

Other theoretical models indicate a different nature for gravity, without introducing a new kind of interaction.

One of these, known as 'chamaleon' gravity [20], admits the existence of a scalar field, whose mass, however, depends on the matter density with which it interacts; namely, its mass increases in presence of big masses and decreases in the opposite case, up to become very small in vacuum. Such a scalar field could explain the current accelerated expansion of the universe, since it contributes to gravity at cosmological scales, where it becomes a long-range field, as a consequence of the above mechanism. Clearly, its mass depends on time, since the total matter density decreases with the expansion of the universe; therefore, from some time onwards, its range overcomes the Hubble radius, so becoming insensible to matter.

Another 'exotic' possibility is offered by 'fat gravitons' [21], particles having finite size, given by a characteristic length L_g. This hypothesis is strictly connected with the necessity of avoiding ultraviolet divergences, raising when one considers GR in a quantum regime. In particular, it implies a potential, whose behavior, below the critical value L_g, runs to zero when distance goes to zero, showing a clear deviation from the Newtonian potential. However, this has not been observed within the current experimental limits $\sim 10^{-2}\, cm$.

Moreover, this characteristic length makes it possible to make an estimate for the cosmological constant, $\Lambda_g = E_g/L_g^3 = 1/L_g^4$, other than that corresponding to the Planck's scale, $\Lambda_P = E_P/L_P^3 = 1/L_P^4 = M_P^4$ ($M_P \sim 10^{19}\, GeV$ is the Planck's mass), whose value is completely in disagreement with that observed, being 120 orders of magnitude higher than this. In fact, hypothesizing a value $L_g \sim 2 \times 10^{-3}\, cm$, just below the current limit, it would be obtained $\Lambda_g \sim (10^{-2}\, eV)^4$, that is the same order of magnitude as that observed.

Finally, I want to quote, briefly, two scenarios connected with multidimensional spaces.

The necessity of giving a unified description of all fundamental interactions, forces the use of a multi-dimensional geometry, containing *extra*-dimensions besides the ordinary ones, namely the three extended and flat spatial dimensions and time.

In fact, in the early eighties, *The String Theory*, recovering the old T.Kaluza - O.Klein idea [22] [23], postulates a 'factored' geometry, given by the product of two spaces; the first, having the three ordinary spatial dimensions and the second, containing the extras ones, warped, at a scale supposed comparable to the Planck's one $L_P \sim 10^{-33}\, cm$. The properties of such a geometry must match the symmetry properties of the gauge fields, Abelian and non-Abelian. But the non-Abelian ones need to be added in a non-geometrical way, unlike the former, according to a 'spontaneous compactification' mechanism, implying the same factored structure, with two different energy-momentum tensors; the first one depending on the three spatial dimensions and the second one on the extras, without mixing each other. On the other hand, the Einstein's equations, working with any number of dimensions, establish a strict relationship between geometry and matter fields. Ultimately, the matter fields determine which dimensions are warped and which not.

In a manifold with $D = 4 + n$ dimensions, the gravitational coupling in D-space is given by $G_D = L_c^n \times G_N$, L_c being the mean radius of compactification and $G_N = L_P^2$. On the other hand, it is possible to define a characteristic lenght also in D-space, L_D, where $G_D = L_D^{2+n}$. So, if the gravitational force has the same intensity in D-space, then $L_D \sim L_P$ and consequently

$L_c \sim L_P$, completely undetectable. But, if it is stronger, there would be a more extended compactification volume, due to a higher value of L_D and consequently of L_c evidently.

Conversely, by using the current experimental limit for compact dimensions, $L_c \lesssim 10^{-2}\, cm$, it is possible to infer their minimum number [24] [25]; for instance, gravity could give detectable effects, according to the energy scale achievable by LHC, $E_D \sim 1/L_D \sim TeV$, only if $n \geq 2$. Actually, nothing has been observed at the smallest length scale nowadays reached, that is $\sim\ 10^{-16}\, cm$; so, admitting $L_c \lesssim 10^{-16}\, cm$, at the same energy scale as above, there would be visible effects only for $n \geq 15$.

I want to conclude, citing the 'partial confinement' of gravity in a multidimensional space.

In *String Theory*, the ends of the open strings, containing the 'charges' associated with the gauge fields, are located on a 3-dimensional brane (our ordinary space), which, together with other extra dimensions, constitutes a n-dimensional manifold. The gravitational interaction is the unique exception, since its source is essentially the energy, which is not a charge; so it is given by a closed string, capable of moving in the total space. This is a problem, since the gravitational force goes as $1/r^{n-1}$ in it, unlike the Newtonian one, going as $1/r^2$. But it can be solved through a suitable geometry. In fact, in the simplest model, the energy contained in the 3-dimensional brane distorts the external space, in accordance with the laws of GR, although it remains flat, as a geometry called anti-De Sitter [26]. According to this geometry, the gravitational interaction, generated by the matter standing on the brane, is due to various components: the first, corresponding to a massless graviton, with long range, assuming exactly the Newtonian form, but trapped on the brane; countless others, having mass that grows, in a continuous way, from a minimum to infinity (remembering the Kaluza-Kline tower), with short range and propagating only in extra dimensions. Furthermore, the intensity of interaction transmitted by the last ones, depends on the mass of graviton and, generally, increases with it, at low energies.

These new massive components involve corrections, at the short distances, to the Newtonian force. For instance, with only one extra dimension, there is a corrective term going as $1/r^4$, but it reaches the ordinary gravitational intensity, at distances of order of the curvature radius of the external dimension, while, at higher distances, it becomes negligible.

This aspect is very interesting, since the corrections to the Newtonian force depend on the curvature radius of extra dimensions, namely on their local geometric properties, not on their global form or extension. In particular, if external dimensions have curvature radius $\rho \lesssim 10^{-2}\, cm$, then, they can be also arbitrarily extended, without contradicting the present experimental results.

Ultimately, only further measures can give indications on the behavior of the Newtonian force at smaller scales, compatibly with technology available in the near future.

1.3.2 Gravity at large scale

I have presented, in the previous section, some theoretical models, which expect modifications in the behavior of the Newtonian force at small distances. They intervene in various ways on gravity; for example, by admitting the existence of a new interaction, by hypothesizing a different nature of gravity, or by introducing extra spatial dimensions in addition to the four ones of the GR, moreover, without any experimental evidence, at least up to distances of the order $r \sim\ 10^{-2}\, cm$.

And what is happening about large-scale gravity?

The Newtonian force, valid in regime of low velocities and weak fields and its relativistic extension, the GR, seem to provide the right answers to all the observed phenomena, at least up to the scale of the Solar System.

But, as is known, two major problems arose, since the early thirties, concerning the dynamics of objects at astrophysical and cosmological scales. On one side, the rotation curves of galaxies and the dynamics of clusters and superclusters of galaxies; on the other side, the current accelerated expansion of the universe. To date, they have not yet found an answer.

It is my intention, here, to make a picture of the situation in both areas.

The Dark Matter problem

In the early thirties of the last century, some astronomical observations, concerning the dynamics of large structures, had detected gravitational effects not attributable to the visible matter. It seems that a Swedish astronomer, K. Lundmark, around 1930, first, had noted a lack of matter in the galaxies he had observed, estimating a ratio missing matter/visible matter close to the currently measured value ~ 5. Two years later, the Dutch astronomer J.H. Oort [27], was able to measure, through the Doppler red-shift, the velocities of stars, moving on the plane of our galaxy, that he noticed to be higher than expected, on the basis of the Newtonian gravity referable to the only visible matter, claiming that they were due to the presence of non-luminous ordinary matter. But, it was F. Zwicky [28], in 1933, first, to hypothesize the existence of the *Dark Matter* (DM), of unknown nature, as a consequence of his observations, regarding the velocities of the galaxies belonging to the Coma cluster. In fact, using the *virial theorem*, he came to the conclusion that these velocities resulted justifiable only by admitting a mass value 400 times higher than the visible one [2]. The work was ignored for more than twenty years, until L. Volders, in 1959, proved that even the spiral galaxy M33 did not rotate according to Newtonian physics [29].

Zwicky's idea was back in vogue in the sixties, thanks to the studies concerning the satellite galaxies, namely small galaxies in the vicinity of larger galaxies, whose contribution is not sufficient to explain the dynamics of the latter, except in terms of additional matter of an unknown nature. In this way, the dark matter became a real field of investigation.

In particular, it was the measurements performed, during the seventies, by V. Rubin and his colleague K. Ford, on the tangential velocity of the stars in spiral galaxies (starting from Andromeda galaxy), that gave new impetus to the subject [30]. In fact, they noted that, outside the galactic center, the velocity of the stars was the same, at any distance along the radial direction, far beyond the region containing luminous matter, what they explained admitting the presence of an amount of matter, of which the detectable one corresponded only to about 1/6.

In this regard, M. Milgrom, in 1981, has given an alternative explanation of the galactic rotation curves, known as MOND theory, *Modified Newtonian Dynamics*, proposing a modification of the Newton's law, according to which, in the limit of low accelerations, the Newtonian force $F \propto a^2$ [31]. Going into detail, below a certain acceleration value a_0, this force goes like,

$$F = m\,\mu\Big(\frac{a}{a_0}\Big)\,a, \tag{1.3}$$

where $\mu(x)$ is an interpolation function, defined as follows,

$$\mu(x) = \begin{cases} 1 & x \gg 1 \\ x & x \ll 1. \end{cases}$$

Other evidences of anomalous gravitational effects have arrived from observations on speed dispersion of stars in elliptical galaxies, from the *gravitational lensing*, another Zwicky's idea, by galaxies and clusters of galaxies, to get to the most recent satellite measurements, more and more precise, COBE (1989), WMAP (2001), PLANCK (2009), related to the *Cosmic Microwave Background* (CMB), thanks to which, among other things, it has been possible to evaluate the contributions to the total energy density of the universe, coming from baryon matter $\sim 5\%$, dark matter $\sim 26\%$ and dark energy $\sim 69\%$ as shown in fig. 1.1.

All that one can say about the properties of the dark matter, is that it interacts very weakly with baryon matter and itself, being capable of producing only gravitational effects; for the rest, it

[2] In the case of an isolated system of particles, held together by internal conservative forces only, the average over time of the kinetic energy and the average potential energy, are linked by the relationship

$$< T > = -\frac{1}{2} \sum_{all\,pairs} < \boldsymbol{F}_{ij} \cdot \boldsymbol{r}_{ij} >$$

where $\boldsymbol{F}_{ij}$ indicates the force that j-particle exerts on i-particle and $\boldsymbol{r}_{ij}$ is the corresponding relative position.

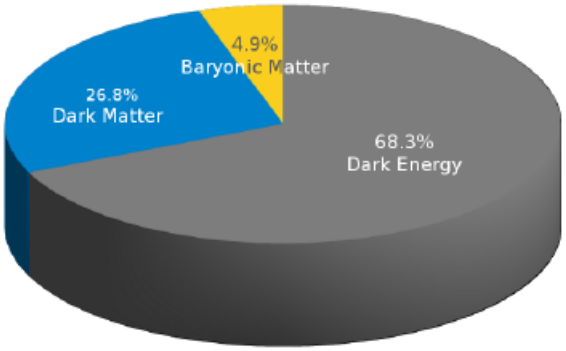

Figure 1.1: The latest results from the Planck satellite on the energy densities

is not subject to any other type of interaction. Furthermore, the constituent particles have very low velocities $v \lesssim 0.1\,c$; in this sense, physicists speak of *Cold Dark Matter* (CDM). Finally, it has no electric charge.

There are several possible candidates for the DM, among which the most discussed are the following: the WIMPs, *Weakly Interacting Massive Particles*, whose mass should be, roughly, of the same order of that of the Higgs boson ($m_H \sim 126\,GeV$); more specifically, recent estimates place their mass in a rather large interval ($1 - 10^2\,GeV$). Furthermore, they interact very weakly with the particles of the Standard Model, besides to gravity; appropriate calculations indicate that the energy carried by these particles, is approximately equal to that associated with the dark matter currently present in the universe. But the various direct detection experiments, currently running, have not yet detected anything about them.

The so-called *asymmetric dark matter*, such as the breaking of matter-antimatter symmetry, hitherto unexplained, with the consequent *baryogenesis*, has determined the current abundance of matter, something similar could have happened about dark matter-dark antimatter annihilation process, being comparable the current abundances of ordinary matter and dark matter.

The *neutrinos*, initially proposed as the components of the *Hot Dark Matter* (HDM), are not good candidates, having too high speed to give the CDM, $v_\nu \sim c$, and too small mass to justify the energy density related to the DM.

The MACHOs, *Massive Compact Halo Objects*, massive, non-radiating and non-reflecting objects, such as brown dwarfs, neutron stars and black holes, all objects made up of ordinary matter, which, for various reasons upon which I do not dwell, do not justify the effects produced by the DM. Furthermore, it is not easy to find an explanation for how these objects ended up in the halo rather than on the galactic disk.

Finally, the AMS experiment, *Alpha Magnetic Spectrometer*, collocated on ISS (*International Space Station*) in 2011, still in progress, may have seen an excess of positrons, from cosmic rays, constant over time and isotropic, whose distribution with energy seems to be the same as that expected in the hypothesis that they are produced by collisions of dark matter particles. However, it is necessary to see what happens at higher energies, far beyond the current limit $\sim 250\,GeV$, where their number is expected to go to zero. To date, no reduction has been observed yet. Furthermore, the detected signals would require so much DM in the early universe, to produce distortions in the CMB, that PLANCK satellite should have been observed.

The Dark Energy problem

The first indications regarding an anomalous gravitational repulsion, observable at large scale and responsible for the current accelerated expansion of the universe, have arrived from measures concerning the Supernovae Ia, better known as *standard candles*, observed in the most distant galaxies, carried out in 1998 by two different teams, the first led by S. Perlmutter [32], the second by A. Riess and B. Schmidt [33], which result in an even lower light intensity than expected on

the basis of what is predicted by the standard cosmological model. The physicists agree that the observed accelerated expansion is attributable to a new kind of energy, which has nothing to do with the ordinary matter and energy; it is, rather, a kind of fluid that permeates the whole universe. For these reasons it is called *Dark Energy* (DE), present since the first moments of life of the universe.

It is generally preferred to identify it with the *cosmological constant*, Λ, back in vogue after A. Einstein had first introduced it into his field equations, in order to justify the then presumed static universe and then eliminated it, following Hubble's observations, attesting to a universe in slow expansion.

Its energy density turns out to be isotropic and constant over time, implying a contribution of negative pressure within Einstein's field equations; moreover, it is dominant in the current era when compared with that associated with dark matter and baryon matter. Both of these aspects contribute to justify the current accelerated expansion on cosmic scale.

There are, however, two open issues.

The first one, known as the *smallness problem*, concerns its very small value, given by $\Lambda \sim 1/{L_P}^2{L_H}^2 \sim 10^{-8}\, eV^4$ (where $L_H \sim 10^{28}\, cm$ is the Hubble's radius, which gives the maximum extent of the observable universe); the computations taking into account the quantum vacuum fluctuations, give a ratio $\Lambda_P/\Lambda \sim 10^{122}$ ($\Lambda_P \sim 1/{L_P}^4$ where $L_P = \sqrt{G_N}$ in natural units), an enormity! If it was an arbitrary (classical) constant, it would be difficult to justify its current value; in fact, if it was a little smaller, it would be negligible (there would be no accelerated expansion), while if it was a little larger, the universe would be completely different.

The second one, the *coincidence problem*, is related to the fact that such energy density has remained constant over time, while those associated with radiation and matter (ordinary and dark), have diluted over the course of expansion. Currently, these densities are comparable, according to the ratio $\Lambda_{DE}/\Lambda_{matter} \sim 7/3$, what makes the present era, in a sense, highly 'privileged'.

There exist, however, alternative models, aimed at explaining the current accelerated expansion, without resorting to the concept of dark energy, with regard to some of which I would like to speak briefly.

The first of them, known as DGP theory [34], from the name of the proposers[3], is based on the concept of multidimensional space containing a 3-dimensional brane. Unlike the model proposed in Section 1.3.1, where a high vacuum energy is present in the total space, here all matter and energy, vacuum included, are on the 3-dimensional brane, while the outer space is completely empty and flat (Euclidean) and the gravitational interaction is everywhere. The brane itself generates a field, the so-called *induced gravity*, due to the matter contained therein, which contributes to the total one. In this situation, the gravity on the brane 'feels' the extra-dimensions, what produces changes on it, only at low energies and consequently at large distances.The intriguing aspect of the model, is due to the fact that the corrections to the gravitational field equations on the brane, under a suitable energy scale, foresee an accelerated expansion of the 3-dimensional space, what is happening in the universe today. This model, however, has shortcomings: first, a very high value of the coupling constant related to the total gravitational field, difficult to justify; furthermore, in its quantum version, it predicts non-physical states having a negative probability!

Another possibility is given by the hypothesis that the dark energy density is not constant, but varies over time. In this case, it could be well described by a force field, for which the dilaton field, of which we have already spoken in Section 1.3.1, results to be a good candidate.

The dilaton potential, indeed, as shown in Fig. 1.2 [35], tends to zero for weak field and shows two maxima with increasing the intensity of the field. As it results from string cosmology, the field increases over time, starting from very low values, to reach, sooner or later, the area between the two peaks. At this point, there are two possible situations:

[3] G. Dvali, G. Gabadadze, M. Porrati.

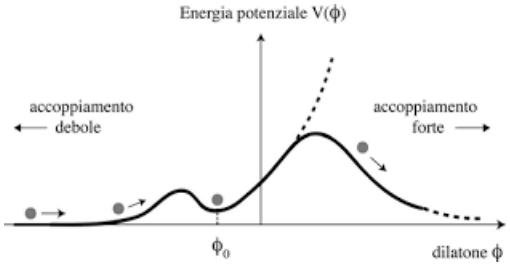

Figure 1.2: Dilaton potential

1. the field reaches the minimum and settles there when the universe is in the radiation era, with which it interacts very weakly; during transition from the radiative to the matter phase, it tends to move from the minimum, as a consequence of its greater coupling with dark matter (the field 'tracks' the evolution of matter [36]). To stay there, the potential well has to be deep enough and the coupling with matter not too strong. In this way, the universe evolves towards a final phase, characterized by this value, which acts as cosmological constant, *i.e.* $\Lambda = V(\phi_0)$. This implies that the *coincidence* between the two densities, occurs within a time window, determined by the depth of the potential well, what makes our epoch not so special.Thus the *coincidence problem* is only partially solved [37];

2. the field manages to exceed the minima, to reach the strong coupling region, in which it is assumed that the potential rapidly falls to zero; the field, clearly, does not remain fixed at a constant value. In the transition from the radiative phase to the matter phase, the potential energy density of dilaton is negligible with respect to the kinetic one, which, in turn, is negligible with respect to the dark matter one; but the coupling between dilaton and dark matter is sufficient to force the dilaton to follow the evolution of dark matter, the so-called *dragging phase*. A *freezing phase* follows, when the potential energy density of dilaton becomes comparable with the kinetic one and the dark matter one, in which the ratios between the various energy densities remain fixed at values of order of unity, for the rest of the time. It should be noted that the dark matter energy density, alone, would dilute rapidly and produce a decelerated expansion, while the dilaton energy density, alone, dominated by the potential, would produce an accelerated expansion; this model, on the other hand, predicts that the two densities, thanks to the coupling, dilute sufficiently slowly, determining in any case an accelerated expansion. In this way, it is evident that our epoch is not privileged and we are not experiencing any *coincidence* [38].

In any case, most of the information acquired to date [39, 40, 41], has found a synthesis in the so-called *concordance model*, ΛCDM *model* for short, which provides a good fit to the high quality data coming from the most recent measurements, concerning the cluster properties as the mass, the correlation function and the evolution with red-shift of their abundance [42, 43, 44, 45], the Hubble diagram of Type Ia Supernovae [46, 47, 48], the optical surveys of large scale structure [49, 50, 51], the anisotropies of the cosmic microwave background [52, 53], the cosmic shear measured from weak gravitational lensing surveys [54, 55] and the Lyman-α forest adsorption [56, 57]. It returns a picture of the universe spatially flat, with insufficient baryon matter content to explain the dynamics observed at astrophysical scale and beyond and that undergoes a phase of accelerated expansion on cosmic scale, for the explanation of which DM and DE are invoked, to which, however, a precise identity has not yet been attributed.

Alternative proposals

General Relativity has more than a century of life behind, during which it has proven to be a consistent theory, capable of providing adequate answers to a plurality of issues belonging to the

cosmological ambit, once these are framed in the context of a curved and dynamic space-time as a consequence of the presence of matter and energy contained therein.

Fundamental is the fact that, in the weak fields and low velocities limit, it manages to recover the Newtonian gravity, in terms of Poisson or Laplace equations, depending on whether there is matter or not.

It should be mentioned another brilliant result among others, that is to have predicted (A. Einstein 1916) the existence of gravitational waves (GW), through a proper linearization procedure of the field equations. They are ripples in the fabric of space-time, generated by accelerated masses and propagating at the speed of light, just as the electromagnetic waves, thus carrying energy in the form of radiation. They were revealed, for the first time, in 2016, by VIRGO and LIGO collaboration, through the event GW150914, the result of a binary black holes merge. Since then, other 6 events have been revealed, one of which, the event GW170817, derived from a binary neutron stars merge and detected also in the electromagnetic window, what gave rise to *multi-messenger astronomy*.

However, as the classical tests show and the PPN data confirm (constraints on Post-Newtonian Parameters) [58, 59], it seems that GR works very well only at the Solar System scale.

Since no precision experiments have been carried out outside the Solar System, with the exception of measurements made on binary pulsars since the first half of the 1970s, which are however systems of small extension, much smaller than that of the Solar System, the fact that it can work at larger scales is only a hypothesis and nothing more. In this regard, it should be noted that a large amount of experimental data obtained from satellites, such as Galileo and Ulysses, deactivated for some time and Pioneer 10/11, now out of the Solar System, seem to have detected anomalous accelerations, at long range, difficult to explain only through the Newtonian potential.

As already mentioned in the previous sections, the GR does not account for the dynamics of structures at galactic and extra-galactic scales, that particle physics would like to explain in terms of exotic to date undetected particles, in order to justify the 'missing' mass necessary to account for the observed excess of gravity. Furthermore, the current accelerated expansion, detected at cosmic scale, which the GR attempts to describe, in the simplest way, through the *cosmological constant*, it is likely that it needs an alternative description; in fact, it could be due to an appropriate scalar field, introduced in the context of an effective field Lagrangian, which would constitute an extension of that of Hilbert-Einstein, that seems a more acceptable solution to the problem of DE.

The strong gravitational fields, such as those generated by black holes, constitute another unsolved issue within the GR theory. As known, black holes are the result of collapsed massive stars, in which space-time is so curved that not even light can escape. Also in the simplest model, that of Schwarzschild, static, uncharged, with spherical symmetry, the solution of the field equations is given by a metric, containing a singularity in correspondence with the *Schwarzschild radius*, which constitutes the so-called *event horizon*, at which it produces an infinity. It has been proved, however, that this singularity is only apparent, since it can be removed by means of a suitable coordinate transformation. Actually, the 'true' singularity lies at the center of the object, which is point-like for a static black hole and it contains the whole mass, having therefore infinite density. But the existence of a region of the space-time with infinite curvature and density does not seem to describe a real situation, that it appears to be the breakdown of GR. However, some theories including quantum effects (loop quantum gravity), exclude the presence of singularities.

Finally, there remains the problem of quantization of the gravitational field. Since the fifties, many efforts have been made to describe all fundamental interactions, matter included, at quantum level, through a single symmetry scheme; the goal has been achieved in part, with the unification of the electromagnetic field with the weak and strong nuclear forces.

The approach is based on the fact that the physical states are described by vectors in a Hilbert space $\mathbf{H}$, while the physical fields are given by linear operators acting on $\mathbf{H}$ and defined at

every point of a flat space-time, given *a priori*. The GR, on the other hand, is essentially a classical theory, where the texture of the space-time is described by the field itself, given by the metric tensor $g_{\mu\nu}(x)$, which is a dynamic entity, solution of the Einstein's field equations. Furthermore, it assumes a hydrodynamic description of the fluid, matter and energy, contained in the universe, totally inadequate at quantum level.

These issues have led to a critical review of Quantum Mechanics and GR, in order to provide a unitary and fundamental description of matter and space-time. In the covariant approach, more suitable for perturbative calculations, it is preferred to use the standard techniques of Quantum Field Theory; in fact, thanks to R.P. Feynman, it has been possible to extend the perturbative methods of Quantum Electrodynamics to gravity. In this regard, the metric $g_{\mu\nu}$ is decomposed into two contributions as follows,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1\,, \tag{1.4}$$

where $\eta_{\mu\nu}$ is the kinematic part, corresponding to the flat geometric background and $h_{\mu\nu}$ the dynamic part, constituting the perturbative term, represents the field, to be quantized. The corresponding quanta are particles with spin 2, called gravitons, massless, propagating on the flat space-time given by $\eta_{\mu\nu}$.

Such a model, treated as a Yang-Mills theory, results to be renormalizable, by perturbative series, only up to the 1-loop level. Therefore, the fact that GR is not renormalizable at all orders, means that its validity fails at high energies (small scales),while it works at low energies (large scales). In other words, the *correct theory of gravity*, to date not available, has to be invoked at the Planck scale, $E_P \sim 10^{19}\, GeV$, while, far below it, the GR is a good approximation.

In this context, the *Extended Theories of Gravity* (ETG) play a key role, as they try to bridge the correct theory of gravity and GR. These theories involve the addition, in the Hilbert-Einstein Lagrangian, of higher order terms in the curvature invariants and additional terms, containing non-minimal couplings with matter fields. The former, of geometry-geometry type, are created starting from the Riemann tensor, the Ricci tensor and the corresponding scalar, that is to say R^2, $R_{\mu\nu}R^{\mu\nu}$, $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$, $R\,\Box\, R, \ldots,\ R\,\Box^k\, R$, or more general analytical functions of the Ricci scalar $f(R)$, while the latter, of geometry-matter type, contain contributions such as $\phi^2 R$, in case of scalar fields and the like for the spinorial matter fields. They describe, at quantum level, by (1.4), the graviton self-interactions and the graviton-matter interactions respectively.

It should be emphasized that, from a variational point of view, such theories can be treated differently, depending on whether one prefers the Lagrangian approach, in which the only field in play is that represented by the metric tensor $g_{\mu\nu}(x)$, containing all information regarding the structure of space-time and geodesics, typical of purely metric theories; or the Hamiltonian approach, in the bi-metric or metric-affine formalism developed by A. Palatini in 1919, based on two independent fields, corresponding to the 'physical' tensor $g_{\mu\nu}(x)$ and to the Levi-Civita connection Γ, expressed through the first derivatives of a second tensor $\hat{g}_{\mu\nu}(x)$ [60, 61, 62]. These are non-equivalent methods, in the sense that the same theory, treated in the two different formalisms, returns different field equations and therefore different results. This work is part of the purely metric theories.

What is important to note, regarding the ETG metric theories, is that the models based on quadratic Lagrangians and higher-order, have been developed, to date, only up to the second order of the perturbation metric, the so-called Post-Newtonian approximation 0.5PN, giving back, in the weak field and low velocity limit, modified Newtonian potentials of the type,

$$U(r) = -\frac{G_N M}{r}\left[1 + \sum_{k=1}^{n} \alpha_k\, e^{-\frac{r}{r_k}}\right], \tag{1.5}$$

(n indicates the number of quadratic and higher-order terms appearing in the Lagrangian) where, in addition to the Newtonian potential, Yukawa-like corrections appear, containing characteristic

lengths r_k, indicative of the distance scales to which these terms give a non-negligible contribution (this kind of potential was originally proposed by H. Yukawa, in 1935, to describe the meson field [63]). Thus, the presence of these corrective terms seems to indicate a different behavior of gravity at different distance scales, being also capable of providing attractive or repulsive contributions, depending on the sign of the coefficients α_k.

Specifically, as evidenced by the abundant literature on the subject, the model based on the generic analytical function $f(R)$, expanded up to the second order, namely,

$$\mathcal{L} = f(R) = a_1 R + a_2 R^2 \,,$$

which returns, in the aforementioned approximation, the following modified potentials, generated by a point-like matter distribution,

$$\begin{aligned} U(r) &= -\frac{G_N M}{a_1 r}\left[1 + \frac{1}{3}e^{-\frac{r}{r_1}}\right], \\ V(r) &= -\frac{G_N M}{a_1 r}\left[1 - \frac{1}{3}e^{-\frac{r}{r_1}}\right], \\ r_1 &= \sqrt{-\frac{6a_2}{a_1}} \quad a_1 > 0,\, a_2 < 0\,, \end{aligned}$$

appears to be the only one to have been tested on the available observational data. Therefore, to complete the studies performed so far, it would be interesting, in the light of new potentials acquired through further extended quadratic Lagrangians, to carry out new tests, to verify whether the potentials thus obtained are able to give definitive answers to the problems emerged at the different distance scales, starting from the astrophysical ones and up.

In conclusion, the ETGs propose an alternative explanation of the dynamics of structures at astrophysical scales and beyond, as well as of the current accelerated expansion at cosmic scale, to the DM and the DE, exotic ingredients, so to speak, having peculiar properties, but of whose identity nothing is known to date; in fact, tests performed on stellar clusters, spiral and elliptical galaxies, clusters of galaxies, as discussed, for example, in [64, 65, 66, 67], as well as, at local scale, in addition to the classic ones, on De Sitter's precession (geodetic effect) [69] and on Lense-Thirring gravitomagnetic precession (frame-dragging effect) [13, 14, 68, 69], seem to show that these theories work well in both ambits.

Incidentally, it should be observed that the Hamiltonians of these theories do not admit minimum energy states, namely they are unlimited from below, therefore unstable. Furthermore, unitarity is violated, that implies the non-conservation of probability. Supergravity, for instance, in the same way as the electroweak theory, tries to describe gravity together with matter fields, with the result that the theory is partially renormalizable; in fact, the infinities related to the bosons of the gravitational field and the fermions of the matter fields cancel out each other, up to two-loop level. In this case, the Hamiltonian is positive defined and unitarity is preserved.

1.4 Thesis topic and work plan

The first part of this work consists of a detailed analysis aimed at developing tensors and scalars of interest in higher-order metric theories of gravity, such as the Riemann tensor, the Ricci tensor and the corresponding curvature scalars, all those curvature invariants appearing in quadratic and higher-order Lagrangians; in particular, the following Lagrangian is taken into consideration,

$$\mathcal{L} = a_1 R + a_2 R^2 + b_1 R_{\mu\nu} R^{\mu\nu} + c_1 R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \,, \tag{1.6}$$

into which the quadratic terms are introduced as perturbative corrections to that of Hilbert-Einstein, when considered in a low curvature regime, namely in the weak-field and low velocity

limit, specifically by using the fourth-order approximation of the perturbation metric, the so-called Post-Newtonian approximation 1PN, expressed through three gravitational potentials $U(\mathbf{x}), V(\mathbf{x})$ and $\Phi(\mathbf{x})$, an analysis that has not yet been carried out in the context of these theoretical models.

However, it has to be observed that, thanks to the Gauss-Bonnet identity, valid only in a four dimensional space-time [70, 71, 72, 73], the Lagrangian (1.6) is reducible, without loss of generality, to the following,

$$\mathcal{L} = a_1 R + a_2 R^2 + b_1 R_{\mu\nu} R^{\mu\nu} \,. \tag{1.7}$$

Incidentally, such identity has greatly simplified the computation of the curvature scalar related to the Riemann tensor, corresponding to the last term of (1.6), in the aforementioned approximation.

The central part exposes the development technique of the corresponding field equations, implemented in a more direct way, as it will be widely explained after and the search for the relative solutions by using the Green function method, on the basis of the above superior perturbative scheme of the metric tensor.

The use of the metric tensor in such an approximation, involves the expansion of the field equations up to the order $O(\epsilon^4)$, where $\epsilon = v/c \ll 1, v$ being the velocity of the bodies, in the above three gravitational potentials. This implies to have to deal with two systems of coupled fourth-order differential equations, the first of which corresponding to the order $O(\epsilon^2)$ and the second to the order $O(\epsilon^4)$.

Regarding these two systems, only the first has been found to be solvable; its solutions have been calculated for the case of point-like sources, although it is possible to give them also for the case of sources with spherical symmetry.

Later, I move on to compare the solutions thus obtained with those deriving from a previous work by S. Capozziello and A. Stabile in 2009 [74], in which the field equations related to (1.7) have been developed and solved in the lower Post-Newtonian approximation 0.5PN of the metric tensor, in order to find out possible variations in the behavior of the modified gravitational potentials found there. Basically, it is a question of knowing whether the superior approximation of the metric tensor, used here, leads to a further refinement of the corrective terms of the Yukawa type appearing in the modified Newtonian potentials.

Finally, all the curvature scalars developed in the above approximation of the metric tensor, are tested by using the gravitational potentials of point-like sources thus obtained, having the same form as that given in (1.5), in order to study their behavior in the weak-field and low-velocity limit, specifically to verify that such quadratic curvature scalars are negligible when compared to the Hilbert-Einstein term, as consequence of the role covered by these ones in the Lagrangian (1.7), as perturbative contributions in the low curvature regime.

All is done in view of tests to carry out on the basis of the most recent observational data, either at astrophysical scale, concerning the rotation curves of spiral galaxies, the dispersion of velocity within elliptical galaxies, or at higher scales, inherent the dynamics of clusters and superclusters of galaxies, for which the gravitational potentials obtained in the previous theoretical models, both in the case of point-like sources and in the case of sources with spherical symmetry, the more realistic case, seem to give promising results, as proved in [64, 65, 66, 67].

The work plan is as follows. I have opened the current introductory chapter with a brief historical summary, followed by a treatment of the open issues concerning gravity, both on small and large scale; in the latter case, the problems concerning the excess of gravitation observed in structures at astrophysical scale and beyond, along with the current accelerated expansion of the universe, are treated in the context of the Extended Theories of Gravity, proposing themselves as alternative theories, as they exclude the existence of exotic matter as well as exotic energy, better known as Dark Matter and Dark Energy respectively, as expected in the field of particle physics, the nature of which is totally unknown nowadays. The chapter ends by outlining the topic of the thesis.

After introducing, in Chapter 2, the essential tools of the General Relativity, necessary for the subsequent investigations, in Chapter 3, I move on to a broad discussion of the metric theories of gravity, based on quadratic and higher-order Lagrangians in the curvature invariants, constructed using the Riemann and Ricci tensors; such curvature scalars play the role of perturbative corrections with respect to the Hilbert-Einstein term, given by the Ricci scalar. In this sense, they constitute an extension of GR. In the second part of the chapter, the procedure necessary to derive the metric tensor in the Post-Newtonian approximation 1PN is discussed.

The Chapter 4 concerns the expansion of all the scalars and tensors of interest in such higher-order metric theories of gravity, based on the aforementioned approximation of the metric tensor, whose perturbative development is calculated in the weak field and low velocity limit up to the order $O(\epsilon^4)$.

On the basis of the perturbative developments derived in the previous chapter, in Chapter 5 I move on to expand the field equations related to the quadratic Lagrangian (1.7), what implying to have to deal with two systems of coupled fourth-order differential equations in the $U(\mathbf{x}), V(\mathbf{x})$ and $\Phi(\mathbf{x})$ gravitational potentials, corresponding to the orders $\mathrm{O}(\epsilon^2)$ and $\mathrm{O}(\epsilon^4)$ respectively. Actually, only the former admits solutions, giving back modified Newtonian potentials, computed for the case of point-like sources; such potentials result to be given by a purely Newtonian term plus two corrective contributions of Yukawa type, the latter containing *characteristic lengths* specifying the scales of distance at which they are not negligible.

These results are part of a larger work, which consists in making explicit the coupled differential equations system relating to the order $O(\epsilon^2)$ as the conditions placed on the coupling constants appearing in (1.7) vary and in the consequent research and analysis of the relative solutions, where they exist, in order to give a picture as general as possible of the theoretical model under study.

Finally, in Chapter 6, the expansions of the curvature scalars acquired in Chapter 4, are tested by using the modified Newtonian potentials obtained in Chapter 5, in order to verify that such quadratic curvature invariants result to be only small corrections with respect to the leading term, that of the Hilbert-Einstein, as the quadratic Lagrangian (1.7) requires in the low curvature regime. The conclusions are drawn at the end of the chapter.

Chapter 2

General Relativity: foundations

2.1 From Special Relativity to General Relativity

The Special Relativity theory, as known, concerns the inertial reference frames only and it does not consider gravity. It was natural to think about an extension of it, so as to include the accelerated frames, not discriminating them from the inertial ones and the gravitation, drawing inspiration from *Mach principle* (see Section 1.2.2 on page 4), that took A. Einstein ten years of work, from 1905 to 1915, to achieve the goal.

The two issues, actually, are closely connected, since the effects of gravity cannot be described without referring to non-inertial reference frames. The starting point of GR is represented by the relationship between inertia and gravitation, in particular the equality between inertial mass and gravitational mass, of which we have the first evidence in the observations made by G. Filipono (V or VI century a.C.); subsequently, G. Benedetti proposed it again in 1553, S. Stevin gave an experimental verification in 1586, to finish with the accurate experimental analyses conducted by G. Galilei (1564 - 1642), which were definitively encoded in the Newton's law, according to which the acceleration is the same for any freely-falling body, regardless of its nature, that is,

$$a = \frac{G_N M_T}{{R_T}^2} \frac{m_g}{m_i} = \frac{G_N M_T}{{R_T}^2} = g \quad \Rightarrow \quad m_g = m_i \,. \tag{2.1}$$

This statement, known as the *Weak Equivalence Principle* (WEP), the core of the *Equivalence Principle* (EP) in its various forms, constitutes the foundation of GR. It should be observed, however, that, this equivalence, tested since the late of nineteenth century by R. E*ö*tv*ö*s (1890) and in more recent years by R.H. Dicke (1964) [76] and V.B. Braginsky (1972) [77], up to the current high precision tests, as discussed in [59, 78], in the Newtonian physics is only a fact, without any theoretical justification, while GR provides an explanation of it through the concept of space-time curvature, as already schematically represented in the diagram (1.2) on page 4.

Another important aspect related to the enlargement of SR to form GR, concerns the new relationship between geometry and physics, in the sense that the space-time is no longer just the stage where phenomena take place, the *spatium absolutum* according to the Newton's idea, but there exist a mutual interaction between space-time and matter-energy. In other words, the space-time acquires a dynamics, given by the *metric tensor* $g_{\mu\nu}(x)$, as a consequence of the distribution of matter-energy and its motion. In this way, gravity is expressed geometrically through the curvature of space-time.

In conclusion, the equivalence between inertial mass and gravitational mass, inertial observers and observers subject to gravitational fields, the metric understood as a field and gravity as curvature of the space-time [75], are the essential ingredients for building a self-consistent metric theory of gravity, which I will outline in the next section.

2.2 The basic elements for a metric theory of gravity

A coherent theory of gravity, GR *in primis*, satisfying all the above requirements, must be based on the subsequent assumptions:

- the *Relativity Principle*, according to which all the reference frames (by frame I mean a generic coordinate system, definable anywhere in the four-dimensional curved space-time), accelerated or not, are good frames for physics;
- the *Equivalence Principle* establishes that the inertial effects are locally indistinguishable from the gravitational ones, that corresponds to the equivalence between inertial and gravitational mass. In other words, any gravitational field can be locally cancelled out by a suitable inertial force;
- the *General Covariance Principle* requires that the field equations must be covariant in form, that means that they must be invariant under the action of any space-time diffeomorphism;
- the *Causality Principle* states that each point of space-time admits a universal notion of past, present and future.

Therefore, the background is constituted by a structure in which space and time are entangled, to form a four dimensional Riemannian manifold $\mathcal{M}$, differentiable, whose metric, *i.e.* the relation that expresses the *separation* between two infinitely close events, is given by,

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu, \tag{2.2}$$

for short the space-time $\{\mathcal{M}, g\}$, where the metric tensor $g_{\mu\nu}(x)$, the gravitational field, is the fundamental object, describing both the geometric properties of the background and its dynamics, with *pseudo-Euclidean* or *Lorentzian* signature $(+,-,-,-)$[1]. In this sense we speak of a *purely metric theory*. It follows immediately, from (2.2), that it is symmetric, so to have ten independent components only. Moreover, the space-time is curved due to the presence of matter-energy distribution, but in the absence of gravitational sources it has to restore the Minkowsky flat space-time,whose metric tensor, in Cartesian coordinates, is given by $\eta_{\mu\nu} = diag\,(+1,-1,-1,-1)$.

However, the metric tensor $g_{\mu\nu}(x)$, which fixes the causal structure of the space-time, *i.e.* the light-cones (and with it all the measuring operations carried out through rods and clocks), is not enough to define completely the background; it is necessary, also, a linear connection Γ, which gives the geodesic structure, suitable to describe the trajectories of the local inertial observers (the free-fall of test particles) in the four dimensional curved space-time, that is,

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}u^\mu u^\nu = 0\,. \tag{2.3}$$

In this context, the linear (or affine) connection, better known as the Levi-Civita connection, is not an independent field, but it is given in terms of the metric tensor and its derivatives, according to the following relation,

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\,g^{\alpha\lambda}(g_{\lambda\mu,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda})\,, \tag{2.4}$$

where the comma denotes the ordinary partial derivative. It should be noted that it is not a tensor and its dynamic is determined by the metric dynamics. It is essential to define the covariant derivative of tensors in a curved manifold, on the basis of the concept of *parallel transport*, through which the covariant derivative of a contravariant four-vector is given by,

[1] In literature it is also used, alternatively, the opposite signature $(-,+,+,+)$.

$$A^{\alpha}_{;\mu} = A^{\alpha}_{,\mu} + \Gamma^{\alpha}_{\mu\nu} A^{\nu} \,, \tag{2.5}$$

while in case of a covariant four-vector,

$$A_{\alpha;\mu} = A_{\alpha,\mu} - \Gamma^{\nu}_{\alpha\mu} A_{\nu} \,. \tag{2.6}$$

Furthermore, Γ is symmetric with respect to its covariant indices (so to have 40 independent components only), that makes Γ torsionless, namely,

$$T^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu} = 0 \,; \tag{2.7}$$

in other words, this means that the spin of the particles is not considered as a possible source of gravitation, as it happens in the models of Teleparallel Gravity.

All these elements, as already noticed, are based on the WEP, according to which it is impossible to distinguish, locally, between the effects of a gravitational field from those experienced in a uniformly accelerated frame, by the observation of a free-falling object.

As a consequence of the equivalence between mass and energy, from the SR, Einstein extends this principle, stating that SR is locally valid, *i.e.* it is impossible to distinguish a uniform acceleration from an external gravitational field, not only for free-falling objects, but also for any other experiment. This is the basis of the *Einstein Equivalence Principle* (EEP), which states that:

- the WEP is valid;
- the outcome of any local[2] non-gravitational test experiment is independent of the velocity of the free-falling apparatus (*local Lorentz invariance)*;
- the outcome of any local non-gravitational test experiment is independent of where and when it is performed in the universe (*local position invariance*).

This means that gravity is given by the curvature of the space-time, having also to satisfy the above postulates (see on the facing page); in other words,

- the space-time is endowed with a metric $g_{\mu\nu}(x)$ constituting the dynamical variable;
- the world-line of test bodies are time-like geodesics of the metric;
- in local freely-falling frames, *i.e.* the local Lorentz frames, the non-gravitational laws of physics are those of the SR.

In order to include also the gravitational interaction, the above postulates can be reformulated as follows:

- the WEP is valid for self-gravitating bodies as well as for test particles (gravitational WEP);
- the outcome of any local test is independent of the velocity of the free-falling apparatus;
- the outcome of any local test is independent of where and when it is performed in the universe.

to give the *Strong Equivalence Principle* (SEP). It should be noted that EEP is recovered from SEP in absence of gravitational fields.

However, it should be stressed, again, that any metric theory of gravity, in order to be consistent, has to recover the Newtonian physics in the low energy regime, namely in the weak field and low velocity limit.

[2]The "local" adjective alludes to a small laboratory, so as to avoid the tidal effects associated with the inhomogeneities of the gravitational field, which are inevitably present when considered over large spaces.

2.3 Toolkit needed for next investigations

The analysis that I will carry out in the next two chapters, requires the use of suitable tensors and scalars, starting from the Riemann tensor [79, 80], a four rank tensor, deriving from the procedure of parallel transport of a four-vector along a closed path, in a differentiable manifold, in our case the four dimensional space-time, essential for describing the curvature of the latter, namely,

$$R^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\sigma}_{\beta\nu}\Gamma^{\alpha}_{\sigma\mu} - \Gamma^{\sigma}_{\beta\mu}\Gamma^{\alpha}_{\sigma\nu}\,, \tag{2.8}$$

which is expressed in terms of the Christoffel's symbols (see 2.4 on page 20) and their derivatives. It obeys the symmetry properties, given by the following relations,

$$\begin{gathered} R^{\alpha}{}_{\beta\mu\nu} = -R^{\alpha}{}_{\beta\nu\mu}\,, \\ R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}, \quad R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}, \quad R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}\,, \\ R^{\alpha}{}_{\beta\mu\nu} + R^{\alpha}{}_{\nu\beta\mu} + R^{\alpha}{}_{\mu\nu\beta} = 0 \quad \textit{cyclic property.} \end{gathered} \tag{2.9}$$

These properties imply that $R^{\alpha}{}_{\beta\mu\nu}$ has only 20 linearly independent components. Moreover, it satisfies also the Bianchi identity,

$$R^{\alpha}_{\beta\mu\nu;\lambda} + R^{\alpha}_{\beta\lambda\mu;\nu} + R^{\alpha}_{\beta\nu\lambda;\mu} = 0\,, \tag{2.10}$$

in which the semicolon stands for the covariant derivative. From (2.8), by contracting the first index with the third, the Ricci tensor is obtained of rank two,

$$R_{\beta\nu} = R^{\alpha}_{\beta\alpha\nu} = \Gamma^{\alpha}_{\beta\nu,\alpha} - \Gamma^{\alpha}_{\beta\alpha,\nu} + \Gamma^{\sigma}_{\beta\nu}\Gamma^{\alpha}_{\sigma\alpha} - \Gamma^{\sigma}_{\beta\alpha}\Gamma^{\alpha}_{\sigma\nu}\,, \tag{2.11}$$

or more explicitly,

$$R_{\beta\nu} = \frac{\partial\Gamma^{\alpha}_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial^2}{\partial x^{\beta}\partial x^{\nu}}\ln\sqrt{-g} + \Gamma^{\sigma}_{\beta\nu}\frac{\partial}{\partial x^{\sigma}}\ln\sqrt{-g} - \Gamma^{\sigma}_{\beta\alpha}\Gamma^{\alpha}_{\sigma\nu}\,, \tag{2.12}$$

($g = det\, g_{\mu\nu}$) which is symmetric in its covariant indices; the contraction of the Ricci tensor with the metric tensor gives the corresponding scalar $R = g^{\mu\nu}R_{\mu\nu}$.

Furthermore, I want to recall, briefly, the Hilbert-Einstein action, which induces the Einstein's field equations, namely,

$$S = \int \mathcal{L}\sqrt{-g}\, d^4x = \int \big(\mathcal{L}_{HE} + 2\chi\,\mathcal{L}_m\big)\sqrt{-g}\, d^4x\,, \tag{2.13}$$

where $\chi = 8\pi G_N/c^4$ and $\sqrt{-g}\, d^4x$ denotes the invariant volume element [81]. Here, the geometric contribution is given by the Hilbert-Einstein term $\mathcal{L}_{HE} = R$, the simplest assumption that one could make, in order to write tensor equations, capable of satisfying the general covariance principle and returning non-linear second-order differential equations, through which to recover the Newtonian physics in the low velocity and weak field limit, while $\mathcal{L}_m$ is the matter Lagrangian.

The application of the minimal action principle, $\delta S = 0$, through variation of the metric tensor, provides the well-known Einstein's field equations,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = \chi\, T^{(m)}_{\mu\nu}\,, \tag{2.14}$$

where, having assumed the matter as a perfect fluid, the tensor on the right-hand side of such equation takes the form,

$$T^{(m)}_{\mu\nu} = \big(\rho c^2 + p\big)u_{\mu}u_{\nu} - p\, g_{\mu\nu}\,; \tag{2.15}$$

it is the energy-momentum tensor, symmetric, responsible for the curvature of space-time, in which ρc^2 and p indicate the rest-energy density and the pressure of the fluid respectively, while $u^\mu = dx^\mu(s)/ds$ is the four-velocity of the particles. This tensor is derived from the matter Lagrangian through the following relation,

$$T^{(m)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta g^{\mu\nu}}\,. \tag{2.16}$$

Moreover, both members of eq. (2.14), must satisfy the corresponding conservation laws, given by suitable continuity equations, the first of which,

$$G_{\mu\nu}{}^{;\mu} = 0\,, \tag{2.17}$$

is easily proved through Bianchi identity (2.10), while the second,

$$T^{(m);\mu}_{\mu\nu} = 0\,, \tag{2.18}$$

is obtained directly by simple computation.

These are the basic tools needed to build not only the GR theory, whose correctness at local scales[3] has been widely proven, but also other interesting extensions of it, the ETGs, capable of obviating the shortcomings from which the GR is affected, concerning both the rotation curves of the galaxies, as well as the dynamics of clusters and super-clusters of galaxies and the current accelerated expansion of the universe, by introducing higher-order curvature invariants into the Hilbert-Einstein Lagrangian, properly constructed by using the above objects, issues that I will discuss in depth in the next chapter.

[3]Here the adjective local indicates the rather broad range $\sim 10^{-2}cm - 10^{13}cm$, involving laboratory as well as solar system experiments.

Chapter 3

Metric theories of extended gravity through quadratic Lagrangians

3.1 Towards extended Lagrangians

The Einstein's field equations, as discussed in Section 2.3 on page 22, have their theoretical foundation in the Hilbert-Einstein Lagrangian density,

$$\mathcal{L}_{HE} = R\,, \tag{3.1}$$

linear in the Ricci's curvature scalar, which is an invariant, indeed; this means that the tensor equations, obtained from the variational principle, correspond to non-linear second-order differential equations in the metric tensor $g_{\mu\nu}(x)$, which, in the low velocities and weak fields regime, are able to restore the Poisson or Laplace equation, depending on whether there is matter or not. At Einstein's time, this choice was the simplest that could be made, both from a mathematical and a physical point of view, although it was completely arbitrary.

Despite the immediate successes achieved by the GR, of which those that today go by the name of *classic tests* have been a clear evidence, it was immediately thought of its possible generalizations, the first of which dictated by the intent to unify gravity with the electromagnetism, thanks to a revolutionary idea proposed by the Polish mathematician T. Kaluza, in 1919, then further developed by the Swedish mathematician O. Klein, in 1926, based on the fact that there could be a fourth warped spatial dimension, together with the three infinitely extended ordinary dimensions. This idea was later taken up by Einstein himself along with others, including Eddington, Weyl, Schrodinger to name the main ones. But the times were not yet ripe, as other fundamental interactions, the weak and strong nuclear ones, had yet be discovered and subsequently incorporated into a single gauge framework, along with the electromagnetic one, which was only possible from the 1950s.

During the first half of the eighties, these arguments flowed into the nascent String Theories, aimed at unifying the four fundamental interactions, also providing a coherent version of quantum gravity, as they predict the existence of the graviton, the quantum of the gravitational interaction [82], although to date they are still all to be verified. In particular, it turns out that gravity is configured as a low-energy limit of such string theories.

On the other hand, the large amount of observational data acquired over the past ninety years, clearly shows the presence of anomalies in the dynamics of structures on astrophysical and cosmological scales, of which the GR is able to explain the nature only in the context of the *concordance model* ΛCDM, in which the existence of DM and DE is admitted. Those properties and effects are well known, but of their nature nothing is known yet. In other words, this model does not foresee changes to the geometric structure of gravity as formulated in the GR.

In reality, the fact of being forced to hypothesize the existence of such exotic ingredients could be a clear indication of the inadequacy of this geometric construction at least at astrophysical

scales and beyond, that leads us to think that gravity behaves differently at different distance scales.

All these reasons have favored the idea that the Hilbert-Einstein Lagrangian could be "corrected", adding further higher-order curvature invariants or non-linear function of them, thus giving rise to the development of the so-called *Extended Theories of Gravity* (ETG). In this way, only the "geometric structure" of gravity changes, at the origin of which would be all the observable content in the universe, namely baryonic matter and radiation substantially. All this translates into the fact that only the left hand-side of the Einstein's field equations is modified (see (2.14) on page 22), leaving the energy-momentum tensor unchanged, not wanting to endorse the hypothesis of the existence of exotic matter as well as of exotic energy, as postulated in DM and DE ambit.

To complete this discussion, a clarification should be made in order to give a theoretical motivation regarding the choice of the Lagrangians, which will be discussed in the next section. Assuming that it is possible to describe the metric field in quantum terms, one may expect that the *vacuum polarization* effects can induce non-linear terms in the Lagrangian, which, in the lowest order of a perturbation theory, could contain quadratic invariants, such as R^2, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ or some linear combination of them.

3.2 Quadratic Lagrangians and higher-order

To date, the field equations obtained through quadratic Lagrangians, of which I will make a brief survey below, as well as of higher order, have been studied in the weak fields and low velocities limit, only in the Post-Newtonian approximation 0.5PN of the metric tensor $g_{\mu\nu}(x)$, which is therefore expressed through a perturbative expansion extended up to the order $O(\epsilon^2)$. The two resulting potentials, $U(\mathbf{x})$ and $V(\mathbf{x})$, which define the metric tensor at such order, are not purely Newtonian, as they appear to be corrected by Yukawa-like perturbative terms. These results are promising; in fact, as various tests carried out on the most recent experimental observations show, they seem to describe well the dynamics of objects on astrophysical, galactic and extra-galactic scales, as widely discussed in [64, 65, 66, 67]. Therefore, it seems natural to investigate further, always remaining in the context of the theories based on quadratic Lagrangians, but developed in the higher perturbation scheme 1PN, containing the expansion of the metric tensor up to the order $O(\epsilon^4)$ in the three gravitational potentials $U(\mathbf{x}), V(\mathbf{x})$ and $\Phi(\mathbf{x})$.

This analysis, however, must necessarily include a study (see Chapter 6) concerning the development of the Ricci's tensor and scalar, as well as of the other curvature invariants, necessary for the construction of such quadratic Lagrangians, in the higher perturbation scheme mentioned above, to test their behavior in order to ascertain that the quadratic contributions do not contain leading terms, that is of the Newtonian type, but only corrections of Yukawa type and "higher" so to speak, as such theories require in the low energy regime.

At this point, I move on to outline the main aspects of the most important quadratic Lagrangians, of which one is central for the subsequent analysis constituting the topic of the next chapter, before going into the perturbation procedure that returns the metric tensor expanded up to the order $O(\epsilon^4)$.

3.2.1 Purely quadratic Lagrangian

Among the first suggestions for modifying the gravitational Lagrangian proposed by D. Hilbert and A. Einstein, linear in the Ricci scalar, some of which offered immediately after the publication of the GR, between the sixties and seventies, there was that of considering only the quadratic curvature scalar,

$$\mathcal{L} = R^2, \tag{3.2}$$

the fourth-order gravity action of which,

$$S = \int d^4x\sqrt{-g}\left(R^2 + 2\chi\mathcal{L}_m\right), \tag{3.3}$$

returns, by varying with respect to the metric $g_{\mu\nu}(x)$, the corresponding field equations, whose left-hand side is given by,

$$H_{\mu\nu} = 2RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^2 - 2R_{;\mu\nu} + 2g_{\mu\nu}\Box R\,, \tag{3.4}$$

and so[1],

$$H_{\mu\nu} = \frac{8\pi G_N}{c^4}\,T^{(m)}_{\mu\nu}, \tag{3.5}$$

when one considers also the contribution coming from the matter-energy Lagrangian; they correspond to fourth-order differential equations in the potentials $U(\mathbf{x})$ and $V(\mathbf{x})$, appearing in the 0.5PN perturbative expansion of the metric tensor, through which this theory was studied. But, it was immediately clear that, such a theory, did not allow to restore the classical results in the weak field regime, as discussed in depth in [83, 84, 85, 86, 87], as well as it did not guarantee the existence of asymptotically flat solutions [83].

3.2.2 The $f(R)$ Lagrangians

The next step was to express the gravitational Lagrangian through an unspecified analytic function of the curvature scalar,

$$\mathcal{L} = f(R)\,, \tag{3.6}$$

from which,

$$H_{\mu\nu} = f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - f'(R)_{;\mu\nu} + g_{\mu\nu}\Box f'(R)\,, \tag{3.7}$$

where $f'(R) = df(R)/dR$; of the corresponding field equations (which result to be of fourth-order in the derivatives of the metric) have been given solutions with spherical symmetry in the Newtonian limit in [88, 89, 90, 91].

In this regard, it is interesting to take a look to such equation, which can be rewritten as follows,

$$G_{\mu\nu} = T^{(curv)}_{\mu\nu} = \frac{1}{f'(R)}\left\{\frac{1}{2}\,g_{\mu\nu}\Big[f(R) - f'(R)R\Big] + f'(R)_{;\mu\nu} - g_{\mu\nu}\Box f'(R)\right\},$$

where the left-hand side of it coincides with the Einstein's tensor, while the right-hand side corresponds to a purely geometric tensor due to the curvature of the space-time, in absence of the matter-energy contribution.

On the other hand, there are other interesting alternatives, regarding Lagrangians expressed in terms of the curvature scalar only, which will be discussed below.

Perturbative expansion

Remaining in this context, the simplest form that the analytic function $f(R)$ can take, is that given by a Taylor's expansion, in the limit in which R runs to zero (low curvature regime), namely,

$$\mathcal{L} = f(R) = \sum_{k=0}^{\infty} a_k R^k = a_0 + a_1 R + a_2 R^2 + \ldots \tag{3.8}$$

[1] $\Box := \frac{1}{\sqrt{-g}}\partial_\alpha\big(\sqrt{-g}g^{\alpha\beta}\partial_\beta\big)$

where a_k are constants having physical dimension $[L^{2k-2}]$. However, one may expect that the coefficients a_k can be expressed in terms of some fundamental length l, as

$$a_k = c_k \, l^{2k-2} \quad k \geq 0 \,,$$

where the constants c_k are dimensionless and of the order of unity. The first term is generally expressed through the cosmological constant, $a_0 = -2\Lambda$, to be neglected here, and one can fix $a_1 = 1$ so as to recover Einstein's theory at first order.

To justify this choice, it can be assumed that the non-linear terms are induced by quantum effects, either in the sources or in the gravitational field or both. It is clear that these terms must be small in ordinary situations, i.e. for low curvature of space-time, which occurs in the universe when it is considered on a large scale.

Expansion truncated to the second order

The perturbative expansion (3.8), stopped at the second order in R,

$$\mathcal{L} = a_0 + a_1 R + a_2 R^2 \,, \tag{3.9}$$

representing the simplest extension of the GR, already studied and tested in the Post-Newtonian approximation 0.5PN, could offer a first test for the metric tensor expressed in the higher approximation 1PN, to date not yet carried out, in order to discover if and how the Yukawa-type contributions change within the gravitational potentials $U(\mathbf{x})$ and $V(\mathbf{x})$ previously obtained; it would then be a question of verifying whether the potentials thus obtained are able to provide a better description of the dynamics observed in structures on an astrophysical scale and beyond, where an excess of gravitation is detected, otherwise inexplicable if one excludes the existence of DM.

It should be noted, however, that the field equations coming from it, as well as from other Lagrangians of the same order, more general, implies solving coupled fourth-order differential equations in the three gravitational potentials $U(\mathbf{x}), V(\mathbf{x})$ and $\Phi(\mathbf{x})$, when one considers the perturbative expansion of the metric tensor up to the order $O(\epsilon^4)$ corresponding to the Post-Newtonian approximation 1PN.

3.2.3 A more general quadratic Lagrangian

The theoretical model considered here, to which the analysis carried out in this thesis refers, constitutes a further extension of the work performed by S. Capozziello and A. Stabile in 2009 [74], of which I am going to outline the fundamental aspects and the corresponding results.

The more general quadratic Lagrangian, involving further quadratic curvature invariants, besides the Ricci scalar and its square, reads,

$$\mathcal{L} = a_0 + a_1 R + a_2 R^2 + b_1 R_{\mu\nu} R^{\mu\nu} + c_1 R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}, \tag{3.10}$$

which can be reduced to the following,

$$\mathcal{L} = a_0 + a_1 R + a_2 R^2 + b_1 R_{\mu\nu} R^{\mu\nu}, \tag{3.11}$$

by virtue of the Gauss-Bonnet identity[2], valid only in a four-dimensional manifold, as proved in [71], concerning the possibility, in the variational context, of expressing the field equations in vacuum corresponding to the fourth invariant of (3.10), through a linear combination of those associated with the second and third invariant (see Appendix B on page 95), namely,

[2]It is clear that, as a consequence of the Gauss-Bonnet identity, the coefficients of the terms of curvature change; i preferred, however, to keep the same symbols for simplicity.

$$ {}^1H_{\mu\nu} - 4\,{}^2H_{\mu\nu} + {}^3H_{\mu\nu} = 0\,, \tag{3.12} $$

where, the left-hand sides of the corresponding field equations, are given by (I disregarded the cosmological term a_0):

- $\mathcal{L}_0 = R$

$$ {}^0H_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\,; $$

- $\mathcal{L}_1 = R^2$

$$ {}^1H_{\mu\nu} = 2RR_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^2 - 2R_{;\mu\nu} + 2g_{\mu\nu}\Box R\,; $$

- $\mathcal{L}_2 = R_{\mu\nu}R^{\mu\nu}$

$$ {}^2H_{\mu\nu} = 2R_{\mu}{}^{\alpha} R_{\nu\alpha} - \frac{1}{2} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} - 2R_{(\mu|\,\alpha;|\nu)\alpha} + \Box R_{\mu\nu} + g_{\mu\nu} R^{\alpha\beta}{}_{;\alpha\beta}\,; $$

- $\mathcal{L}_3 = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$

$$ {}^3H_{\mu\nu} = 2R_{\mu\alpha\beta\gamma} R_{\nu}{}^{\alpha\beta\gamma} - \frac{1}{2} g_{\mu\nu} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} + 4R_{\mu\ \ \nu\ \ ;(\alpha\beta)}^{\ \alpha\ \ \beta}\,. $$

The first three terms are those that contribute to the overall field equation, given by the following relation,

$$ H_{\mu\nu} = \sum_{k=0}^{2} c_k\, {}^k H_{\mu\nu} = \frac{8\pi G_N}{c^4}\, T^{(m)}_{\mu\nu}\,, \tag{3.13} $$

where, $c_0 = a_1, c_1 = a_2, c_2 = b_1$.

In the Post-Newtonian approximation 0.5PN of the metric tensor[3], expanded up to the order $O(\epsilon^2)$ in the $U(\mathbf{x})$ and $V(\mathbf{x})$ potentials,

$$ \begin{aligned} g_{00} &= 1 - \frac{2U}{c^2} + O(\epsilon^4)\,, \\ g_{0i} &= \frac{h_{0i}}{c^3} + O(\epsilon^5)\,, \\ g_{ij} &= -\left(1 + \frac{2V}{c^2}\right)\delta_{ij} + O(\epsilon^4)\,, \end{aligned} \tag{3.14} $$

with the corresponding expansion of the energy-momentum tensor (see (2.15) on page 22),

$$ \begin{aligned} T^{(m)}_{00} &= \rho c^2 + O(\epsilon^2)\,, \\ T^{(m)}_{0i} &= -\rho c v^i + O(\epsilon^3)\,, \\ T^{(m)}_{ij} &= \rho v^i v^j + p\delta_{ij} + O(\epsilon^2)\,, \end{aligned} \tag{3.15} $$

the left-hand sides of the field equations take the following form[4]:

[3] The greek indices μ, ν, $\ldots = 0, 1, 2, 3$; the latin indices i, j, k, $\ldots = 1, 2, 3$.

[4] $\nabla^2 := \delta^{ij}\frac{\partial^2}{\partial x^i \partial x^j}$; $\nabla^4 := \nabla^2\nabla^2$.

- $\mathcal{L}_0 = R$

$$\begin{aligned} {}^0H_{00} &= -\frac{2}{c^2}\nabla^2 V\,, \\ {}^0H_{0i} &= 0\,, \\ {}^0H_{ij} &= \frac{1}{c^2}\Big[\big(\nabla^2 V - \nabla^2 U\big)\delta_{ij} - (V-U)_{,ij}\Big]\,; \end{aligned} \tag{3.16a}$$

- $\mathcal{L}_1 = R^2$

$$\begin{aligned} {}^1H_{00} &= \frac{4}{c^2}\big(2\nabla^4 V - \nabla^4 U\big)\,, \\ {}^1H_{0i} &= 0\,, \\ {}^1H_{ij} &= \frac{4}{c^2}\Big[\big(\nabla^4 U - 2\nabla^4 V\big)\delta_{ij} + \big(2\nabla^2 V - \nabla^2 U\big)_{,ij}\Big]\,; \end{aligned} \tag{3.16b}$$

- $\mathcal{L}_2 = R_{\mu\nu}R^{\mu\nu}$

$$\begin{aligned} {}^2H_{00} &= \frac{2}{c^2}\big(\nabla^4 V - \nabla^4 U\big)\,, \\ {}^2H_{0i} &= 0\,, \\ {}^2H_{ij} &= \frac{1}{c^2}\Big[\big(\nabla^4 U - 3\nabla^4 V\big)\delta_{ij} + \big(3\nabla^2 V - \nabla^2 U\big)_{,ij}\Big]\,. \end{aligned} \tag{3.16c}$$

The resulting field equations, written in compact form,

$$\sum_{k=0}^{2} c_k\, {}^kH_{00} = \frac{8\pi G_N}{c^4}\, T^{(m)}_{00}\,, \tag{3.17a}$$

$$\sum_{k=0}^{2} c_k\, {}^kH_{0i} = \frac{8\pi G_N}{c^4}\, T^{(m)}_{0i}\,, \tag{3.17b}$$

$$\sum_{k=0}^{2} c_k\, {}^kH_{ij} = \frac{8\pi G_N}{c^4}\, T^{(m)}_{ij}\,, \tag{3.17c}$$

actually, correspond to two coupled fourth-order differential equations in the $U(\mathbf{x}), V(\mathbf{x})$ potentials; in fact, expanding the (3.17b) up to the second-order in ϵ, that is,

$$0 = \frac{8\pi G_N}{c^4}\big(-\rho c v^i\big)\,,$$

it results that $v^i = 0$, where $i = 1, 2, 3$. Furthermore, it can be assumed that the matter currently present in the universe, is substantially constituted by non-relativistic particles, without collision, whose pressure is negligible with respect to their energy density, namely something comparable to *dust*, at any scale of distance; in this case, it makes sense to put $p \sim 0$. In view of these conditions, the equations (3.17a) and (3.17c) take the following explicit form,

$$-2a_1\Delta V + \big(8a_2 + 2b_1\big)\Delta^2 V - \big(4a_2 + 2b_1\big)\Delta^2 U = 8\pi G_N \rho\,, \tag{3.18}$$

$$\begin{aligned} \Big[a_1\big(\Delta V - \Delta U\big) - \big(8a_2 + 3b_1\big)\Delta^2 V + \big(4a_2 + b_1\big)\Delta^2 U\Big]\delta_{ij} + \Big[\big(8a_2 + 3b_1\big)\Delta V - \big(4a_2 + b_1\big)\Delta U \\ + a_1\big(U - V\big)\Big]_{,ij} = 0\,. \end{aligned} \tag{3.19}$$

In the hypothesis to consider point-like gravitational sources, the density of which is given by $\rho(\mathbf{x}) = M\,\delta^{(3)}(\mathbf{x})$[5], the above equations admit solutions of the type,

$$U(\mathbf{x}) = \frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\left(1 - \frac{4}{3}e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}e^{-\lambda_2|\mathbf{x}|}\right), \tag{3.20}$$

$$V(\mathbf{x}) = \frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\left(1 - \frac{2}{3}e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}e^{-\lambda_2|\mathbf{x}|}\right), \tag{3.21}$$

where,

$$\lambda_1 = \sqrt{-\frac{a_1}{b_1}}\,,$$
$$\lambda_2 = \sqrt{\frac{a_1}{2(3a_2 + b_1)}}\,,$$

and they have physical meaning under the following conditions,

$$a_1 > 0,$$
$$b_1 < 0\,,$$
$$3a_2 + b_1 > 0\,.$$

They are *characteristic lengths* having dimension $[L^{-1}]$, apt to define the distance scales at which the Yukawa-like corrections to the Newtonian potential become appreciable. These values, not theoretically evaluable, can be determined through a fitting of the observational data relating to galactic scales and beyond, for which it is thought that such potentials can provide a key to interpret the excess of gravitation detected at these distance scales, where the GR fails, unless large amounts of dark matter are admitted, as required in the concordance model ΛCDM.

The modified Newtonian potentials thus obtained, show a typical aspect of these theories, which consists in the fact that they contain as many Yukawa contributions as there are terms of higher-order than the Ricci scalar present in the corresponding Lagrangian; moreover, each term contributes to increase or decrease the intensity of the gravitational interaction, depending on the sign associated with the corresponding coefficient.

3.2.4 Higher-order Lagrangians

This discussion would not be complete, if I did not consider an even more general Lagrangian, involving terms containing higher-order derivatives in the curvature scalar, useful to show a further aspect concerning the modified Newtonian potentials, with regard to the study of the corresponding field equations of higher order than the fourth in the metric, when analyzed in the Newtonian limit, that is to say for weak gravitational fields and low velocities, when a static metric is considered, $g_{\mu\nu,0} = 0$, as argued extensively in [92].

It is the following Lagrangian,

$$\mathcal{L} = R + \sum_{k=0}^{p} a_k\, R\Box^k R \qquad a_p \neq 0\,, \tag{3.22}$$

whose coefficients a_k have dimension $[L^{2k+2}]$.

From a quantum point of view, this choice implies adding higher-loop corrections in the Hilbert-Einstein Lagrangian, in comparison with the one-loop corrections related to the quadratic terms of the Lagrangians mentioned above.

[5] $\delta^{(3)}(\mathbf{x})$ is the 3-dimensional Dirac's delta.

The exact field equations, deriving from the application of the minimal action principle [93, 94], once linearized [95], reads,

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + 2\sum_{k=0}^{p} a_k \Big[g^{\mu\nu}\Box^{k+1}R - \Box^k R^{;\mu\nu} \Big] = \frac{8\pi G_N}{c^4} T^{\mu\nu}, \tag{3.23}$$

corresponding to $(2k+4)$th-order differential equations and whose trace is given by the subsequent relation,

$$-R + 6\sum_{k=0}^{p} a_k \Box^{k+1} R = \frac{8\pi G_N}{c^4} T\,. \tag{3.24}$$

What is important to emphasize here, is that, in the 0.5PN approximation of the metric tensor (see (3.14) on page 29), for point-like sources with density $\rho(\boldsymbol{x}) = M\delta^{(3)}(\boldsymbol{x})$, in the hypothesis that the solutions of the field equations (3.23) and (3.24) are Newton + Yukawa combinations, namely,

$$U(r) = \frac{G_N M}{r}\left(1 + \sum_{i=0}^{q} c_i\, e^{-\frac{r}{l_i}}\right) \qquad l_i > 0\,, \tag{3.25}$$

and placing, without loss of generality, $l_i \neq l_j$ for $i \neq j$, ($V(r)$ has a similar expression), there exists always a unique solution $(a_0, \ldots, a_p)$ for the a_k coefficients of the Lagrangian, given as a function of prescribed characteristic lengths l_i, when $q = p$. Moreover, the c_i coefficients, appearing in (3.25), are expressed in terms of the a_k.

Just to give some examples:

- second-order gravity

$$\begin{aligned} a_k &= 0 \quad \forall\, k = 0, \ldots, p\,, \\ \mathcal{L} &= R\,, \end{aligned}$$

  in which Newton's theory is evidently recovered,

$$\begin{aligned} \Phi(r) &= \frac{G_N M}{r}\,, \\ \Delta\Phi &= -4\pi G_N \rho\,; \end{aligned}$$

- fourth-order gravity [87, 96]

$$\begin{aligned} p &= 0 \quad a_0 < 0\,, \\ \mathcal{L} &= R + a_0 R^2\,, \end{aligned}$$

$$\begin{aligned} U(r) &= \frac{G_N M}{r}\left(1 + \frac{1}{3}e^{-r/\sqrt{-6a_0}}\right), \\ V(r) &= \frac{G_N M}{r}\left(1 - \frac{1}{3}e^{-r/\sqrt{-6a_0}}\right). \end{aligned}$$

  For $a_0 > 0$ the Newtonian limit of the theory does not exist;

- sixth-order gravity

$$\begin{aligned} p &= 1\,, \\ \mathcal{L} &= R + a_0 R^2 + a_1 R \Box R\,, \end{aligned}$$

$$U(r) = \frac{G_N M}{r}\left(1 + c_0\, e^{-r/l_0} + c_1\, e^{-r/l_1}\right),$$
$$V(r) = \frac{G_N M}{r}\left(1 - c_0\, e^{-r/l_0} - c_1\, e^{-r/l_1}\right),$$

where

$$c_{0,1} = \frac{1}{6} \mp \frac{a_0}{2\sqrt{9a_0^2 + 6a_1}},$$
$$l_{0,1} = \sqrt{-3a_0 \pm \sqrt{9a_0^2 + 6a_1}};$$

- $(2p+4)$-order gravity

$$p > 1\,,$$
$$\mathcal{L} = R + \sum_{k=0}^{p} a_k\, R\, \Box^k R\,,$$

or, more explicitly,

$$\mathcal{L} = R - \frac{R}{6}\Big[\big(l_0^2 + \cdots + l_p^2\big)R + \big(l_0^2 l_1^2 + l_0^2 l_2^2 + \cdots + l_{p-1}^2 l_p^2\big)\Box\, R + \big(l_0^2 l_1^2 l_2^2 + \cdots + l_{p-2}^2 l_{p-1}^2 l_p^2\big)\Box^2\, R + \\ + l_0^2 l_1^2 \ldots l_p^2\, \Box^p\, R\Big]\,,$$

with $0 < l_0 < l_1 < \cdots < l_p$,the potentials take the following form,

$$U(r) = \frac{G_N M}{r}\left[1 + \frac{1}{3}\sum_{i=0}^{p}(-1)^{i+1}\prod_{j\neq i}\left|\frac{l_j^2}{l_i^2} - 1\right|^{-1} e^{-\frac{r}{l_i}}\right],$$
$$V(r) = \frac{G_N M}{r}\left[1 - \frac{1}{3}\sum_{i=0}^{p}(-1)^{i+1}\prod_{j\neq i}\left|\frac{l_j^2}{l_i^2} - 1\right|^{-1} e^{-\frac{r}{l_i}}\right].$$

It is therefore evident that the higher order Lagrangians in the curvature scalar R, within the limits set by the above conditions, return gravitational potentials containing, in addition to the Newtonian term, as many Yukawa contributions as there are terms of higher order than the first. This becomes almost a theorem.

3.3 Metric tensor in the Post-Newtonian approximation 1PN

Before moving on to outline the salient steps, necessary to obtain the perturbative development of the metric tensor $g_{\mu\nu}(x)$ up to the order $O(\epsilon^4)$, technique widely documented in literature, see for instance [97, 98], it is useful to make a couple of general observations, which are helpful in assessing the weight of the various contributions in the perturbative expansions of the metric tensor components.

It is important, first of all, to make an assessment of the gravitational potential in terms of the aforementioned parameter; in the simplest case of a test-particle with mass m, which is moving in a circular orbit of radius r, with velocity v, about a point-like source having mass M, from the Newton's law it follows that,

$$F_G = m\, a_N \quad \Rightarrow \quad \frac{G_N M m}{r^2} = m\frac{v^2}{r}\,,$$

from which

$$v^2 = \frac{G_N M}{r} \quad \Rightarrow \quad \frac{|U(r)|}{c^2} = \left(\frac{v}{c}\right)^2 \sim O(\epsilon^2)\,,$$

where $U(r) = -G_N M/r$ is the gravitational potential. In the most general case of a virialized particle system, for which the well-known equation is valid,

$$2\,\overline{T} = -\overline{V} \quad \Rightarrow \quad 2\,\overline{T} = -m\,\overline{U}(r)\,,$$

the same result as above is obtained,

$$\bar{v}^2 = \frac{G_N \overline{M}}{\bar{r}} \quad \Rightarrow \quad \frac{|\overline{U}(r)|}{c^2} = \left(\frac{\bar{v}}{c}\right)^2 \sim O(\epsilon^2)\,,$$

where $\bar{M}$ is the total mass of the system and $\bar{r}, \bar{v}$ are typical average values of the separations and velocities of the particles.

Another technical detail to consider, concerns the fact that the time derivative of any function, defined in a four dimensional manifold, turns out to be of the first order in ϵ, according to the following relation,

$$\frac{\partial}{\partial t} = \mathbf{v}\cdot\nabla \quad \Rightarrow \quad \frac{\partial}{\partial x^0} = \frac{\mathbf{v}}{c}\cdot\nabla \quad \Rightarrow \quad \frac{|\frac{\partial}{\partial x^0}|}{|\nabla|} \sim O(\epsilon)\,.$$

Precisely, the Einstein's field equations (see (2.14) on page 22), as well known, are equivalent to a system of ten second-order non-linear differential equations, for which there is no analytical solution capable of describing the structure of curved space-time and defining the geodesic of a test-particle in motion in a gravitational field, for which reason the only feasible thing is to express the metric tensor $g_{\mu\nu}(x)$ through a perturbative expansion, built on the basis of reasonable hypotheses.

These criteria are expressions of the so-called Post-Newtonian limit, according to which:

- for weak gravitational fields, the curved four dimensional space-time can be described by a perturbative term with respect to the flat Minkowski's one, namely,

  $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,,$$

  where $\eta_{\mu\nu} = diag(1,-1,-1,-1)$ and $|h_{\mu\nu}| \ll 1$;

- for low velocities of the bodies,

  $$\epsilon = \frac{v}{c} \ll 1\,.$$

On the basis of these assumptions, it is possible to express the metric tensor as well as the energy-momentum tensor, by means of the following expansions,

$$\begin{aligned} g_{\mu\nu} &= {}^{(0)}g_{\mu\nu} + {}^{(1)}g_{\mu\nu} + {}^{(2)}g_{\mu\nu} + {}^{(3)}g_{\mu\nu} + {}^{(4)}g_{\mu\nu} + \dots\,, \\ T_{\mu\nu} &= {}^{(0)}T_{\mu\nu} + {}^{(1)}T_{\mu\nu} + {}^{(2)}T_{\mu\nu} + {}^{(3)}T_{\mu\nu} + {}^{(4)}T_{\mu\nu} + \dots\,, \end{aligned}$$

where ${}^{(i)}g_{\mu\nu}$, ${}^{(i)}T_{\mu\nu} \sim O(\epsilon^i)$.

- Under time-reversal transformation,

  $$t \xrightarrow{\mathcal{T}} -t$$

for which it is admitted that the system remains unchanged, the velocity v and consequently the ϵ parameter change sign; this implies that $g_{00}, g_{ij}, T_{00}, T_{ij}$ do not change sign, while g_{0i}, T_{0i} change it; therefore, only the following developments are admissible,

$$\begin{aligned} g_{00} &= {}^{(0)}g_{00} + {}^{(2)}g_{00} + {}^{(4)}g_{00} + \dots , \\ g_{0i} &= {}^{(1)}g_{0i} + {}^{(3)}g_{0i} + {}^{(5)}g_{0i} + \dots , \\ g_{ij} &= {}^{(0)}g_{ij} + {}^{(2)}g_{ij} + {}^{(4)}g_{ij} + \dots ; \end{aligned}$$

$$\begin{aligned} T_{00} &= {}^{(0)}T_{00} + {}^{(2)}T_{00} + {}^{(4)}T_{00} + \dots , \\ T_{0i} &= {}^{(1)}T_{0i} + {}^{(3)}T_{0i} + {}^{(5)}T_{0i} + \dots , \\ T_{ij} &= {}^{(0)}T_{ij} + {}^{(2)}T_{ij} + {}^{(4)}T_{ij} + \dots . \end{aligned}$$

At this point a clarification is in order. Because I am interested in expansions up to the fourth order in ϵ, the system I mentioned above, can be considered conservative. Actually, this is not true, as even in the simplest case of a test-particle in motion in a gravitational field, it emits gravitational waves, which carry momentum and energy, what entails a recoil of the test-particle, the so-called *radiation reaction*, the effects of which are no longer negligible on orders ${}^{(7)}g_{00}, {}^{(6)}g_{0i}, {}^{(5)}g_{ij}$. It is clear, therefore, that these higher-order contributions do not appear in the final expansions.

- In the low velocity limit the flat space-time has to be recovered,

$$\lim_{\epsilon \to 0} g_{\mu\nu} = \eta_{\mu\nu} ,$$

what means that, at the lowest order,

$${}^{(0)}g_{00} = \eta_{00} = 1 \quad {}^{(1)}g_{0i} = \eta_{0i} = 0 \quad {}^{(0)}g_{ij} = \eta_{ij} = -\delta_{ij} ,$$

namely,

$$\begin{aligned} g_{00} &= 1 + {}^{(2)}g_{00} + {}^{(4)}g_{00} + \dots , \\ g_{0i} &= {}^{(3)}g_{0i} + {}^{(5)}g_{0i} + \dots , \\ g_{ij} &= -\delta_{ij} + {}^{(2)}g_{ij} + {}^{(4)}g_{ij} + \dots . \end{aligned}$$

Definitely, the expansions sought for take the form,

$$\begin{aligned} g_{00} &= 1 + {}^{(2)}g_{00} + {}^{(4)}g_{00} + O(\epsilon^6) , \\ g_{0i} &= {}^{(3)}g_{0i} + O(\epsilon^5) , \\ g_{ij} &= -\delta_{ij} + {}^{(2)}g_{ij} + O(\epsilon^4) , \end{aligned} \tag{3.26}$$

$$\begin{aligned} T_{00} &= {}^{(0)}T_{00} + {}^{(2)}T_{00} + {}^{(4)}T_{00} + O(\epsilon^6) , \\ T_{0i} &= {}^{(1)}T_{0i} + {}^{(3)}T_{0i} + O(\epsilon^5) , \\ T_{ij} &= {}^{(0)}T_{ij} + {}^{(2)}T_{ij} + O(\epsilon^4) , \end{aligned} \tag{3.27}$$

of which the terms to be developed are ${}^{(2)}g_{00}, {}^{(4)}g_{00}, {}^{(3)}g_{0i}, {}^{(2)}g_{ij}$, by using the Einstein's field equations, which, for the sake of convenience, can be rewritten as follows,

$$R_{\mu\nu} = \frac{8\pi G_N}{c^4} S_{\mu\nu} = \frac{8\pi G_N}{c^4} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) , \tag{3.28}$$

where I set $T = g_{\mu\nu}T^{\mu\nu}$. In short, it is a matter of inserting (3.26) and (3.27) into this equation and matching the terms of corresponding order, after having calculated, with some algebra, the expansions of the Christoffel's symbols and the Ricci's tensor components (see (2.4) and (2.11)), whose final expressions are listed below:

- Christoffel's symbols components[6]

$$
\begin{aligned}
\Gamma^0_{00} &= {}^{(3)}\Gamma^0_{00} + {}^{(5)}\Gamma^0_{00} + O(\epsilon^7)\,, \\
{}^{(3)}\Gamma^0_{00} &= \frac{1}{2}\,{}^{(2)}g_{00,0}\,, \\
{}^{(5)}\Gamma^0_{00} &= \frac{1}{2}\Big[\,{}^{(2)}g^{00}\,{}^{(2)}g_{00,0} + {}^{(4)}g_{00,0} - {}^{(3)}g^{0i}\,{}^{(2)}g_{00,i}\Big]\,;
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^0_{0i} &= {}^{(2)}\Gamma^0_{0i} + {}^{(4)}\Gamma^0_{0i} + O(\epsilon^6)\,, \\
{}^{(2)}\Gamma^0_{0i} &= \frac{1}{2}\,{}^{(2)}g_{00,i}\,, \\
{}^{(4)}\Gamma^0_{0i} &= \frac{1}{2}\Big[\,{}^{(4)}g_{00,i} + {}^{(2)}g^{00}\,{}^{(2)}g_{00,i}\Big]\,;
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^0_{ik} &= {}^{(3)}\Gamma^0_{ik} + {}^{(5)}\Gamma^0_{ik} + O(\epsilon^7)\,, \\
{}^{(3)}\Gamma^0_{ik} &= \frac{1}{2}\Big[\,{}^{(3)}g_{0i,k} + {}^{(3)}g_{k0,i} - {}^{(2)}g_{ik,0}\Big]\,, \\
{}^{(5)}\Gamma^0_{ik} &= \frac{1}{2}\,{}^{(2)}g^{00}\Big[\,{}^{(3)}g_{0i,k} + {}^{(3)}g_{k0,i} - {}^{(2)}g_{ik,0}\Big] + \frac{1}{2}\,{}^{(3)}g^{0j}\Big[\,{}^{(2)}g_{ji,k} + {}^{(2)}g_{kj,i} - {}^{(2)}g_{ik,j}\Big]\,;
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^i_{00} &= {}^{(2)}\Gamma^i_{00} + {}^{(4)}\Gamma^i_{00} + O(\epsilon^6)\,, \\
{}^{(2)}\Gamma^i_{00} &= \frac{1}{2}\,\delta^{ik}\,{}^{(2)}g_{00,k}\,, \\
{}^{(4)}\Gamma^i_{00} &= \frac{1}{2}\,\delta^{ij}\Big[\,{}^{(4)}g_{00,j} + {}^{(2)}g_{jk}\,{}^{(2)}g_{00,k} - 2\,{}^{(3)}g_{j0,0}\Big]\,;
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^i_{0j} &= {}^{(3)}\Gamma^i_{0j} + {}^{(5)}\Gamma^i_{0j} + O(\epsilon^7)\,, \\
{}^{(3)}\Gamma^i_{0j} &= -\frac{1}{2}\,\delta^{ik}\Big[\,{}^{(3)}g_{k0,j} + {}^{(2)}g_{jk,0} - {}^{(3)}g_{0j,k}\Big]\,, \\
{}^{(5)}\Gamma^i_{0j} &= \frac{1}{2}\,{}^{(2)}g^{ik}\Big[\,{}^{(3)}g_{k0,j} + {}^{(2)}g_{jk,0} - {}^{(3)}g_{0j,k}\Big] + \frac{1}{2}\,{}^{(3)}g^{i0}\,{}^{(2)}g_{00,j}\,;
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^i_{jk} &= {}^{(2)}\Gamma^i_{jk} + {}^{(4)}\Gamma^i_{jk} + O(\epsilon^6)\,, \\
{}^{(2)}\Gamma^i_{jk} &= -\frac{1}{2}\,\delta^{il}\Big[\,{}^{(2)}g_{lj,k} + {}^{(2)}g_{kl,j} - {}^{(2)}g_{jk,l}\Big]\,, \\
{}^{(4)}\Gamma^i_{jk} &= \frac{1}{2}\,{}^{(2)}g^{im}\Big[\,{}^{(2)}g_{mj,k} + {}^{(2)}g_{km,j} - {}^{(2)}g_{jk,m}\Big]\,;
\end{aligned}
$$

- Ricci's tensor components

$$
R_{00} = {}^{(2)}R_{00} + {}^{(4)}R_{00} + O(\epsilon^6)\,,
$$

[6] I want to remember here that the time derivative of a term increases its order by one unit.

$$
\begin{aligned}
{}^{(2)}R_{00} &= \frac{1}{2}\,\nabla^2\,{}^{(2)}g_{00}\,,\\
{}^{(4)}R_{00} &= \frac{1}{2}\Big[\nabla^2\,{}^{(4)}g_{00} - 2\,{}^{(3)}g_{0i,0i} + {}^{(2)}g_{ii,00} + {}^{(2)}g_{ik}\,{}^{(2)}g_{00,ik} + {}^{(2)}g_{ik,i}\,{}^{(2)}g_{00,k} +\\
&\quad - \frac{1}{2}\,{}^{(2)}g_{00,k}\,{}^{(2)}g_{ii,k} - \frac{1}{2}\,{}^{(2)}g_{00,i}\,{}^{(2)}g_{00,i}\Big]\,;
\end{aligned}
$$

$$
\begin{aligned}
R_{0i} &= {}^{(3)}R_{0i} + O(\epsilon^5)\,,\\
{}^{(3)}R_{0i} &= -\frac{1}{2}\Big[{}^{(2)}g_{ij,0j} - \nabla^2\,{}^{(3)}g_{0i} - {}^{(2)}g_{jj,0i} + {}^{(3)}g_{0j,ji}\Big]\,;
\end{aligned}
$$

$$
\begin{aligned}
R_{ij} &= {}^{(2)}R_{ij} + O(\epsilon^4)\,,\\
{}^{(2)}R_{ij} &= -\frac{1}{2}\Big[{}^{(2)}g_{ki,jk} + {}^{(2)}g_{kj,ik} - \nabla^2\,{}^{(2)}g_{ji} + {}^{(2)}g_{00,ij} - {}^{(2)}g_{kk,ij}\Big]\,.
\end{aligned}
$$

With a suitable gauge choice, it is possible to simplify the $R_{\mu\nu}$ expressions; specifically, it is convenient to use the harmonic gauge, $g^{\alpha\beta}\,\Gamma^{\mu}_{\alpha\beta} = 0$, through which the above expansions reduce to the following,

$$
\begin{aligned}
{}^{(2)}R_{00} &= \frac{1}{2}\,\nabla^2\,{}^{(2)}g_{00}\,,\\
{}^{(4)}R_{00} &= \frac{1}{2}\Big[\nabla^2\,{}^{(4)}g_{00} + {}^{(2)}g_{ij}\,{}^{(2)}g_{00,ij} - {}^{(2)}g_{00,00} - |\boldsymbol{\nabla}\,{}^{(2)}g_{00}\,|^2\Big]\,,\\
{}^{(3)}R_{0i} &= \frac{1}{2}\,\nabla^2\,{}^{(3)}g_{0i}\,,\\
{}^{(2)}R_{ij} &= \frac{1}{2}\,\nabla^2\,{}^{(2)}g_{ij}\,.
\end{aligned}
$$

Furthermore, the expansions of the components of the energy-momentum tensor (see (2.15) on page 22) that follow, whose different orders of approximation are evaluated for the case of systems on a large scale, the most common ones in the universe, comparable to dust for which $p \sim 0$,

$$
\begin{aligned}
T_{00} &= {}^{(0)}T_{00} + {}^{(2)}T_{00} + O(\epsilon^4)\,,\\
{}^{(0)}T_{00} &= \rho\,c^2\,,\\
{}^{(2)}T_{00} &= \Big(\rho\,c^2 + p\Big)\epsilon^2 - p\,{}^{(2)}g_{00} \sim \rho\,c^2\epsilon^2\,,
\end{aligned}
$$

$$
\begin{aligned}
T_{0i} &= {}^{(1)}T_{0i} + {}^{(3)}T_{0i} + O(\epsilon^5)\,,\\
{}^{(1)}T_{0i} &= -\frac{v^i}{c}\,\Big(\rho\,c^2 + p\Big) \sim -\,\rho\,c\,v^i,\\
{}^{(3)}T_{0i} &= -\frac{v^i}{c}\,\Big(\rho\,c^2 + p\Big)\epsilon^2 - p\,{}^{(3)}g_{0i} \sim -\,\rho\,c\,v^i\,\epsilon^2\,,
\end{aligned}
$$

$$
\begin{aligned}
T_{ij} &= {}^{(0)}T_{ij} + {}^{(2)}T_{ij} + O(\epsilon^4)\,,\\
{}^{(0)}T_{ij} &= p\,\delta_{ij} \sim 0\,,\\
{}^{(2)}T_{ij} &= \frac{v^i v^j}{c^2}\,\Big(\rho\,c^2 + p\Big) - p\,{}^{(2)}g_{ij} \sim \rho\,v^i\,v^j\,,
\end{aligned}
$$

along with the expansions of the corresponding contravariant components, i.e.,

$$
\begin{aligned}
T^{00} &= {}^{(0)}T^{00} + {}^{(2)}T^{00} + O(\epsilon^4)\,, \\
{}^{(0)}T^{00} &= \rho\, c^2\,, \\
{}^{(2)}T^{00} &= \Big(\rho\, c^2 + p\Big)\epsilon^2 - p\, {}^{(2)}g^{00} \sim \rho\, c^2 \epsilon^2\,,
\end{aligned}
$$

$$
\begin{aligned}
T^{0i} &= {}^{(1)}T^{0i} + {}^{(3)}T^{0i} + O(\epsilon^5)\,, \\
{}^{(1)}T^{0i} &= \Big(\rho\, c^2 + p\Big)\left(\frac{v^i}{c}\right) \sim\ \rho\, c\, v^i, \\
{}^{(3)}T^{0i} &= \Big(\rho\, c^2 + p\Big)\left(\frac{v^i}{c}\right)\epsilon^2 - p\, {}^{(3)}g^{0i} \sim \rho\, c\, v^i\, \epsilon^2\,,
\end{aligned}
$$

$$
\begin{aligned}
T^{ij} &= {}^{(0)}T^{ij} + {}^{(2)}T^{ij} + O(\epsilon^4)\,, \\
{}^{(0)}T^{ij} &= p\, \delta_{ij} \sim 0\,, \\
{}^{(2)}T^{ij} &= \Big(\rho\, c^2 + p\Big)\left(\frac{v^i}{c}\right)\left(\frac{v^j}{c}\right) - p\, {}^{(2)}g^{ij} \sim \rho\, v^i\, v^j\,,
\end{aligned}
$$

allow to compute those relating to the $S_{\mu\nu}$ tensor, namely,

$$
\begin{aligned}
S_{00} &= T_{00} - \frac{1}{2}\, g_{00}\, g_{\alpha\beta}\, T^{\alpha\beta} = {}^{(0)}S_{00} + {}^{(2)}S_{00} + O(\epsilon^4)\,, \\
{}^{(0)}S_{00} &= \frac{1}{2}\Big[\, {}^{(0)}T^{00} + {}^{(0)}T^{ii}\Big] \sim \frac{1}{2}\, {}^{(0)}T^{00}\,, \\
{}^{(2)}S_{00} &= \frac{1}{2}\Big[\, {}^{(2)}T^{00} + 2\, {}^{(2)}g_{00}\, {}^{(0)}T^{00} + {}^{(2)}T^{ii} - {}^{(2)}g_{ij}\, {}^{(0)}T^{ij} + \delta_{ij}\, {}^{(2)}g_{00}\, {}^{(0)}T^{ij}\Big] \\
&\sim \frac{1}{2}\Big[\, {}^{(2)}T^{00} + 2\, {}^{(2)}g_{00}\, {}^{(0)}T^{00} + {}^{(2)}T^{ii}\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
S_{0i} &= T_{0i} - \frac{1}{2}\, g_{0i}\, g_{\alpha\beta}\, T^{\alpha\beta} = {}^{(1)}\ S_{0i} + O(\epsilon^3)\,, \\
{}^{(1)}S_{0i} &= {}^{(1)}T_{0i}\,,
\end{aligned}
$$

$$
\begin{aligned}
S_{ij} &= T_{ij} - \frac{1}{2}\, g_{ij}\, g_{\alpha\beta}\, T^{\alpha\beta} = {}^{(0)}S_{ij} + O(\epsilon^2)\,, \\
{}^{(0)}S_{ij} &= {}^{(0)}T_{ij} + \frac{1}{2}\, \delta_{ij}\Big[\, {}^{(0)}T^{00} - {}^{(0)}T^{jj}\Big] \sim \frac{1}{2}\, \delta_{ij}\, {}^{(0)}T^{00}\,.
\end{aligned}
$$

The field equations, therefore, are as follows,

$$
\begin{aligned}
{}^{(2)}R_{00} &= \frac{8\pi G_N}{c^4}\, {}^{(0)}S_{00}\,, \\
{}^{(4)}R_{00} &= \frac{8\pi G_N}{c^4}\, {}^{(2)}S_{00}\,, \\
{}^{(3)}R_{0i} &= \frac{8\pi G_N}{c^4}\, {}^{(1)}S_{0i}\,, \\
{}^{(2)}R_{ij} &= \frac{8\pi G_N}{c^4}\, {}^{(0)}S_{ij}\,,
\end{aligned}
$$

which, once developed, become,

$$
\begin{aligned}
\nabla^2\, {}^{(2)}g_{00} &= \frac{8\pi G_N}{c^4}\, {}^{(0)}T^{00}\,,\\
\nabla^2\, {}^{(4)}g_{00} &= -\,{}^{(2)}g_{ij}\, {}^{(2)}g_{00,ij} + {}^{(2)}g_{00,00} + |\boldsymbol{\nabla}\, {}^{(2)}g_{00}\,|^2 + \frac{8\pi G_N}{c^4}\Big[\,{}^{(2)}T^{00} + 2\,{}^{(2)}g_{00}\,{}^{(0)}T^{00} + {}^{(2)}T^{ii}\Big]\,,\\
\nabla^2\, {}^{(3)}g_{0i} &= \frac{16\pi G_N}{c^4}\, {}^{(1)}T_{0i}\,,\\
\nabla^2\, {}^{(2)}g_{ij} &= \frac{8\pi G_N}{c^4}\delta_{ij}\, {}^{(0)}T^{00}\,,
\end{aligned}
$$

whose solutions are given by the following functions,

$$
\begin{aligned}
{}^{(2)}g_{00}\big(x^0,\,\mathbf{x}\big) &= -\frac{2}{c^2}U(\mathbf{x})\,,\\
{}^{(4)}g_{00}\big(x^0,\,\mathbf{x}\big) &= \frac{2}{c^4}\Big[U(\mathbf{x})^2 - 2\,\Phi(\mathbf{x})\Big]\,,\\
{}^{(3)}g_{0i}\big(x^0,\,\mathbf{x}\big) &= \frac{h_{0i}}{c^3}\,,\\
{}^{(2)}g_{ij}\big(x^0,\,\mathbf{x}\big) &= -\frac{2}{c^2}V(\mathbf{x})\,\delta_{ij}\,.
\end{aligned}
$$

where $h_{0i} = P_i, P_i$ being the i-th component of the classical momentum of a body, that is to say $P_i = mv_i$.

In this regard, it should be pointed out that the P_i are constant. This is a direct consequence of the *geodesic equation*,

$$
\frac{Du^\alpha}{ds} = \frac{du^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}u^\mu u^\nu = 0\,, \tag{3.29}
$$

which corresponds to the covariant derivative of the tangent vector $u^\alpha = dx^\alpha/ds$ along the curve itself (world-line), $x^\alpha = x^\alpha(s)$, in a Riemannian manifold. This means that the v_i are constant, what implies that the h_{0i} are constant too.
The equation (3.29) defines a *self-parallel curve*, otherwise said *affine geodesic*; it is a generalization of the condition $du^\alpha/ds = 0$, typical of the straight trajectories in the Minkowski space-time. In this sense, the self-parallel curve generalizes the notion of straight line to the case of manifolds equipped with an affine connection.

Ultimately, the covariant metric tensor, in the Post-Newtonian approximation 1PN, as defined in (3.26) on page 35, takes the following form:

$$
\begin{aligned}
g_{00}\big(x^0,\,\mathbf{x}\big) &= 1 - \frac{2}{c^2}\,U\big(\mathbf{x}\big) + \frac{2}{c^4}\Big[U(\mathbf{x})^2 - 2\,\Phi(\mathbf{x})\Big] + O\big(\epsilon^6\big)\,,\\
g_{0i}\big(x^0,\,\mathbf{x}\big) &= \frac{h_{0i}}{c^3} + O\big(\epsilon^5\big)\,,\\
g_{ij}\big(x^0,\,\mathbf{x}\big) &= -\Big[1 + \frac{2}{c^2}V(\mathbf{x})\Big]\delta_{ij} + O\big(\epsilon^4\big).
\end{aligned} \tag{3.30}
$$

With a similar procedure, one can obtain the contravariant metric tensor, the components of which are given by the following expansions,

$$
\begin{aligned}
g^{00}\big(x^0,\,\mathbf{x}\big) &= 1 + \frac{2}{c^2}\,U\big(\mathbf{x}\big) + \frac{2}{c^4}\Big[U(\mathbf{x})^2 + 2\,\Phi(\mathbf{x})\Big] + O\big(\epsilon^6\big)\,,\\
g^{0i}\big(x^0,\,\mathbf{x}\big) &= -\frac{h_{0i}}{c^3} + O\big(\epsilon^5\big)\,,
\end{aligned} \tag{3.31}
$$

$$g^{ij}\big(x^0,\,\mathbf{x}\big) = -\bigg[1 - \frac{2}{c^2}\,V(\mathbf{x})\bigg]\delta_{ij} + O\big(\epsilon^4\big).$$

With these tools in hand, I will go on to compute all basic quantities, starting from the Ricci's tensor and the corresponding scalar, necessary to carry out a survey, aimed at developing the curvature scalars appearing in the Lagrangians mentioned above, in particular in the Lagrangian referred to in (3.11) on page 28; this will allow me, later on, to develop the associated field equations, by expanding, time by time, all the other objects of different nature appearing in them.

Chapter 4

Curvature scalars of interest in 1PN approximation

In this chapter, I will describe in detail the procedure necessary to obtain the perturbative expansions of the curvature scalars appearing in the Lagrangians under examination (3.10) - (3.11), using the 1PN perturbation scheme of the metric tensor that I discussed in the previous chapter.

Stated in advance that, for sakes of convenience, I made use of a suitable symbology to work out the computations, for which I refer to Appendix B on page 81, I proceeded, first of all, to develop the Christoffel's symbols, through which it has been then possible to obtain the preliminary expressions of the Ricci's tensor components. The latter have been definitively expanded in the potentials $U(\mathbf{x}), V(\mathbf{x})$ and $\Phi(\mathbf{x})$, so as to be able to develop all other curvature scalars appearing in the aforementioned Lagrangians, as well as the other mathematical tools necessary to expand the field equations related to (3.11).

4.1 Ricci tensor components

Starting from the metric tensor $g_{\mu\nu}(x)$ and its contravariant version, referred to in (3.30) - (3.31), it is possible to derive directly the explicit expressions of the Christoffel's symbols (2.4), of which I give the results in Appendix C, so as not to burden the discussion too much.

4.1.1 Preliminary results

Once the Christoffel's connections $\Gamma^{\alpha}_{\mu\nu}$ have been computed, it is immediate to determine the expressions of the $R_{\mu\nu}$ terms, according to the relation given in (2.12) on page 22, which I rewrite here for convenience,

$$R_{\mu\nu} = \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} - \frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}} \ln\sqrt{-g} + \Gamma^{\sigma}_{\mu\nu}\frac{\partial}{\partial x^{\sigma}} \ln\sqrt{-g} - \Gamma^{\sigma}_{\mu\alpha}\Gamma^{\alpha}_{\sigma\nu}\,,$$

producing the following basic results, which could not be further compacted, given their mathematical structure (all details are given in Appendix D on page 101):

- R_{00} component

$$R_{00} = \frac{2}{c^6}\big(PD_1V_{,1} + PD_2V_{,2} + PD_3V_{,3}\big) - \frac{1}{c^6}\big(c^2 - 2V\big)\Big\{\Big[PD_{11} - 2\big(U_{,1}\big)^2\Big] + \Big[PD_{22} - 2\big(U_{,2}\big)^2\Big] + \Big[PD_{33} - 2\big(U_{,3}\big)^2\Big]\Big\} + \frac{1}{c^6\big(c^2+2V\big)D}\Big[\big(c^2-2V\big)\big(c^2+2V\big)^2\big(PD_1^2 + PD_2^2$$

$$+ PD_3^2\big) - \big(c^2 - 2V\big)\big(2D + P\big)\big(PD_1V_{,1} + PD_2V_{,2} + PD_3V_{,3}\big)\Big] - \frac{1}{c^{14}}\Big[2Q\big(PD_1^2 + PD_2^2 + PD_3^2\big) + \big(h_{01}PD_1 + h_{02}PD_2 + h_{03}PD_3\big)^2 + 2h_{01}h_{02}PD_1PD_2 + 2h_{01}h_{03}PD_1PD_3 + 2h_{02}h_{03}PD_2PD_3 + h_{01}^2PD_1^2 + h_{02}^2PD_2^2 + h_{03}^2PD_3^2\Big];$$

- R_{01} component

$$R_{01} = \frac{1}{c^7}\Big\{h_{01}\Big[PD_{11} - 2\big(U_{,1}\big)^2\Big] + h_{02}\big(PD_{12} - 2\,U_{,1}U_{,2}\big) + h_{03}\big(PD_{13} - 2\,U_{,1}U_{,3}\big)\Big\} + \frac{1}{c^7\big(c^2 + 2V\big)D}\Big\{h_{01}\Big[-\big(c^2 + 2V\big)^2PD_1^2 + \big(2D + P\big)PD_1V_{,1}\Big] + h_{02}\Big[-\big(c^2 + 2V\big)^2 PD_1PD_2 + \big(2D + P\big)PD_1V_{,2}\Big] + h_{03}\Big[-\big(c^2 + 2V\big)^2PD_1PD_3 + \big(2D + P\big)PD_1V_{,3}\Big]\Big\} + \frac{2}{c^{11}}\big(c^2 - 2V\big)\Big\{h_{01}\big(PD_2V_{,2} + PD_3V_{,3}\big) - h_{02}PD_1V_{,2} - h_{03}PD_1V_{,3}\Big\};$$

- R_{02} component

$$R_{02} = \frac{1}{c^7}\Big\{h_{01}\big(PD_{12} - 2\,U_{,1}U_{,2}\big) + h_{02}\Big[PD_{22} - 2\big(U_{,2}\big)^2\Big] + h_{03}\big(PD_{23} - 2\,U_{,2}U_{,3}\big)\Big\} + \frac{1}{c^7\big(c^2 + 2V\big)D}\Big\{h_{01}\Big[-\big(c^2 + 2V\big)^2PD_1PD_2 + \big(2D + P\big)PD_2V_{,1}\Big] + h_{02}\Big[-\big(c^2 + 2V\big)^2 PD_2^2 + \big(2D + P\big)PD_2V_{,2}\Big] + h_{03}\Big[-\big(c^2 + 2V\big)^2PD_2PD_3 + \big(2D + P\big)PD_2V_{,3}\Big]\Big\} - \frac{2}{c^{11}}\big(c^2 - 2V\big)\Big[h_{01}PD_2V_{,1} - h_{02}\big(PD_1V_{,1} + PD_3V_{,3}\big) + h_{03}PD_2V_{,3}\Big];$$

- R_{03} component

$$R_{03} = \frac{1}{c^7}\Big\{h_{01}\big(PD_{13} - 2\,U_{,1}U_{,3}\big) + h_{02}\big(PD_{23} - 2\,U_{,2}U_{,3}\big) + h_{03}\Big[PD_{33} - 2\big(U_{,3}\big)^2\Big]\Big\} + \frac{1}{c^7\big(c^2 + 2V\big)D}\Big\{h_{01}\Big[-\big(c^2 + 2V\big)^2PD_1PD_3 + \big(2D + P\big)PD_3V_{,1}\Big] + h_{02}\Big[-\big(c^2 + 2V\big)^2 PD_2PD_3 + \big(2D + P\big)PD_3V_{,2}\Big] + h_{03}\Big[-\big(c^2 + 2V\big)^2PD_3^2 + \big(2D + P\big)PD_3V_{,3}\Big]\Big\} - \frac{2}{c^{11}}\big(c^2 - 2V\big)\Big[h_{01}PD_3V_{,1} + h_{02}PD_3V_{,2} - h_{03}\big(PD_1V_{,1} + PD_2V_{,2}\big)\Big];$$

- R_{11} component

$$R_{11} = \Big[-\frac{2}{c^4} + \frac{\big(c^2 - 2V\big)\big(2D + P\big)}{c^4\big(c^2 + 2V\big)D}\Big]\Big[\big(V_{,1}\big)^2 - \big(V_{,2}\big)^2 - \big(V_{,3}\big)^2\Big] + \frac{\big(c^2 - 2V\big)}{c^4}\big(V_{,11} - V_{,22} - V_{,33}\big) + \frac{\big(c^4 - 4V^2\big)}{c^4D}\big(-PD_1V_{,1} + PD_2V_{,2} + PD_3V_{,3}\big) + \frac{2}{\big(c^2 + 2V\big)^2D^2}\Big[\big(2D + P\big)V_{,1} - \big(c^2 + 2V\big)^2PD_1\Big]^2 - \frac{1}{\big(c^2 + 2V\big)^2D}\Big[2\big(c^2 + 2V\big)^3\big(U_{,1}\big)^2 - 12\big(c^2 + 2V\big)^2PD_1V_{,1} + 4\big(D$$

$$+ 2P\big)\big(V_{,1}\big)^2 - \big(c^2+2V\big)^3 PD_{11} + \big(c^2+2V\big)\big(2D+P\big)V_{,11}\Big] - \frac{1}{c^{16}}\, q^2 PD_1^2 - \frac{2}{c^{12}}\, h_{123}^2 PD_1$$
$$V_{,1} - \frac{1}{c^8}\big(c^2-2V\big)^2\Big[3\big(V_{,1}\big)^2 - 2\big(V_{,2}\big)^2 - 2\big(V_{,3}\big)^2\Big]\,;$$

- R_{12} component

$$R_{12} = \frac{2}{c^4}\Big[\big(c^2-2V\big)V_{,12} - 2\,V_{,1}V_{,2}\Big] + \frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(2D+P\big)V_{,1} - \big(c^2+2V\big)^2 PD_1\Big]$$
$$\Big[\big(2D+P\big)V_{,2} - \big(c^2+2V\big)^2 PD_2\Big] - \frac{1}{\big(c^2+2V\big)^2 D}\Big\{2\,U_{,2}\big(c^2+2V\big)^2\Big[\big(c^2+2V\big)U_{,1} - 3\big(c^2$$
$$- 2U\big)V_{,1}\Big] + 2\,V_{,2}\Big[2\big(D+2P\big)V_{,1} - 3\big(c^2+2V\big)^2 PD_1\Big] - \big(c^2+2V\big)\Big[12\big(c^2+2V\big)V_{,1}\Phi_{,2}$$
$$- \big(2D+P\big)V_{,12} + \big(c^2+2V\big)^2 PD_{12}\Big]\Big\} + \frac{2\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\, V_{,1}V_{,2} - \Bigg[\frac{\big(c^4-4V^2\big)}{c^4 D}$$
$$+ \frac{h_{123}^2}{c^{12}}\Bigg]\Big(PD_1V_{,2} + PD_2V_{,1}\Big) - \frac{1}{c^{16}}\, q^2 PD_1PD_2 - \frac{5}{c^8}\big(c^2-2V\big)^2 V_{,1}V_{,2}\,;$$

- R_{13} component

$$R_{13} = \frac{2}{c^4}\Big[\big(c^2-2V\big)V_{,13} - 2\,V_{,1}V_{,3}\Big] + \frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(2D+P\big)V_{,1} - \big(c^2+2V\big)^2 PD_1\Big]$$
$$\Big[\big(2D+P\big)V_{,3} - \big(c^2+2V\big)^2 PD_3\Big] - \frac{1}{\big(c^2+2V\big)^2 D}\Big\{2\,U_{,3}\big(c^2+2V\big)^2\Big[\big(c^2+2V\big)U_{,1} - 3\big(c^2$$
$$- 2U\big)V_{,1}\Big] + 2\,V_{,3}\Big[2\big(D+2P\big)V_{,1} - 3\big(c^2+2V\big)^2 PD_1\Big] - \big(c^2+2V\big)\Big[12\big(c^2+2V\big)V_{,1}\Phi_{,3}$$
$$- \big(2D+P\big)V_{,13} + \big(c^2+2V\big)^2 PD_{13}\Big]\Big\} + \frac{2\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\, V_{,1}V_{,3} - \Bigg[\frac{\big(c^4-4V^2\big)}{c^4 D}$$
$$+ \frac{h_{123}^2}{c^{12}}\Bigg]\Big(PD_1V_{,3} + PD_3V_{,1}\Big) - \frac{1}{c^{16}}\, q^2 PD_1PD_3 - \frac{5}{c^8}\big(c^2-2V\big)^2 V_{,1}V_{,3}\,;$$

- R_{22} component

$$R_{22} = \Big[-\frac{2}{c^4} + \frac{\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\Big]\Big[-\big(V_{,1}\big)^2 + \big(V_{,2}\big)^2 - \big(V_{,3}\big)^2\Big] + \frac{\big(c^2-2V\big)}{c^4}\big(-V_{,11}$$
$$+ V_{,22} - V_{,33}\big) + \frac{\big(c^4-4V^2\big)}{c^4 D}\big(PD_1V_{,1} - PD_2V_{,2} + PD_3V_{,3}\big) + \frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(2D$$
$$+ P\big)V_{,2} - \big(c^2+2V\big)^2 PD_2\Big]^2 - \frac{1}{\big(c^2+2V\big)^2 D}\Big[2\big(c^2+2V\big)^3\big(U_{,2}\big)^2 - 12\big(c^2+2V\big)^2 PD_2V_{,2}$$
$$+ 4\big(D+2P\big)\big(V_{,2}\big)^2 - \big(c^2+2V\big)^3 PD_{22} + \big(c^2+2V\big)\big(2D+P\big)V_{,22}\Big] - \frac{1}{c^{16}}\, q^2 PD_2^2 - \frac{2}{c^{12}}$$
$$h_{123}^2 PD_2V_{,2} - \frac{1}{c^8}\big(c^2-2V\big)^2\Big[-2\big(V_{,1}\big)^2 + 3\big(V_{,2}\big)^2 - 2\big(V_{,3}\big)^2\Big]\,;$$

- R_{23} component

$$R_{23} = \frac{2}{c^4}\Big[\big(c^2-2V\big)V_{,23} - 2\,V_{,2}V_{,3}\Big] + \frac{2}{\big(c^2+2V\big)^2D^2}\Big[\big(2D+P\big)V_{,2} - \big(c^2+2V\big)^2PD_2\Big]$$
$$\Big[\big(2D+P\big)V_{,3} - \big(c^2+2V\big)^2PD_3\Big] - \frac{1}{\big(c^2+2V\big)^2D}\Big\{2\,U_{,3}\big(c^2+2V\big)^2\Big[\big(c^2+2V\big)U_{,2} - 3\big(c^2$$
$$-2U\big)V_{,2}\Big] + 2\,V_{,3}\Big[2\big(D+2P\big)V_{,2} - 3\big(c^2+2V\big)^2PD_2\Big] - \big(c^2+2V\big)\Big[12\big(c^2+2V\big)V_{,2}\Phi_{,3}$$
$$-\big(2D+P\big)V_{,23} + \big(c^2+2V\big)^2PD_{23}\Big]\Big\} + \frac{2\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\,V_{,2}V_{,3} - \Bigg[\frac{\big(c^4-4V^2\big)}{c^4D}$$
$$+\frac{h_{123}^2}{c^{12}}\Bigg]\Big(PD_2V_{,3} + PD_3V_{,2}\Big) - \frac{1}{c^{16}}\,q^2PD_2PD_3 - \frac{5}{c^8}\big(c^2-2V\big)^2V_{,2}V_{,3}\,;$$

- R_{33} component

$$R_{33} = \Big[-\frac{2}{c^4} + \frac{\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\Big]\Big[-\big(V_{,1}\big)^2 - \big(V_{,2}\big)^2 + \big(V_{,3}\big)^2\Big] + \frac{\big(c^2-2V\big)}{c^4}\big(-V_{,11}$$
$$-V_{,22}+V_{,33}\big) + \frac{\big(c^4-4V^2\big)}{c^4D}\big(PD_1V_{,1} + PD_2V_{,2} - PD_3V_{,3}\big) + \frac{2}{\big(c^2+2V\big)^2D^2}\Big[\big(2D$$
$$+P\big)V_{,3} - \big(c^2+2V\big)^2PD_3\Big]^2 - \frac{1}{\big(c^2+2V\big)^2D}\Big[2\big(c^2+2V\big)^3\big(U_{,3}\big)^2 - 12\big(c^2+2V\big)^2PD_3V_{,3}$$
$$+4\big(D+2P\big)\big(V_{,3}\big)^2 - \big(c^2+2V\big)^3PD_{33} + \big(c^2+2V\big)\big(2D+P\big)V_{,33}\Big] - \frac{1}{c^{16}}\,q^2PD_3^2 - \frac{2}{c^{12}}$$
$$h_{123}^2PD_3V_{,3} - \frac{1}{c^8}\big(c^2-2V\big)^2\Big[-2\big(V_{,1}\big)^2 - 2\big(V_{,2}\big)^2 + 3\big(V_{,3}\big)^2\Big]\,.$$

4.1.2 Final expansions

The expressions thus obtained have been further developed in order to express them in the potentials $U(\mathbf{x}), V(\mathbf{x})$ and $\Phi(\mathbf{x})$ and then compacted; in this regard, it may be useful to recall the assessments (A.1), (A.2) made in Appendix A on page 82.

The resulting expansions are as follows (see for details Appendix D on page 117):

$$\begin{aligned} R_{00} &= {}^{(2)}R_{00} + {}^{(4)}R_{00} + O(\epsilon^6)\,,\\ {}^{(2)}R_{00} &= -\frac{1}{c^2}\,\Delta U\,,\\ {}^{(4)}R_{00} &= \frac{1}{c^4}\Big[2\big(U+V\big)\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla}U\cdot\big(\boldsymbol{\nabla}U - \boldsymbol{\nabla}V\big)\Big]\,; \end{aligned} \tag{4.1}$$

$$\begin{aligned} R_{0i} &= {}^{(5)}R_{0i} + O(\epsilon^7)\,,\\ {}^{(5)}R_{0i} &= \frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,i}\big)\,; \end{aligned} \tag{4.2}$$

$$\begin{aligned} R_{ij} &= {}^{(2)}R_{ij} + {}^{(4)}R_{ij} + O(\epsilon^6)\,,\\ {}^{(2)}R_{ij} &= \frac{1}{c^2}\big(U_{,ij} - V_{,ij} - \Delta V\,\delta_{ij}\big)\,, \end{aligned} \tag{4.3}$$

$$
{}^{(4)}R_{ij} = \frac{1}{c^4}\Big[3\,V_{,i}V_{,j} + \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,V\Delta V\big)\,\delta_{ij} + 2\,VV_{,ij} - \big(U_{,i}V_{,j} + U_{,j}V_{,i}\big) + 2\,\Phi_{,ij} - U_{,i}U_{,j}\Big]\,.
$$

It should be noted that the R_{0i} components, having the first non-negligible term of the fifth order in ϵ, do not give any contribution to the expansions of the curvature scalars, of which I will speak in the following sections, as I am interested only in those up to the fourth order. Therefore, they can be disregarded.

4.2 Ricci curvature scalar

From its definition, that is,

$$
R = g^{\mu\nu}R_{\mu\nu}\,,
$$

the corresponding explicit expression follows,

$$
R = g^{00}R_{00} + 2\sum_{i=1}^{3} g^{0i}R_{0i} + g^{11}R_{11} + 2\big(g^{12}R_{12} + g^{13}R_{13}\big) + g^{22}R_{22} + 2\,g^{23}R_{23} + g^{33}R_{33}\,,
$$

in which the well-known symmetry properties of both tensors have been used, both being symmetrical, $g^{\mu\nu} = g^{\nu\mu}$ and $R_{\mu\nu} = R_{\nu\mu}$. Actually, on the basis of the expansions relating to the components of the contravariant metric tensor (3.31) on page 39, the previous expression reduces to the following,

$$
R = g^{00}R_{00} + 2\sum_{i=1}^{3} g^{0i}R_{0i} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33}\,.
$$

The insertion of the expansions concerning the components of the contravariant metric tensor and the Ricci's tensor (4.1), (4.2), (4.3), has led to the following result (all details are in Appendix E on page 167),

$$
\begin{aligned}
R &= {}^{(2)}R + {}^{(4)}R + O(\epsilon^6)\,,\\
{}^{(2)}R &= -\frac{2}{c^2}\big(\Delta U - 2\,\Delta V\big)\,,\\
{}^{(4)}R &= \frac{2}{c^4}\Big[\boldsymbol{\nabla}U\cdot\big(\boldsymbol{\nabla}U - \boldsymbol{\nabla}V\big) - 3\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 2\,V\big(\Delta U - 4\,\Delta V\big) - 2\,\Delta\Phi\Big]\,.
\end{aligned}
\tag{4.4}
$$

4.3 R^2 curvature scalar

The expansion of the R^2 scalar follows directly from that concerning R, given in (4.4), that is,

$$
\begin{aligned}
R^2 &= {}^{(4)}R^2 + O(\epsilon^6)\,,\\
{}^{(4)}R^2 &= \frac{4}{c^4}\Big[\big(\Delta U\big)^2 + 4\big(\Delta V\big)^2 - 4\,\Delta U\Delta V\Big]\,,
\end{aligned}
\tag{4.5}
$$

what has taken only a short calculation, as shown in Appendix E on page 170.

4.4 $R_{\mu\nu}R^{\mu\nu}$ curvature scalar

The perturbative development of this curvature scalar requires, first of all, the computation of the Ricci's contravariant tensor, according to the following relation,

$$g^{\alpha\mu}g^{\beta\nu}R_{\alpha\beta} = R^{\mu\nu}\,,$$

taking into account the expansions regarding the contravariant metric tensor components (3.31) on page 39 and the Ricci's covariant tensor components (4.1), (4.2), (4.3).

The details of these computations can be found in Appendix E on page 170, of which I give here, for the sake of brevity, only the final perturbative expansion:

$$\begin{aligned}
R_{\mu\nu}R^{\mu\nu} &= {}^{(4)}R_{\mu\nu}R^{\mu\nu} + O(\epsilon^6)\,,\\
{}^{(4)}R_{\mu\nu}R^{\mu\nu} &= \frac{1}{c^4}\Big\{\big(\Delta U\big)^2 + 5\big(\Delta V\big)^2 - 2\,\Delta U\Delta V + \Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big)\\
&\quad + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big] + \Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(V_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\\
&\quad - 2\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}\,.
\end{aligned} \tag{4.6}$$

4.5 $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ curvature scalar

By virtue of the Gauss-Bonnet identity (see Appendix B on page 95), which I rewrite here for convenience,

$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = 4R_{\mu\nu}R^{\mu\nu} - R^2\,,$$

it is immediate to obtain the perturbative expansion of this scalar, in the light of the two previous results (see Appendix E on page 179), which is given by the following,

$$\begin{aligned}
R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} &= {}^{(4)}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} + O(\epsilon^6)\,,\\
{}^{(4)}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} &= \frac{4}{c^4}\Big\{\big(\Delta V\big)^2 + 2\,\Delta U\Delta V + \Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\\
&\quad \cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big] + \Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(V_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\\
&\quad - 2\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}\,.
\end{aligned} \tag{4.7}$$

Chapter 5

Solving of the field equations of quadratic Lagrangian in Post-Newtonian approximation 1PN

5.1 Expansion of the left-hand sides of the field equations

It is interesting, at this point, to develop the field equations deriving from the most general quadratic Lagrangian, referred to in the Chapter 3, that I quote here, again, for convenience,

$$\mathcal{L} = a_1 R + a_2 R^2 + b_1 R_{\mu\nu} R^{\mu\nu}, \tag{5.1}$$

with the associated left-hand sides of the corresponding field equations,

$$\begin{aligned}
{}^0H_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R\,, \\
{}^1H_{\mu\nu} &= 2RR_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R^2 - 2R_{;\mu\nu} + 2g_{\mu\nu}\Box_{GR} R\,, \\
{}^2H_{\mu\nu} &= 2R_{\mu}^{\ \alpha} R_{\nu\alpha} - \frac{1}{2}\, g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} - 2R_{(\mu|\,\alpha;|\nu)\alpha} + \Box_{GR} R_{\mu\nu} + g_{\mu\nu} R^{\alpha\beta}{}_{;\alpha\beta}\,,
\end{aligned} \tag{5.2}$$

armed of the 1PN approximation of the metric tensor, through which I have developed, in the previous chapter, the Ricci's tensor, the Ricci's scalar and other curvature scalars of interest, in order to investigate on the corrections to the Newtonian term, appearing in the gravitational potentials $U(\mathbf{x}), V(\mathbf{x})$ and eventually in $\Phi(\mathbf{x})$, characterizing the fourth-order expansion of the metric tensor (3.30), which has not been yet tested to date. It is matter of verifying if this further approximation involves corrective terms of Yukawa type, as it results from the job of S. Capozziello and A. Stabile [74], based on the second-order expansion of the metric tensor, in the $U(\mathbf{x})$ and $V(\mathbf{x})$ potentials only, namely in the Post-Newtonian approximation 0.5PN. In particular, it would be interesting to see how the corresponding coefficients and their signs change, in order to understand how much weakened or strengthened the resulting gravitational interaction results to be in consequence of these corrections.

The procedure to achieve this goal consists of expanding the field equations without computing these last ones exactly, but developing them in the low velocity and weak field limit directly; a short cut due to the enormous amount of computations and time required by the approximation in use. In some sense, this job constitutes a first attempt to probe the aforementioned field equations in the Post-Newtonian approximation 1PN.

In other words, this means to develop the above field equations treating "*a priori*" the space-time as it was roughly flat, a hypothesis certainly valid in the universe when considered on a large scale, that constitutes the ambit of application of such a theory; this involves making in (5.2) the following substitutions,

$$\text{covariant derivative ";"} \rightarrow \text{ordinary derivative ","}\,,$$
$$\Box_{GR} = \frac{1}{\sqrt{-g}}\,\partial_\alpha\Big(\sqrt{-g}\,g^{\alpha\beta}\partial_\beta\Big) \rightarrow \Box_{SR} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\,,$$

where the $\Box_{SR}$, in this case, is reduced simply to $-\Delta$, since all the gravitational potentials and as consequence all other curvature tensors and scalars are independent of time in stationary conditions, as I have pointed out previously; therefore,

$$\begin{aligned}
{}^0H_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R\,,\\
{}^1H_{\mu\nu} &= 2RR_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R^2 - 2R_{,\mu\nu} - 2g_{\mu\nu}\Delta R\,,\\
{}^2H_{\mu\nu} &= 2R_\mu^{\ \alpha}R_{\nu\alpha} - \frac{1}{2}\,g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - 2R_{(\mu|\,\alpha,|\nu)\alpha} - \Delta R_{\mu\nu} + g_{\mu\nu}R^{\alpha\beta}{}_{,\alpha\beta}\,,
\end{aligned} \tag{5.3}$$

are the final expressions which I am going to expand.

5.1.1 Left-hand sides of the Einstein field equations

About the well-known Einstein's field equations (see also (2.14) on page 22), corresponding to the first term of (5.1),

$${}^0H_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R\,,$$

of which I consider, here, the left-hand side only, I give the results of the perturbative expansions of the time-time, time-space and space-space components (all details are shown in Appendix F on page 181 and following):

$$\begin{aligned}
{}^0H_{00} &= {}^{(2)\,0}H_{00} + {}^{(4)\,0}H_{00} + O(\epsilon^6)\,,\\
{}^{(2)\,0}H_{00} &= -\frac{2}{c^2}\,\Delta V\,,\\
{}^{(4)\,0}H_{00} &= \frac{1}{c^4}\Big[3\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 4\big(2\,V+U\big)\Delta V\Big]\,;
\end{aligned}$$

$$\begin{aligned}
{}^0H_{0i} &= {}^{(5)\,0}H_{0i} + O(\epsilon^7)\,,\\
{}^{(5)\,0}H_{0i} &= \frac{1}{c^5}\Big[\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,i}\big) + h_{0i}\big(\Delta U - 2\,\Delta V\big)\Big]\,;
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
{}^0H_{ij} &= {}^{(2)\,0}H_{ij} + {}^{(4)\,0}H_{ij} + O(\epsilon^6)\,,\\
{}^{(2)\,0}H_{ij} &= \frac{1}{c^2}\Big[\big(\Delta V - \Delta U\big)\delta_{ij} - \big(V-U\big)_{,ij}\Big]\,,\\
{}^{(4)\,0}H_{ij} &= \frac{1}{c^4}\Big\{2\Big[V_{,i}V_{,j} - \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big)\delta_{ij}\Big] + \big(V_{,i}V_{,j} - V_{,i}U_{,j} - U_{,i}V_{,j}\big) - \Big[U_{,i}U_{,j}\\
&\quad - \big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)\delta_{ij}\Big] + 2\,V\Big[V_{,ij} - \big(\Delta V\big)\delta_{ij}\Big] + 2\Big[\Phi_{,ij} - \big(\Delta\Phi\big)\delta_{ij}\Big]\Big\}\,.
\end{aligned}$$

5.1.2 Left-hand sides of the field equations related to R^2 term

The left-hand sides of the field equations associated with the R^2 term in (5.1),

$$ {}^{1}H_{\mu\nu} = 2RR_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R^2 - 2R_{,\mu\nu} - 2g_{\mu\nu}\Delta R\,, $$

have given the following results (for details see Appendix F on page 185 and following):

$$
\begin{aligned}
{}^{1}H_{00} &= {}^{(2)\,1}H_{00} + {}^{(4)\,1}H_{00} + O\big(\epsilon^6\big)\,,\\
{}^{(2)\,1}H_{00} &= \frac{1}{c^2}\big(4\,\Delta^2 U - 8\,\Delta^2 V\big)\,,\\
{}^{(4)\,1}H_{00} &= \frac{1}{c^4}\Big\{2\big(\Delta U\big)^2 + 24\big(\Delta V\big)^2 - 8\,\Delta U\Delta V + 8\,\Delta^2\Phi - 8V\big(\Delta^2 U - 4\,\Delta^2 V\big) + 4U\big(\Delta^2 U\\
&\quad - 2\,\Delta^2 V\big) - 4\Big[\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big) - \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big) - 3\,\Delta\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big) + 4\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U\\
&\quad - 4\,\Delta V\big)\Big]\Big\}\,;
\end{aligned}
$$

$$
\begin{aligned}
{}^{1}H_{0i} &= {}^{(5)\,1}H_{0i} + O\big(\epsilon^7\big)\,,\\
{}^{(5)\,1}H_{0i} &= \frac{1}{c^5}\Big[h_{0i}\big(4\,\Delta^2 U - 8\,\Delta^2 V\big)\Big]\,;
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
{}^{1}H_{ij} &= {}^{(2)\,1}H_{ij} + {}^{(4)\,1}H_{ij} + O\big(\epsilon^6\big)\,,\\
{}^{(2)\,1}H_{ij} &= \frac{1}{c^2}\Big[4\big(\Delta U - 2\,\Delta V\big)_{,ij} - 4\big(\Delta^2 U - 2\,\Delta^2 V\big)\delta_{ij}\Big]\,,\\
{}^{(4)\,1}H_{ij} &= \frac{1}{c^4}\Big\{-4\,\Delta U\big(U+V\big)_{,ij} + 8\,\Delta V\big(U+3V\big)_{,ij} - 4\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)_{,ij} + 4\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,ij}\\
&\quad + 12\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big)_{,ij} - 8V_{,j}\big(\Delta U - 4\,\Delta V\big)_{,i} - 8V_{,i}\big(\Delta U - 4\,\Delta V\big)_{,j} - 8V\big(\Delta U - 4\,\Delta V\big)_{,ij}\\
&\quad + 8\,\Delta\Phi_{,ij} + \Big[4\,\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big) - 4\,\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big) - 12\,\Delta\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big) + 16\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U\\
&\quad - 4\,\Delta V\big) - 16V\Delta^2 V - 8\,\Delta^2\Phi + 4\,\Delta U\Delta V - 32\big(\Delta V\big)^2 + 2\big(\Delta U\big)^2\Big]\delta_{ij}\Big\}\,.
\end{aligned}
$$

5.1.3 Left-hand sides of the field equations related to $R_{\mu\nu}R^{\mu\nu}$ term

Finally, the left-hand sides of the field equations related to the $R_{\mu\nu}R^{\mu\nu}$ term of (5.1),

$$ {}^{2}H_{\mu\nu} = 2R_{\mu}^{\ \alpha}R_{\nu\alpha} - \frac{1}{2}\,g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - 2R_{(\mu|\,\alpha,|\nu)\alpha} - \Delta R_{\mu\nu} + g_{\mu\nu}R^{\alpha\beta}{}_{,\alpha\beta}\,, $$

have produced what follows (all details are shown in Appendix F on page 189 and following):

$$
\begin{aligned}
{}^{2}H_{00} &= {}^{(2)\,2}H_{00} + {}^{(4)\,2}H_{00} + O\big(\epsilon^6\big)\,,\\
{}^{(2)\,2}H_{00} &= \frac{1}{c^2}\big(2\,\Delta^2 U - 2\,\Delta^2 V\big)\,,\\
{}^{(4)\,2}H_{00} &= \frac{1}{c^4}\Big\{-\frac{3}{2}\big(\Delta U\big)^2 + \frac{13}{2}\big(\Delta V\big)^2 - 3\,\Delta U\Delta V - 4\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(\Delta U\big) - 13\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U\big)\\
&\quad - 2\big(2U+3V\big)\Delta^2 U + 4\,\Delta^2\Phi - 2\,\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big) + \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big) + 4\big(U+3V\big)\Delta^2 V\\
&\quad + 32\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta V\big) - \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(\Delta V\big) + \frac{21}{2}\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(V_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big)\\
&\quad + \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] + \frac{1}{2}\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big]\\
&\quad - 3\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}\,;
\end{aligned}
$$

$$
\begin{aligned}
{}^{2}H_{0i} &= {}^{(5)\,2}H_{0i} + O\big(\epsilon^{7}\big)\,,\\
{}^{(5)\,2}H_{0i} &= \frac{1}{c^{5}}\Big\{-\Delta\Big[\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,i}\big)\Big] + h_{0i}\big(\Delta^{2}U - 2\,\Delta^{2}V\big)\Big\}\,;
\end{aligned}
\tag{5.6}
$$

$$
\begin{aligned}
{}^{2}H_{ij} &= {}^{(2)\,2}H_{ij} + {}^{(4)\,2}H_{ij} + O\big(\epsilon^{6}\big)\,,\\
{}^{(2)\,2}H_{ij} &= \frac{1}{c^{2}}\Big[\big(\Delta U - 3\,\Delta V\big)_{,ij} + \big(3\,\Delta^{2}V - \Delta^{2}U\big)\delta_{ij}\Big]\,,\\
{}^{(4)\,2}H_{ij} &= \frac{1}{c^{4}}\Big\{-2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(U_{,j}\big) + 10\,\boldsymbol{\nabla}\big(V_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + 2\big(U+4V\big)_{,ij}\Delta V + 14\,\boldsymbol{\nabla}\big(V\big)\\
&\quad\cdot\boldsymbol{\nabla}\big(V_{,ij}\big) + 12V_{,i}\Delta V_{,j} + 12V_{,j}\Delta V_{,i} + 14V\Delta V_{,ij} - 4\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) + 2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big)\\
&\quad + 2\,\boldsymbol{\nabla}\big(U_{,j}\big)\cdot\boldsymbol{\nabla}\big(V_{,i}\big) - 2V_{,i}\Delta U_{,j} - 2V_{,j}\Delta U_{,i} - 2\big(U+V\big)_{,ij}\Delta U - 4V\Delta U_{,ij} + 2\,\Delta\Phi_{,ij}\\
&\quad - 2\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) + \Big\{-\frac{21}{2}\big(\Delta V\big)^{2} + \frac{3}{2}\big(\Delta U\big)^{2} + \Delta U\Delta V - \Delta\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big) - \Delta\big(\boldsymbol{\nabla}U\\
&\quad\cdot\boldsymbol{\nabla}V\big) - 36\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta V\big) - 10V\Delta^{2}V + 2V\Delta^{2}U - 2\,\Delta^{2}\Phi + \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big) + 9\,\boldsymbol{\nabla}V\\
&\quad\cdot\boldsymbol{\nabla}\big(\Delta U\big) + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(\Delta V\big) - \frac{1}{2}\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\\
&\quad\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big] - \frac{21}{2}\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(V_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] + 3\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\\
&\quad\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}\delta_{ij}\Big\}\,.
\end{aligned}
$$

5.2 Complete field equations

Before going into the merits of the matter, it has to be pointed out that the time-space components of (5.4), (5.5), (5.6), are of the fifth-order in ϵ and therefore must be set equal to zero, by virtue of the fact of wanting to study the field equations up to the fourth-order in ϵ, in consequence of the choice of using the fourth-order approximation of the metric tensor.

Therefore, the left-hand sides of the field equations, rewritten more compactly, are as follows:

- $\mathcal{L}_{0} = R$

$$
\begin{aligned}
{}^{0}H_{00} &= -\frac{2}{c^{2}}\,\Delta V + \frac{1}{c^{4}}\Big[3\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 4\big(2\,V+U\big)\Delta V\Big]\,,\\
{}^{0}H_{0i} &= 0\,,\\
{}^{0}H_{ij} &= \frac{1}{c^{2}}\Big[\big(\Delta V - \Delta U\big)\delta_{ij} - \big(V-U\big)_{,ij}\Big] + \frac{1}{c^{4}}\Big\{2\Big[V_{,i}V_{,j} - \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big)\delta_{ij}\Big] + \big(V_{,i}V_{,j}\\
&\quad - V_{,i}U_{,j} - U_{,i}V_{,j}\big) - \Big[U_{,i}U_{,j} - \big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)\delta_{ij}\Big] + 2\,V\Big[V_{,ij} - \big(\Delta V\big)\delta_{ij}\Big] + 2\Big[\Phi_{,ij}\\
&\quad - \big(\Delta\Phi\big)\delta_{ij}\Big]\Big\}\,;
\end{aligned}
\tag{5.7}
$$

- $\mathcal{L}_1 = R^2$

$$
\begin{aligned}
{}^1H_{00} &= \frac{1}{c^2}\big(4\,\Delta^2 U - 8\,\Delta^2 V\big) + \frac{1}{c^4}\Big\{2\big(\Delta U\big)^2 + 24\big(\Delta V\big)^2 - 8\,\Delta U\Delta V + 8\,\Delta^2\Phi - 8V \\
&\quad \big(\Delta^2 U - 4\,\Delta^2 V\big) + 4U\big(\Delta^2 U - 2\,\Delta^2 V\big) - 4\Big[\Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\big) - \Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V\big) \\
&\quad - 3\,\Delta\big(\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V\big) + 4\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\big(\Delta U - 4\,\Delta V\big)\Big]\Big\}, \\
{}^1H_{0i} &= 0\,, \qquad (5.8)\\
{}^1H_{ij} &= \frac{1}{c^2}\Big[4\big(\Delta U - 2\,\Delta V\big)_{,ij} - 4\big(\Delta^2 U - 2\,\Delta^2 V\big)\delta_{ij}\Big] + \frac{1}{c^4}\Big\{-4\,\Delta U\big(U+V\big)_{,ij} + 8\,\Delta V \\
&\quad \big(U+3V\big)_{,ij} - 4\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\big)_{,ij} + 4\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V\big)_{,ij} + 12\big(\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V\big)_{,ij} - 8V_{,j}\big(\Delta U \\
&\quad - 4\,\Delta V\big)_{,i} - 8V_{,i}\big(\Delta U - 4\,\Delta V\big)_{,j} - 8V\big(\Delta U - 4\,\Delta V\big)_{,ij} + 8\,\Delta\Phi_{,ij} + \Big[4\,\Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\big) \\
&\quad - 4\,\Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V\big) - 12\,\Delta\big(\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V\big) + 16\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\big(\Delta U - 4\,\Delta V\big) - 16V\Delta^2 V \\
&\quad - 8\,\Delta^2\Phi + 4\,\Delta U\Delta V - 32\big(\Delta V\big)^2 + 2\big(\Delta U\big)^2\Big]\delta_{ij}\Big\};
\end{aligned}
$$

- $\mathcal{L}_2 = R_{\mu\nu}R^{\mu\nu}$

$$
\begin{aligned}
{}^2H_{00} &= \frac{1}{c^2}\big(2\,\Delta^2 U - 2\,\Delta^2 V\big) + \frac{1}{c^4}\Big\{-\frac{3}{2}\big(\Delta U\big)^2 + \frac{13}{2}\big(\Delta V\big)^2 - 3\,\Delta U\Delta V - 4\,\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla}\big(\Delta U\big) \\
&\quad - 13\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\big(\Delta U\big) - 2\big(2U+3V\big)\Delta^2 U + 4\,\Delta^2\Phi - 2\,\Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\big) + \Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V\big) \\
&\quad + 4\big(U+3V\big)\Delta^2 V + 32\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\big(\Delta V\big) - \boldsymbol{\nabla} U\cdot\boldsymbol{\nabla}\big(\Delta V\big) + \frac{21}{2}\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) \\
&\quad + \boldsymbol{\nabla}\big(V_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] + \frac{1}{2}\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big) \\
&\quad + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big] - 3\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}, \\
{}^2H_{0i} &= 0\,, \qquad (5.9)\\
{}^2H_{ij} &= \frac{1}{c^2}\Big[\big(\Delta U - 3\,\Delta V\big)_{,ij} + \big(3\,\Delta^2 V - \Delta^2 U\big)\delta_{ij}\Big] + \frac{1}{c^4}\Big\{-2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(U_{,j}\big) \\
&\quad + 10\,\boldsymbol{\nabla}\big(V_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + 2\big(U+4V\big)_{,ij}\Delta V + 14\,\boldsymbol{\nabla}\big(V\big)\cdot\boldsymbol{\nabla}\big(V_{,ij}\big) + 12V_{,i}\Delta V_{,j} \\
&\quad + 12V_{,j}\Delta V_{,i} + 14V\Delta V_{,ij} - 4\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) + 2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + 2\,\boldsymbol{\nabla}\big(U_{,j}\big)\cdot\boldsymbol{\nabla}\big(V_{,i}\big) \\
&\quad - 2V_{,i}\Delta U_{,j} - 2V_{,j}\Delta U_{,i} - 2\big(U+V\big)_{,ij}\Delta U - 4V\Delta U_{,ij} + 2\,\Delta\Phi_{,ij} - 2\,\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) \\
&\quad + \Big\{-\frac{21}{2}\big(\Delta V\big)^2 + \frac{3}{2}\big(\Delta U\big)^2 + \Delta U\Delta V - \Delta\big(\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V\big) - \Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V\big) - 36\,\boldsymbol{\nabla} V \\
&\quad \cdot\,\boldsymbol{\nabla}\big(\Delta V\big) - 10V\Delta^2 V + 2V\Delta^2 U - 2\,\Delta^2\Phi + \Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\big) + 9\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\big(\Delta U\big) + \boldsymbol{\nabla} U \\
&\quad \cdot\,\boldsymbol{\nabla}\big(\Delta V\big) - \frac{1}{2}\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big] - \frac{21}{2}\Big[\boldsymbol{\nabla}\big(V_{,1}\big) \\
&\quad \cdot\,\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(V_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] + 3\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big) \\
&\quad \cdot\,\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}\delta_{ij}\Big\}.
\end{aligned}
$$

The next step consists of solving the general field equation,

$$a_1\,{}^0H_{\mu\nu} + a_2\,{}^1H_{\mu\nu} + b_1\,{}^2H_{\mu\nu} = \frac{8\pi G_N}{c^4}\,T^{(m)}_{\mu\nu}\,, \tag{5.10}$$

which, once made explicit in the time-time, time-space and space-space components, corresponds to three coupled fourth-order differential equations, everyone of which expanded up to the fourth-order in ϵ.

In this regard, it is therefore necessary to expand also the energy-momentum tensor up to the fourth-order in ϵ, just like the ${}^iH_{\mu\nu}, i = 0, 1, 2$. Thus, from (2.15) on page 22 and remembering that,

$$\begin{aligned} u_\mu &= \gamma\left(1, -\frac{\mathbf{v}}{c}\right), \\ \gamma &= \frac{1}{\sqrt{1-\beta^2}}\,, \\ \beta &= \frac{v}{c}\,, \end{aligned}$$

of which it has to be given, for what follows, the Taylor expansion of γ^2 in the low-velocity limit, $\beta \ll 1$,namely,

$$\gamma^2 = 1 + \beta^2 + \beta^4 + O\big(\beta^6\big)\,,$$

it follows:

- Time - Time component

$$\begin{aligned} T^{(m)}_{00} &= \big(\rho c^2 + p\big)u_0^2 - p\,g_{00} \\ &= \big(\rho c^2 + p\big)\gamma^2 - p\,g_{00} \\ &= \big(\rho c^2 + p\big)\Big[1 + \beta^2 + \beta^4 + O\big(\beta^6\big)\Big] - p\left[1 - \frac{2\,U}{c^2} + \frac{2}{c^4}\big(U^2 - 2\,\Phi\big) + O\big(\epsilon^6\big)\right], \end{aligned}$$

  that is,

$$\begin{aligned} T^{(m)}_{00} &= {}^{(0)}T^{(m)}_{00} + {}^{(2)}T^{(m)}_{00} + {}^{(4)}T^{(m)}_{00} + O\big(\epsilon^6\big)\,, \\ {}^{(0)}T^{(m)}_{00} &= \rho c^2\,, \\ {}^{(2)}T^{(m)}_{00} &= \big(\rho c^2 + p\big)\beta^2 + p\left(\frac{2\,U}{c^2}\right), \\ {}^{(4)}T^{(m)}_{00} &= \big(\rho c^2 + p\big)\beta^4 - \frac{2p}{c^4}\big(U^2 - 2\Phi\big)\,; \end{aligned}$$

- Time - Space component

$$\begin{aligned} T^{(m)}_{0i} &= \big(\rho c^2 + p\big)u_0 u_i - p\,g_{0i} \\ &= \big(\rho c^2 + p\big)\gamma^2\left(-\frac{v^i}{c}\right) - p\left[\frac{h_{0i}}{c^3} + O\big(\epsilon^5\big)\right] \\ &= -\frac{v^i}{c}\big(\rho c^2 + p\big)\Big[1 + \beta^2 + \beta^4 + O\big(\beta^6\big)\Big] - p\left[\frac{h_{0i}}{c^3} + O\big(\epsilon^5\big)\right], \end{aligned}$$

that is,

$$
\begin{aligned}
T^{(m)}_{0i} &= {}^{(1)}T^{(m)}_{0i} + {}^{(3)}T^{(m)}_{0i} + O\big(\epsilon^5\big)\,,\\
{}^{(1)}T^{(m)}_{0i} &= \big(\rho c^2 + p\big)\left(\frac{v_i}{c}\right),\\
{}^{(3)}T^{(m)}_{0i} &= \big(\rho c^2 + p\big)\left(\frac{v_i}{c}\right)\beta^2 - p\left(\frac{h_{0i}}{c^3}\right);
\end{aligned}
$$

- Space - Space component

$$
\begin{aligned}
T^{(m)}_{ij} &= \big(\rho c^2 + p\big)u_i u_j - p\, g_{ij}\\
&= \big(\rho c^2 + p\big)\gamma^2\left(\frac{v^i}{c}\right)\left(\frac{v^j}{c}\right) - p\left[-\left(1 + \frac{2\,V}{c^2}\right)\delta_{ij} + O\big(\epsilon^4\big)\right]\\
&= \left(\frac{v^i}{c}\right)\left(\frac{v^j}{c}\right)\big(\rho c^2 + p\big)\Big[1 + \beta^2 + \beta^4 + O\big(\beta^6\big)\Big] - p\left[-\left(1 + \frac{2\,V}{c^2}\right)\delta_{ij} + O\big(\epsilon^4\big)\right],
\end{aligned}
$$

that is,

$$
\begin{aligned}
T^{(m)}_{ij} &= {}^{(0)}T^{(m)}_{ij} + {}^{(2)}T^{(m)}_{ij} + {}^{(4)}T^{(m)}_{ij} + O\big(\epsilon^6\big)\,,\\
{}^{(0)}T^{(m)}_{ij} &= p\,\delta_{ij}\,,\\
{}^{(2)}T^{(m)}_{ij} &= \big(\rho c^2 + p\big)\left(\frac{v_i}{c}\right)\left(\frac{v_j}{c}\right) + p\left(\frac{2\,V}{c^2}\right)\delta_{ij}\,,\\
{}^{(4)}T^{(m)}_{ij} &= \big(\rho c^2 + p\big)\left(\frac{v_i}{c}\right)\left(\frac{v_j}{c}\right)\beta^2\,.
\end{aligned}
$$

Therefore, starting from the time - space component of the field equations (5.10), that is to say,

$$
a_1\,{}^0H_{0i} + a_2\,{}^1H_{0i} + b_1\,{}^2H_{0i} = \frac{8\pi G_N}{c^4}\,T^{(m)}_{0i}\,,
$$

this one assumes the following form,

$$
0 = \frac{8\pi G_N}{c^4}\big(-\rho c v^i\big)\,,
$$

where I have considered only the first term of the $T^{(m)}_{0i}$ component, the first-order one, disregarding the other for evident reasons of consistency and setting $p \sim 0$ for the same reasons adduced on page 31; it follows, evidently, that $v^i = 0$, where $i = 1, 2, 3$, that implies $\mathbf{v} = 0$. This allows the components of the energy-momentum tensor to be rewritten in an approximate form as follows:

$$
\begin{aligned}
T^{(m)}_{00} &\sim \rho c^2\,,\\
T^{(m)}_{0i} &\sim 0\,,\\
T^{(m)}_{ij} &\sim 0\,.
\end{aligned}
$$

In light of this result, the other two field equations of (5.10), the time - time and space - space ones, assume the following form:

$$a_1\,{}^0H_{00} + a_2\,{}^1H_{00} + b_1\,{}^2H_{00} = \frac{8\pi G_N}{c^4}\,T^{(m)}_{00}\,,$$

$$a_1\Big\{-\frac{2}{c^2}\,\Delta V + \frac{1}{c^4}\Big[3\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 4\big(2\,V+U\big)\Delta V\Big]\Big\} + a_2\Big\{\frac{1}{c^2}\big(4\,\Delta^2U - 8\,\Delta^2V\big) + \frac{1}{c^4}\Big\{2\big(\Delta U\big)^2$$
$$+ 24\big(\Delta V\big)^2 - 8\,\Delta U\Delta V + 8\,\Delta^2\Phi - 8V\big(\Delta^2U - 4\,\Delta^2V\big) + 4U\big(\Delta^2U - 2\,\Delta^2V\big) - 4\Big[\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)$$
$$- \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big) - 3\,\Delta\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big) + 4\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U - 4\,\Delta V\big)\Big]\Big\}\Big\} + b_1\Big\{\frac{1}{c^2}\big(2\,\Delta^2U - 2\,\Delta^2V\big)$$
$$+ \frac{1}{c^4}\Big\{-\frac{3}{2}\big(\Delta U\big)^2 + \frac{13}{2}\big(\Delta V\big)^2 - 3\,\Delta U\Delta V - 4\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(\Delta U\big) - 13\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U\big) - 2\big(2U+3V\big)$$
$$\Delta^2U + 4\,\Delta^2\Phi - 2\,\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big) + \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big) + 4\big(U+3V\big)\Delta^2V + 32\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta V\big)$$
$$- \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(\Delta V\big) + \frac{21}{2}\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(V_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] + \frac{1}{2}\Big[\boldsymbol{\nabla}\big(U_{,1}\big)$$
$$\cdot\,\boldsymbol{\nabla}\big(U_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big] - 3\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big)$$
$$+ \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}\Big\} = \frac{8\pi\rho\,G_N}{c^2}\,;$$

$$a_1\,{}^0H_{ij} + a_2\,{}^1H_{ij} + b_1\,{}^2H_{ij} = \frac{8\pi G_N}{c^4}\,T^{(m)}_{ij}\,,$$

$$a_1\Big\{\frac{1}{c^2}\Big[\big(\Delta V - \Delta U\big)\delta_{ij} - \big(V-U\big)_{,ij}\Big] + \frac{1}{c^4}\Big\{2\Big[V_{,i}V_{,j} - \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big)\delta_{ij}\Big] + \big(V_{,i}V_{,j} - V_{,i}U_{,j}$$
$$- U_{,i}V_{,j}\big) - \Big[U_{,i}U_{,j} - \big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)\delta_{ij}\Big] + 2\,V\Big[V_{,ij} - \big(\Delta V\big)\delta_{ij}\Big] + 2\Big[\Phi_{,ij} - \big(\Delta\Phi\big)\delta_{ij}\Big]\Big\}\Big\}$$
$$+ a_2\Big\{\frac{1}{c^2}\Big[4\big(\Delta U - 2\,\Delta V\big)_{,ij} - 4\big(\Delta^2U - 2\,\Delta^2V\big)\delta_{ij}\Big] + \frac{1}{c^4}\Big\{-4\,\Delta U\big(U+V\big)_{,ij} + 8\,\Delta V\big(U$$
$$+ 3V\big)_{,ij} - 4\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)_{,ij} + 4\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,ij} + 12\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big)_{,ij} - 8V_{,j}\big(\Delta U - 4\,\Delta V\big)_{,i}$$
$$- 8V_{,i}\big(\Delta U - 4\,\Delta V\big)_{,j} - 8V\big(\Delta U - 4\,\Delta V\big)_{,ij} + 8\,\Delta\Phi_{,ij} + \Big[4\,\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big) - 4\,\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)$$
$$- 12\,\Delta\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big) + 16\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U - 4\,\Delta V\big) - 16V\Delta^2V - 8\,\Delta^2\Phi + 4\,\Delta U\Delta V - 32\big(\Delta V\big)^2$$
$$+ 2\big(\Delta U\big)^2\Big]\delta_{ij}\Big\}\Big\} + b_1\Big\{\frac{1}{c^2}\Big[\big(\Delta U - 3\,\Delta V\big)_{,ij} + \big(3\,\Delta^2V - \Delta^2U\big)\delta_{ij}\Big] + \frac{1}{c^4}\Big\{-2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(U_{,j}\big)$$
$$+ 10\,\boldsymbol{\nabla}\big(V_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + 2\big(U+4V\big)_{,ij}\Delta V + 14\,\boldsymbol{\nabla}\big(V\big)\cdot\boldsymbol{\nabla}\big(V_{,ij}\big) + 12V_{,i}\Delta V_{,j} + 12V_{,j}\Delta V_{,i}$$
$$+ 14V\Delta V_{,ij} - 4\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) + 2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + 2\,\boldsymbol{\nabla}\big(U_{,j}\big)\cdot\boldsymbol{\nabla}\big(V_{,i}\big) - 2V_{,i}\Delta U_{,j} - 2V_{,j}\Delta U_{,i}$$
$$- 2\big(U+V\big)_{,ij}\Delta U - 4V\Delta U_{,ij} + 2\,\Delta\Phi_{,ij} - 2\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) + \Big\{-\frac{21}{2}\big(\Delta V\big)^2 + \frac{3}{2}\big(\Delta U\big)^2$$
$$+ \Delta U\Delta V - \Delta\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big) - \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big) - 36\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta V\big) - 10V\Delta^2V + 2V\Delta^2U - 2\,\Delta^2\Phi$$
$$+ \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big) + 9\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U\big) + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(\Delta V\big) - \frac{1}{2}\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big)$$
$$+ \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big] - \frac{21}{2}\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(V_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] + 3\Big[\boldsymbol{\nabla}\big(U_{,1}\big)$$

$$\cdot\,\boldsymbol{\nabla}(V_{,1})+\boldsymbol{\nabla}(U_{,2})\cdot\boldsymbol{\nabla}(V_{,2})+\boldsymbol{\nabla}(U_{,3})\cdot\boldsymbol{\nabla}(V_{,3})\Big]\Big\}\delta_{ij}\Big\}\Big\}=0\,.$$

At this point, matching the terms of the same order in ϵ, two systems of differential equations follow, both of fourth order, the first one related to the terms of $O(\epsilon^2)$ order and the second associated with those of $O(\epsilon^4)$ order, namely:

$$-2a_1\Delta V+\big(4a_2+2b_1\big)\Delta^2U-\big(8a_2+2b_1\big)\Delta^2V=8\pi G_N\rho\,, \tag{5.11a}$$

$$\Big[a_1\big(\Delta V-\Delta U\big)-\big(4a_2+b_1\big)\Delta^2U+\big(8a_2+3b_1\big)\Delta^2V\Big]\delta_{ij}$$
$$+\Big[a_1\big(U-V\big)+\big(4a_2+b_1\big)\Delta U-\big(8a_2+3b_1\big)\Delta V\Big]_{,ij}=0\,; \tag{5.11b}$$

$$a_1\Big[3\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V+4\big(2\,V+U\big)\Delta V\Big]+a_2\Big\{2\big(\Delta U\big)^2+24\big(\Delta V\big)^2-8\,\Delta U\Delta V+8\,\Delta^2\Phi$$
$$-8V\big(\Delta^2U-4\,\Delta^2V\big)+4U\big(\Delta^2U-2\,\Delta^2V\big)-4\Big[\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)-\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)$$
$$-3\,\Delta\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big)+4\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U-4\,\Delta V\big)\Big]\Big\}+b_1\Big\{-\frac{3}{2}\big(\Delta U\big)^2+\frac{13}{2}\big(\Delta V\big)^2-3\,\Delta U\Delta V$$
$$-4\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(\Delta U\big)-13\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U\big)-2\big(2U+3V\big)\Delta^2U+4\,\Delta^2\Phi-2\,\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)$$
$$+\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)+4\big(U+3V\big)\Delta^2V+32\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta V\big)-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(\Delta V\big)+\frac{21}{2}\Big[\boldsymbol{\nabla}\big(V_{,1}\big)$$
$$\cdot\,\boldsymbol{\nabla}\big(V_{,1}\big)+\boldsymbol{\nabla}\big(V_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big)+\boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]+\frac{1}{2}\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big)+\boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big)$$
$$+\boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big]-3\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big)+\boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big)+\boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}=0\,, \tag{5.12a}$$

$$a_1\Big\{2\Big[V_{,i}V_{,j}-\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big)\delta_{ij}\Big]+\big(V_{,i}V_{,j}-V_{,i}U_{,j}-U_{,i}V_{,j}\big)-\Big[U_{,i}U_{,j}-\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)\delta_{ij}\Big]$$
$$+2\,V\Big[V_{,ij}-\big(\Delta V\big)\delta_{ij}\Big]+2\Big[\Phi_{,ij}-\big(\Delta\Phi\big)\delta_{ij}\Big]\Big\}+a_2\Big\{-4\,\Delta U\big(U+V\big)_{,ij}+8\,\Delta V\big(U+3V\big)_{,ij}$$
$$-4\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)_{,ij}+4\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,ij}+12\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big)_{,ij}-8V_{,j}\big(\Delta U-4\,\Delta V\big)_{,i}-8V_{,i}\big(\Delta U$$
$$-4\,\Delta V\big)_{,j}-8V\big(\Delta U-4\,\Delta V\big)_{,ij}+8\,\Delta\Phi_{,ij}+\Big[4\,\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)-4\,\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)$$
$$-12\,\Delta\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big)+16\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U-4\,\Delta V\big)-16V\Delta^2V-8\,\Delta^2\Phi+4\,\Delta U\Delta V-32\big(\Delta V\big)^2$$
$$+2\big(\Delta U\big)^2\Big]\delta_{ij}\Big\}+b_1\Big\{-2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(U_{,j}\big)+10\,\boldsymbol{\nabla}\big(V_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big)+2\big(U+4V\big)_{,ij}\Delta V$$
$$+14\,\boldsymbol{\nabla}\big(V\big)\cdot\boldsymbol{\nabla}\big(V_{,ij}\big)+12V_{,i}\Delta V_{,j}+12V_{,j}\Delta V_{,i}+14V\Delta V_{,ij}-4\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(U_{,ij}\big)+2\,\boldsymbol{\nabla}\big(U_{,i}\big)$$
$$\cdot\,\boldsymbol{\nabla}\big(V_{,j}\big)+2\,\boldsymbol{\nabla}\big(U_{,j}\big)\cdot\boldsymbol{\nabla}\big(V_{,i}\big)-2V_{,i}\Delta U_{,j}-2V_{,j}\Delta U_{,i}-2\big(U+V\big)_{,ij}\Delta U-4V\Delta U_{,ij}+2\,\Delta\Phi_{,ij}$$
$$-2\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(U_{,ij}\big)+\Big\{-\frac{21}{2}\big(\Delta V\big)^2+\frac{3}{2}\big(\Delta U\big)^2+\Delta U\Delta V-\Delta\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big)-\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)$$
$$-36\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta V\big)-10V\Delta^2V+2V\Delta^2U-2\,\Delta^2\Phi+\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)+9\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U\big)$$
$$+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(\Delta V\big)-\frac{1}{2}\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big)+\boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big)+\boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big]-\frac{21}{2}\Big[\boldsymbol{\nabla}\big(V_{,1}\big)$$
$$\cdot\,\boldsymbol{\nabla}\big(V_{,1}\big)+\boldsymbol{\nabla}\big(V_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big)+\boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]+3\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big)+\boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big)$$

$$+\boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}\delta_{ij}\Big\}=0\,. \tag{5.12b}$$

For what concerns the system of differential equations given by (5.11a) and (5.11b), in the $U(\mathbf{x})$ and $V(\mathbf{x})$ potentials, it is solvable through a proper decoupling technic, but an observation is in order in this regard. In fact, the contributions coming from ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$ have opposite sign with respect to those given in (3.16b), (3.16c), related to the lower Post-Newtonian approximation 0.5PN, while ${}^{(0)}H_{\mu\nu}$ results unchanged, what has required a new computation.

On the other hand, it is immediate to verify that the system of differential equation given by (5.12a) and (5.12b), corresponding to $O(\epsilon^4)$ order, cannot be solved, since the insertion of (5.7), (5.8) and (5.9) into (5.10) gives two differential equations only, those ones relating to time-time and space-space components, in consequence of the fact that the time-space components are of $O(\epsilon^5)$ order, therefore negligible with respect to the maximum degree of approximation here considered, namely $O(\epsilon^4)$, while three are the gravitational potentials $U(\mathbf{x}), V(\mathbf{x})$ and $\Phi(\mathbf{x})$ to determine.

5.2.1 A survey on the field equations depending on the coupling constants

It is interesting, before to face the problem of solving the differential equation system (5.11a), (5.11b), to give a description of them with varying the values of the coupling constants; in particular, the conditions set on these last ones are such to make null the determinant of the transformation given by (5.13a), (5.13b). The final results are shown in table 5.1 on the facing page (see Appendix G for details on page 213).

It has to be observed that the first three cases correspond to the terms R, R^2 and $R_{\mu\nu}R^{\mu\nu}$ of the quadratic Lagrangian (5.1) respectively. In particular, the case 1, related to the Hilbert-Einstein Lagrangian, in the weak field regime, gives back second order differential equations, with a unique gravitational potential, restoring the Poisson equation with $a_1=1$, whose solution is given by the Newtonian potential exactly, as expected.

On the other hand, all other cases (2-8) give fourth order differential equations, in consequence of the contributions coming from the quadratic terms R^2 and/or $R_{\mu\nu}R^{\mu\nu}$ in the Lagrangian (5.1).

5.2.2 Decoupling of the differential equations

The equations (5.11a), (5.11b), can be modified, preliminary, by using the following transformation,

$$A=a_1\big(V-U\big)\,, \tag{5.13a}$$
$$B=\big(4a_2+b_1\big)U-\big(8a_2+3b_1\big)V\,, \tag{5.13b}$$

with the condition that it will be not singular, namely $2a_1\big(2a_2+b_1\big)\neq 0$, being the corresponding determinant. Consequently, they take the form,

$$\frac{\big(4a_2+b_1\big)}{\big(2a_2+b_1\big)}\,\Delta A+\frac{a_1}{\big(2a_2+b_1\big)}\,\Delta B-\frac{2b_1\big(3a_2+b_1\big)}{a_1\big(2a_2+b_1\big)}\,\Delta^2 A+\frac{2a_2}{\big(2a_2+b_1\big)}\,\Delta^2 B=8\pi G_N\rho\,, \tag{5.14a}$$
$$\Delta\big(A-\Delta B\big)\delta_{ij}-\big(A-\Delta B\big)_{,ij}=0\,; \tag{5.14b}$$

at this point, with the introduction of the auxiliary function,

$$\Phi=A-\Delta B\,, \tag{5.15}$$

the above equations are decoupled as follows,

Table 5.1: Field equations corresponding to different choices of the coupling constants.

Cases	**Coupling constants values**	**Field equations**
Case 1	$a_2 = 0$ $b_1 = 0$	$\Delta V = -\frac{4\pi G_N}{a_1}\rho\,,$ $\big(\Delta V - \Delta U\big)\delta_{ij} - \big(V - U\big)_{,ij} = 0\,;$
Case 2	$a_1 = 0$ $b_1 = 0$	$2\,\Delta^2 V - \Delta^2 U = -\frac{2\pi G_N}{a_2}\rho\,,$ $\big(2\,\Delta^2 V - \Delta^2 U\big)\delta_{ij} - \big(2\,\Delta V - \Delta U\big)_{,ij} = 0\,;$
Case 3	$a_1 = 0$ $a_2 = 0$	$\Delta^2 U - \Delta^2 V = \frac{4\pi G_N}{b_1}\rho\,,$ $-\big(\Delta^2 U - 3\,\Delta^2 V\big)\delta_{ij} + \big(\Delta U - 3\,\Delta V\big)_{,ij} = 0\,;$
Case 4	$b_1 = -2a_2$	$a_1\Delta V + 2a_2\Delta^2 V = -4\pi G_N\rho\,,$ $\Big[a_1\big(\Delta V - \Delta U\big) + 2\,a_2\big(\Delta^2 V - \Delta^2 U\big)\Big]\delta_{ij} - \Big[a_1\big(V - U\big) +$ $+2\,a_2\big(\Delta V - \Delta U\big)\Big]_{,ij} = 0\,;$
Case 5	$a_1 = 0$ $b_1 = -\,4a_2$	$\Delta^2 U = -\frac{2\pi G_N}{a_2}\rho\,,$ $\Delta^2 V\,\delta_{ij} - \Delta V_{,ij} = 0\,;$
Case 6	$a_1 = 0$ $b_1 = -2a_2$	$\Delta^2 V = -\frac{2\pi G_N}{a_2}\rho\,,$ $\big(\Delta^2 V - \Delta^2 U\big)\delta_{ij} - \big(\Delta V - \Delta U\big)_{,ij} = 0\,;$
Case 7	$a_1 = 0$ $b_1 = -\frac{8a_2}{3}$	$2\,\Delta^2 V + \Delta^2 U = -\frac{6\pi G_N}{a_2}\rho\,,$ $\Delta^2 U\,\delta_{ij} - \Delta U_{,ij} = 0\,.$

$$-\frac{2b_1\big(3a_2+b_1\big)}{a_1\big(2a_2+b_1\big)}\,\Delta^3 B+\frac{\big(6a_2+b_1\big)}{\big(2a_2+b_1\big)}\,\Delta^2 B+\frac{a_1}{\big(2a_2+b_1\big)}\,\Delta B=8\pi G_N\rho-\Delta f\,, \tag{5.16a}$$

$$\Delta\Phi\,\delta_{ij}-\Phi_{,ij}=0\,, \tag{5.16b}$$

where,

$$f=\frac{\big(4a_2+b_1\big)}{\big(2a_2+b_1\big)}\,\Phi-\frac{2b_1\big(3a_2+b_1\big)}{a_1\big(2a_2+b_1\big)}\,\Delta\Phi\,. \tag{5.17}$$

5.2.3 Resolution of the differential equations through Green's method

With a redefinition of $B(\mathbf{x})$ function through the integral,

$$B\big(\mathbf{x}\big)=Y_I\int\mathcal{G}\big(\mathbf{x},\mathbf{x}'\big)g\big(\mathbf{x}'\big)d^3\mathbf{x}'\,, \tag{5.18}$$

in which it has been adjoint the Y_I constant for dimensional reasons and

$$g\big(\mathbf{x}\big)=8\pi G_N\rho-\Delta f\big(\mathbf{x}\big)\,, \tag{5.19}$$

the system given by (5.16a), (5.16b) becomes,

$$\frac{2b_1\big(3a_2+b_1\big)}{a_1\big(2a_2+b_1\big)}\,\Delta^3_{\mathbf{x}}\mathcal{G}\big(\mathbf{x},\mathbf{x}'\big)-\frac{\big(6a_2+b_1\big)}{\big(2a_2+b_1\big)}\,\Delta^2_{\mathbf{x}}\mathcal{G}\big(\mathbf{x},\mathbf{x}'\big)-\frac{a_1}{\big(2a_2+b_1\big)}\,\Delta_{\mathbf{x}}\mathcal{G}\big(\mathbf{x},\mathbf{x}'\big)=-Y_I^{-1}\delta^3\big(\mathbf{x}-\mathbf{x}'\big)\,, \tag{5.20a}$$

$$\Delta\Phi\big(\mathbf{x}\big)\,\delta_{ij}-\Phi\big(\mathbf{x}\big)_{,ij}=0\,. \tag{5.20b}$$

By virtue of (5.18) and (5.15), that is to say,

$$B\big(\mathbf{x}\big)=Y_I\int\mathcal{G}\big(\mathbf{x},\mathbf{x}'\big)g\big(\mathbf{x}'\big)d^3\mathbf{x}'\,,$$
$$A=\Phi+\Delta B\,,$$

the inverse transformation of (5.13a), (5.13b), given by the following expressions,

$$U=-\frac{\big(8a_2+3b_1\big)}{2a_1\big(2a_2+b_1\big)}\,A-\frac{1}{2\big(2a_2+b_1\big)}\,B\,,$$
$$V=-\frac{\big(4a_2+b_1\big)}{2a_1\big(2a_2+b_1\big)}\,A-\frac{1}{2\big(2a_2+b_1\big)}\,B\,,$$

allows to obtain the general solution of the problem through the Green's function $\mathcal{G}\big(\mathbf{x},\mathbf{x}'\big)$ and the auxiliary function $\Phi\big(\mathbf{x}\big)$, that is the gravitational potentials $U(\mathbf{x})$ and $V(\mathbf{x})$:

$$U\big(\mathbf{x}\big)=-Y_I\bigg[\frac{\big(8a_2+3b_1\big)\Delta_{\mathbf{x}}+a_1}{2a_1\big(2a_2+b_1\big)}\bigg]\int\mathcal{G}\big(\mathbf{x},\mathbf{x}'\big)\bigg[8\pi G_N\rho\big(\mathbf{x}'\big)-\frac{\big(4a_2+b_1\big)}{\big(2a_2+b_1\big)}\Delta_{\mathbf{x}'}\Phi\big(\mathbf{x}'\big)$$
$$+\frac{2b_1\big(3a_2+b_1\big)}{a_1\big(2a_2+b_1\big)}\Delta^2_{\mathbf{x}'}\Phi\big(\mathbf{x}'\big)\bigg]d^3\mathbf{x}'-\frac{\big(8a_2+3b_1\big)}{2a_1\big(2a_2+b_1\big)}\,\Phi\big(\mathbf{x}\big)\,, \tag{5.21a}$$

$$V\big(\mathbf{x}\big)=-Y_I\bigg[\frac{\big(4a_2+b_1\big)\Delta_{\mathbf{x}}+a_1}{2a_1\big(2a_2+b_1\big)}\bigg]\int\mathcal{G}\big(\mathbf{x},\mathbf{x}'\big)\bigg[8\pi G_N\rho\big(\mathbf{x}'\big)-\frac{\big(4a_2+b_1\big)}{\big(2a_2+b_1\big)}\Delta_{\mathbf{x}'}\Phi\big(\mathbf{x}'\big)$$
$$+\frac{2b_1\big(3a_2+b_1\big)}{a_1\big(2a_2+b_1\big)}\Delta^2_{\mathbf{x}'}\Phi\big(\mathbf{x}'\big)\bigg]d^3\mathbf{x}'-\frac{\big(4a_2+b_1\big)}{2a_1\big(2a_2+b_1\big)}\,\Phi\big(\mathbf{x}\big)\,. \tag{5.21b}$$

5.2.4 Fourier's transform of the Green's function

At this point, it is convenient to express the Green's function through Fourier's transform,

$$\mathcal{G}\big(\mathbf{x},\mathbf{x}'\big)=\frac{1}{\big(2\pi\big)^{\frac{3}{2}}}\int\mathcal{G}\big(\mathbf{k}\big)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}d^3\mathbf{k}\,, \tag{5.22}$$

putting it into (5.20a), in order to obtain the explicit expression of this last one:

$$\mathcal{G}\big(\mathbf{k}\big)=Y_I^{-1}\frac{a_1\big(2a_2+b_1\big)}{k^2\Big[2b_1\big(3a_2+b_1\big)k^4+a_1\big(6a_2+b_1\big)k^2-a_1^2\Big]}\,.$$

Furthermore, it is useful to expand the denominator of $\mathcal{G}\big(\mathbf{k}\big)$ in sight of integrals in complex field to compute, to get the $\mathcal{G}\big(\mathbf{x},\mathbf{x}'\big)$ function; it is matter of factoring the bi-quadratic expression, to obtain then, by using the indeterminate coefficients method, a linear combination of rational algebraic functions, namely,

$$\mathcal{G}\big(\mathbf{k}\big)=Y_I^{-1}\left\{-\left(\frac{2a_2+b_1}{a_1}\right)\frac{1}{k^2}+\frac{2}{3}\left(\frac{3a_2+b_1}{a_1}\right)\frac{1}{\Big[k^2-\frac{a_1}{2\big(3a_2+b_1\big)}\Big]}+\frac{1}{3}\left(\frac{b_1}{a_1}\right)\frac{1}{\Big[k^2+\frac{a_1}{b_1}\Big]}\right\}. \tag{5.23}$$

It is useful, for what follows in the discussion, introducing two *characteristic lengths*, so defined,

$$\lambda_1^2=\frac{a_1}{b_1}\,, \tag{5.24a}$$

$$\lambda_2^2=-\frac{a_1}{2\big(3a_2+b_1\big)}\,, \tag{5.24b}$$

which have to be necessarily real and positive numbers ($\lambda_1,\ \lambda_2\in\mathbb{R}^+$). It has to be pointed out, in this regard, that these lengths, actually, have dimension $[L^{-1}]$, by virtue of the fact that a_1 has dimension $[L^0]$, while a_2 and b_1 have dimension $[L^2]$, as it results from (5.1).

The relations (5.24a) and (5.24b) suggest the possibility that such characteristic lengths are real and positive for different choices of the coupling constants, to which correspond different expressions for what concerns the Green's function; the results of such an investigation are shown in table 5.2 on the next page. I have assumed, in any case, $a_1>0$.

The computations developed in the next section, refer to the Green's function relating to Case 1 of table 5.2, since the other two expressions, due to the presence of oscillating terms, do not seem suitable to describe the behavior of the gravitational interaction.

5.2.5 Gravitational potentials of point-like sources

In what follows, I am going to compute the gravitational potentials, as given in (5.21a) and (5.21b), by using the Green's functions shown in table 5.2, for point-like sources with matter density $\rho\big(\mathbf{x}\big)=(2\pi)^{-\frac{3}{2}}M\delta\big(\mathbf{x}\big)$[1], in which the presence of the multiplying constant is due to reasons of consistency, concerning the overall factor appearing in the final expressions of the two gravitational potentials $U(\mathbf{x})$ and $V(\mathbf{x})$.

[1] $\delta(\mathbf{x})$ is the 3-dimensional Dirac's delta.

Table 5.2: Green functions relating to different conditions placed on the coupling constants.

Cases	**Conditions**	$\mathcal{G}(\mathbf{k})$ **Fourier transform**	$\mathcal{G}\left(\mathbf{x},\mathbf{x}'\right)$ **Green function**
Case 1	$b_1 > 0$ $\left(3a_2 + b_1\right) < 0$	$\frac{Y_I^{-1}}{3}\left[\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2\lambda_2^2}\frac{1}{k^2} + \frac{1}{\lambda_1^2}\frac{1}{\left(k^2+\lambda_1^2\right)} - \frac{1}{\lambda_2^2}\frac{1}{\left(k^2+\lambda_2^2\right)}\right]$	$Y_I^{-1}\sqrt{\frac{\pi}{18}}\frac{1}{\lvert\mathbf{x}-\mathbf{x}'\rvert}\left[\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2\lambda_2^2} + \frac{e^{-\lambda_1\lvert\mathbf{x}-\mathbf{x}'\rvert}}{\lambda_1^2} - \frac{e^{-\lambda_2\lvert\mathbf{x}-\mathbf{x}'\rvert}}{\lambda_2^2}\right]$
Case 2	$b_1 < 0$ $\left(3a_2 + b_1\right) > 0$	$\frac{Y_I^{-1}}{3}\left[\frac{\lambda_2^2-\lambda_1^2}{\lambda_1^2\lambda_2^2}\frac{1}{k^2} - \frac{1}{\lambda_1^2}\frac{1}{\left(k^2-\lambda_1^2\right)} + \frac{1}{\lambda_2^2}\frac{1}{\left(k^2-\lambda_2^2\right)}\right]$	$Y_I^{-1}\sqrt{\frac{\pi}{18}}\frac{1}{\lvert\mathbf{x}-\mathbf{x}'\rvert}\left[\frac{\lambda_2^2-\lambda_1^2}{\lambda_1^2\lambda_2^2} - \frac{\cos\left(\lambda_1\lvert\mathbf{x}-\mathbf{x}'\rvert\right)}{\lambda_1^2} + \frac{\cos\left(\lambda_2\lvert\mathbf{x}-\mathbf{x}'\rvert\right)}{\lambda_2^2}\right]$
Case 3	$b_1 < 0$ $\left(3a_2 + b_1\right) < 0$	$\frac{Y_I^{-1}}{3}\left[\frac{\lambda_1^2+\lambda_2^2}{\lambda_1^2\lambda_2^2}\frac{1}{k^2} - \frac{1}{\lambda_1^2}\frac{1}{\left(k^2-\lambda_1^2\right)} - \frac{1}{\lambda_2^2}\frac{1}{\left(k^2+\lambda_2^2\right)}\right]$	$Y_I^{-1}\sqrt{\frac{\pi}{18}}\frac{1}{\lvert\mathbf{x}-\mathbf{x}'\rvert}\left[\frac{\lambda_1^2+\lambda_2^2}{\lambda_1^2\lambda_2^2} - \frac{\cos\left(\lambda_1\lvert\mathbf{x}-\mathbf{x}'\rvert\right)}{\lambda_1^2} - \frac{e^{-\lambda_2\lvert\mathbf{x}-\mathbf{x}'\rvert}}{\lambda_2^2}\right]$

Results relating to Case 1

I rewrite, here, for convenience, the corresponding Green's function,

$$\mathcal{G}\big(\mathbf{x},\mathbf{x}'\big)=Y_I^{-1}\sqrt{\frac{\pi}{18}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\left[\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2\lambda_2^2}+\frac{e^{-\lambda_1|\mathbf{x}-\mathbf{x}'|}}{\lambda_1^2}-\frac{e^{-\lambda_2|\mathbf{x}-\mathbf{x}'|}}{\lambda_2^2}\right].$$

In this regard, I remind that the conditions on the coupling constants,

$$\begin{aligned}&a_1>0\,,\\&b_1>0\,,\\&\big(3a_2+b_1\big)<0\,,\end{aligned}$$

force to define the two characteristic lengths as follows,

$$\begin{aligned}\lambda_1&=\sqrt{\frac{a_1}{b_1}}\,,\\\lambda_2&=\sqrt{-\frac{a_1}{2\big(3a_2+b_1\big)}}\,,\end{aligned}$$

in order to be real and positive. Therefore, regarding the general expression (5.21a), placing $\Phi(\mathbf{x})=0$, it follows,

$$\begin{aligned}U\big(\mathbf{x}\big)&=-\frac{8\pi}{(2\pi)^{\frac{3}{2}}}G_NY_I\left[\frac{\big(8a_2+3b_1\big)\Delta_{\mathbf{x}}+a_1}{2a_1\big(2a_2+b_1\big)}\right]\int Y_I^{-1}\sqrt{\frac{\pi}{18}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\left[\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2\lambda_2^2}+\frac{e^{-\lambda_1|\mathbf{x}-\mathbf{x}'|}}{\lambda_1^2}\right.\\&\left.-\frac{e^{-\lambda_2|\mathbf{x}-\mathbf{x}'|}}{\lambda_2^2}\right]M\delta\big(\mathbf{x}'\big)d^3\mathbf{x}'\\&=-\frac{8\pi}{(2\pi)^{\frac{3}{2}}}G_NM\sqrt{\frac{\pi}{18}}\left[\frac{\big(8a_2+3b_1\big)\Delta+a_1}{2a_1\big(2a_2+b_1\big)}\right]\left[\frac{1}{|\mathbf{x}|}\left(\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2\lambda_2^2}+\frac{e^{-\lambda_1|\mathbf{x}|}}{\lambda_1^2}-\frac{e^{-\lambda_2|\mathbf{x}|}}{\lambda_2^2}\right)\right]\\&=-\frac{2}{3}G_NM\left\{\frac{\big(8a_2+3b_1\big)}{2a_1\big(2a_2+b_1\big)}\Delta\left[\frac{1}{|\mathbf{x}|}\left(\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2\lambda_2^2}+\frac{e^{-\lambda_1|\mathbf{x}|}}{\lambda_1^2}-\frac{e^{-\lambda_2|\mathbf{x}|}}{\lambda_2^2}\right)\right]\right.\\&\left.+\frac{1}{2\big(2a_2+b_1\big)}\frac{1}{|\mathbf{x}|}\left(\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2\lambda_2^2}+\frac{e^{-\lambda_1|\mathbf{x}|}}{\lambda_1^2}-\frac{e^{-\lambda_2|\mathbf{x}|}}{\lambda_2^2}\right)\right\},\end{aligned}$$

where the calculation,

$$\Delta\left[\frac{1}{|\mathbf{x}|}\Big(\cdots\Big)\right],$$

has been performed starting from one dimension and then generalizing it to three dimensions, as follows:

- 1-st step

$$\partial_i\left[\frac{1}{|\mathbf{x}|}\Big(\cdots\Big)\right]=-\frac{x^i}{|\mathbf{x}|^3}\Big(\cdots\Big)+\frac{x^i}{|\mathbf{x}|^2}\left(-\frac{e^{-\lambda_1|\mathbf{x}|}}{\lambda_1}+\frac{e^{-\lambda_2|\mathbf{x}|}}{\lambda_2}\right);$$

- 2-nd step

$$\partial_{ii}\left[\frac{1}{|\mathbf{x}|}\Big(\cdots\Big)\right]=-\frac{1}{|\mathbf{x}|^3}\Big(\cdots\Big)+\frac{3x^{i^2}}{|\mathbf{x}|^5}\Big(\cdots\Big)-\frac{3x^{i^2}}{|\mathbf{x}|^4}\Big(-\frac{e^{-\lambda_1|\mathbf{x}|}}{\lambda_1}+\frac{e^{-\lambda_2|\mathbf{x}|}}{\lambda_2}\Big)+\frac{1}{|\mathbf{x}|^2}\Big(-\frac{e^{-\lambda_1|\mathbf{x}|}}{\lambda_1}+\frac{e^{-\lambda_2|\mathbf{x}|}}{\lambda_2}\Big)+\frac{x^{i^2}}{|\mathbf{x}|^3}\Big(e^{-\lambda_1|\mathbf{x}|}-e^{-\lambda_2|\mathbf{x}|}\Big);$$

- 3-rd step

$$\begin{aligned}\Delta\left[\frac{1}{|\mathbf{x}|}\Big(\cdots\Big)\right]&=-\frac{3}{|\mathbf{x}|^3}\Big(\cdots\Big)+\frac{3\big(x^{1^2}+x^{2^2}+x^{3^2}\big)}{|\mathbf{x}|^5}\Big(\cdots\Big)-\frac{3\big(x^{1^2}+x^{2^2}+x^{3^2}\big)}{|\mathbf{x}|^4}\Big(-\frac{e^{-\lambda_1|\mathbf{x}|}}{\lambda_1}\\&+\frac{e^{-\lambda_2|\mathbf{x}|}}{\lambda_2}\Big)+\frac{3}{|\mathbf{x}|^2}\Big(-\frac{e^{-\lambda_1|\mathbf{x}|}}{\lambda_1}+\frac{e^{-\lambda_2|\mathbf{x}|}}{\lambda_2}\Big)+\frac{\big(x^{1^2}+x^{2^2}+x^{3^2}\big)}{|\mathbf{x}|^3}\Big(e^{-\lambda_1|\mathbf{x}|}\\&-e^{-\lambda_2|\mathbf{x}|}\Big)\\&=\frac{1}{|\mathbf{x}|}\Big(e^{-\lambda_1|\mathbf{x}|}-e^{-\lambda_2|\mathbf{x}|}\Big).\end{aligned}$$

Thus, with some algebra,

$$\begin{aligned}U\big(\mathbf{x}\big)&=-\frac{2}{3}G_NM\left[\frac{\big(8a_2+3b_1\big)}{2a_1\big(2a_2+b_1\big)}\frac{1}{|\mathbf{x}|}\Big(e^{-\lambda_1|\mathbf{x}|}-e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{1}{2\big(2a_2+b_1\big)}\frac{1}{|\mathbf{x}|}\Big(\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2\lambda_2^2}\right.\\&\left.+\frac{e^{-\lambda_1|\mathbf{x}|}}{\lambda_1^2}-\frac{e^{-\lambda_2|\mathbf{x}|}}{\lambda_2^2}\Big)\right]\\&=-\frac{2}{3}G_NM\frac{1}{|\mathbf{x}|}\left\{\frac{1}{2\big(2a_2+b_1\big)}\left[-\frac{3\big(2a_2+b_1\big)}{a_1}\right]+\left[\frac{\big(8a_2+3b_1\big)}{2a_1\big(2a_2+b_1\big)}+\frac{b_1}{2a_1\big(2a_2+b_1\big)}\right]e^{-\lambda_1|\mathbf{x}|}\right.\\&\left.-\left[\frac{\big(8a_2+3b_1\big)}{2a_1\big(2a_2+b_1\big)}-\frac{\big(3a_2+b_1\big)}{a_1\big(2a_2+b_1\big)}\right]e^{-\lambda_2|\mathbf{x}|}\right\}\\&=-\frac{2}{3}\frac{G_NM}{a_1}\frac{1}{|\mathbf{x}|}\Big(-\frac{3}{2}+2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{2}\,e^{-\lambda_2|\mathbf{x}|}\Big),\end{aligned}$$

namely,

$$U\big(\mathbf{x}\big)=\frac{G_NM}{a_1}\frac{1}{|\mathbf{x}|}\Big(1-\frac{4}{3}e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}e^{-\lambda_2|\mathbf{x}|}\Big).\tag{5.25}$$

In the same way, concerning the general expression (5.21b), with the same condition on $\Phi(\mathbf{x})$, it follows,

$$\begin{aligned}V\big(\mathbf{x}\big)&=-\frac{8\pi}{(2\pi)^{\frac{3}{2}}}G_NY_I\left[\frac{\big(4a_2+b_1\big)\Delta_{\mathbf{x}}+a_1}{2a_1\big(2a_2+b_1\big)}\right]\int Y_I^{-1}\sqrt{\frac{\pi}{18}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\left[\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2\lambda_2^2}+\frac{e^{-\lambda_1|\mathbf{x}-\mathbf{x}'|}}{\lambda_1^2}\right.\\&\left.-\frac{e^{-\lambda_2|\mathbf{x}-\mathbf{x}'|}}{\lambda_2^2}\right]M\delta\big(\mathbf{x}'\big)d^3\mathbf{x}'\end{aligned}$$

$$
\begin{aligned}
&= -\frac{8\pi}{(2\pi)^{\frac{3}{2}}} G_N M \sqrt{\frac{\pi}{18}} \left[\frac{(4a_2 + b_1)\Delta + a_1}{2a_1(2a_2 + b_1)} \right] \left[\frac{1}{|\mathbf{x}|} \left(\frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 \lambda_2^2} + \frac{e^{-\lambda_1 |\mathbf{x}|}}{\lambda_1^2} - \frac{e^{-\lambda_2 |\mathbf{x}|}}{\lambda_2^2} \right) \right] \\
&= -\frac{2}{3} G_N M \left\{ \frac{(4a_2 + b_1)}{2a_1(2a_2 + b_1)} \Delta \left[\frac{1}{|\mathbf{x}|} \left(\frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 \lambda_2^2} + \frac{e^{-\lambda_1 |\mathbf{x}|}}{\lambda_1^2} - \frac{e^{-\lambda_2 |\mathbf{x}|}}{\lambda_2^2} \right) \right] \right. \\
&\left. + \frac{1}{2(2a_2 + b_1)} \frac{1}{|\mathbf{x}|} \left(\frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 \lambda_2^2} + \frac{e^{-\lambda_1 |\mathbf{x}|}}{\lambda_1^2} - \frac{e^{-\lambda_2 |\mathbf{x}|}}{\lambda_2^2} \right) \right\} \\
&= -\frac{2}{3} G_N M \left[\frac{(4a_2 + b_1)}{2a_1(2a_2 + b_1)} \frac{1}{|\mathbf{x}|} \left(e^{-\lambda_1 |\mathbf{x}|} - e^{-\lambda_2 |\mathbf{x}|} \right) + \frac{1}{2(2a_2 + b_1)} \frac{1}{|\mathbf{x}|} \left(\frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 \lambda_2^2} \right. \right. \\
&\left. \left. + \frac{e^{-\lambda_1 |\mathbf{x}|}}{\lambda_1^2} - \frac{e^{-\lambda_2 |\mathbf{x}|}}{\lambda_2^2} \right) \right] \\
&= -\frac{2}{3} G_N M \frac{1}{|\mathbf{x}|} \left\{ \frac{1}{2(2a_2 + b_1)} \left[-\frac{3(2a_2 + b_1)}{a_1} \right] + \left[\frac{(4a_2 + b_1)}{2a_1(2a_2 + b_1)} + \frac{b_1}{2a_1(2a_2 + b_1)} \right] e^{-\lambda_1 |\mathbf{x}|} \right. \\
&\left. - \left[\frac{(4a_2 + b_1)}{2a_1(2a_2 + b_1)} - \frac{(3a_2 + b_1)}{a_1(2a_2 + b_1)} \right] e^{-\lambda_2 |\mathbf{x}|} \right\} \\
&= -\frac{2}{3} \frac{G_N M}{a_1} \frac{1}{|\mathbf{x}|} \left(-\frac{3}{2} + e^{-\lambda_1 |\mathbf{x}|} + \frac{1}{2} e^{-\lambda_2 |\mathbf{x}|} \right),
\end{aligned}
$$

that is,

$$
V(\mathbf{x}) = \frac{G_N M}{a_1} \frac{1}{|\mathbf{x}|} \left(1 - \frac{2}{3} e^{-\lambda_1 |\mathbf{x}|} - \frac{1}{3} e^{-\lambda_2 |\mathbf{x}|} \right). \tag{5.26}
$$

Results relating to Case 2

Starting from the corresponding Green's function,

$$
\mathcal{G}(\mathbf{x}, \mathbf{x}') = Y_I^{-1} \sqrt{\frac{\pi}{18}} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \left[\frac{\lambda_2^2 - \lambda_1^2}{\lambda_1^2 \lambda_2^2} - \frac{\cos(\lambda_1 |\mathbf{x} - \mathbf{x}'|)}{\lambda_1^2} + \frac{\cos(\lambda_2 |\mathbf{x} - \mathbf{x}'|)}{\lambda_2^2} \right],
$$

related to the following conditions,

$$
\begin{aligned}
&a_1 > 0\,, \\
&b_1 < 0\,, \\
&(3a_2 + b_1) > 0\,,
\end{aligned}
$$

for which the lengths,

$$
\begin{aligned}
\lambda_1 &= \sqrt{-\frac{a_1}{b_1}}\,, \\
\lambda_2 &= \sqrt{\frac{a_1}{2(3a_2 + b_1)}}\,,
\end{aligned}
$$

are real and positive again, it follows:

$$
U(\mathbf{x}) = -\frac{8\pi}{(2\pi)^{\frac{3}{2}}} G_N Y_I \left[\frac{(8a_2 + 3b_1)\Delta_{\mathbf{x}} + a_1}{2a_1(2a_2 + b_1)} \right] \int Y_I^{-1} \sqrt{\frac{\pi}{18}} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \left[\frac{\lambda_2^2 - \lambda_1^2}{\lambda_1^2 \lambda_2^2} - \frac{\cos(\lambda_1 |\mathbf{x} - \mathbf{x}'|)}{\lambda_1^2} \right.
$$

$$
\begin{aligned}
&+\frac{\cos\big(\lambda_2|\mathbf{x}-\mathbf{x}'|\big)}{\lambda_2^2}\Big]M\delta\big(\mathbf{x}'\big)d^3\mathbf{x}'\\
&=-\frac{8\pi}{(2\pi)^{\frac{3}{2}}}G_NM\sqrt{\frac{\pi}{18}}\Big[\frac{\big(8a_2+3b_1\big)\Delta+a_1}{2a_1\big(2a_2+b_1\big)}\Big]\Bigg\{\frac{1}{|\mathbf{x}|}\Big[\frac{\lambda_2^2-\lambda_1^2}{\lambda_1^2\lambda_2^2}-\frac{\cos\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1^2}+\frac{\cos\big(\lambda_2|\mathbf{x}|\big)}{\lambda_2^2}\Big]\Bigg\}\\
&=-\frac{2}{3}G_NM\Bigg\{\frac{\big(8a_2+3b_1\big)}{2a_1\big(2a_2+b_1\big)}\Delta\Big[\frac{1}{|\mathbf{x}|}\Big(\frac{\lambda_2^2-\lambda_1^2}{\lambda_1^2\lambda_2^2}-\frac{\cos\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1^2}+\frac{\cos\big(\lambda_2|\mathbf{x}|\big)}{\lambda_2^2}\Big)\Big]\\
&+\frac{1}{2\big(2a_2+b_1\big)}\frac{1}{|\mathbf{x}|}\Big[\frac{\lambda_2^2-\lambda_1^2}{\lambda_1^2\lambda_2^2}-\frac{\cos\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1^2}+\frac{\cos\big(\lambda_2|\mathbf{x}|\big)}{\lambda_2^2}\Big]\Bigg\};
\end{aligned}
$$

for what concerns,

$$
\Delta\Big[\frac{1}{|\mathbf{x}|}\Big(\cdots\Big)\Big],
$$

it results that,

- 1-st step

$$
\partial_i\Big[\frac{1}{|\mathbf{x}|}\Big(\cdots\Big)\Big]=-\frac{x^i}{|\mathbf{x}|^3}\Big(\cdots\Big)+\frac{x^i}{|\mathbf{x}|^2}\Big[\frac{\sin\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1}-\frac{\sin\big(\lambda_2|\mathbf{x}|\big)}{\lambda_2}\Big];
$$

- 2-nd step

$$
\begin{aligned}
\partial_{ii}\Big[\frac{1}{|\mathbf{x}|}\Big(\cdots\Big)\Big]&=-\frac{1}{|\mathbf{x}|^3}\Big(\cdots\Big)+\frac{3{x^i}^2}{|\mathbf{x}|^5}\Big(\cdots\Big)-\frac{3{x^i}^2}{|\mathbf{x}|^4}\Big[\frac{\sin\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1}-\frac{\sin\big(\lambda_2|\mathbf{x}|\big)}{\lambda_2}\Big]\\
&+\frac{1}{|\mathbf{x}|^2}\Big[\frac{\sin\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1}-\frac{\sin\big(\lambda_2|\mathbf{x}|\big)}{\lambda_2}\Big]+\frac{{x^i}^2}{|\mathbf{x}|^3}\Big[\cos\big(\lambda_1|\mathbf{x}|\big)-\cos\big(\lambda_2|\mathbf{x}|\big)\Big];
\end{aligned}
$$

- 3-rd step

$$
\begin{aligned}
\Delta\Big[\frac{1}{|\mathbf{x}|}\Big(\cdots\Big)\Big]&=-\frac{3}{|\mathbf{x}|^3}\Big(\cdots\Big)+\frac{3\big({x^1}^2+{x^2}^2+{x^3}^2\big)}{|\mathbf{x}|^5}\Big(\cdots\Big)-\frac{3\big({x^1}^2+{x^2}^2+{x^3}^2\big)}{|\mathbf{x}|^4}\Big[\frac{\sin\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1}\\
&-\frac{\sin\big(\lambda_2|\mathbf{x}|\big)}{\lambda_2}\Big]+\frac{3}{|\mathbf{x}|^2}\Big[\frac{\sin\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1}-\frac{\sin\big(\lambda_2|\mathbf{x}|\big)}{\lambda_2}\Big]+\frac{\big({x^1}^2+{x^2}^2+{x^3}^2\big)}{|\mathbf{x}|^3}\Big[\cos\big(\lambda_1|\mathbf{x}|\big)\\
&-\cos\big(\lambda_2|\mathbf{x}|\big)\Big]\\
&=\frac{1}{|\mathbf{x}|}\Big[\cos\big(\lambda_1|\mathbf{x}|\big)-\cos\big(\lambda_2|\mathbf{x}|\big)\Big].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
U\big(\mathbf{x}\big)=-\frac{2}{3}G_NM\Bigg\{&\frac{\big(8a_2+3b_1\big)}{2a_1\big(2a_2+b_1\big)}\frac{1}{|\mathbf{x}|}\Big[\cos\big(\lambda_1|\mathbf{x}|\big)-\cos\big(\lambda_2|\mathbf{x}|\big)\Big]+\frac{1}{2\big(2a_2+b_1\big)}\frac{1}{|\mathbf{x}|}\Big[\frac{\lambda_2^2-\lambda_1^2}{\lambda_1^2\lambda_2^2}\\
&-\frac{\cos\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1^2}+\frac{\cos\big(\lambda_2|\mathbf{x}|\big)}{\lambda_2^2}\Big]\Bigg\}
\end{aligned}
$$

$$= -\frac{2}{3}G_N M\frac{1}{|\mathbf{x}|}\Bigg\{\frac{1}{2\big(2a_2+b_1\big)}\Bigg[\frac{-3\big(2a_2+b_1\big)}{a_1}\Bigg]+\Bigg[\frac{\big(8a_2+3b_1\big)}{2a_1\big(2a_2+b_1\big)}+\frac{b_1}{2a_1\big(2a_2+b_1\big)}\Bigg]\cos\big(\lambda_1|\mathbf{x}|\big)$$
$$+\Bigg[-\frac{\big(8a_2+3b_1\big)}{2a_1\big(2a_2+b_1\big)}+\frac{\big(3a_2+b_1\big)}{a_1\big(2a_2+b_1\big)}\Bigg]\cos\big(\lambda_2|\mathbf{x}|\big)\Bigg\}$$
$$= -\frac{2}{3}\frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\Bigg[-\frac{3}{2}+2\cos\big(\lambda_1|\mathbf{x}|\big)-\frac{1}{2}\cos\big(\lambda_2|\mathbf{x}|\big)\Bigg],$$

that is,

$$U\big(\mathbf{x}\big) = \frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\Bigg[1-\frac{4}{3}\cos\big(\lambda_1|\mathbf{x}|\big)+\frac{1}{3}\cos\big(\lambda_2|\mathbf{x}|\big)\Bigg]. \tag{5.27}$$

In the same way,

$$V\big(\mathbf{x}\big) = -\frac{8\pi}{(2\pi)^{\frac{3}{2}}}G_N Y_I\Bigg[\frac{\big(4a_2+b_1\big)\Delta_{\mathbf{x}}+a_1}{2a_1\big(2a_2+b_1\big)}\Bigg]\int Y_I^{-1}\sqrt{\frac{\pi}{18}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\Bigg[\frac{\lambda_2^2-\lambda_1^2}{\lambda_1^2\lambda_2^2}-\frac{\cos\big(\lambda_1|\mathbf{x}-\mathbf{x}'|\big)}{\lambda_1^2}$$
$$+\frac{\cos\big(\lambda_2|\mathbf{x}-\mathbf{x}'|\big)}{\lambda_2^2}\Bigg]M\delta\big(\mathbf{x}'\big)d^3\mathbf{x}'$$
$$= -\frac{8\pi}{(2\pi)^{\frac{3}{2}}}G_N M\sqrt{\frac{\pi}{18}}\Bigg[\frac{\big(4a_2+b_1\big)\Delta+a_1}{2a_1\big(2a_2+b_1\big)}\Bigg]\Bigg\{\frac{1}{|\mathbf{x}|}\Bigg[\frac{\lambda_2^2-\lambda_1^2}{\lambda_1^2\lambda_2^2}-\frac{\cos\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1^2}+\frac{\cos\big(\lambda_2|\mathbf{x}|\big)}{\lambda_2^2}\Bigg]\Bigg\}$$
$$= -\frac{2}{3}G_N M\Bigg\{\frac{\big(4a_2+b_1\big)}{2a_1\big(2a_2+b_1\big)}\Delta\Bigg[\frac{1}{|\mathbf{x}|}\Bigg(\frac{\lambda_2^2-\lambda_1^2}{\lambda_1^2\lambda_2^2}-\frac{\cos\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1^2}+\frac{\cos\big(\lambda_2|\mathbf{x}|\big)}{\lambda_2^2}\Bigg)\Bigg]$$
$$+\frac{1}{2\big(2a_2+b_1\big)}\frac{1}{|\mathbf{x}|}\Bigg[\frac{\lambda_2^2-\lambda_1^2}{\lambda_1^2\lambda_2^2}-\frac{\cos\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1^2}+\frac{\cos\big(\lambda_2|\mathbf{x}|\big)}{\lambda_2^2}\Bigg]\Bigg\}$$
$$= -\frac{2}{3}G_N M\Bigg\{\frac{\big(4a_2+b_1\big)}{2a_1\big(2a_2+b_1\big)}\frac{1}{|\mathbf{x}|}\Big[\cos\big(\lambda_1|\mathbf{x}|\big)-\cos\big(\lambda_2|\mathbf{x}|\big)\Big]+\frac{1}{2\big(2a_2+b_1\big)}\frac{1}{|\mathbf{x}|}\Bigg[\frac{\lambda_2^2-\lambda_1^2}{\lambda_1^2\lambda_2^2}$$
$$-\frac{\cos\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1^2}+\frac{\cos\big(\lambda_2|\mathbf{x}|\big)}{\lambda_2^2}\Bigg]\Bigg\}$$
$$= -\frac{2}{3}G_N M\frac{1}{|\mathbf{x}|}\Bigg\{\frac{1}{2\big(2a_2+b_1\big)}\Bigg[-\frac{3\big(2a_2+b_1\big)}{a_1}\Bigg]+\Bigg[\frac{\big(4a_2+b_1\big)}{2a_1\big(2a_2+b_1\big)}+\frac{b_1}{2a_1\big(2a_2+b_1\big)}\Bigg]\cos\big(\lambda_1|\mathbf{x}|\big)$$
$$+\Bigg[-\frac{\big(4a_2+b_1\big)}{2a_1\big(2a_2+b_1\big)}+\frac{\big(3a_2+b_1\big)}{a_1\big(2a_2+b_1\big)}\Bigg]\cos\big(\lambda_2|\mathbf{x}|\big)\Bigg\}$$
$$= -\frac{2}{3}\frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\Bigg[-\frac{3}{2}+\cos\big(\lambda_1|\mathbf{x}|\big)+\frac{1}{2}\cos\big(\lambda_2|\mathbf{x}|\big)\Bigg],$$

that is,

$$V\big(\mathbf{x}\big) = \frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\Bigg[1-\frac{2}{3}\cos\big(\lambda_1|\mathbf{x}|\big)-\frac{1}{3}\cos\big(\lambda_2|\mathbf{x}|\big)\Bigg]. \tag{5.28}$$

Results relating to Case 3

Finally, from the function,

$$\mathcal{G}(\mathbf{x},\,\mathbf{x}') = Y_I^{-1}\sqrt{\frac{\pi}{18}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\Bigg[\frac{\lambda_1^2+\lambda_2^2}{\lambda_1^2\lambda_2^2}-\frac{\cos\big(\lambda_1|\mathbf{x}-\mathbf{x}'|\big)}{\lambda_1^2}-\frac{e^{-\lambda_2|\mathbf{x}-\mathbf{x}'|}}{\lambda_2^2}\Bigg]$$

with,

$$
\begin{aligned}
&a_1 > 0\,,\\
&b_1 < 0\,,\\
&(3a_2 + b_1) < 0\,,
\end{aligned}
$$

for which the lengths,

$$
\begin{aligned}
\lambda_1 &= \sqrt{-\frac{a_1}{b_1}}\,,\\
\lambda_2 &= \sqrt{-\frac{a_1}{2\big(3a_2 + b_1\big)}}\,,
\end{aligned}
$$

turn out to be real and positive, it follows:

$$
\begin{aligned}
U\big(\mathbf{x}\big) &= -\frac{8\pi}{(2\pi)^{\frac{3}{2}}} G_N Y_I \bigg[\frac{\big(8a_2 + 3b_1\big)\Delta_{\mathbf{x}} + a_1}{2a_1\big(2a_2 + b_1\big)}\bigg] \int Y_I^{-1} \sqrt{\frac{\pi}{18}} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \bigg[\frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 \lambda_2^2} - \frac{\cos\big(\lambda_1 |\mathbf{x} - \mathbf{x}'|\big)}{\lambda_1^2}\\
&- \frac{e^{-\lambda_2 |\mathbf{x} - \mathbf{x}'|}}{\lambda_2^2}\bigg] M\delta\big(\mathbf{x}'\big) d^3\mathbf{x}'\\
&= -\frac{8\pi}{(2\pi)^{\frac{3}{2}}} G_N M \sqrt{\frac{\pi}{18}} \bigg[\frac{\big(8a_2 + 3b_1\big)\Delta + a_1}{2a_1\big(2a_2 + b_1\big)}\bigg] \bigg\{\frac{1}{|\mathbf{x}|}\bigg[\frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 \lambda_2^2} - \frac{\cos(\lambda_1 |\mathbf{x}|)}{\lambda_1^2} - \frac{e^{-\lambda_2 |\mathbf{x}|}}{\lambda_2^2}\bigg]\bigg\}\\
&= -\frac{2}{3} G_N M \bigg\{\frac{\big(8a_2 + 3b_1\big)}{2a_1\big(2a_2 + b_1\big)} \Delta\bigg[\frac{1}{|\mathbf{x}|}\bigg(\frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 \lambda_2^2} - \frac{\cos(\lambda_1 |\mathbf{x}|)}{\lambda_1^2} - \frac{e^{-\lambda_2 |\mathbf{x}|}}{\lambda_2^2}\bigg)\bigg]\\
&+ \frac{1}{2\big(2a_2 + b_1\big)} \frac{1}{|\mathbf{x}|}\bigg[\frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 \lambda_2^2} - \frac{\cos(\lambda_1 |\mathbf{x}|)}{\lambda_1^2} - \frac{e^{-\lambda_2 |\mathbf{x}|}}{\lambda_2^2}\bigg]\bigg\}\,;
\end{aligned}
$$

for what concerns,

$$
\Delta\bigg[\frac{1}{|\mathbf{x}|}\Big(\cdots\Big)\bigg]\,,
$$

it results that,

- 1-st step

$$
\partial_i\bigg[\frac{1}{|\mathbf{x}|}\Big(\cdots\Big)\bigg] = -\frac{x^i}{|\mathbf{x}|^3}\Big(\cdots\Big) + \frac{x^i}{|\mathbf{x}|^2}\bigg[\frac{\sin\big(\lambda_1 |\mathbf{x}|\big)}{\lambda_1} + \frac{e^{-\lambda_2 |\mathbf{x}|}}{\lambda_2}\bigg]\,;
$$

- 2-nd step

$$
\begin{aligned}
\partial_{ii}\bigg[\frac{1}{|\mathbf{x}|}\Big(\cdots\Big)\bigg] &= -\frac{1}{|\mathbf{x}|^3}\Big(\cdots\Big) + \frac{3{x^i}^2}{|\mathbf{x}|^5}\Big(\cdots\Big) - \frac{3{x^i}^2}{|\mathbf{x}|^4}\bigg[\frac{\sin\big(\lambda_1 |\mathbf{x}|\big)}{\lambda_1} + \frac{e^{-\lambda_2 |\mathbf{x}|}}{\lambda_2}\bigg]\\
&+ \frac{1}{|\mathbf{x}|^2}\bigg[\frac{\sin\big(\lambda_1 |\mathbf{x}|\big)}{\lambda_1} + \frac{e^{-\lambda_2 |\mathbf{x}|}}{\lambda_2}\bigg] + \frac{{x^i}^2}{|\mathbf{x}|^3}\Big[\cos\big(\lambda_1 |\mathbf{x}|\big) - e^{-\lambda_2 |\mathbf{x}|}\Big]\,;
\end{aligned}
$$

- 3-rd step

$$\begin{aligned}\Delta\left[\frac{1}{|\mathbf{x}|}\Big(\cdots\Big)\right] &= -\frac{3}{|\mathbf{x}|^3}\Big(\cdots\Big) + \frac{3\big(x^{1^2}+x^{2^2}+x^{3^2}\big)}{|\mathbf{x}|^5}\Big(\cdots\Big) - \frac{3\big(x^{1^2}+x^{2^2}+x^{3^2}\big)}{|\mathbf{x}|^4}\left[\frac{\sin\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1}\right.\\ &\quad\left.+\frac{e^{-\lambda_2|\mathbf{x}|}}{\lambda_2}\right] + \frac{3}{|\mathbf{x}|^2}\left[\frac{\sin\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1}+\frac{e^{-\lambda_2|\mathbf{x}|}}{\lambda_2}\right] + \frac{\big(x^{1^2}+x^{2^2}+x^{3^2}\big)}{|\mathbf{x}|^3}\Big[\cos\big(\lambda_1|\mathbf{x}|\big)\\ &\quad - e^{-\lambda_2|\mathbf{x}|}\Big]\\ &= \frac{1}{|\mathbf{x}|}\Big[\cos\big(\lambda_1|\mathbf{x}|\big) - e^{-\lambda_2|\mathbf{x}|}\Big].\end{aligned}$$

Thus,

$$\begin{aligned}U\big(\mathbf{x}\big) &= -\frac{2}{3}G_NM\left\{\frac{\big(8a_2+3b_1\big)}{2a_1\big(2a_2+b_1\big)}\frac{1}{|\mathbf{x}|}\Big[\cos\big(\lambda_1|\mathbf{x}|\big)-e^{-\lambda_2|\mathbf{x}|}\Big] + \frac{1}{2\big(2a_2+b_1\big)}\frac{1}{|\mathbf{x}|}\left[\frac{\lambda_1^2+\lambda_2^2}{\lambda_1^2\lambda_2^2}\right.\right.\\ &\quad\left.\left.-\frac{\cos\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1^2}-\frac{e^{-\lambda_2|\mathbf{x}|}}{\lambda_2^2}\right]\right\}\\ &= -\frac{2}{3}G_NM\frac{1}{|\mathbf{x}|}\left\{\frac{1}{2\big(2a_2+b_1\big)}\left[\frac{-3\big(2a_2+b_1\big)}{a_1}\right]+\left[\frac{\big(8a_2+3b_1\big)}{2a_1\big(2a_2+b_1\big)}+\frac{b_1}{2a_1\big(2a_2+b_1\big)}\right]\cos\big(\lambda_1|\mathbf{x}|\big)\right.\\ &\quad\left.-\left[\frac{\big(8a_2+3b_1\big)}{2a_1\big(2a_2+b_1\big)}-\frac{\big(3a_2+b_1\big)}{a_1\big(2a_2+b_1\big)}\right]e^{-\lambda_2|\mathbf{x}|}\right\}\\ &= -\frac{2}{3}\frac{G_NM}{a_1}\frac{1}{|\mathbf{x}|}\left[-\frac{3}{2}+2\cos\big(\lambda_1|\mathbf{x}|\big)-\frac{1}{2}e^{-\lambda_2|\mathbf{x}|}\right],\end{aligned}$$

that is,

$$U\big(\mathbf{x}\big) = \frac{G_NM}{a_1}\frac{1}{|\mathbf{x}|}\left[1-\frac{4}{3}\cos\big(\lambda_1|\mathbf{x}|\big)+\frac{1}{3}e^{-\lambda_2|\mathbf{x}|}\right]. \tag{5.29}$$

Similarly,

$$\begin{aligned}V\big(\mathbf{x}\big) &= -\frac{8\pi}{(2\pi)^{\frac{3}{2}}}G_NY_I\left[\frac{\big(4a_2+b_1\big)\Delta_{\mathbf{x}}+a_1}{2a_1\big(2a_2+b_1\big)}\right]\int Y_I^{-1}\sqrt{\frac{\pi}{18}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\left[\frac{\lambda_1^2+\lambda_2^2}{\lambda_1^2\lambda_2^2}-\frac{\cos\big(\lambda_1|\mathbf{x}-\mathbf{x}'|\big)}{\lambda_1^2}\right.\\ &\quad\left.-\frac{e^{-\lambda_2|\mathbf{x}-\mathbf{x}'|}}{\lambda_2^2}\right]M\delta\big(\mathbf{x}'\big)d^3\mathbf{x}'\\ &= -\frac{8\pi}{(2\pi)^{\frac{3}{2}}}G_NM\sqrt{\frac{\pi}{18}}\left[\frac{\big(4a_2+b_1\big)\Delta+a_1}{2a_1\big(2a_2+b_1\big)}\right]\left\{\frac{1}{|\mathbf{x}|}\left[\frac{\lambda_1^2+\lambda_2^2}{\lambda_1^2\lambda_2^2}-\frac{\cos\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1^2}-\frac{e^{-\lambda_2|\mathbf{x}|}}{\lambda_2^2}\right]\right\}\\ &= -\frac{2}{3}G_NM\left\{\frac{\big(4a_2+b_1\big)}{2a_1\big(2a_2+b_1\big)}\Delta\left[\frac{1}{|\mathbf{x}|}\left(\frac{\lambda_1^2+\lambda_2^2}{\lambda_1^2\lambda_2^2}-\frac{\cos\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1^2}-\frac{e^{-\lambda_2|\mathbf{x}|}}{\lambda_2^2}\right)\right]\right.\\ &\quad\left.+\frac{1}{2\big(2a_2+b_1\big)}\frac{1}{|\mathbf{x}|}\left[\frac{\lambda_1^2+\lambda_2^2}{\lambda_1^2\lambda_2^2}-\frac{\cos\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1^2}-\frac{e^{-\lambda_2|\mathbf{x}|}}{\lambda_2^2}\right]\right\}\\ &= -\frac{2}{3}G_NM\left\{\frac{\big(4a_2+b_1\big)}{2a_1\big(2a_2+b_1\big)}\frac{1}{|\mathbf{x}|}\Big[\cos\big(\lambda_1|\mathbf{x}|\big)-e^{-\lambda_2|\mathbf{x}|}\Big] + \frac{1}{2\big(2a_2+b_1\big)}\frac{1}{|\mathbf{x}|}\left[\frac{\lambda_1^2+\lambda_2^2}{\lambda_1^2\lambda_2^2}\right.\right.\end{aligned}$$

$$
\begin{aligned}
&- \frac{\cos\big(\lambda_1|\mathbf{x}|\big)}{\lambda_1^2} - \frac{e^{-\lambda_2|\mathbf{x}|}}{\lambda_2^2}\Bigg]\Bigg\} \\
&= -\frac{2}{3}G_N M\frac{1}{|\mathbf{x}|}\Bigg\{\frac{1}{2\big(2a_2+b_1\big)}\Bigg[-\frac{3\big(2a_2+b_1\big)}{a_1}\Bigg]+\Bigg[\frac{\big(4a_2+b_1\big)}{2a_1\big(2a_2+b_1\big)}+\frac{b_1}{2a_1\big(2a_2+b_1\big)}\Bigg]\cos\big(\lambda_1|\mathbf{x}|\big) \\
&- \Bigg[\frac{\big(4a_2+b_1\big)}{2a_1\big(2a_2+b_1\big)}-\frac{\big(3a_2+b_1\big)}{a_1\big(2a_2+b_1\big)}\Bigg]e^{-\lambda_2|\mathbf{x}|}\Bigg\} \\
&= -\frac{2}{3}\frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\Bigg[-\frac{3}{2}+\cos\big(\lambda_1|\mathbf{x}|\big)+\frac{1}{2}e^{-\lambda_2|\mathbf{x}|}\Bigg],
\end{aligned}
$$

that is,

$$
V\big(\mathbf{x}\big) = \frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\Bigg[1-\frac{2}{3}\cos\big(\lambda_1|\mathbf{x}|\big)-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Bigg]. \tag{5.30}
$$

The potentials obtained in (5.25) and (5.26) are evidently identical to those ones derived in [74], describing a Newtonian potential modified by virtue of the presence of corrective terms of Yukawa type, the coefficients of which, in particular their signs, indicate how much the gravitational interaction is strengthened or weakened. Moreover, the characteristic lengths λ_1 and λ_2, which appear in them, although defined in a different way than in [74], still give the distance scales at which these corrections are non negligible.

Similar considerations also apply to the potentials (5.27),(5.28) and (5.29), (5.30), relating to the other two Green's functions of table 5.2 on page 60, although in the last two potentials (5.29) and (5.30), the functions corresponding to the corrective terms are inverted with respect to those ones acquired in [74], in consequence of a different definition of the *characteristic lengths* there appearing, as I have already outlined previously. The gravitational potentials thus obtained are resumed in table 5.3 on the next page.

However, it has to be pointed out that there is a quite difference between the two approaches; in fact, in this work, the left-hand sides of the field equations (5.2) have been developed in the weak-field limit, *a priori*, considering the space-time roughly flat, within the Post-Newtonian approximation 1PN, which provides the perturbative expansion of the metric tensor up to the order $O(\epsilon^4)$; in the cited paper, on the other hand, the aforementioned field equations, *in primis*, have been developed in the curved space-time and subsequently evaluated in the weak-field regime, contextually to the lower Post-Newtonian approximation 0.5PN, which gives the perturbative expansion of the metric tensor only up to the order $O(\epsilon^2)$. This has involved an inversion of the sign of ${}^1H_{\mu\nu}$ and ${}^2H_{\mu\nu}$ corresponding to the quadratic terms R^2 and $R_{\mu\nu}R^{\mu\nu}$ of the Lagrangian (5.1), at the order $O(\epsilon^2)$, while that one corresponding to the Hilbert-Einstein term results to be unchanged.

Nevertheless, the results achieved by using the aforementioned procedure are the same as those shown in [74], although the scale-lengths are defined with opposite signs. In other words, one can conclude that there exists an invariance of the gravitational potentials under sign-reversal of the second and third term of the general field equations (5.10).

5.2.6 Gravitational potentials expressed through Green's function

It is useful, to conclude the current discussion, to investigate further the study of the field equations given in table (5.1) on page 57, in order to understand which are the cases able to give back acceptable solutions, in particular which provide for a behavior of Newtonian type.

The results of this analysis are shown in table 5.4 on page 72; it should be noted that the potentials are expressed in the form as general as possible, that is through the corresponding Green's functions, for a generic matter density $\rho(\mathbf{x})$. What emerges from the computations, is that

Table 5.3: Gravitational potentials as the Green functions vary.

Cases	**Conditions**	$\mathcal{G}(\mathbf{x},\mathbf{x}')$ **Green function**	**Gravitational potentials**
Case 1	$b_1 > 0$ $(3a_2 + b_1) < 0$	$Y_I^{-1}\sqrt{\frac{\pi}{18}}\frac{1}{\lvert\mathbf{x}-\mathbf{x}'\rvert}\left[\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2\lambda_2^2} + \frac{e^{-\lambda_1\lvert\mathbf{x}-\mathbf{x}'\rvert}}{\lambda_1^2} - \frac{e^{-\lambda_2\lvert\mathbf{x}-\mathbf{x}'\rvert}}{\lambda_2^2}\right]$	$U(\mathbf{x}) = \frac{G_N M}{a_1}\frac{1}{\lvert\mathbf{x}\rvert}\left(1 - \frac{4}{3}e^{-\lambda_1\lvert\mathbf{x}\rvert} + \frac{1}{3}e^{-\lambda_2\lvert\mathbf{x}\rvert}\right)$ $V(\mathbf{x}) = \frac{G_N M}{a_1}\frac{1}{\lvert\mathbf{x}\rvert}\left(1 - \frac{2}{3}\,e^{-\lambda_1\lvert\mathbf{x}\rvert} - \frac{1}{3}\,e^{-\lambda_2\lvert\mathbf{x}\rvert}\right)$
Case 2	$b_1 < 0$ $(3a_2 + b_1) > 0$	$Y_I^{-1}\sqrt{\frac{\pi}{18}}\frac{1}{\lvert\mathbf{x}-\mathbf{x}'\rvert}\left[\frac{\lambda_2^2-\lambda_1^2}{\lambda_1^2\lambda_2^2} - \frac{\cos\left(\lambda_1\lvert\mathbf{x}-\mathbf{x}'\rvert\right)}{\lambda_1^2} + \frac{\cos\left(\lambda_2\lvert\mathbf{x}-\mathbf{x}'\rvert\right)}{\lambda_2^2}\right]$	$U(\mathbf{x}) = \frac{G_N M}{a_1}\frac{1}{\lvert\mathbf{x}\rvert}\left[1 - \frac{4}{3}\cos\left(\lambda_1\lvert\mathbf{x}\rvert\right) + \frac{1}{3}\cos\left(\lambda_2\lvert\mathbf{x}\rvert\right)\right]$ $V(\mathbf{x}) = \frac{G_N M}{a_1}\frac{1}{\lvert\mathbf{x}\rvert}\left[1 - \frac{2}{3}\cos\left(\lambda_1\lvert\mathbf{x}\rvert\right) - \frac{1}{3}\cos\left(\lambda_2\lvert\mathbf{x}\rvert\right)\right]$
Case 3	$b_1 < 0$ $(3a_2 + b_1) < 0$	$Y_I^{-1}\sqrt{\frac{\pi}{18}}\frac{1}{\lvert\mathbf{x}-\mathbf{x}'\rvert}\left[\frac{\lambda_1^2+\lambda_2^2}{\lambda_1^2\lambda_2^2} - \frac{\cos\left(\lambda_1\lvert\mathbf{x}-\mathbf{x}'\rvert\right)}{\lambda_1^2} - \frac{e^{-\lambda_2\lvert\mathbf{x}-\mathbf{x}'\rvert}}{\lambda_2^2}\right]$	$U(\mathbf{x}) = \frac{G_N M}{a_1}\frac{1}{\lvert\mathbf{x}\rvert}\left[1 - \frac{4}{3}\cos\left(\lambda_1\lvert\mathbf{x}\rvert\right) + \frac{1}{3}e^{-\lambda_2\lvert\mathbf{x}\rvert}\right]$ $V(\mathbf{x}) = \frac{G_N M}{a_1}\frac{1}{\lvert\mathbf{x}\rvert}\left[1 - \frac{2}{3}\cos\left(\lambda_1\lvert\mathbf{x}\rvert\right) - \frac{1}{3}\,e^{-\lambda_2\lvert\mathbf{x}\rvert}\right]$

only from the case 1 and the case 4 it is possible to extract physically acceptable solutions, since all the others, having divergent Green's functions, result to be unsolvable.

I am going to show, briefly, the results concerning the aforementioned cases:

- Case 1

$$\begin{aligned} a_2 &= 0\,, \\ b_1 &= 0\,, \\ \mathcal{L} &= a_1 R\,, \end{aligned}$$

for which,

$$U(\mathbf{x}) = V(\mathbf{x}) = -\frac{4\pi G_N}{a_1}\int \mathcal{G}_{(\Delta)}(\mathbf{x},\,\mathbf{x}')\rho(\mathbf{x}')d^3\mathbf{x}'\,.$$

The differential equation related to $\mathcal{G}_{(\Delta)}(\mathbf{x},\,\mathbf{x}')$, namely,

$$\Delta\,\mathcal{G}(\mathbf{x},\,\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')\,,$$

where,

$$\delta(\mathbf{x}-\mathbf{x}') = \frac{1}{(2\pi)^{\frac{3}{2}}}\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}d^3\mathbf{k}\,,$$

through (5.22), gives back the following Fourier's transform,

$$\mathcal{G}_{(\Delta)}(\mathbf{k}) = -\frac{1}{k^2}\,,$$

which, with some work in complex field (using the matter density relative to a point-like source referred to in the previous section on page 59), returns the following result,

$$\begin{aligned} U(\mathbf{x}) = V(\mathbf{x}) &= (2\pi)^{\frac{3}{2}}\frac{G_N}{a_1}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'\,, \\ &= \frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\,; \end{aligned}$$

- Case 4

$$\begin{aligned} b_1 &= -2a_2\,, \\ \mathcal{L} &= a_1 R + a_2 R^2 - 2a_2 R_{\mu\nu}R^{\mu\nu}\,, \end{aligned}$$

having,

$$U(\mathbf{x}) = V(\mathbf{x}) = -4\pi G_N\int \mathcal{G}_{(a_1\Delta+2a_2\Delta^2)}(\mathbf{x},\,\mathbf{x}')\rho(\mathbf{x}')\,d^3\mathbf{x}'\,.$$

As above, the Green's function $\mathcal{G}_{(a_1\Delta+2a_2\Delta^2)}(\mathbf{x},\,\mathbf{x}')$ has to satisfy the condition,

$$\left(a_1\Delta + 2a_2\Delta^2\right)\mathcal{G}(\mathbf{x},\,\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')\,,$$

from which,

$$\mathcal{G}_{(a_1\Delta+2a_2\Delta^2)}(\mathbf{k}) = \frac{1}{\left(2a_2k^4 - a_1k^2\right)}\,;$$

hence,

$$\begin{aligned} U(\mathbf{x}) = V(\mathbf{x}) &= (2\pi)^{\frac{3}{2}}\frac{G_N}{a_1}\int \frac{1}{|\mathbf{x}-\mathbf{x}'|}\left[1-\cos\left(\sqrt{\frac{a_1}{2a_2}}|\mathbf{x}-\mathbf{x}'|\right)\right]\rho(\mathbf{x}')\,d^3\mathbf{x}'\,, \\ &= \frac{G_NM}{a_1}\frac{1}{|\mathbf{x}|}\left[1-\cos\left(\sqrt{\frac{a_1}{2a_2}}|\mathbf{x}|\right)\right]. \end{aligned}$$

Here, the factor appearing in the argument of the cosine,

$$\lambda = \sqrt{\frac{a_1}{2a_2}}\,,$$

is a new *characteristic length* with dimension $[L^{-1}]$, which, having to return a real positive number, by virtue of the choice $a_1 > 0$, involves the other coupling constant to be positive too.

Evidently, it is possible to recover, in both cases, the Newtonian behavior of the potential, which results to be unique in the weak-field regime; in Case 1, the result corresponds exactly to the Newtonian potential, as expected in the Newtonian limit of the field equations deriving from the Hilbert-Einstein Lagrangian; in Case 4, on the other hand, the Newtonian potential contains an oscillating perturbative term, due to the presence of the quadratic terms in the Lagrangian, which introduce, besides the Laplacian, also the Δ^2 operator in the corresponding field equations. From a physical point of view, this last result does not seem interesting, since it has been never observed a gravitational field whose behavior is periodic in space.

In all other cases, the Green's function to consider, is that one corresponding to Δ^2 operator only, which gives back the following Fourier's transform,

$$\mathcal{G}_{(\Delta^2)}(\mathbf{k}) = \frac{1}{k^4}\,,$$

that makes the integral giving the solution divergent.

Table 5.4: Gravitational potentials expressed through Green's function as the coupling constants vary.

Cases	**Coupling constants values**	**Gravitational potentials**
Case 1	$a_2 = 0$ $b_1 = 0$	$U(\mathbf{x}) = V(\mathbf{x}) = -\frac{4\pi G_N}{a_1} \int \mathcal{G}_{(\Delta)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')d^3\mathbf{x}'$;
Case 2	$a_1 = 0$ $b_1 = 0$	$2V(\mathbf{x}) - U(\mathbf{x}) = -\frac{2\pi G_N}{a_2} \int \mathcal{G}_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')\, d^3\mathbf{x}'$, $2V(\mathbf{x}) - U(\mathbf{x}) = \frac{C}{\lvert\mathbf{x}\rvert}$;
Case 3	$a_1 = 0$ $a_2 = 0$	$U(\mathbf{x}) = -\frac{C}{2\lvert\mathbf{x}\rvert} + \frac{6\pi G_N}{b_1} \int \mathcal{G}_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')\, d^3\mathbf{x}'$, $V(\mathbf{x}) = -\frac{C}{2\lvert\mathbf{x}\rvert} + \frac{2\pi G_N}{b_1} \int \mathcal{G}_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')\, d^3\mathbf{x}'$;
Case 4	$b_1 = -2a_2$	$U(\mathbf{x}) = V(\mathbf{x}) = -4\pi G_N \int \mathcal{G}_{(a_1\Delta + 2a_2\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')\, d^3\mathbf{x}'$;
Case 5	$a_1 = 0$ $b_1 = -\,4a_2$	$U(\mathbf{x}) = -\frac{2\pi G_N}{a_2} \int \mathcal{G}_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')\, d^3\mathbf{x}'$, $V(\mathbf{x}) = \frac{C}{\lvert\mathbf{x}\rvert}$;
Case 6	$a_1 = 0$ $b_1 = -2a_2$	$U(\mathbf{x}) = -\frac{C}{\lvert\mathbf{x}\rvert} - \frac{2\pi G_N}{a_2} \int \mathcal{G}_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')\, d^3\mathbf{x}'$, $V(\mathbf{x}) = -\frac{2\pi G_N}{a_2} \int \mathcal{G}_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')\, d^3\mathbf{x}'$;
Case 7	$a_1 = 0$ $b_1 = -\frac{8a_2}{3}$	$U(\mathbf{x}) = \frac{C}{\lvert\mathbf{x}\rvert}$, $V(\mathbf{x}) = -\frac{1}{2}\frac{C}{\lvert\mathbf{x}\rvert} - \frac{3\pi G_N}{a_2} \int \mathcal{G}_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')\, d^3\mathbf{x}'$.

Chapter 6

Curvature scalars in 1PN approximation tested by using point-like sources potentials

6.1 Modified Newtonian potentials used for testing

I am going to test, in this chapter, the curvature scalars appearing in the quadratic Lagrangians of which I spoke in Chapter 3, in particular in subsection 3.2.3, through the Newtonian potentials modified by means of corrective terms of Yukawa type, relating to point-like sources, with density $\rho(\mathbf{x}) = (2\pi)^{-\frac{3}{2}} M\delta^{(3)}(\mathbf{x})$, obtained in the previous chapter, which I rewrite here for convenience,

$$U(\mathbf{x}) = \frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\left(1 - \frac{4}{3}e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}e^{-\lambda_2|\mathbf{x}|}\right), \tag{6.1}$$

$$V(\mathbf{x}) = \frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\left(1 - \frac{2}{3}e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}e^{-\lambda_2|\mathbf{x}|}\right), \tag{6.2}$$

where,

$$\lambda_1 = \sqrt{\frac{a_1}{b_1}}\,,$$
$$\lambda_2 = \sqrt{-\frac{a_1}{2(3a_2 + b_1)}}\,,$$

under the conditions,

$$a_1 > 0,$$
$$b_1 > 0\,,$$
$$(3a_2 + b_1) < 0\,.$$

It should be emphasized, in this regard, that,although potentials coming from a specific quadratic Lagrangian have been used herein, these ones are typical solutions of field equations associated with more general quadratic and higher-order Lagrangians in the curvature scalars, as discussed widely in Chapter 3. Therefore, the results given in the following section, can be considered, in some sense, of general validity.

The purpose of this analysis, is to study the behavior of the aforementioned curvature scalars under the action of these modified potentials, deduced in the Post-Newtonian approximation 1PN of the metric tensor, namely in the weak-field and low-velocity limit, at the different orders in ϵ in the expansions thus obtained.

6.2 Test results on curvature scalars of interest

In what follows, I give the results deriving from the application of the gravitational potentials (6.1) and (6.2) to the expansions of the curvature scalars previously developed:

- R scalar

  for what concerns the Ricci scalar, the corresponding expression (4.4) on page 45, has produced the following result,

$$
\begin{aligned}
R &= {}^{(2)}R + {}^{(4)}R + O(\epsilon^6)\,, \\
{}^{(2)}R(\mathbf{x}) &= -\frac{2}{c^2}\left(\frac{G_N M}{a_1}\right)\lambda_2^2\,\frac{1}{|\mathbf{x}|}\,e^{-\lambda_2|\mathbf{x}|}\,, \\
{}^{(4)}R(\mathbf{x}) &= ?\,.
\end{aligned}
\tag{6.3}
$$

  It is not possible to develop the fourth-order contribution, since it contains the $\Phi(\mathbf{x})$ potential, currently unknown. All details are shown in Appendix G on page 225;

- R^2 scalar

  from (4.5) on page 45,

$$
\begin{aligned}
R^2 &= {}^{(4)}R^2 + O(\epsilon^6)\,, \\
{}^{(4)}R^2(\mathbf{x}) &= \frac{4}{c^4}\left(\frac{G_N M}{a_1}\right)^2\lambda_2^4\,\frac{1}{|\mathbf{x}|^2}e^{-2\,\lambda_2|\mathbf{x}|}\,,
\end{aligned}
\tag{6.4}
$$

  whose details are shown in Appendix G on page 226;

- $R_{\mu\nu}R^{\mu\nu}$ scalar

  from (4.6) on page 46,

$$
\begin{aligned}
R_{\mu\nu}R^{\mu\nu} &= {}^{(4)}R_{\mu\nu}R^{\mu\nu} + O(\epsilon^6)\,, \\
{}^{(4)}R_{\mu\nu}R^{\mu\nu}(\mathbf{x}) &= \frac{1}{c^4}\left(\frac{G_N M}{a_1}\right)^2\left\{\frac{4}{3}\,\frac{1}{|\mathbf{x}|^2}\left[2\,\lambda_1^4\,e^{-2\,\lambda_1|\mathbf{x}|} + \lambda_2^4\,e^{-2\,\lambda_2|\mathbf{x}|}\right] + \frac{16}{9}\,\frac{1}{|\mathbf{x}|^3}\left[\lambda_1^3\,e^{-2\,\lambda_1|\mathbf{x}|}\right.\right. \\
&\quad \left. - \lambda_1\lambda_2\Big(\lambda_1+\lambda_2\Big)e^{-\left(\lambda_1+\lambda_2\right)|\mathbf{x}|} + \lambda_2^3\,e^{-2\,\lambda_2|\mathbf{x}|}\right] + \frac{8}{3}\,\frac{1}{|\mathbf{x}|^4}\left[\frac{5}{3}\Big(\lambda_1^2\,e^{-2\,\lambda_1|\mathbf{x}|}\right. \\
&\quad \left.\left. + \lambda_2^2\,e^{-2\,\lambda_2|\mathbf{x}|}\Big) - 2\left(\lambda_1\,\lambda_2 + \frac{1}{3}\,\lambda_1^2 + \frac{1}{3}\,\lambda_2^2\right)e^{-\left(\lambda_1+\lambda_2\right)|\mathbf{x}|}\right] + \frac{16}{3}\,\frac{1}{|\mathbf{x}|^5}\left[\lambda_1\,e^{-2\,\lambda_1|\mathbf{x}|}\right. \\
&\quad + \lambda_2\,e^{-2\,\lambda_2|\mathbf{x}|} - \Big(\lambda_1+\lambda_2\Big)e^{-\left(\lambda_1+\lambda_2\right)|\mathbf{x}|}\Big] + \frac{8}{3}\,\frac{1}{|\mathbf{x}|^6}\Big[e^{-2\,\lambda_1|\mathbf{x}|} + e^{-2\,\lambda_2|\mathbf{x}|} \\
&\quad \left.\left. - 2\,e^{-\left(\lambda_1+\lambda_2\right)|\mathbf{x}|}\Big]\right\}\,,
\end{aligned}
\tag{6.5}
$$

  for details of which it is referred to Appendix G on page 226 and following;

- $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ scalar

  finally, from (4.7) on page 46,

$$
\begin{aligned}
R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} &= {}^{(4)}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} + O(\epsilon^6)\,,\\
{}^{(4)}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}(\mathbf{x}) &= \frac{1}{c^4}\left(\frac{G_N M}{a_1}\right)^2\left\{\frac{4}{3}\,\frac{1}{|\mathbf{x}|^2}\left[8\,\lambda_1^4\,e^{-2\,\lambda_1|\mathbf{x}|} + \lambda_2^4\,e^{-2\,\lambda_2|\mathbf{x}|}\right] + \frac{64}{9}\,\frac{1}{|\mathbf{x}|^3}\left[\lambda_1^3\,e^{-2\,\lambda_1|\mathbf{x}|}\right.\right.\\
&\quad - \lambda_1\lambda_2\Big(\lambda_1+\lambda_2\Big)e^{-\left(\lambda_1+\lambda_2\right)|\mathbf{x}|} + \lambda_2^3\,e^{-2\,\lambda_2|\mathbf{x}|}\Big] + \frac{32}{3}\,\frac{1}{|\mathbf{x}|^4}\left[\frac{5}{3}\Big(\lambda_1^2\,e^{-2\,\lambda_1|\mathbf{x}|}\right.\\
&\quad \left.\left.+\,\lambda_2^2\,e^{-2\,\lambda_2|\mathbf{x}|}\Big) - 2\left(\lambda_1\,\lambda_2 + \frac{1}{3}\,\lambda_1^2 + \frac{1}{3}\,\lambda_2^2\right)e^{-\left(\lambda_1+\lambda_2\right)|\mathbf{x}|}\right] + \frac{64}{3}\,\frac{1}{|\mathbf{x}|^5}\left[\lambda_1\,e^{-2\,\lambda_1|\mathbf{x}|}\right.\\
&\quad + \lambda_2\,e^{-2\,\lambda_2|\mathbf{x}|} - \Big(\lambda_1+\lambda_2\Big)e^{-\left(\lambda_1+\lambda_2\right)|\mathbf{x}|}\Big] + \frac{32}{3}\,\frac{1}{|\mathbf{x}|^6}\left[e^{-2\,\lambda_1|\mathbf{x}|} + e^{-2\,\lambda_2|\mathbf{x}|}\right.\\
&\quad \left.\left.\left. - 2\,e^{-\left(\lambda_1+\lambda_2\right)|\mathbf{x}|}\right]\right\}\,, \qquad (6.6)
\end{aligned}
$$

whose details are shown in Appendix G on page 238.

These results prove that such curvature scalars, expressed in the Post-Newtonian approximation 1PN, that is by using the metric tensor expanded up to the order $O(\epsilon^4)$, in the low-energy regime, when tested by means of Newtonian potentials modified by virtue of the presence of corrective terms of Yukawa type, all show trends similar to the Yukawa's one or even more rapidly decreasing, that is to say,

$$
\frac{1}{|\mathbf{x}|^n}\,e^{-c_i\,\lambda_i|\mathbf{x}|} \quad 1 \leq n \leq 6\,, \quad i = 1, 2\,,
$$

where c_i are dimensionless constants. In conclusion, in the context of the most general quadratic theory of gravity, analyzed in the aforementioned approximation, all the terms of the perturbative expansions concerning the above scalars, up to the fourth-order in ϵ, show a Yukawa type trend or further decreasing so to speak, an aspect which, as it appears evident from the expressions thus found, is accentuated in passing from the R scalar to higher-order scalars up to the Riemann scalar $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$.

6.3 Conclusions and outlook

The efforts made over the past ninety years, starting from the observations made by F. Zwicky in the early thirties, to get to the satellite observation campaigns of the nineties onwards, to remedy the shortcomings manifested by the GR, both regarding the excess of gravitation observed in the dynamics of structures at astrophysical scale and beyond, both the recent experimental acquisitions relating to the current accelerated expansion on cosmic scale, have essentially concerned two areas of research.

In fact, it should be pointed out that the GR explains correctly, with a high degree of accuracy, many phenomena of different nature observed on local scale, more precisely at the Solar System scale, as all the experiments performed during these last ninety years show, roughly speaking, leaving unanswered all the other issues to which I referred here above.

The first, inherent the quantum field theory, proposes the existence of exotic particles, completely unknown to date, constituting the so-called *dark matter*, through which to explain the dynamics of large-scale structures, not justifiable in the field of classical physics, nor in that of the GR. Furthermore, the current accelerated expansion it is thought that it can be connected to a particle responsible for the behavior of the universe from the earliest evolutionary stages, for which the best candidates could be the dilaton or the Higgs' boson, unless one wants to recover the cosmological constant,introduced in GR, however peculiar, being precisely a constant and not a quantum field, in order to give an identity to what goes under the name of *dark energy*.

The second research area deals with critically revisiting gravity, according to two opposing but equivalent points of view, as widely demonstrated in the literature: the metric theories based on the concept of curvature, without torsion, in which the field associated with the metric tensor plays a central role; the Teleparallel Gravity (TG), built on the concept of torsion, in absence of curvature, focused instead on the tetrad field.

The metric theories, here considered, constitute a natural extension, so to speak, of the Hilbert Einstein Lagrangian, this last one based on the Ricci's curvature scalar only, generalizing it in two different ways. In the first one, through a generic analytical function of the Ricci's scalar, actually an expansion in R that can be truncated to the desired order; however, starting from the third-order onwards, this implies the solution of coupled sixth-order differential equations or higher, very difficult to solve. In the second, through quadratic and higher-order curvature scalars in the Riemann's and Ricci's tensors. In any case, it should be noted that, in both approaches, it is always possible to recover the GR at the lowest order, that is to say in the low-energy regime.

The work carried out in this thesis, evidently, fits into the latter context and finds its *raison d'etre* in the attempt of providing an accurate analysis of the aforementioned theories, performed by using a higher-order perturbative expansion of the metric tensor, the so-called Post-Newtonian approximation 1PN, not yet tested in any way to date.

In the first part of it, starting from the Christoffel's connection components, I have been gone on to develop the curvature tensors and the corresponding curvature scalars of interest in those metric theories concerning quadratic and higher-order Lagrangians, widely treated in Chapter 3, in particular the most general quadratic Lagrangian,

$$\mathcal{L} = a_1 R + a_2 R^2 + b_1 R_{\mu\nu} R^{\mu\nu} + c_1 R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}, \tag{6.7}$$

which I remember to be reducible, without loss of generality, by virtue of the Gauss - Bonnet identity and through a proper redefinition of the coupling constants (here given with the same symbols), to the following expression,

$$\mathcal{L} = a_1 R + a_2 R^2 + b_1 R_{\mu\nu} R^{\mu\nu}, \tag{6.8}$$

in the Post-Newtonian approximation 1PN, exactly, that is to say by using the metric tensor $g_{\mu\nu}(x)$ expanded up to the fourth-order in ϵ, where $\epsilon = v/c$, in the weak-field regime and as consequence in the low-velocity limit of the bodies, $v \ll c$, whose final expressions, they too expanded up to the fourth-order in ϵ, are given in terms of the three gravitational potentials $U(\mathbf{x}), V(\mathbf{x})$ and $\Phi(\mathbf{x})$, characterizing the perturbative expansion of the metric tensor at the aforementioned order, which has not any dependence on time in stationary regime and consequently the above gravitational potentials too.

This preliminary analysis did allow, subsequently, to develop the left-hand sides of the field equations, related to the curvature invariants appearing in (6.8), necessary to expand the complete field equations given by,

$$H_{\mu\nu} = \sum_{k=0}^{2} c_k \, {}^{k}H_{\mu\nu} = \frac{8\pi G_N}{c^4} T^{(m)}_{\mu\nu}, \tag{6.9}$$

which gave back two systems of coupled fourth-order differential equations, related to the orders $O(\epsilon^2)$ and $O(\epsilon^4)$ respectively.

In doing this, I preferred to follow a more direct approach, so to speak, non canonical, a sort of a preliminary test on such a theory, delaying to a next future the canonical analysis of it. In fact, I have developed the aforesaid field equations relative to an approximately flat space-time; in other words, I treated these ones locally and then expanded them in the weak-field and low-velocity limit, in order to see what happens to gravitational potentials under such conditions.

It turned out, at first glance, that only the first of the aforementioned systems is solvable, being a system of two fourth-order differential equations in the gravitational potentials $U(\mathbf{x})$ and

$V(\mathbf{x})$, whereas the second consists of the same number of equations, but in the three gravitational potentials $U(\mathbf{x}), V(\mathbf{x})$ and $\Phi(\mathbf{x})$. Therefore, the fourth-order perturbative expansion of the above quadratic metric theory of gravity, given by the Post-Newtonian approximation 1PN of the metric tensor, does not add anything new to the second-order one, relative to the 0.5PN approximation of the metric tensor, analyzed in the work of S. Capozziello and A. Stabile (2009) [74], of which this thesis is a further development.

For a discussion which is as general as possible, i wrote the differential equations relating to the order $O(\epsilon^2)$ with varying the conditions on the coupling constants, as shown in table 5.1 on page 57. It is evident that when one turns off the quadratic terms in the Lagrangian (6.8), as it happens in Case 1, immediately one recovers the Poisson's equation giving back the classical Newtonian potential; in all other cases, one always has to deal with fourth-order differential equations, from which, where possible, only modified Newtonian potentials are derived.

However, it should be noted that, for what concerns the first system of differential equations, that one corresponding to the $O(\epsilon^2)$ order, the terms ${}^1H_{\mu\nu}$ and ${}^2H_{\mu\nu}$ of the complete equation (6.9), show both an overall sign minus, which does not appear in the computations of the aforementioned paper, whereas the term ${}^0H_{\mu\nu}$, that of the Hilbert-Einstein, results unchanged.

This has implied to have to solve it again, with the result of obtaining the same modified Newtonian potentials, computed through the Green's method, for the case of point-like sources. Again it has been possible to define different Green's functions with varying the coupling constants, as shown in table 5.3 on page 69, the first of which, corresponding to Case 1, has produced the following potentials,

$$U(\mathbf{x}) = \frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\left(1 - \frac{4}{3}e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}e^{-\lambda_2|\mathbf{x}|}\right), \tag{6.10a}$$

$$V(\mathbf{x}) = \frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\left(1 - \frac{2}{3}e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}e^{-\lambda_2|\mathbf{x}|}\right), \tag{6.10b}$$

obtained through a different definition of the two *characteristic lengths*,

$$\lambda_1 = \sqrt{\frac{a_1}{b_1}}\,,$$
$$\lambda_2 = \sqrt{-\frac{a_1}{2(3a_2 + b_1)}}\,,$$

where,

$$a_1 > 0\,,$$
$$b_1 > 0\,,$$
$$(3a_2 + b_1) < 0\,,$$

appearing in the perturbative terms of Yukawa type, by virtue of an exchanging of sign with respect to those ones derived in [74].

The effect produced by such perturbative terms is that of accentuating the slope of the Newtonian curve, thus raising the strength of the gravitational interaction; more precisely, these terms strengthen or weaken the Newtonian gravity, depending on the sign of the corresponding coefficients. However, it should be emphasized that such contributions result to be not negligible only on distance scales given by the *characteristic lengths* appearing therein, corresponding to astrophysical scales and beyond. More specifically, the Yukawa's perturbative terms, at small scales, such as those of the Solar System, where the GR works very well, show a typically Newtonian behavior,

$$\frac{1}{|\mathbf{x}|}\, e^{-\lambda_i |\mathbf{x}|} \sim \frac{1}{|\mathbf{x}|} \qquad |\mathbf{x}| \ll \lambda_i^{-1}\,,$$

while at very large scales, namely at cosmic scales, they don't give any contribution; in fact, it results that,

$$\frac{1}{|\mathbf{x}|}\, e^{-\lambda_i |\mathbf{x}|} \sim 0 \qquad |\mathbf{x}| \gg \lambda_i^{-1}\,,$$

which is why these modified potentials show the same asymptotic behavior as the pure Newtonian one. In this sense, they can not say anything about the current accelerated expansion of the universe.

However, at intermediate scales, such as the galactic ones (10 - 20 Kpc), up to those typical of clusters and super-clusters of galaxies (1 - 10^2 Mpc), such corrective terms seem to be suitable for explaining the observed dynamic behaviors, without requiring the presence of DM.

Specifically, to date, it has been possible to test only the simplest quadratic Lagrangian,

$$\mathcal{L} = f(R) = a_1 R + a_2 R^2\,,$$

obtained through Taylor's expansion of a generic differentiable function f(R), truncated at the second order (disregarding the a_0 term), studied in the Post-Newtonian approximation 0.5PN of the metric tensor, that is to say up to the second-order in ϵ of its perturbative expansion, whose parametrized modified potential is given by the following expression,

$$U(r) = -\frac{G_N M}{(1+\delta) r}\left(1 + \delta e^{-\frac{r}{L}}\right), \tag{6.11}$$

where $1 + \delta = a_1$ and L is the usual length-scale.

In this way, it has been possible to reproduce approximately the flat behavior of the rotation curves $v_{circ}(r)$ of spiral galaxies [99], on the basis of a sample of fifty objects, with $\delta < 0$, although L has been evaluated case by case, what is not the best fit, since a unique value of it is expected. Some interesting simulations, based on Monte Carlo techniques, are given in [64].

The velocity dispersion curves, in a sample of three elliptical galaxies, are successfully discussed in [66], in the context of such a quadratic theory predicting the aforementioned modified Newtonian potentials.

Moreover, at higher scales, such theory seems to offer a good description of the mass radial distribution of a sample of twelve clusters of galaxies, consisting of three contributions,

$$M_{clust}(r) = M_{gas}(r) + M_{gal}(r) + M_{CDgal}(r)\,,$$

where M_{gas} indicates the mass associated with the intergalactic gas, M_{gal} the overall mass of the galaxies present in the cluster and M_{CDgal}[1] that relating to the CD-type galaxies located in the central areas of the clusters, as widely discussed in [65, 67]; in such a case, unlike the sample of spiral galaxies, it is possible to recognize a length-scale corresponding to a value of about 150 Kpc, beyond which the theoretical curves give a good description of those observed. In the innermost regions, however, a systematical discrepancy between the two curves appears, what can due either to a breaking of the hydrostatic equilibrium or, more simply, to limits in the series expansion of the f(R) function.

Finally, a hint is in order, concerning the matching between the computation of the *Jeans' critical mass*[2] based on the modified potential (6.11) and that one predicted by the GR, on the basis of a sample of twenty-six nebulae, as reported in [64]; it results that,

[1] The acronym *CDgal* stands for "central dominant galaxies", specifically supergiant elliptical galaxies, according to the classification of the *Yerkes galaxy classification scheme*.

[2] The *Jeans' mass* indicates the critical value above which there is instability and therefore gravitational collapse in a gas cloud with consequent star formation.

$$M_J\big[f(R)\big] < M_J\big[R\big]\,,$$

namely, in such f(R) model the star formation process is favored more than in the GR theory, being the former lower than the latter systematically.

Therefore, in the light of the new gravitational potentials so obtained, it would be interesting to verify if they are able to give a better description of the observed trends in the aforementioned ambits; in particular, the $U(\mathbf{x})$ potential, given by (6.10a), containing two perturbative corrections, which play a different role, attractive or repulsive, depending on the sign of the coefficients that appear there.

For what concerns the other two Green's functions of table 5.3 on page 69, corresponding to Case 2 and Case 3, they give back gravitational potentials containing oscillating perturbative corrections of the type,

$$\frac{1}{|\mathbf{x}|}\cos\big(\lambda_i|\mathbf{x}|\big)\,,$$

which does not seem to describe correctly gravity at any scale of distance, since any periodic trend in space of it has been never observed.

One could extend this discussion to the more realistic case of gravitational sources with spherical symmetry, having a given radius, by using the Green's functions already mentioned, in order to compute the corresponding inner and outer potentials; this would permit also to verify that the outer potentials give back the results obtained for the case of point-like sources, in the limit of large distances, as already done in [74] for the lower approximation 0.5PN of the metric tensor.

The expansions concerning the curvature scalars appearing in the Lagrangian (6.7), achieved in Chapter 4, have been subsequently tested by means of the above gravitational potentials, in order to understand how they behave under the action of such modified potentials. What emerged from the results, is that they contain, at the different orders in ϵ, only perturbative terms of Yukawa type and further decreasing, so to speak, namely,

$$\frac{1}{|\mathbf{x}|^n}\,e^{-c_i\,\lambda_i|\mathbf{x}|}\quad 1\leq n\leq 6\,,\quad i=1,2\,.$$

This confirms the fact that the quadratic curvature scalars appearing in (6.7), are only perturbative corrections of the General Relativity, corresponding to the Hilbert-Einstein term, as expected by such extended theories, since the second-order contribution to the Ricci's scalar, see (6.3), the only one evaluable, shows a trend of Yukawa type, while all the other scalars have a trend even more rapidly decreasing than the Yukawa's one, as shown by the results given in (6.4), (6.5), (6.6).

A more comprehensive analysis of the metric theories of gravity, extended up to the quadratic order in the curvature scalars, could be offered by the study, to be completed in a next future, of the field equations related to the Lagrangian constituting the most direct and natural extension of that of Hilbert-Einstein, that is,

$$\mathcal{L} = a_1 R + a_2 R^2\,,$$

by using the canonical approach, consisting of developing the corresponding field equations in the curved space-time and subsequently expressing the results thus obtained in the Post-Newtonian approximation 1PN of the metric tensor, namely up to the order $O(\epsilon^4)$, in the weak-field and low-velocity limit, in order to check whether the modified potentials contain a further refinement of the corrective terms of Yukawa type to the Newtonian potential.

A further step would be given by the application of the above method to the most general quadratic Lagrangian, which I have talked about widely,

$$\mathcal{L} = a_1 R + a_2 R^2 + b_1 R_{\mu\nu} R^{\mu\nu}\,.$$

It should be pointed out that this discussion has concerned only second-order theories in the curvature invariants, since the higher-order ones, from the third onwards, involve coupled sixth-order differential equations, even more difficult to solve than those just obtained in this work.

Furthermore, an estimate of the *characteristic lengths*, acquired on the basis of the most updated observational data, as already mentioned, would allow also to evaluate the coupling constants a_2 and b_1 of the above Lagrangian, setting $a_1 = 1$ in order to restore the General Relativity, in the attempt to find the 'correct' theory of gravity, capable of responding to the most urgent questions currently posed, concerning the excess of gravity observed outside the Solar System, which drives the dynamics of the large-scale structures, starting from galaxies up to the clusters and the super-clusters of galaxies, to which the GR does not manage to give any answer, although it works very well at the Solar System scale, as many experiments of different nature have proved during the last ninety years, allowing to recover the Newtonian physics in presence of weak gravitational fields and for low velocities of the bodies.

In conclusion, such models based on quadratic and higher-order Lagrangians in the curvature scalars, seem to show a different behavior of gravity at different scales of distance, to which belong the various structures observed in the universe, excluding the existence of any type of exotic matter, of which it has been not detected any trace up to now. Thus one can conclude that, although the GR works in excellent way on the Solar System scale, the dynamic observed at galactic and cluster scales require a further refinement of it, for which the Yukawa perturbative corrections to the Newtonian potential could be a valid proposal.

Appendix A

Toolkit

A.1 Symbols

$$
\begin{aligned}
\alpha, \beta, \ldots &= 0, 1, 2, 3\,, \\
i, j, \ldots &= 1, 2, 3\,, \\
\eta_{\alpha\beta} &= diag(1, -1, -1, -1)\,, \\
\epsilon &= \frac{v}{c}\,, \\
\mathbf{x} &= \boldsymbol{i}(x^1) + \boldsymbol{j}(x^2) + \boldsymbol{k}(x^3) = \boldsymbol{i}(x) + \boldsymbol{j}(y) + \boldsymbol{k}(z)\,, \\
\mathbf{v} &= \boldsymbol{i}(v^1) + \boldsymbol{j}(v^2) + \boldsymbol{k}(v^3) = \boldsymbol{i}(v_x) + \boldsymbol{j}(v_y) + \boldsymbol{k}(v_z)\,, \\
P_i &= mv_i\,, \\
h_{0i} &= P_i\,, \\
\mathbf{h} &= \boldsymbol{i}(h_{01}) + \boldsymbol{j}(h_{02}) + \boldsymbol{k}(h_{03})\,, \\
h_{123}^2 &= h_{01}^2 + h_{02}^2 + h_{03}^2\,, \\
h_{12}^2 &= h_{01}^2 + h_{02}^2\,, \\
h_{13}^2 &= h_{01}^2 + h_{03}^2\,, \\
h_{23}^2 &= h_{02}^2 + h_{03}^2\,, \\
p &= \big(c^4 + 2U^2 - 2c^2U - 4\Phi\big)\,, \\
q &= \big(c^4 + 2U^2 + 2c^2U + 4\Phi\big)\,, \\
P &= \big(c^2 + 2V\big)\big(c^4 + 2U^2 - 2c^2U - 4\Phi\big) = \big(c^2 + 2V\big)p\,, \\
Q &= \big(c^2 - 2V\big)\big(c^4 + 2U^2 + 2c^2U + 4\Phi\big) = \big(c^2 - 2V\big)q\,, \\
D &= h_{01}^2 + h_{02}^2 + h_{03}^2 + \big(c^2 + 2V\big)\big(c^4 + 2U^2 - 2c^2U - 4\Phi\big) = h_{123}^2 + P\,, \\
PD_i &= \big(c^2 - 2U\big)\partial_i U + 2\,\partial_i\Phi\,, \\
PD_{ij} &= \big(c^2 - 2U\big)\partial_{ij} U + 2\,\partial_{ij}\Phi\,, \\
\boldsymbol{\nabla} &= \boldsymbol{i}\left(\frac{\partial}{\partial x^1}\right) + \boldsymbol{j}\left(\frac{\partial}{\partial x^2}\right) + \boldsymbol{k}\left(\frac{\partial}{\partial x^3}\right) = \boldsymbol{i}\left(\frac{\partial}{\partial x}\right) + \boldsymbol{j}\left(\frac{\partial}{\partial y}\right) + \boldsymbol{k}\left(\frac{\partial}{\partial z}\right)\,, \\
\Delta &= \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,, \\
\Delta^2 &= \Delta\Delta\,, \\
\Box_{GR} &= \frac{1}{\sqrt{-g}}\,\partial_\alpha\Big(\sqrt{-g}\,g^{\alpha\beta}\,\partial_\beta\Big)\,, \\
\Box_{SR} &= \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\,,
\end{aligned}
$$

$$
\begin{aligned}
\partial_i U &= U_{,i} = \frac{\partial U}{\partial x^i}\,, \\
\partial_{ij} U &= U_{,ij} = \frac{\partial^2 U}{\partial x^i \partial x^j}\,, \\
\partial_i |\mathbf{x}| &= \frac{x^i}{|\mathbf{x}|}\,, \\
\partial_i \frac{1}{|\mathbf{x}|} &= -\frac{x^i}{|\mathbf{x}|^3}\,, \\
\partial_i \left(\frac{x^i}{|\mathbf{x}|^2} \right) &= \frac{1}{|\mathbf{x}|^2} - \frac{2{x^i}^2}{|\mathbf{x}|^4}\,, \\
\partial_i \left(\frac{x^i}{|\mathbf{x}|^3} \right) &= \frac{1}{|\mathbf{x}|^3} - \frac{3{x^i}^2}{|\mathbf{x}|^5}\,.
\end{aligned}
$$

A.2 P and Q expansions

- $$P = \big(c^2 + 2V\big)\big(c^4 + 2U^2 - 2c^2 U - 4\Phi\big) = c^6 \left(1 + \frac{2V}{c^2}\right)\left[1 - \frac{2U}{c^2} + \frac{1}{c^4}\big(2U^2 - 4\Phi\big)\right],$$

  where,

$$
\begin{aligned}
&\frac{2U}{c^2} \sim O\big(\epsilon^2\big)\,, \\
&\frac{2V}{c^2} \sim O\big(\epsilon^2\big)\,, \\
&\frac{1}{c^4}(2U^2 - 4\Phi) \sim O\big(\epsilon^4\big)\,;
\end{aligned}
\tag{A.1}
$$

- $$Q = \big(c^2 - 2V\big)\big(c^4 + 2U^2 + 2c^2 U + 4\Phi\big) = c^6 \left(1 - \frac{2V}{c^2}\right)\left[1 + \frac{2U}{c^2} + \frac{1}{c^4}\big(2U^2 + 4\Phi\big)\right],$$

  where,

$$
\frac{1}{c^4}\big(2U^2 + 4\Phi\big) \sim O\big(\epsilon^4\big)\,. \tag{A.2}
$$

A.3 Symbols and their higher-order derivatives

$$
\begin{aligned}
PD_i &= \big(c^2 - 2U\big)\partial_i U + 2\,\partial_i \Phi = c^2 \left(1 - \frac{2U}{c^2}\right)\partial_i U + 2\partial_i \Phi\,, \\
PD_{ij} &= \big(c^2 - 2U\big)\partial_{ij} U + 2\,\partial_{ij} \Phi = c^2 \left(1 - \frac{2U}{c^2}\right)\partial_{ij} U + 2\partial_{ij} \Phi\,, \\
\partial_i PD_j &= \partial_i \Big[\big(c^2 - 2U\big)\partial_j U + 2\partial_j \Phi\Big] \\
&= -2\partial_i U \partial_j U + c^2 \left(1 - \frac{2U}{c^2}\right)\partial_{ij} U + 2\partial_{ij} \Phi\,,
\end{aligned}
$$

$$
\begin{aligned}
\partial_i PD_{jk} &= \partial_i\Big[\big(c^2-2U\big)\partial_{jk}U + 2\partial_{jk}\Phi\Big] \\
&= -2\partial_i U\partial_{jk}U + c^2\left(1-\frac{2U}{c^2}\right)\partial_{ijk}U + 2\partial_{ijk}\Phi\,, \\
\partial_{ii} PD_j &= \partial_i(\partial_i PD_j) = \partial_i\Big[-2\partial_i U\partial_j U + \big(c^2-2U\big)\partial_{ij}U + 2\partial_{ij}\Phi\Big] \\
&= -2\partial_{ii}U\partial_j U - 4\partial_i U\partial_{ij}U + c^2\left(1-\frac{2U}{c^2}\right)\partial_{iij}U + 2\partial_{iij}\Phi\,, \\
\partial_{ii} PD_{jk} &= \partial_i(\partial_i PD_{jk}) = \partial_i\Big[-2\partial_i U\partial_{jk}U + \big(c^2-2U\big)\partial_{ijk}U + 2\partial_{ijk}\Phi\Big] \\
&= -2\partial_{ii}U\partial_{jk}U - 4\partial_i U\partial_{ijk}U + c^2\left(1-\frac{2U}{c^2}\right)\partial_{iijk}U + 2\partial_{iijk}\Phi\,;
\end{aligned}
$$

A.4 Recurrent higher-order differential expressions

In what follows, the f_1, f_2 and f functions stand for the U, V or Φ potentials;

- $$
\begin{aligned}
(f_1 f_2)_{,ij} &= f_{1,ij}\, f_2 + f_{1,i}\, f_{2,j} + f_{1,j}\, f_{2,i} + f_1\, f_{2,ij}\,, \\
\Delta(f_1 f_2) &= f_2\,\Delta f_1 + 2\,\boldsymbol{\nabla} f_1\cdot\boldsymbol{\nabla} f_2 + f_1\,\Delta f_2\,;
\end{aligned}
\tag{A.3}
$$

- $$
\begin{aligned}
\boldsymbol{\nabla} f &= \boldsymbol{i}\big(f_{,1}\big) + \boldsymbol{j}\big(f_{,2}\big) + \boldsymbol{k}\big(f_{,3}\big)\,, \\
\boldsymbol{\nabla}\big(f_{,i}\big) &= \boldsymbol{i}\big(f_{,i1}\big) + \boldsymbol{j}\big(f_{,i2}\big) + \boldsymbol{k}\big(f_{,i3}\big)\,, \\
\boldsymbol{\nabla}\big(f_{,i}\big)\cdot\boldsymbol{\nabla}\big(f_{,i}\big) &= \big(f_{,i1}\big)^2 + \big(f_{,i2}\big)^2 + \big(f_{,i3}\big)^2\,;
\end{aligned}
\tag{A.4}
$$

- $$
\Delta\big(\boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} f\big) = \Delta\Big[\big(f_{,1}\big)^2 + \big(f_{,2}\big)^2 + \big(f_{,3}\big)^2\Big]\,,
$$

$$
\begin{aligned}
\Big[\big(f_{,1}\big)^2\Big]_{,11} &= 2\Big[\big(f_{,11}\big)^2 + f_{,1}f_{,111}\Big]\,, \\
\Big[\big(f_{,1}\big)^2\Big]_{,22} &= 2\Big[\big(f_{,12}\big)^2 + f_{,1}f_{,122}\Big]\,, \\
\Big[\big(f_{,1}\big)^2\Big]_{,33} &= 2\Big[\big(f_{,13}\big)^2 + f_{,1}f_{,133}\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
\Big[\big(f_{,2}\big)^2\Big]_{,11} &= 2\Big[\big(f_{,12}\big)^2 + f_{,2}f_{,112}\Big]\,, \\
\Big[\big(f_{,2}\big)^2\Big]_{,22} &= 2\Big[\big(f_{,22}\big)^2 + f_{,2}f_{,222}\Big]\,, \\
\Big[\big(f_{,2}\big)^2\Big]_{,33} &= 2\Big[\big(f_{,23}\big)^2 + f_{,2}f_{,233}\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
\Big[\big(f_{,3}\big)^2\Big]_{,11} &= 2\Big[\big(f_{,13}\big)^2 + f_{,3}f_{,113}\Big]\,, \\
\Big[\big(f_{,3}\big)^2\Big]_{,22} &= 2\Big[\big(f_{,23}\big)^2 + f_{,3}f_{,223}\Big]\,, \\
\Big[\big(f_{,3}\big)^2\Big]_{,33} &= 2\Big[\big(f_{,33}\big)^2 + f_{,3}f_{,333}\Big]\,,
\end{aligned}
$$

from which,

$$\Delta\big(\boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} f\big)=2\Big[\big(f_{,11}\big)^2+f_{,1}f_{,111}+2\big(f_{,12}\big)^2+f_{,1}f_{,122}+2\big(f_{,13}\big)^2+f_{,1}f_{,133}+f_{,2}f_{,112}+\big(f_{,22}\big)^2 \\ +f_{,2}f_{,222}+2\big(f_{,23}\big)^2+f_{,2}f_{,233}+f_{,3}f_{,113}+f_{,3}f_{,223}+\big(f_{,33}\big)^2+f_{,3}f_{,333}\Big]\,; \tag{A.5}$$

- $$\Delta\big(\boldsymbol{\nabla} f_1\cdot\boldsymbol{\nabla} f_2\big)=\Delta\big(f_{1,1}f_{2,1}+f_{1,2}f_{2,2}+f_{1,3}f_{2,3}\big)\,,$$

$$\begin{aligned}
\big(f_{1,1}f_{2,1}\big)_{,11}&=f_{1,111}f_{2,1}+2f_{1,11}f_{2,11}+f_{1,1}f_{2,111}\,,\\
\big(f_{1,1}f_{2,1}\big)_{,22}&=f_{1,122}f_{2,1}+2f_{1,12}f_{2,12}+f_{1,1}f_{2,122}\,,\\
\big(f_{1,1}f_{2,1}\big)_{,33}&=f_{1,133}f_{2,1}+2f_{1,13}f_{2,13}+f_{1,1}f_{2,133}\,,
\end{aligned}$$

$$\begin{aligned}
\big(f_{1,2}f_{2,2}\big)_{,11}&=f_{1,211}f_{2,2}+2f_{1,21}f_{2,21}+f_{1,2}f_{2,211}\,,\\
\big(f_{1,2}f_{2,2}\big)_{,22}&=f_{1,222}f_{2,2}+2f_{1,22}f_{2,22}+f_{1,2}f_{2,222}\,,\\
\big(f_{1,2}f_{2,2}\big)_{,33}&=f_{1,233}f_{2,2}+2f_{1,23}f_{2,23}+f_{1,2}f_{2,233}\,,
\end{aligned}$$

$$\begin{aligned}
\big(f_{1,3}f_{2,3}\big)_{,11}&=f_{1,311}f_{2,3}+2f_{1,31}f_{2,31}+f_{1,3}f_{2,311}\,,\\
\big(f_{1,3}f_{2,3}\big)_{,22}&=f_{1,322}f_{2,3}+2f_{1,32}f_{2,32}+f_{1,3}f_{2,322}\,,\\
\big(f_{1,3}f_{2,3}\big)_{,33}&=f_{1,333}f_{2,3}+2f_{1,33}f_{2,33}+f_{1,3}f_{2,333}\,,
\end{aligned}$$

from which,

$$\Delta\big(\boldsymbol{\nabla} f_1\cdot\boldsymbol{\nabla} f_2\big)=f_{1,111}f_{2,1}+2f_{1,11}f_{2,11}+f_{1,1}f_{2,111}+f_{1,122}f_{2,1}+4f_{1,12}f_{2,12}+f_{1,1}f_{2,122} \\ +f_{1,133}f_{2,1}+4f_{1,13}f_{2,13}+f_{1,1}f_{2,133}+f_{1,211}f_{2,2}+f_{1,2}f_{2,211}+f_{1,222}f_{2,2} \\ +2f_{1,22}f_{2,22}+f_{1,2}f_{2,222}+f_{1,233}f_{2,2}+4f_{1,23}f_{2,23}+f_{1,2}f_{2,233}+f_{1,311}f_{2,3} \\ +f_{1,3}f_{2,311}+f_{1,322}f_{2,3}+f_{1,3}f_{2,322}+f_{1,333}f_{2,3}+2f_{1,33}f_{2,33}+f_{1,3}f_{2,333}\,; \tag{A.6}$$

- $$\begin{aligned}
\boldsymbol{\nabla} f\cdot\boldsymbol{\nabla}\big(\Delta f\big)&=f_{,1}\,\Delta f_{,1}+f_{,2}\,\Delta f_{,2}+f_{,3}\,\Delta f_{,3}\\
&=f_{,1}\big(f_{,111}+f_{,122}+f_{,133}\big)+f_{,2}\big(f_{,211}+f_{,222}+f_{,233}\big)\\
&\quad+f_{,3}\big(f_{,311}+f_{,322}+f_{,333}\big)\,.
\end{aligned} \tag{A.7}$$

- $$\begin{aligned}
\boldsymbol{\nabla} f_1\cdot\boldsymbol{\nabla}\big(\Delta f_2\big)&=f_{1,1}\,\Delta f_{2,1}+f_{1,2}\,\Delta f_{2,2}+f_{1,3}\,\Delta f_{2,3}\\
&=f_{1,1}\big(f_{2,111}+f_{2,122}+f_{2,133}\big)+f_{1,2}\big(f_{2,211}+f_{2,222}+f_{2,233}\big)\\
&\quad+f_{1,3}\big(f_{2,311}+f_{2,322}+f_{2,333}\big)\,.
\end{aligned} \tag{A.8}$$

- $$\begin{aligned}
&\boldsymbol{\nabla}\big(f_{,1}\big)\cdot\boldsymbol{\nabla}\big(f_{,1}\big)+\boldsymbol{\nabla}\big(f_{,2}\big)\cdot\boldsymbol{\nabla}\big(f_{,2}\big)+\boldsymbol{\nabla}\big(f_{,3}\big)\cdot\boldsymbol{\nabla}\big(f_{,3}\big)\\
&=\big(f_{,11}\big)^2+2\big(f_{,12}\big)^2+2\big(f_{,13}\big)^2+\big(f_{,22}\big)^2+2\big(f_{,23}\big)^2+\big(f_{,33}\big)^2\,;
\end{aligned} \tag{A.9}$$

•

$$\begin{aligned}&\boldsymbol{\nabla}(f_{1,1})\cdot\boldsymbol{\nabla}(f_{2,1})+\boldsymbol{\nabla}(f_{1,2})\cdot\boldsymbol{\nabla}(f_{2,2})+\boldsymbol{\nabla}(f_{1,3})\cdot\boldsymbol{\nabla}(f_{2,3})\\&=f_{1,11}f_{2,11}+2f_{1,12}f_{2,12}+2f_{1,13}f_{2,13}+f_{1,22}f_{2,22}+2f_{1,23}f_{1,23}+f_{1,33}f_{2,33}\,.\end{aligned}\tag{A.10}$$

A.5 $g = det\, g_{\mu\nu}$ computation

$$\begin{aligned}g&=\frac{1}{c^{10}}\Big\{-\big(c^2+2V\big)^2\Big[h_{123}^2+\big(c^4+2U^2-2c^2U\big)\big(c^2+2V\big)\Big]+4\Phi\big(c^2+2V\big)^3\Big\}\\&=-\frac{1}{c^{10}}\Big\{\big(c^2+2V\big)^2\Big[h_{123}^2+\big(c^4+2U^2-2c^2U\big)\big(c^2+2V\big)\Big]-4\Phi\big(c^2+2V\big)^3\Big\},\end{aligned}$$

from which,

$$\begin{aligned}-g&=\frac{1}{c^{10}}\big(c^2+2V\big)^2\Big\{\Big[h_{123}^2+\big(c^4+2U^2-2c^2U\big)\big(c^2+2V\big)\Big]-4\Phi\big(c^2+2V\big)\Big\}\\&=\frac{1}{c^{10}}\big(c^2+2V\big)^2\Big[h_{123}^2+\big(c^2+2V\big)\big(c^4+2U^2-2c^2U-4\Phi\big)\Big],\end{aligned}$$

that is,

$$-g=\frac{1}{c^{10}}\big(c^2+2V\big)^2D\,.\tag{A.11}$$

A.6 Taylor's expansions at first order of functions of interest

For sakes of simplicity of reading, i have preferred to introduce the following new variables:

$$\begin{aligned}x&=\frac{h_{01}}{c^3}\,,\\y&=\frac{h_{02}}{c^3}\,,\\z&=\frac{h_{03}}{c^3}\,,\\s&=\frac{U}{c^2}\,,\\t&=\frac{V}{c^2}\,,\\v&=\frac{\Phi}{c^4}\,.\end{aligned}$$

A.6.1 $f_1(x,y,z,s,t,v)$ expansion

$$\begin{aligned}\frac{1}{\big(c^2+2V\big)D}&=\frac{1}{c^8}\,f_1\bigg(\frac{h_{01}}{c^3},\frac{h_{02}}{c^3},\frac{h_{03}}{c^3},\frac{U}{c^2},\frac{V}{c^2},\frac{\Phi}{c^4}\bigg)\\&=\frac{1}{c^8}\bigg\{\bigg(1+\frac{2V}{c^2}\bigg)\bigg[\frac{h_{123}^2}{c^6}+\bigg(1+\frac{2V}{c^2}\bigg)\bigg(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\bigg)\bigg]\bigg\}^{-1},\end{aligned}$$

where,

$$f_1\Big(\frac{h_{01}}{c^3},\dots,\frac{\Phi}{c^4}\Big)=\left\{\left(1+\frac{2V}{c^2}\right)\left[\frac{h_{123}^2}{c^6}+\left(1+\frac{2V}{c^2}\right)\left(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\right)\right]\right\}^{-1},$$

$$f_1(x,\,y,\,z,\,s,\,t,\,v)=\frac{1}{\big(2t+1\big)\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\big)\Big]},$$

from which,

$$\begin{aligned}
\frac{\partial f_1}{\partial x}\bigg|_0 &= -\frac{2x}{\big(2t+1\big)\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2}\bigg|_0=0\,,\\
\frac{\partial f_1}{\partial y}\bigg|_0 &= -\frac{2y}{\big(2t+1\big)\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2}\bigg|_0=0\,,\\
\frac{\partial f_1}{\partial z}\bigg|_0 &= -\frac{2z}{\big(2t+1\big)\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2}\bigg|_0=0\,,\\
\frac{\partial f_1}{\partial s}\bigg|_0 &= -\frac{\big(4s-2\big)}{\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2}\bigg|_0=2\,,\\
\frac{\partial f_1}{\partial t}\bigg|_0 &= \left\{-\frac{2\big(2s^2-2s-4v+1\big)}{\big(2t+1\big)\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2}\right.\\
&\quad\left.-\frac{2}{\big(2t+1\big)^2\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]}\right\}\bigg|_0=-4\,,\\
\frac{\partial f_1}{\partial v}\bigg|_0 &= \frac{4}{\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2}\bigg|_0=4\,;
\end{aligned}$$

therefore,

$$\begin{aligned}
f_1(x,\,y,\,z,\,s,\,t,\,v)&=1+2s-4t+4v+O\big(\epsilon^4\big)\,,\\
f_1\Big(\frac{h_{01}}{c^3},\frac{h_{02}}{c^3},\frac{h_{03}}{c^3},\frac{U}{c^2},\frac{V}{c^2},\frac{\Phi}{c^4}\Big)&=1+\frac{2U}{c^2}-\frac{4V}{c^2}+\frac{4\Phi}{c^4}+O\big(\epsilon^4\big)\,.
\end{aligned}$$

A.6.2 $f_2(x,y,z,s,t,v)$ expansion

$$\begin{aligned}
\frac{1}{D}&=\frac{1}{c^6}\,f_2\Big(\frac{h_{01}}{c^3},\frac{h_{02}}{c^3},\frac{h_{03}}{c^3},\frac{U}{c^2},\frac{V}{c^2},\frac{\Phi}{c^4}\Big)\\
&=\frac{1}{c^6}\left[\frac{h_{123}^2}{c^6}+\left(1+\frac{2V}{c^2}\right)\left(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\right)\right]^{-1},
\end{aligned}$$

where,

$$f_2\Big(\frac{h_{01}}{c^3},\ldots,\frac{\Phi}{c^4}\Big)=\left[\frac{h_{123}^2}{c^6}+\Big(1+\frac{2V}{c^2}\Big)\Big(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\Big)\right]^{-1},$$
$$f_2(x,\,y,\,z,\,s,\,t,\,v)=\frac{1}{\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]},$$

from which,

$$\frac{\partial f_2}{\partial x}\bigg|_0=-\frac{2x}{\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2}\bigg|_0=0\,,$$
$$\frac{\partial f_2}{\partial y}\bigg|_0=-\frac{2y}{\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2}\bigg|_0=0\,,$$
$$\frac{\partial f_2}{\partial z}\bigg|_0=-\frac{2z}{\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2}\bigg|_0=0\,,$$
$$\frac{\partial f_2}{\partial s}\bigg|_0=-\frac{\big(4s-2\big)\big(2t+1\big)}{\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2}\bigg|_0=2\,,$$
$$\frac{\partial f_2}{\partial t}\bigg|_0=-\frac{2\big(2s^2-2s-4v+1\big)}{\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2}\bigg|_0=-2\,,$$
$$\frac{\partial f_2}{\partial v}\bigg|_0=\frac{4\big(2t+1\big)}{\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2}\bigg|_0=4\,;$$

therefore,

$$f_2(x,\,y,\,z,\,s,\,t,\,v)=1+2s-2t+4v+O\big(\epsilon^4\big)\,,$$
$$f_2\Big(\frac{h_{01}}{c^3},\frac{h_{02}}{c^3},\frac{h_{03}}{c^3},\frac{U}{c^2},\frac{V}{c^2},\frac{\Phi}{c^4}\Big)=1+\frac{2U}{c^2}-\frac{2V}{c^2}+\frac{4\Phi}{c^4}+O\big(\epsilon^4\big)\,.$$

A.6.3 $f_3(x,y,z,s,t,v)$ expansion

$$\frac{1}{\big(c^2+2V\big)^2D^2}=\frac{1}{c^{16}}\,f_3\Big(\frac{h_{01}}{c^3},\frac{h_{02}}{c^3},\frac{h_{03}}{c^3},\frac{U}{c^2},\frac{V}{c^2},\frac{\Phi}{c^4}\Big)$$
$$=\frac{1}{c^{16}}\left\{\Big(1+\frac{2V}{c^2}\Big)^2\left[\frac{h_{123}^2}{c^6}+\Big(1+\frac{2V}{c^2}\Big)\Big(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\Big)\right]^2\right\}^{-1},$$

where,

$$f_3\Big(\frac{h_{01}}{c^3},\ldots,\frac{\Phi}{c^4}\Big)=\left\{\Big(1+\frac{2V}{c^2}\Big)^2\left[\frac{h_{123}^2}{c^6}+\Big(1+\frac{2V}{c^2}\Big)\Big(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\Big)\right]^2\right\}^{-1},$$
$$f_3(x,\,y,\,z,\,s,\,t,\,v)=\frac{1}{\big(2t+1\big)^2\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2},$$

from which,

$$\left.\frac{\partial f_3}{\partial x}\right|_0 = -\left.\frac{4x}{\big(2t+1\big)^2\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^3}\right|_0 = 0\,,$$

$$\left.\frac{\partial f_3}{\partial y}\right|_0 = -\left.\frac{4y}{\big(2t+1\big)^2\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^3}\right|_0 = 0\,,$$

$$\left.\frac{\partial f_3}{\partial z}\right|_0 = -\left.\frac{4z}{\big(2t+1\big)^2\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^3}\right|_0 = 0\,,$$

$$\left.\frac{\partial f_3}{\partial s}\right|_0 = -\left.\frac{2\big(4s-2\big)}{\big(2t+1\big)\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^3}\right|_0 = 4\,,$$

$$\left.\frac{\partial f_3}{\partial t}\right|_0 = \left\{-\frac{4\big(2s^2-2s-4v+1\big)}{\big(2t+1\big)^2\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^3}\right.$$
$$\left.-\frac{4}{\big(2t+1\big)^3\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2}\right\}\Bigg|_0 = -8\,,$$

$$\left.\frac{\partial f_3}{\partial v}\right|_0 = \left.\frac{8}{\big(2t+1\big)\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^3}\right|_0 = 8\,;$$

therefore,

$$f_3(x,\,y,\,z,\,s,\,t,\,v) = 1+4s-8t+8v+O\big(\epsilon^4\big)\,,$$
$$f_3\left(\frac{h_{01}}{c^3},\frac{h_{02}}{c^3},\frac{h_{03}}{c^3},\frac{U}{c^2},\frac{V}{c^2},\frac{\Phi}{c^4}\right) = 1+\frac{4U}{c^2}-\frac{8V}{c^2}+\frac{8\Phi}{c^4}+O\big(\epsilon^4\big)\,.$$

A.6.4 $f_4(x,y,z,s,t,v)$ expansion

$$\frac{1}{\big(c^2+2V\big)^2 D} = \frac{1}{c^{10}}\,f_4\left(\frac{h_{01}}{c^3},\frac{h_{02}}{c^3},\frac{h_{03}}{c^3},\frac{U}{c^2},\frac{V}{c^2},\frac{\Phi}{c^4}\right)$$
$$= \frac{1}{c^{10}}\left\{\left(1+\frac{2V}{c^2}\right)^2\left[\frac{h_{123}^2}{c^6}+\left(1+\frac{2V}{c^2}\right)\left(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\right)\right]\right\}^{-1}$$

where,

$$f_4\left(\frac{h_{01}}{c^3},\ldots,\frac{\Phi}{c^4}\right) = \left\{\left(1+\frac{2V}{c^2}\right)^2\left[\frac{h_{123}^2}{c^6}+\left(1+\frac{2V}{c^2}\right)\left(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\right)\right]\right\}^{-1},$$
$$f_4(x,\,y,\,z,\,s,\,t,\,v) = \frac{1}{\big(2t+1\big)^2\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]}\,,$$

from which,

$$
\begin{aligned}
\left.\frac{\partial f_4}{\partial x}\right|_0 &= -\left.\frac{2x}{\big(2t+1\big)^2\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2}\right|_0 = 0\,,\\
\left.\frac{\partial f_4}{\partial y}\right|_0 &= -\left.\frac{2y}{\big(2t+1\big)^2\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2}\right|_0 = 0\,,\\
\left.\frac{\partial f_4}{\partial z}\right|_0 &= -\left.\frac{2z}{\big(2t+1\big)^2\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2}\right|_0 = 0\,,\\
\left.\frac{\partial f_4}{\partial s}\right|_0 &= -\left.\frac{\big(4s-2\big)}{\big(2t+1\big)\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2}\right|_0 = 2\,,\\
\left.\frac{\partial f_4}{\partial t}\right|_0 &= \left\{-\frac{2\big(2s^2-2s-4v+1\big)}{\big(2t+1\big)^2\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2}\right.\\
&\quad\left.-\frac{4}{\big(2t+1\big)^3\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]}\right\}\Bigg|_0 = -6\,,\\
\left.\frac{\partial f_4}{\partial v}\right|_0 &= \left.\frac{4}{\big(2t+1\big)\Big[\big(2t+1\big)\big(2s^2-2s-4v+1\big)+x^2+y^2+z^2\Big]^2}\right|_0 = 4\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
f_4(x,\,y,\,z,\,s,\,t,\,v) &= 1+2s-6t+4v+O\big(\epsilon^4\big)\,,\\
f_4\left(\frac{h_{01}}{c^3},\frac{h_{02}}{c^3},\frac{h_{03}}{c^3},\frac{U}{c^2},\frac{V}{c^2},\frac{\Phi}{c^4}\right) &= 1+\frac{2U}{c^2}-\frac{6V}{c^2}+\frac{4\Phi}{c^4}+O\big(\epsilon^4\big)\,.
\end{aligned}
$$

A.7 Gradients applied to point-like source potentials along with their first-order derivatives and Laplacians

$$
\begin{aligned}
U(\mathbf{x}) &= \frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right),\\
V(\mathbf{x}) &= \frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right).
\end{aligned}
$$

A.7.1 $\boldsymbol{\nabla} U(\mathbf{x})$

$$
\begin{aligned}
&\boldsymbol{\nabla} U(\mathbf{x}) = \boldsymbol{i}(\partial_1 U)+\boldsymbol{j}(\partial_2 U)+\boldsymbol{k}(\partial_3 U)\,,\\
&\partial_1 U = \frac{G_N M}{a_1}\frac{x}{|\mathbf{x}|^2}\left[-\frac{1}{|\mathbf{x}|}\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right],\\
&\partial_2 U = \frac{G_N M}{a_1}\frac{y}{|\mathbf{x}|^2}\left[-\frac{1}{|\mathbf{x}|}\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right],
\end{aligned}
$$

$$\partial_3 U = \frac{G_N M}{a_1}\frac{z}{|\mathbf{x}|^2}\left[-\frac{1}{|\mathbf{x}|}\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right],$$

$$\boldsymbol{\nabla} U(\mathbf{x}) = \frac{G_N M}{a_1}\frac{\mathbf{x}}{|\mathbf{x}|^2}\left[-\frac{1}{|\mathbf{x}|}\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right].$$

A.7.2 $\boldsymbol{\nabla} V(\mathbf{x})$

$$\boldsymbol{\nabla} V(\mathbf{x}) = \boldsymbol{i}(\partial_1 V)+\boldsymbol{j}(\partial_2 V)+\boldsymbol{k}(\partial_3 V),$$

$$\partial_1 V = \frac{G_N M}{a_1}\frac{x}{|\mathbf{x}|^2}\left[-\frac{1}{|\mathbf{x}|}\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right],$$

$$\partial_2 V = \frac{G_N M}{a_1}\frac{y}{|\mathbf{x}|^2}\left[-\frac{1}{|\mathbf{x}|}\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right],$$

$$\partial_3 V = \frac{G_N M}{a_1}\frac{z}{|\mathbf{x}|^2}\left[-\frac{1}{|\mathbf{x}|}\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right],$$

$$\boldsymbol{\nabla} V(\mathbf{x}) = \frac{G_N M}{a_1}\frac{\mathbf{x}}{|\mathbf{x}|^2}\left[-\frac{1}{|\mathbf{x}|}\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right].$$

A.7.3 $\Delta U(\mathbf{x})$

$$\Delta U = \partial_{11}U+\partial_{22}U+\partial_{33}U,$$

$$\partial_{11}U = \frac{G_N M}{a_1}\left[-\frac{1}{|\mathbf{x}|^3}\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{1}{|\mathbf{x}|^2}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\right.$$
$$+\frac{3x^2}{|\mathbf{x}|^5}\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{3x^2}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)$$
$$\left.-\frac{x^2}{|\mathbf{x}|^3}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\right],$$

$$\partial_{22}U = \frac{G_N M}{a_1}\left[-\frac{1}{|\mathbf{x}|^3}\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{1}{|\mathbf{x}|^2}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\right.$$
$$+\frac{3y^2}{|\mathbf{x}|^5}\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{3y^2}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)$$
$$\left.-\frac{y^2}{|\mathbf{x}|^3}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\right],$$

$$\partial_{33}U = \frac{G_N M}{a_1}\left[-\frac{1}{|\mathbf{x}|^3}\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{1}{|\mathbf{x}|^2}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\right.$$
$$+\frac{3z^2}{|\mathbf{x}|^5}\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{3z^2}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)$$
$$\left.-\frac{z^2}{|\mathbf{x}|^3}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\right],$$

$$\Delta U(\mathbf{x}) = -\frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right).$$

A.7.4 $\Delta V(\mathbf{x})$

$$
\begin{aligned}
&\Delta V = \partial_{11}V + \partial_{22}V + \partial_{33}V\,,\\
&\partial_{11}V = \frac{G_N M}{a_1}\Bigg[-\frac{1}{|\mathbf{x}|^3}\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{1}{|\mathbf{x}|^2}\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\\
&+\frac{3x^2}{|\mathbf{x}|^5}\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{3x^2}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\\
&-\frac{x^2}{|\mathbf{x}|^3}\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\Bigg]\,,\\
&\partial_{22}V = \frac{G_N M}{a_1}\Bigg[-\frac{1}{|\mathbf{x}|^3}\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{1}{|\mathbf{x}|^2}\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\\
&+\frac{3y^2}{|\mathbf{x}|^5}\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{3y^2}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\\
&-\frac{y^2}{|\mathbf{x}|^3}\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\Bigg]\,,\\
&\partial_{33}V = \frac{G_N M}{a_1}\Bigg[-\frac{1}{|\mathbf{x}|^3}\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{1}{|\mathbf{x}|^2}\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\\
&+\frac{3z^2}{|\mathbf{x}|^5}\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{3z^2}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\\
&-\frac{z^2}{|\mathbf{x}|^3}\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\Bigg]\,,\\
&\Delta V(\mathbf{x}) = -\frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right).
\end{aligned}
$$

A.7.5 $\boldsymbol{\nabla}\big(\partial_1 U\big)$

$$
\begin{aligned}
&\boldsymbol{\nabla}(\partial_1 U) = \boldsymbol{i}(\partial_{11}U)+\boldsymbol{j}(\partial_{21}U)+\boldsymbol{k}(\partial_{31}U)\,,\\
&\quad\partial_{11}U = \text{see above}\,,\\
&\quad\partial_{21}U = \frac{G_N M}{a_1}\Bigg[\frac{3xy}{|\mathbf{x}|^5}\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{3xy}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\\
&\qquad-\frac{xy}{|\mathbf{x}|^3}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\Bigg]\,,\\
&\quad\partial_{31}U = \frac{G_N M}{a_1}\Bigg[\frac{3xz}{|\mathbf{x}|^5}\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{3xz}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\\
&\qquad-\frac{xz}{|\mathbf{x}|^3}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\Bigg].
\end{aligned}
$$

A.7.6 $\boldsymbol{\nabla}\big(\partial_2 U\big)$

$$
\boldsymbol{\nabla}(\partial_2 U) = \boldsymbol{i}(\partial_{12}U)+\boldsymbol{j}(\partial_{22}U)+\boldsymbol{k}(\partial_{32}U)\,,
$$

$$
\begin{aligned}
\partial_{12}U &= \partial_{21}U\,,\\
\partial_{22}U &= \text{see above}\,,\\
\partial_{32}U &= \frac{G_N M}{a_1}\left[\frac{3yz}{|\mathbf{x}|^5}\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{3yz}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\right.\\
&\quad\left.-\frac{yz}{|\mathbf{x}|^3}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\right].
\end{aligned}
$$

A.7.7 $\boldsymbol{\nabla}\big(\partial_3 U\big)$

$$
\begin{aligned}
\boldsymbol{\nabla}(\partial_3 U) &= \boldsymbol{i}(\partial_{13}U)+\boldsymbol{j}(\partial_{23}U)+\boldsymbol{k}(\partial_{33}U)\,,\\
\partial_{13}U &= \partial_{31}U\,,\\
\partial_{23}U &= \partial_{32}U\,,\\
\partial_{33}U &= \text{see above}\,.
\end{aligned}
$$

A.7.8 $\boldsymbol{\nabla}\big(\partial_1 V\big)$

$$
\begin{aligned}
\boldsymbol{\nabla}(\partial_1 V) &= \boldsymbol{i}(\partial_{11}V)+\boldsymbol{j}(\partial_{21}V)+\boldsymbol{k}(\partial_{31}V)\,,\\
\partial_{11}V &= \text{see above}\,,\\
\partial_{21}V &= \frac{G_N M}{a_1}\left[\frac{3xy}{|\mathbf{x}|^5}\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{3xy}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\right.\\
&\quad\left.-\frac{xy}{|\mathbf{x}|^3}\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\right],\\
\partial_{31}V &= \frac{G_N M}{a_1}\left[\frac{3xz}{|\mathbf{x}|^5}\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{3xz}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+.\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\right.\\
&\quad\left.-\frac{xz}{|\mathbf{x}|^3}\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\right].
\end{aligned}
$$

A.7.9 $\boldsymbol{\nabla}\big(\partial_2 V\big)$

$$
\begin{aligned}
\boldsymbol{\nabla}(\partial_2 V) &= \boldsymbol{i}(\partial_{12}V)+\boldsymbol{j}(\partial_{22}V)+\boldsymbol{k}(\partial_{32}V)\,,\\
\partial_{12}V &= \partial_{21}V\,,\\
\partial_{22}V &= \text{see above}\,,\\
\partial_{32}V &= \frac{G_N M}{a_1}\left[\frac{3yz}{|\mathbf{x}|^5}\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{3yz}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\right.\\
&\quad\left.-\frac{yz}{|\mathbf{x}|^3}\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\right].
\end{aligned}
$$

A.7.10 $\boldsymbol{\nabla}\big(\partial_3 V\big)$

$$
\boldsymbol{\nabla}(\partial_3 V) = \boldsymbol{i}(\partial_{13}V)+\boldsymbol{j}(\partial_{23}V)+\boldsymbol{k}(\partial_{33}V)\,,
$$

$$\partial_{13}V = \partial_{31}V\,,$$
$$\partial_{23}V = \partial_{32}V\,,$$
$$\partial_{33}V = \text{see above}\,.$$

Appendix B

Gauss-Bonnet identity

Leaving aside the proof, for which reference is made to C. Lanczos' paper [71], it turns out, from the variational calculus, for four-dimensional manifolds only, that the following quadratic Lagrangians produce the same field equations (in vacuum); in fact, from

$$\mathcal{L}_1 = 4R_{\mu\nu}R^{\mu\nu} - R^2\,,$$

follows

$$\delta \int \sqrt{-g}\,\mathcal{L}_1\, d^4x = 0 \quad \Rightarrow \quad 4\,{}^2H_{\mu\nu} - {}^1H_{\mu\nu} = 0\,,$$

from

$$\mathcal{L}_2 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}\,,$$

$$\delta \int \sqrt{-g}\,\mathcal{L}_2\, d^4x = 0 \quad \Rightarrow \quad {}^3H_{\mu\nu} = 4\,{}^2H_{\mu\nu} - {}^1H_{\mu\nu} = 0\,,$$

(the explicit expressions of the left-hand sides of the field equations can be found on page 28), that is,

$${}^1H_{\mu\nu} - 4\,{}^2H_{\mu\nu} + {}^3H_{\mu\nu} = 0\,,$$

better known as the *Gauss-Bonnet identity*. Therefore, it can be concluded that $\mathcal{L}_1$ and $\mathcal{L}_2$ are equivalent, from a variational point of view and consequently,

$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = 4R_{\mu\nu}R^{\mu\nu} - R^2\,.$$

Appendix C

Christoffel symbols

From the general expression

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\, g^{\alpha\lambda}\big(g_{\lambda\mu,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}\big)\,,$$

it follows:

- $\Gamma^{0}_{\mu\nu} = \frac{1}{2}\, g^{0\lambda}\big(g_{\lambda\mu,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}\big)$

$$\begin{aligned}
\Gamma^{0}_{00} &= -\frac{1}{c^7}\left\{h_{01}\Big[\big(c^2-2U\big)\partial_1 U + 2\,\partial_1\Phi\Big] + h_{02}\Big[\big(c^2-2U\big)\partial_2 U + 2\,\partial_2\Phi\Big]\right.\\
&\quad \left. + h_{03}\Big[\big(c^2-2U\big)\partial_3 U + 2\,\partial_3\Phi\Big]\right\} = -\frac{1}{c^7}\left(h_{01}PD_1 + h_{02}PD_2 + h_{03}PD_3\right),\\
\Gamma^{0}_{01} &= -\frac{1}{c^8}\Big[\big(c^2-2U\big)\partial_1 U + 2\,\partial_1\Phi\Big]\big(c^4 + 2U^2 + 2c^2U + 4\Phi\big) = -\frac{q}{c^8}\,PD_1\,,\\
\Gamma^{0}_{02} &= -\frac{1}{c^8}\Big[\big(c^2-2U\big)\partial_2 U + 2\,\partial_2\Phi\Big]\big(c^4 + 2U^2 + 2c^2U + 4\Phi\big) = -\frac{q}{c^8}\,PD_2\,,\\
\Gamma^{0}_{03} &= -\frac{1}{c^8}\Big[\big(c^2-2U\big)\partial_3 U + 2\,\partial_3\Phi\Big]\big(c^4 + 2U^2 + 2c^2U + 4\Phi\big) = -\frac{q}{c^8}\,PD_3\,,\\
\Gamma^{0}_{11} &= -\frac{1}{c^5}\left(h_{03}\,\partial_3 V + h_{02}\,\partial_2 V - h_{01}\,\partial_1 V\right),\\
\Gamma^{0}_{12} &= \frac{1}{c^5}\left(h_{01}\,\partial_2 V + h_{02}\,\partial_1 V\right),\\
\Gamma^{0}_{13} &= \frac{1}{c^5}\left(h_{01}\,\partial_3 V + h_{03}\,\partial_1 V\right),\\
\Gamma^{0}_{22} &= -\frac{1}{c^5}\left(h_{03}\,\partial_3 V - h_{02}\,\partial_2 V + h_{01}\,\partial_1 V\right),\\
\Gamma^{0}_{23} &= \frac{1}{c^5}\left(h_{02}\,\partial_3 V + h_{03}\,\partial_2 V\right),\\
\Gamma^{0}_{33} &= -\frac{1}{c^5}\left(-\,h_{03}\,\partial_3 V + h_{02}\,\partial_2 V + h_{01}\,\partial_1 V\right);
\end{aligned}$$

- $\Gamma^{1}_{\mu\nu} = \frac{1}{2}\, g^{1\lambda}\big(g_{\lambda\mu,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}\big)$

$$\begin{aligned}
\Gamma^{1}_{00} &= -\,\frac{1}{c^6}\left(c^2-2V\right)\Big[\big(c^2-2U\big)\partial_1 U + 2\,\partial_1\Phi\Big] = -\,\frac{1}{c^6}\left(c^2-2V\right)PD_1\,,\\
\Gamma^{1}_{01} &= \frac{1}{c^7}\,h_{01}\Big[\big(c^2-2U\big)\partial_1 U + 2\,\partial_1\Phi\Big] = \frac{h_{01}}{c^7}PD_1\,,
\end{aligned}$$

$$\Gamma^1_{02} = \frac{1}{c^7}\, h_{01}\Big[\big(c^2-2U\big)\partial_2 U + 2\,\partial_2\Phi\Big] = \frac{h_{01}}{c^7} PD_2\,,$$
$$\Gamma^1_{03} = \frac{1}{c^7}\, h_{01}\Big[\big(c^2-2U\big)\partial_3 U + 2\,\partial_3\Phi\Big] = \frac{h_{01}}{c^7} PD_3\,,$$
$$\Gamma^1_{11} = \frac{1}{c^4}\big(c^2-2V\big)\partial_1 V\,,$$
$$\Gamma^1_{12} = \frac{1}{c^4}\big(c^2-2V\big)\partial_2 V\,,$$
$$\Gamma^1_{13} = \frac{1}{c^4}\big(c^2-2V\big)\partial_3 V\,,$$
$$\Gamma^1_{22} = -\,\frac{1}{c^4}\,\big(c^2-2V\big)\partial_1 V\,,$$
$$\Gamma^1_{23} = 0\,,$$
$$\Gamma^1_{33} = -\,\frac{1}{c^4}\,\big(c^2-2V\big)\partial_1 V\,;$$

- $\Gamma^2_{\mu\nu} = \frac{1}{2}\, g^{2\lambda}\big(g_{\lambda\mu,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}\big)$

$$\Gamma^2_{00} = -\,\frac{1}{c^6}\,\big(c^2-2V\big)\Big[\big(c^2-2U\big)\partial_2 U + 2\,\partial_2\Phi\Big] = -\,\frac{1}{c^6}\,\big(c^2-2V\big) PD_2\,,$$
$$\Gamma^2_{01} = \frac{1}{c^7}\, h_{02}\Big[\big(c^2-2U\big)\partial_1 U + 2\,\partial_1\Phi\Big] = \frac{h_{02}}{c^7} PD_1\,,$$
$$\Gamma^2_{02} = \frac{1}{c^7}\, h_{02}\Big[\big(c^2-2U\big)\partial_2 U + 2\,\partial_2\Phi\Big] = \frac{h_{02}}{c^7} PD_2\,,$$
$$\Gamma^2_{03} = \frac{1}{c^7}\, h_{02}\Big[\big(c^2-2U\big)\partial_3 U + 2\,\partial_3\Phi\Big] = \frac{h_{02}}{c^7} PD_3\,,$$
$$\Gamma^2_{11} = -\,\frac{1}{c^4}\big(c^2-2V\big)\partial_2 V\,,$$
$$\Gamma^2_{12} = \frac{1}{c^4}\big(c^2-2V\big)\partial_1 V\,,$$
$$\Gamma^2_{13} = 0\,,$$
$$\Gamma^2_{22} = \frac{1}{c^4}\big(c^2-2V\big)\partial_2 V\,,$$
$$\Gamma^2_{23} = \frac{1}{c^4}\big(c^2-2V\big)\partial_3 V\,,$$
$$\Gamma^2_{33} = -\,\frac{1}{c^4}\big(c^2-2V\big)\partial_2 V\,;$$

- $\Gamma^3_{\mu\nu} = \frac{1}{2}\, g^{3\lambda}\big(g_{\lambda\mu,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}\big)$

$$\Gamma^3_{00} = -\,\frac{1}{c^6}\,\big(c^2-2V\big)\Big[\big(c^2-2U\big)\partial_3 U + 2\,\partial_3\Phi\Big] = -\,\frac{1}{c^6}\,\big(c^2-2V\big) PD_3\,,$$
$$\Gamma^3_{01} = \frac{1}{c^7}\, h_{03}\Big[\big(c^2-2U\big)\partial_1 U + 2\,\partial_1\Phi\Big] = \frac{h_{03}}{c^7} PD_1\,,$$
$$\Gamma^3_{02} = \frac{1}{c^7}\, h_{03}\Big[\big(c^2-2U\big)\partial_2 U + 2\,\partial_2\Phi\Big] = \frac{h_{03}}{c^7} PD_2\,,$$
$$\Gamma^3_{03} = \frac{1}{c^7}\, h_{03}\Big[\big(c^2-2U\big)\partial_3 U + 2\,\partial_3\Phi\Big] = \frac{h_{03}}{c^7} PD_3\,,$$
$$\Gamma^3_{11} = -\,\frac{1}{c^4}\big(c^2-2V\big)\partial_3 V\,,$$
$$\Gamma^3_{12} = 0\,,$$

$$\Gamma^3_{13} = \frac{1}{c^4}\big(c^2 - 2V\big)\partial_1 V\,,$$
$$\Gamma^3_{22} = -\frac{1}{c^4}\big(c^2 - 2V\big)\partial_3 V\,,$$
$$\Gamma^3_{23} = \frac{1}{c^4}\big(c^2 - 2V\big)\partial_2 V\,,$$
$$\Gamma^3_{33} = \frac{1}{c^4}\big(c^2 - 2V\big)\partial_3 V\,.$$

Appendix D

Ricci tensor

D.1 Preliminary results

D.1.1 $R_{0\mu}$ components $\quad \mu = 0, 1, 2, 3.$

R_{00} component

$$R_{00} = \frac{\partial \Gamma^{\alpha}_{00}}{\partial x^{\alpha}} - \frac{\partial^2}{\partial x^{0^2}} \ln\sqrt{-g} + \Gamma^{\sigma}_{00} \frac{\partial}{\partial x^{\sigma}} \ln\sqrt{-g} - \Gamma^{\sigma}_{0\alpha}\Gamma^{\alpha}_{\sigma 0}\,;$$

- 1-st term

$$\begin{aligned}
&\frac{\partial \Gamma^{\alpha}_{00}}{\partial x^{\alpha}} = \frac{\partial \Gamma^{0}_{00}}{\partial x^{0}} + \frac{\partial \Gamma^{1}_{00}}{\partial x^{1}} + \frac{\partial \Gamma^{2}_{00}}{\partial x^{2}} + \frac{\partial \Gamma^{3}_{00}}{\partial x^{3}}\,,\\
&\frac{\partial \Gamma^{0}_{00}}{\partial x^{0}} = 0\,,\\
&\frac{\partial \Gamma^{0}_{00}}{\partial x^{0}} = \frac{1}{c^6}\left\{2\,PD_1\,\partial_1 V - \left(c^2 - 2V\right)\left[PD_{11} - 2\left(\partial_1 U\right)^2\right]\right\},\\
&\frac{\partial \Gamma^{2}_{00}}{\partial x^{2}} = \frac{1}{c^6}\left\{2\,PD_2\,\partial_2 V - \left(c^2 - 2V\right)\left[PD_{22} - 2\left(\partial_2 U\right)^2\right]\right\},\\
&\frac{\partial \Gamma^{3}_{00}}{\partial x^{3}} = \frac{1}{c^6}\left\{2\,PD_3\,\partial_3 V - \left(c^2 - 2V\right)\left[PD_{33} - 2\left(\partial_3 U\right)^2\right]\right\};
\end{aligned}$$

- 2-nd term

$$\frac{\partial^2}{\partial x^{0^2}} \ln\sqrt{-g} = 0\,;$$

- 3-rd term

$$\begin{aligned}
&\Gamma^{\sigma}_{00}\frac{\partial}{\partial x^{\sigma}} \ln\sqrt{-g} = \Gamma^{0}_{00}\frac{\partial}{\partial x^{0}} \ln\sqrt{-g} + \Gamma^{1}_{00}\frac{\partial}{\partial x^{1}} \ln\sqrt{-g} + \Gamma^{2}_{00}\frac{\partial}{\partial x^{2}} \ln\sqrt{-g} + \Gamma^{3}_{00}\frac{\partial}{\partial x^{3}} \ln\sqrt{-g}\,,\\
&\Gamma^{0}_{00}\frac{\partial}{\partial x^{0}} \ln\sqrt{-g} = 0\,,\\
&\Gamma^{1}_{00}\frac{\partial}{\partial x^{1}} \ln\sqrt{-g} = \left(c^2 - 2V\right)\left[\left(c^2 + 2V\right)^2 PD_1^2 - \left(2\,h_{123}^2 + 3P\right)PD_1\,\partial_1 V\right]\Big/ c^6\left(c^2 + 2V\right)D\,,\\
&\Gamma^{2}_{00}\frac{\partial}{\partial x^{2}} \ln\sqrt{-g} = \left(c^2 - 2V\right)\left[\left(c^2 + 2V\right)^2 PD_2^2 - (2\,h_{123}^2 + 3P)PD_2\,\partial_2 V\right]\Big/ c^6\left(c^2 + 2V\right)D\,,\\
&\Gamma^{3}_{00}\frac{\partial}{\partial x^{3}} \ln\sqrt{-g} = \left(c^2 - 2V\right)\left[\left(c^2 + 2V\right)^2 PD_3^2 - \left(2\,h_{123}^2 + 3P\right)PD_3\,\partial_3 V\right]\Big/ c^6\left(c^2 + 2V\right)D\,;
\end{aligned}$$

- 4-th term

$$
\begin{aligned}
&\Gamma^{\sigma}_{0\alpha}\Gamma^{\alpha}_{\sigma 0} = \Big(\Gamma^{0}_{00}\Gamma^{0}_{00} + \cdots + \Gamma^{0}_{03}\Gamma^{3}_{00}\Big) + \Big(\Gamma^{1}_{00}\Gamma^{0}_{10} + \cdots + \Gamma^{1}_{03}\Gamma^{3}_{10}\Big) + \Big(\Gamma^{2}_{00}\Gamma^{0}_{20} + \cdots + \Gamma^{2}_{03}\Gamma^{3}_{20}\Big) \\
&+ \Big(\Gamma^{3}_{00}\Gamma^{0}_{30} + \cdots + \Gamma^{3}_{03}\Gamma^{3}_{30}\Big)\,, \\
&\Big(\Gamma^{0}_{00}\Gamma^{0}_{00} + \cdots + \Gamma^{0}_{03}\Gamma^{3}_{00}\Big) = \frac{1}{c^{14}}\Big[Q\Big(PD_1^2 + PD_2^2 + PD_3^2\Big) + \big(h_{01}PD_1 + h_{02}PD_2 + h_{03}PD_3\big)^2\Big]\,, \\
&\Big(\Gamma^{1}_{00}\Gamma^{0}_{10} + \cdots + \Gamma^{1}_{03}\Gamma^{3}_{10}\Big) = \frac{1}{c^{14}}\Big(h_{01}h_{03}PD_1PD_3 + h_{01}h_{02}PD_1PD_2 + h_{01}^2PD_1^2 + Q\,PD_1^2\Big)\,, \\
&\Big(\Gamma^{2}_{00}\Gamma^{0}_{20} + \cdots + \Gamma^{2}_{03}\Gamma^{3}_{20}\Big) = \frac{1}{c^{14}}\Big(h_{01}h_{02}PD_1PD_2 + h_{02}^2PD_2^2 + h_{02}h_{03}PD_2PD_3 + Q\,PD_2^2\Big)\,, \\
&\Big(\Gamma^{3}_{00}\Gamma^{0}_{30} + \cdots + \Gamma^{3}_{03}\Gamma^{3}_{30}\Big) = \frac{1}{c^{14}}\Big(h_{01}h_{03}PD_1PD_3 + h_{02}h_{03}PD_2PD_3 + h_{03}^2PD_3^2 + Q\,PD_3^2\Big)\,;
\end{aligned}
$$

$$
\begin{aligned}
&R_{00} = \frac{2}{c^6}\big(PD_1\,\partial_1V + PD_2\,\partial_2V + PD_3\,\partial_3V\big) - \frac{\big(c^2-2V\big)}{c^6}\Big\{\Big[PD_{11} - 2\big(\partial_1U\big)^2\Big] + \Big[PD_{22} \\
&- 2\big(\partial_2U\big)^2\Big] + \Big[PD_{33} - 2\big(\partial_3U\big)^2\Big]\Big\} + \frac{1}{c^6\big(c^2+2V\big)D}\Big[\big(c^2-2V\big)\big(c^2+2V\big)^2\big(PD_1^2 + PD_2^2 \\
&+ PD_3^2\big) - \big(c^2-2V\big)\big(2D+P\big)\big(PD_1\,\partial_1V + PD_2\,\partial_2V + PD_3\,\partial_3V\big)\Big] - \frac{1}{c^{14}}\Big[2\,Q\big(PD_1^2 + PD_2^2 \\
&+ PD_3^2\big) + \big(h_{01}PD_1 + h_{02}PD_2 + h_{03}PD_3\big)^2 + 2\,h_{01}h_{02}PD_1PD_2 + 2\,h_{01}h_{03}PD_1PD_3 \\
&+ 2\,h_{02}h_{03}PD_2PD_3 + h_{01}^2PD_1^2 + h_{02}^2PD_2^2 + h_{03}^2PD_3^2\Big]\,.
\end{aligned}
$$

R_{01} component

$$
R_{01} = \frac{\partial\Gamma^{\alpha}_{01}}{\partial x^{\alpha}} - \frac{\partial^2}{\partial x^0\partial x^1}\ln\sqrt{-g} + \Gamma^{\sigma}_{01}\frac{\partial}{\partial x^{\sigma}}\ln\sqrt{-g} - \Gamma^{\sigma}_{0\alpha}\Gamma^{\alpha}_{\sigma 1}\,;
$$

- 1-st term

$$
\begin{aligned}
&\frac{\partial\Gamma^{\alpha}_{01}}{\partial x^{\alpha}} = \frac{\partial\Gamma^{0}_{01}}{\partial x^0} + \frac{\partial\Gamma^{1}_{01}}{\partial x^1} + \frac{\partial\Gamma^{2}_{01}}{\partial x^2} + \frac{\partial\Gamma^{3}_{01}}{\partial x^3}\,, \\
&\frac{\partial\Gamma^{0}_{01}}{\partial x^0} = 0\,, \\
&\frac{\partial\Gamma^{1}_{01}}{\partial x^1} = \frac{h_{01}}{c^7}\Big[PD_{11} - 2\big(\partial_1U\big)^2\Big]\,, \\
&\frac{\partial\Gamma^{2}_{01}}{\partial x^2} = \frac{h_{02}}{c^7}\Big[PD_{12} - 2\,\partial_1U\partial_2U\Big]\,, \\
&\frac{\partial\Gamma^{3}_{01}}{\partial x^3} = \frac{h_{03}}{c^7}\Big[PD_{13} - 2\,\partial_1U\partial_3U\Big]\,;
\end{aligned}
$$

- 2-nd term

$$
\frac{\partial^2}{\partial x^0\partial x^1}\ln\sqrt{-g} = 0\,;
$$

- 3-rd term

$$\Gamma^{\sigma}_{01}\frac{\partial}{\partial x^{\sigma}}\ln\sqrt{-g}=\Gamma^{0}_{01}\frac{\partial}{\partial x^{0}}\ln\sqrt{-g}+\Gamma^{1}_{01}\frac{\partial}{\partial x^{1}}\ln\sqrt{-g}+\Gamma^{2}_{01}\frac{\partial}{\partial x^{2}}\ln\sqrt{-g}+\Gamma^{3}_{01}\frac{\partial}{\partial x^{3}}\ln\sqrt{-g}\,,$$
$$\Gamma^{0}_{01}\frac{\partial}{\partial x^{0}}\ln\sqrt{-g}=0\,,$$
$$\Gamma^{1}_{01}\frac{\partial}{\partial x^{1}}\ln\sqrt{-g}=h_{01}\Big[-\big(c^2+2V\big)^2PD_1^2+\big(2D+P\big)PD_1\,\partial_1V\Big]\Big/c^7\big(c^2+2V\big)D\,,$$
$$\Gamma^{2}_{01}\frac{\partial}{\partial x^{2}}\ln\sqrt{-g}=h_{02}\Big[-\big(c^2+2V\big)^2PD_1PD_2+\big(2D+P\big)PD_1\,\partial_2V\Big]\Big/c^7\big(c^2+2V\big)D\,,$$
$$\Gamma^{3}_{01}\frac{\partial}{\partial x^{3}}\ln\sqrt{-g}=h_{03}\Big[-\big(c^2+2V\big)^2PD_1PD_3+\big(2D+P\big)PD_1\,\partial_3V\Big]\Big/c^7\big(c^2+2V\big)D\,;$$

- 4-th term

$$\Gamma^{\sigma}_{0\alpha}\Gamma^{\alpha}_{\sigma1}=\Big(\Gamma^{0}_{00}\Gamma^{0}_{01}+\cdots+\Gamma^{0}_{03}\Gamma^{3}_{01}\Big)+\Big(\Gamma^{1}_{00}\Gamma^{0}_{11}+\cdots+\Gamma^{1}_{03}\Gamma^{3}_{11}\Big)+\Big(\Gamma^{2}_{00}\Gamma^{0}_{21}+\cdots+\Gamma^{2}_{03}\Gamma^{3}_{21}\Big)$$
$$+\Big(\Gamma^{3}_{00}\Gamma^{0}_{31}+\cdots+\Gamma^{3}_{03}\Gamma^{3}_{31}\Big)\,,$$
$$\Big(\Gamma^{0}_{00}\Gamma^{0}_{01}+\cdots+\Gamma^{0}_{03}\Gamma^{3}_{01}\Big)=0\,,$$
$$\Big(\Gamma^{1}_{00}\Gamma^{0}_{11}+\cdots+\Gamma^{1}_{03}\Gamma^{3}_{11}\Big)=-\frac{\big(c^2-2V\big)}{c^{11}}\Big[h_{01}(PD_2\,\partial_2V+PD_3\,\partial_3V)-PD_1\big(h_{02}\,\partial_2V$$
$$+h_{03}\,\partial_3V\big)\Big]\,,$$
$$\Big(\Gamma^{2}_{00}\Gamma^{0}_{21}+\cdots+\Gamma^{2}_{03}\Gamma^{3}_{21}\Big)=-\frac{\big(c^2-2V\big)}{c^{11}}\,\partial_2V\big(h_{01}PD_2-h_{02}PD_1\big)\,,$$
$$\Big(\Gamma^{3}_{00}\Gamma^{0}_{31}+\cdots+\Gamma^{3}_{03}\Gamma^{3}_{31}\Big)=-\frac{\big(c^2-2V\big)}{c^{11}}\,\partial_3V\big(h_{01}PD_3-h_{03}PD_1\big)\,;$$

$$R_{01}=\frac{1}{c^7}\Big\{h_{01}\Big[PD_{11}-2\big(\partial_1U\big)^2\Big]+h_{02}\big(PD_{12}-2\,\partial_1U\,\partial_2U\big)+h_{03}\big(PD_{13}-2\,\partial_1U\,\partial_3U\big)\Big\}$$
$$+\frac{1}{c^7\big(c^2+2V\big)D}\Big\{h_{01}\Big[-\big(c^2+2V\big)^2PD_1^2+\big(2D+P\big)PD_1\,\partial_1V\Big]+h_{02}\Big[-\big(c^2+2V\big)^2PD_1$$
$$PD_2+\big(2D+P\big)PD_1\,\partial_2V\Big]+h_{03}\Big[-\big(c^2+2V\big)^2PD_1PD_3+\big(2D+P\big)PD_1\,\partial_3V\Big]\Big\}$$
$$+\frac{2\big(c^2-2V\big)}{c^{11}}\Big[h_{01}\big(PD_2\,\partial_2V+PD_3\,\partial_3V\big)-h_{02}PD_1\,\partial_2V-h_{03}PD_1\,\partial_3V\Big]\,.$$

R_{02} component

$$R_{02}=\frac{\partial\Gamma^{\alpha}_{02}}{\partial x^{\alpha}}-\frac{\partial^2}{\partial x^0\partial x^2}\ln\sqrt{-g}+\Gamma^{\sigma}_{02}\frac{\partial}{\partial x^{\sigma}}\ln\sqrt{-g}-\Gamma^{\sigma}_{0\alpha}\Gamma^{\alpha}_{\sigma2}\,;$$

- 1-st term

$$\frac{\partial\Gamma^{\alpha}_{02}}{\partial x^{\alpha}}=\frac{\partial\Gamma^{0}_{02}}{\partial x^{0}}+\frac{\partial\Gamma^{1}_{02}}{\partial x^{1}}+\frac{\partial\Gamma^{2}_{02}}{\partial x^{2}}+\frac{\partial\Gamma^{3}_{02}}{\partial x^{3}}\,,$$
$$\frac{\partial\Gamma^{0}_{02}}{\partial x^{0}}=0\,,$$

$$\frac{\partial\Gamma^1_{02}}{\partial x^1} = \frac{h_{01}}{c^7}\Big[PD_{12} - 2\,\partial_1 U\partial_2 U\Big]\,,$$
$$\frac{\partial\Gamma^2_{02}}{\partial x^2} = \frac{h_{02}}{c^7}\Big[PD_{22} - 2\big(\partial_2 U\big)^2\Big]\,,$$
$$\frac{\partial\Gamma^3_{02}}{\partial x^3} = \frac{h_{03}}{c^7}\Big[PD_{23} - 2\,\partial_2 U\partial_3 U\Big]\,;$$

- 2-nd term

$$\frac{\partial^2}{\partial x^0\partial x^2}\ln\sqrt{-g} = 0\,;$$

- 3-rd term

$$\Gamma^\sigma_{02}\frac{\partial}{\partial x^\sigma}\ln\sqrt{-g} = \Gamma^0_{02}\frac{\partial}{\partial x^0}\ln\sqrt{-g} + \Gamma^1_{02}\frac{\partial}{\partial x^1}\ln\sqrt{-g} + \Gamma^2_{02}\frac{\partial}{\partial x^2}\ln\sqrt{-g} + \Gamma^3_{02}\frac{\partial}{\partial x^3}\ln\sqrt{-g}\,,$$
$$\Gamma^0_{02}\frac{\partial}{\partial x^0}\ln\sqrt{-g} = 0\,,$$
$$\Gamma^1_{02}\frac{\partial}{\partial x^1}\ln\sqrt{-g} = h_{01}\Big[-\big(c^2+2V\big)^2 PD_1PD_2 + \big(2D+P\big)PD_2\,\partial_1 V\Big]\Big/c^7\big(c^2+2V\big)D\,,$$
$$\Gamma^2_{02}\frac{\partial}{\partial x^2}\ln\sqrt{-g} = h_{02}\Big[-\big(c^2+2V\big)^2 PD_2^2 + \big(2D+P\big)PD_2\,\partial_2 V\Big]\Big/c^7\big(c^2+2V\big)D\,,$$
$$\Gamma^3_{02}\frac{\partial}{\partial x^3}\ln\sqrt{-g} = h_{03}\Big[-\big(c^2+2V\big)^2 PD_2PD_3 + \big(2D+P\big)PD_2\,\partial_3 V\Big]\Big/c^7\big(c^2+2V\big)D\,;$$

- 4-th term

$$\Gamma^\sigma_{0\alpha}\Gamma^\alpha_{\sigma 2} = \Big(\Gamma^0_{00}\Gamma^0_{02} + \cdots + \Gamma^0_{03}\Gamma^3_{02}\Big) + \Big(\Gamma^1_{00}\Gamma^0_{12} + \cdots + \Gamma^1_{03}\Gamma^3_{12}\Big) + \Big(\Gamma^2_{00}\Gamma^0_{22} + \cdots + \Gamma^2_{03}\Gamma^3_{22}\Big)$$
$$+ \Big(\Gamma^3_{00}\Gamma^0_{32} + \cdots + \Gamma^3_{03}\Gamma^3_{32}\Big)\,,$$
$$\Big(\Gamma^0_{00}\Gamma^0_{02} + \cdots + \Gamma^0_{03}\Gamma^3_{02}\Big) = 0\,,$$
$$\Big(\Gamma^1_{00}\Gamma^0_{12} + \cdots + \Gamma^1_{03}\Gamma^3_{12}\Big) = \frac{\big(c^2-2V\big)}{c^{11}}\,\partial_1 V\big(h_{01}PD_2 - h_{02}PD_1\big)\,,$$
$$\Big(\Gamma^2_{00}\Gamma^0_{22} + \cdots + \Gamma^2_{03}\Gamma^3_{22}\Big) = \frac{\big(c^2-2V\big)}{c^{11}}\Big[PD_2\big(h_{01}\,\partial_1 V + h_{03}\,\partial_3 V\big) - h_{02}\big(PD_1\,\partial_1 V$$
$$+ PD_3\,\partial_3 V\big)\Big]\,,$$
$$\Big(\Gamma^3_{00}\Gamma^0_{32} + \cdots + \Gamma^3_{03}\Gamma^3_{32}\Big) = \frac{\big(c^2-2V\big)}{c^{11}}\,\partial_3 V\big(h_{03}PD_2 - h_{02}PD_3\big)\,;$$

$$R_{02} = \frac{1}{c^7}\Big\{h_{01}\big(PD_{12} - 2\,\partial_1 U\,\partial_2 U\big) + h_{02}\Big[PD_{22} - 2\big(\partial_2 U\big)^2\Big] + h_{03}\big(PD_{23} - 2\,\partial_2 U\,\partial_3 U\big)\Big\}$$
$$+ \frac{1}{c^7\big(c^2+2V\big)D}\Big\{h_{01}\Big[-\big(c^2+2V\big)^2 PD_1PD_2 + \big(2D+P\big)PD_2\,\partial_1 V\Big] + h_{02}\Big[-\big(c^2+2V\big)^2$$
$$PD_2^2 + \big(2D+P\big)PD_2\,\partial_2 V\Big] + h_{03}\Big[-\big(c^2+2V\big)^2 PD_2PD_3 + \big(2D+P\big)PD_2\,\partial_3 V\Big]\Big\}$$
$$- \frac{2\big(c^2-2V\big)}{c^{11}}\Big[h_{01}PD_2\,\partial_1 V - h_{02}\big(PD_1\,\partial_1 V + PD_3\,\partial_3 V\big) + h_{03}PD_2\,\partial_3 V\Big]\,.$$

***R_{03}* component**

$$R_{03} = \frac{\partial \Gamma^{\alpha}_{03}}{\partial x^{\alpha}} - \frac{\partial^2}{\partial x^0 \partial x^3} \ln\sqrt{-g} + \Gamma^{\sigma}_{03} \frac{\partial}{\partial x^{\sigma}} \ln\sqrt{-g} - \Gamma^{\sigma}_{0\alpha}\Gamma^{\alpha}_{\sigma 3}\,;$$

- 1-st term

$$\begin{aligned}
&\frac{\partial \Gamma^{\alpha}_{03}}{\partial x^{\alpha}} = \frac{\partial \Gamma^{0}_{03}}{\partial x^{0}} + \frac{\partial \Gamma^{1}_{03}}{\partial x^{1}} + \frac{\partial \Gamma^{2}_{03}}{\partial x^{2}} + \frac{\partial \Gamma^{3}_{03}}{\partial x^{3}}\,,\\
&\frac{\partial \Gamma^{0}_{03}}{\partial x^{0}} = 0\,,\\
&\frac{\partial \Gamma^{1}_{03}}{\partial x^{1}} = \frac{h_{01}}{c^7}\Big[PD_{13} - 2\,\partial_1 U \partial_3 U\Big]\,,\\
&\frac{\partial \Gamma^{2}_{03}}{\partial x^{2}} = \frac{h_{02}}{c^7}\Big[PD_{23} - 2\,\partial_2 U \partial_3 U\Big]\,,\\
&\frac{\partial \Gamma^{3}_{03}}{\partial x^{3}} = \frac{h_{03}}{c^7}\Big[PD_{33} - 2\,\big(\partial_3 U\big)^2\Big]\,;
\end{aligned}$$

- 2-nd term

$$\frac{\partial^2}{\partial x^0 \partial x^3} \ln\sqrt{-g} = 0\,;$$

- 3-rd term

$$\begin{aligned}
&\Gamma^{\sigma}_{03} \frac{\partial}{\partial x^{\sigma}} \ln\sqrt{-g} = \Gamma^{0}_{03} \frac{\partial}{\partial x^{0}} \ln\sqrt{-g} + \Gamma^{1}_{03} \frac{\partial}{\partial x^{1}} \ln\sqrt{-g} + \Gamma^{2}_{03} \frac{\partial}{\partial x^{2}} \ln\sqrt{-g} + \Gamma^{3}_{03} \frac{\partial}{\partial x^{3}} \ln\sqrt{-g}\,,\\
&\Gamma^{1}_{03} \frac{\partial}{\partial x^{1}} \ln\sqrt{-g} = h_{01}\Big[-\big(c^2+2V\big)^2 PD_1 PD_3 + \big(2D+P\big)PD_3\,\partial_1 V\Big]\Big/ c^7\big(c^2+2V\big)D\,,\\
&\Gamma^{2}_{03} \frac{\partial}{\partial x^{2}} \ln\sqrt{-g} = h_{02}\Big[-\big(c^2+2V\big)^2 PD_2 PD_3 + \big(2D+P\big)PD_3\,\partial_2 V\Big]\Big/ c^7\big(c^2+2V\big)D\,,\\
&\Gamma^{3}_{03} \frac{\partial}{\partial x^{3}} \ln\sqrt{-g} = h_{03}\Big[-\big(c^2+2V\big)^2 PD_3^2 + \big(2D+P\big)PD_3\,\partial_3 V\Big]\Big/ c^7\big(c^2+2V\big)D\,;
\end{aligned}$$

- 4-th term

$$\begin{aligned}
&\Gamma^{\sigma}_{0\alpha}\Gamma^{\alpha}_{\sigma 3} = \Big(\Gamma^{0}_{00}\Gamma^{0}_{03} + \cdots + \Gamma^{0}_{03}\Gamma^{3}_{03}\Big) + \Big(\Gamma^{1}_{00}\Gamma^{0}_{13} + \cdots + \Gamma^{1}_{03}\Gamma^{3}_{13}\Big) + \Big(\Gamma^{2}_{00}\Gamma^{0}_{23} + \cdots + \Gamma^{2}_{03}\Gamma^{3}_{23}\Big)\\
&+ \Big(\Gamma^{3}_{00}\Gamma^{0}_{33} + \cdots + \Gamma^{3}_{03}\Gamma^{3}_{33}\Big)\,,\\
&\Big(\Gamma^{0}_{00}\Gamma^{0}_{03} + \cdots + \Gamma^{0}_{03}\Gamma^{3}_{03}\Big) = 0\,,\\
&\Big(\Gamma^{1}_{00}\Gamma^{0}_{13} + \cdots + \Gamma^{1}_{03}\Gamma^{3}_{13}\Big) = \frac{\big(c^2-2V\big)}{c^{11}}\,\partial_1 V\big(h_{01}PD_3 - h_{03}PD_1\big)\,,\\
&\Big(\Gamma^{2}_{00}\Gamma^{0}_{23} + \cdots + \Gamma^{2}_{03}\Gamma^{3}_{23}\Big) = \frac{\big(c^2-2V\big)}{c^{11}}\,\partial_2 V\big(h_{02}PD_3 - h_{03}PD_2\big)\,,\\
&\Big(\Gamma^{3}_{00}\Gamma^{0}_{33} + \cdots + \Gamma^{3}_{03}\Gamma^{3}_{33}\Big) = \frac{\big(c^2-2V\big)}{c^{11}}\Big[PD_3\big(h_{01}\,\partial_1 V + h_{02}\,\partial_2 V\big) - h_{03}\big(PD_1\,\partial_1 V\\
&+ PD_2\,\partial_2 V\big)\Big]\,;
\end{aligned}$$

$$R_{03} = \frac{1}{c^7}\Big\{h_{01}\big(PD_{13} - 2\,\partial_1 U\,\partial_3 U\big) + h_{02}\big(PD_{23} - 2\,\partial_2 U\,\partial_3 U\big) + h_{03}\Big[PD_{33} - 2\big(\partial_3 U\big)^2\Big]\Big\}$$
$$+ \frac{1}{c^7\big(c^2+2V\big)D}\Big\{h_{01}\Big[-\big(c^2+2V\big)^2 PD_1 PD_3 + \big(2D+P\big)PD_3\,\partial_1 V\Big] + h_{02}\Big[-\big(c^2+2V\big)^2$$
$$PD_2 PD_3 + \big(2D+P\big)PD_3\,\partial_2 V\Big] + h_{03}\Big[-\big(c^2+2V\big)^2 PD_3^2 + \big(2D+P\big)PD_3\,\partial_3 V\Big]\Big\}$$
$$- \frac{2\big(c^2-2V\big)}{c^{11}}\Big[h_{01}PD_3\,\partial_1 V + h_{02}PD_3\,\partial_2 V - h_{03}\big(PD_1\,\partial_1 V + PD_2\,\partial_2 V\big)\Big]\,.$$

D.1.2 R_{1i} components $\quad i = 1, 2, 3.$

R_{11} component

$$R_{11} = \frac{\partial\Gamma^\alpha_{11}}{\partial x^\alpha} - \frac{\partial^2}{\partial x^{1^2}}\ln\sqrt{-g} + \Gamma^\sigma_{11}\frac{\partial}{\partial x^\sigma}\ln\sqrt{-g} - \Gamma^\sigma_{1\alpha}\Gamma^\alpha_{\sigma 1}\,;$$

- 1-st term

$$\begin{aligned}
&\frac{\partial\Gamma^\alpha_{11}}{\partial x^\alpha} = \frac{\partial\Gamma^0_{11}}{\partial x^0} + \frac{\partial\Gamma^1_{11}}{\partial x^1} + \frac{\partial\Gamma^2_{11}}{\partial x^2} + \frac{\partial\Gamma^3_{11}}{\partial x^3}\,,\\
&\frac{\partial\Gamma^0_{11}}{\partial x^0} = 0\,,\\
&\frac{\partial\Gamma^1_{11}}{\partial x^1} = -\frac{1}{c^4}\Big[2\big(\partial_1 V\big)^2 - \big(c^2-2V\big)\partial_{11}V\Big]\,,\\
&\frac{\partial\Gamma^2_{11}}{\partial x^2} = \frac{1}{c^4}\Big[2\big(\partial_2 V\big)^2 - \big(c^2-2V\big)\partial_{22}V\Big]\,,\\
&\frac{\partial\Gamma^3_{11}}{\partial x^3} = \frac{1}{c^4}\Big[2\big(\partial_3 V\big)^2 - \big(c^2-2V\big)\partial_{33}V\Big]\,;
\end{aligned}$$

- 2-nd term

$$\begin{aligned}
&\frac{\partial^2}{\partial x^{1^2}}\ln\sqrt{-g} = -\Big\{2\Big[\big(2D+P\big)\partial_1 V - \big(c^2+2V\big)^2 PD_1\Big]^2 - D\Big[2\big(c^2+2V\big)^3\big(\partial_1 U\big)^2\\
&- 12\big(c^2+2V\big)^2 PD_1\,\partial_1 V + 4\big(D+2P\big)\big(\partial_1 V\big)^2 - \big(c^2+2V\big)^3 PD_{11} + \big(c^2+2V\big)\\
&\big(2D+P\big)\partial_{11}V\Big]\Big\}\Big/\big(c^2+2V\big)^2 D^2\,;
\end{aligned}$$

- 3-rd term

$$\begin{aligned}
&\Gamma^\sigma_{11}\frac{\partial}{\partial x^\sigma}\ln\sqrt{-g} = \Gamma^0_{11}\frac{\partial}{\partial x^0}\ln\sqrt{-g} + \Gamma^1_{11}\frac{\partial}{\partial x^1}\ln\sqrt{-g} + \Gamma^2_{11}\frac{\partial}{\partial x^2}\ln\sqrt{-g} + \Gamma^3_{11}\frac{\partial}{\partial x^3}\ln\sqrt{-g}\,,\\
&\Gamma^0_{11}\frac{\partial}{\partial x^0}\ln\sqrt{-g} = 0\,,\\
&\Gamma^1_{11}\frac{\partial}{\partial x^1}\ln\sqrt{-g} = \big(c^2-2V\big)\partial_1 V\Big[\big(2D+P\big)\partial_1 V - \big(c^2+2V\big)^2 PD_1\Big]\Big/c^4\big(c^2+2V\big)D\,,\\
&\Gamma^2_{11}\frac{\partial}{\partial x^2}\ln\sqrt{-g} = -\big(c^2-2V\big)\partial_2 V\Big[\big(2D+P\big)\partial_2 V - \big(c^2+2V\big)^2 PD_2\Big]\Big/c^4\big(c^2+2V\big)D\,,\\
&\Gamma^3_{11}\frac{\partial}{\partial x^3}\ln\sqrt{-g} = -\big(c^2-2V\big)\partial_3 V\Big[\big(2D+P\big)\partial_3 V - \big(c^2+2V\big)^2 PD_3\Big]\Big/c^4\big(c^2+2V\big)D\,;
\end{aligned}$$

- 4-th term

$$\Gamma^{\sigma}_{1\alpha}\Gamma^{\alpha}_{\sigma 1} = \Big(\Gamma^{0}_{10}\Gamma^{0}_{01} + \cdots + \Gamma^{0}_{13}\Gamma^{3}_{01}\Big) + \Big(\Gamma^{1}_{10}\Gamma^{0}_{11} + \cdots + \Gamma^{1}_{13}\Gamma^{3}_{11}\Big) + \Big(\Gamma^{2}_{10}\Gamma^{0}_{21} + \cdots + \Gamma^{2}_{13}\Gamma^{3}_{21}\Big) + \Big(\Gamma^{3}_{10}\Gamma^{0}_{31} + \cdots + \Gamma^{3}_{13}\Gamma^{3}_{31}\Big),$$

$$\Big(\Gamma^{0}_{10}\Gamma^{0}_{01} + \cdots + \Gamma^{0}_{13}\Gamma^{3}_{01}\Big) = \frac{1}{c^{16}}\Big(q^2 PD_1^2 + c^4 h_{123}^2\, PD_1\, \partial_1 V\Big),$$

$$\Big(\Gamma^{1}_{10}\Gamma^{0}_{11} + \cdots + \Gamma^{1}_{13}\Gamma^{3}_{11}\Big) = -\frac{1}{c^{12}}\Big\{c^4\big(c^2-2V\big)^2\Big[\big(\partial_3 V\big)^2 + \big(\partial_2 V\big)^2 - \big(\partial_1 V\big)^2\Big] + h_{01}\big(h_{03}\,\partial_3 V + h_{02}\,\partial_2 V - h_{01}\,\partial_1 V\big) PD_1\Big\},$$

$$\Big(\Gamma^{2}_{10}\Gamma^{0}_{21} + \cdots + \Gamma^{2}_{13}\Gamma^{3}_{21}\Big) = \frac{1}{c^{12}}\Big\{c^4\big(c^2-2V\big)^2\Big[\big(\partial_1 V\big)^2 - \big(\partial_2 V\big)^2\Big] + h_{02}\big(h_{01}\,\partial_2 V + h_{02}\,\partial_1 V\big) PD_1\Big\},$$

$$\Big(\Gamma^{3}_{10}\Gamma^{0}_{31} + \cdots + \Gamma^{3}_{13}\Gamma^{3}_{31}\Big) = \frac{1}{c^{12}}\Big\{c^4\big(c^2-2V\big)^2\Big[\big(\partial_1 V\big)^2 - \big(\partial_3 V\big)^2\Big] + h_{03}\big(h_{01}\,\partial_3 V + h_{03}\,\partial_1 V\big) PD_1\Big\}.$$

Further compacting, it follows,

$$\text{1-st terms of } \sum_{i=1}^{3}\frac{\partial\Gamma^{i}_{11}}{\partial x^i} = -\frac{2}{c^4}\Big[\big(\partial_1 V\big)^2 - \big(\partial_2 V\big)^2 - \big(\partial_3 V\big)^2\Big],$$

$$\text{1-st terms of } \sum_{i=1}^{3}\Gamma^{i}_{11}\frac{\partial}{\partial x^i}\ln\sqrt{-g} = \frac{\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\Big[\big(\partial_1 V\big)^2 - \big(\partial_2 V\big)^2 - \big(\partial_3 V\big)^2\Big],$$

$$\text{1-st terms of } \sum_{i=1}^{3}\frac{\partial\Gamma^{i}_{11}}{\partial x^i} + \text{1-st terms of } \sum_{i=1}^{3}\Gamma^{i}_{11}\frac{\partial}{\partial x^i}\ln\sqrt{-g}$$
$$= \left[-\frac{2}{c^4} + \frac{\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\right]\Big[\big(\partial_1 V\big)^2 - \big(\partial_2 V\big)^2 - \big(\partial_3 V\big)^2\Big],$$

$$\text{2-nd terms of } \sum_{i=1}^{3}\frac{\partial\Gamma^{i}_{11}}{\partial x^i} = \frac{\big(c^2-2V\big)}{c^4}\big(\partial_{11}V - \partial_{22}V - \partial_{33}V\big),$$

$$\text{2-nd terms of } \sum_{i=1}^{3}\Gamma^{i}_{11}\frac{\partial}{\partial x^i}\ln\sqrt{-g} = \frac{\big(c^4-4V^2\big)}{c^4 D}\big(-PD_1\,\partial_1 V + PD_2\,\partial_2 V + PD_3\,\partial_3 V\big),$$

$$\text{4-th terms of } \Gamma^{\sigma}_{1\alpha}\Gamma^{\alpha}_{\sigma 1} = \frac{1}{c^{16}}\big(q^2 PD_1^2 + c^4 h_{123}^2\, PD_1\,\partial_1 V\big) + \frac{\big(c^2-2V\big)^2}{c^8}\Big[3\big(\partial_1 V\big)^2 - 2\big(\partial_2 V\big)^2 - 2\big(\partial_3 V\big)^2\Big] + \frac{1}{c^{12}}\,h_{123}^2\, PD_1\,\partial_1 V\,,$$

from which,

$$R_{11} = \left[-\frac{2}{c^4} + \frac{\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\right]\Big[\big(\partial_1 V\big)^2 - \big(\partial_2 V\big)^2 - \big(\partial_3 V\big)^2\Big] + \frac{\big(c^2-2V\big)}{c^4}\big(\partial_{11}V - \partial_{22}V$$

$$- \partial_{33}V) + \frac{(c^4 - 4V^2)}{c^4 D}(- PD_1\,\partial_1 V + PD_2\,\partial_2 V + PD_3\,\partial_3 V) + \frac{2}{(c^2+2V)^2 D^2}\Big[(2D+P)\,\partial_1 V$$
$$- (c^2+2V)^2 PD_1\Big]^2 - \frac{1}{(c^2+2V)^2 D}\Big[2(c^2+2V)^3(\partial_1 U)^2 - 12(c^2+2V)^2 PD_1\,\partial_1 V + 4(D+2P)$$
$$(\partial_1 V)^2 - (c^2+2V)^3 PD_{11} + (c^2+2V)(2D+P)\,\partial_{11}V\Big] - \frac{1}{c^{16}}\,q^2 PD_1^2 - \frac{2}{c^{12}}\,h_{123}^2\,PD_1\,\partial_1 V$$
$$- \frac{(c^2-2V)^2}{c^8}\Big[3(\partial_1 V)^2 - 2(\partial_2 V)^2 - 2(\partial_3 V)^2\Big].$$

R_{12} component

$$R_{12} = \frac{\partial \Gamma^\alpha_{12}}{\partial x^\alpha} - \frac{\partial^2}{\partial x^1 \partial x^2}\ln\sqrt{-g} + \Gamma^\sigma_{12}\frac{\partial}{\partial x^\sigma}\ln\sqrt{-g} - \Gamma^\sigma_{1\alpha}\Gamma^\alpha_{\sigma 2}\,;$$

- 1-st term

$$\begin{aligned}
&\frac{\partial \Gamma^\alpha_{12}}{\partial x^\alpha} = \frac{\partial \Gamma^0_{12}}{\partial x^0} + \frac{\partial \Gamma^1_{12}}{\partial x^1} + \frac{\partial \Gamma^2_{12}}{\partial x^2} + \frac{\partial \Gamma^3_{12}}{\partial x^3}\,,\\
&\frac{\partial \Gamma^0_{12}}{\partial x^0} = 0\,,\\
&\frac{\partial \Gamma^1_{12}}{\partial x^1} = \frac{1}{c^4}\Big[(c^2-2V)\partial_{12}V - 2\,\partial_1 V \partial_2 V\Big]\,,\\
&\frac{\partial \Gamma^2_{12}}{\partial x^2} = \frac{1}{c^4}\Big[(c^2-2V)\partial_{12}V - 2\,\partial_1 V \partial_2 V\Big]\,,\\
&\frac{\partial \Gamma^3_{12}}{\partial x^3} = 0\,;
\end{aligned}$$

- 2-nd term

$$\frac{\partial^2}{\partial x^1 \partial x^2}\ln\sqrt{-g} = -\Bigg\{+2\Big[(2D+P)\,\partial_1 V - (c^2+2V)^2 PD_1\Big]\Big[(2D+P)\,\partial_2 V - (c^2+2V)^2$$
$$PD_2\Big] - D\Bigg\{2\,\partial_2 U\,(c^2+2V)^2\Big[(c^2+2V)\,\partial_1 U - 3(c^2-2U)\,\partial_1 V\Big] + 2\,\partial_2 V\Big[2(D+2P)\,\partial_1 V$$
$$- 3(c^2+2V)^2 PD_1\Big] - (c^2+2V)\Big[12(c^2+2V)\,\partial_1 V\,\partial_2\Phi - (2D+P)\,\partial_{12}V + (c^2+2V)^2$$
$$PD_{12}\Big]\Bigg\}\Bigg\}\Big/(c^2+2V)^2 D^2\,;$$

- 3-rd term

$$\begin{aligned}
&\Gamma^\sigma_{12}\frac{\partial}{\partial x^\sigma}\ln\sqrt{-g} = \Gamma^0_{12}\frac{\partial}{\partial x^0}\ln\sqrt{-g} + \Gamma^1_{12}\frac{\partial}{\partial x^1}\ln\sqrt{-g} + \Gamma^2_{12}\frac{\partial}{\partial x^2}\ln\sqrt{-g} + \Gamma^3_{12}\frac{\partial}{\partial x^3}\ln\sqrt{-g}\,,\\
&\Gamma^0_{12}\frac{\partial}{\partial x^0}\ln\sqrt{-g} = 0\,,\\
&\Gamma^1_{12}\frac{\partial}{\partial x^1}\ln\sqrt{-g} = (c^2-2V)\partial_2 V\Big[(2D+P)\partial_1 V - (c^2+2V)^2 PD_1\Big]\Big/c^4(c^2+2V)D\,,\\
&\Gamma^2_{12}\frac{\partial}{\partial x^2}\ln\sqrt{-g} = (c^2-2V)\partial_1 V\Big[(2D+P)\partial_2 V - (c^2+2V)^2 PD_2\Big]\Big/c^4(c^2+2V)D\,,\\
&\Gamma^3_{12}\frac{\partial}{\partial x^3}\ln\sqrt{-g} = 0\,;
\end{aligned}$$

- 4-th term

$$
\begin{aligned}
&\Gamma^{\sigma}_{1\alpha}\Gamma^{\alpha}_{\sigma 2} = \big(\Gamma^{0}_{10}\Gamma^{0}_{02} + \cdots + \Gamma^{0}_{13}\Gamma^{3}_{02}\big) + \big(\Gamma^{1}_{10}\Gamma^{0}_{12} + \cdots + \Gamma^{1}_{13}\Gamma^{3}_{12}\big) + \big(\Gamma^{2}_{10}\Gamma^{0}_{22} + \cdots + \Gamma^{2}_{13}\Gamma^{3}_{22}\big) \\
&+ \big(\Gamma^{3}_{10}\Gamma^{0}_{32} + \cdots + \Gamma^{3}_{13}\Gamma^{3}_{32}\big)\,, \\
&\big(\Gamma^{0}_{10}\Gamma^{0}_{02} + \cdots + \Gamma^{0}_{13}\Gamma^{3}_{02}\big) = \frac{1}{c^{16}}\big(q^2 PD_1\, PD_2 + c^4 h^2_{123}\, PD_2\, \partial_1 V\big)\,, \\
&\big(\Gamma^{1}_{10}\Gamma^{0}_{12} + \cdots + \Gamma^{1}_{13}\Gamma^{3}_{12}\big) = \frac{2}{c^{8}}\big(c^2 - 2V\big)^2 \partial_1 V\, \partial_2 V + \frac{1}{c^{12}}\, h_{01}\big(h_{01}\, \partial_2 V + h_{02}\, \partial_1 V\big) PD_1\,, \\
&\big(\Gamma^{2}_{10}\Gamma^{0}_{22} + \cdots + \Gamma^{2}_{13}\Gamma^{3}_{22}\big) = \frac{2}{c^{8}}\big(c^2 - 2V\big)^2 \partial_1 V\, \partial_2 V - \frac{1}{c^{12}}\, h_{02}\big(h_{03}\, \partial_3 V - h_{02}\, \partial_2 V + h_{01}\, \partial_1 V\big) PD_1\,, \\
&\big(\Gamma^{3}_{10}\Gamma^{0}_{32} + \cdots + \Gamma^{3}_{13}\Gamma^{3}_{32}\big) = \frac{1}{c^{8}}\big(c^2 - 2V\big)^2 \partial_1 V\, \partial_2 V + \frac{1}{c^{12}}\, h_{03}\big(h_{02}\, \partial_3 V + h_{03}\, \partial_2 V\big) PD_1\,.
\end{aligned}
$$

Further compacting, it follows,

$$
\begin{aligned}
&\sum_{i=1}^{3} \frac{\partial \Gamma^{i}_{12}}{\partial x^i} = \frac{2}{c^4}\Big[\big(c^2 - 2\,V\big)\partial_{12} V - 2\,\partial_1 V\, \partial_2 V\Big]\,, \\
&\sum_{i=1}^{3} \Gamma^{i}_{12} \frac{\partial}{\partial x^i} \ln\sqrt{-g} = \frac{2\big(c^2 - 2V\big)\big(2D + P\big)}{c^4\big(c^2 + 2V\big)D}\, \partial_1 V \partial_2 V - \frac{\big(c^4 - 4V^2\big)}{c^4 D}\big(PD_1\, \partial_2 V + PD_2\, \partial_1 V\big)\,, \\
&\Gamma^{\sigma}_{1\alpha}\Gamma^{\alpha}_{\sigma 2} = \frac{1}{c^{16}}\big(q^2 PD_1 PD_2 + c^4 h^2_{123}\, PD_2\, \partial_1 V\big) + \frac{5\big(c^2 - 2V\big)^2}{c^8}\, \partial_1 V \partial_2 V + \frac{1}{c^{12}} h^2_{123}\, PD_1\, \partial_2 V\,,
\end{aligned}
$$

from which,

$$
\begin{aligned}
&R_{12} = \frac{2}{c^4}\Big[\big(c^2 - 2V\big)\partial_{12} V - 2\,\partial_1 V\, \partial_2 V\Big] + \frac{2}{\big(c^2 + 2V\big)^2 D^2}\Big[\big(2D + P\big)\partial_1 V - \big(c^2 + 2V\big)^2 PD_1\Big] \\
&\Big[\big(2D + P\big)\partial_2 V - \big(c^2 + 2V\big)^2 PD_2\Big] - \frac{1}{\big(c^2 + 2V\big)^2 D}\Big\{2\,\partial_2 U\big(c^2 + 2V\big)^2\Big[\big(c^2 + 2V\big)\partial_1 U - 3\big(c^2 - 2U\big) \\
&\partial_1 V\Big] + 2\,\partial_2 V\Big[2\big(D + 2P\big)\partial_1 V - 3\big(c^2 + 2V\big)^2 PD_1\Big] - \big(c^2 + 2V\big)\Big[12\big(c^2 + 2V\big)\partial_1 V\, \partial_2 \Phi - \big(2D + P\big) \\
&\partial_{12} V + \big(c^2 + 2V\big)^2 PD_{12}\Big]\Big\} + \frac{2\big(c^2 - 2V\big)\big(2D + P\big)}{c^4\big(c^2 + 2V\big)D}\, \partial_1 V \partial_2 V - \frac{\big(c^4 - 4V^2\big)}{c^4 D}\big(PD_1\, \partial_2 V + PD_2\, \partial_1 V\big) \\
&- \frac{1}{c^{16}}\, q^2 PD_1 PD_2 - \frac{1}{c^{12}}\, h^2_{123}\big(PD_2\, \partial_1 V + PD_1\, \partial_2 V\big) - \frac{5\big(c^2 - 2V\big)^2}{c^8}\, \partial_1 V \partial_2 V\,.
\end{aligned}
$$

R_{13} component

$$
R_{13} = \frac{\partial \Gamma^{\alpha}_{13}}{\partial x^{\alpha}} - \frac{\partial^2}{\partial x^1 \partial x^3} \ln\sqrt{-g} + \Gamma^{\sigma}_{13} \frac{\partial}{\partial x^{\sigma}} \ln\sqrt{-g} - \Gamma^{\sigma}_{1\alpha}\Gamma^{\alpha}_{\sigma 3}\,;
$$

- 1-st term

$$
\begin{aligned}
&\frac{\partial \Gamma^{\alpha}_{13}}{\partial x^{\alpha}} = \frac{\partial \Gamma^{0}_{13}}{\partial x^0} + \frac{\partial \Gamma^{1}_{13}}{\partial x^1} + \frac{\partial \Gamma^{2}_{13}}{\partial x^2} + \frac{\partial \Gamma^{3}_{13}}{\partial x^3}\,, \\
&\frac{\partial \Gamma^{0}_{13}}{\partial x^0} = 0\,,
\end{aligned}
$$

$$\frac{\partial\Gamma^1_{13}}{\partial x^1} = \frac{1}{c^4}\Big[\big(c^2-2V\big)\partial_{13}V - 2\,\partial_1 V\partial_3 V\Big]\,,$$
$$\frac{\partial\Gamma^2_{13}}{\partial x^2} = 0\,,$$
$$\frac{\partial\Gamma^3_{13}}{\partial x^3} = \frac{1}{c^4}\Big[\big(c^2-2V\big)\partial_{13}V - 2\,\partial_1 V\partial_3 V\Big]\,;$$

- 2-nd term

$$\frac{\partial^2}{\partial x^1\partial x^3}\ln\sqrt{-g} = -\Bigg\{-2\Big[\big(2D+P\big)\partial_1 V - \big(c^2+2V\big)^2 PD_1\Big]\Big[\big(2D+P\big)\partial_3 V - \big(c^2+2V\big)^2 PD_3\Big] - D\Bigg\{2\,\partial_3 U\big(c^2+2V\big)^2\Big[\big(c^2+2V\big)\,\partial_1 U - 3\big(c^2-2U\big)\partial_1 V\Big] + 2\,\partial_3 V\Big[2\big(D+2P\big)\partial_1 V - 3\big(c^2+2V\big)^2 PD_1\Big] - \big(c^2+2V\big)\Big[12\big(c^2+2V\big)\,\partial_1 V\partial_3\Phi - \big(2D+P\big)\partial_{13}V + \big(c^2+2V\big)^2 PD_{13}\Big]\Bigg\}\Bigg\}\Big/\big(c^2+2V\big)^2 D^2\,;$$

- 3-rd term

$$\Gamma^\sigma_{13}\frac{\partial}{\partial x^\sigma}\ln\sqrt{-g} = \Gamma^0_{13}\frac{\partial}{\partial x^0}\ln\sqrt{-g} + \Gamma^1_{13}\frac{\partial}{\partial x^1}\ln\sqrt{-g} + \Gamma^2_{13}\frac{\partial}{\partial x^2}\ln\sqrt{-g} + \Gamma^3_{13}\frac{\partial}{\partial x^3}\ln\sqrt{-g}\,,$$
$$\Gamma^0_{13}\frac{\partial}{\partial x^0}\ln\sqrt{-g} = 0\,,$$
$$\Gamma^1_{13}\frac{\partial}{\partial x^1}\ln\sqrt{-g} = \big(c^2-2V\big)\partial_3 V\Big[\big(2D+P\big)\partial_1 V - \big(c^2+2V\big)^2 PD_1\Big]\Big/c^4\big(c^2+2V\big)D\,,$$
$$\Gamma^2_{13}\frac{\partial}{\partial x^2}\ln\sqrt{-g} = 0\,,$$
$$\Gamma^3_{13}\frac{\partial}{\partial x^3}\ln\sqrt{-g} = \big(c^2-2V\big)\partial_1 V\Big[\big(2D+P\big)\partial_3 V - \big(c^2+2V\big)^2 PD_3\Big]\Big/c^4\big(c^2+2V\big)D\,;$$

- 4-th term

$$\Gamma^\sigma_{1\alpha}\Gamma^\alpha_{\sigma 3} = \big(\Gamma^0_{10}\Gamma^0_{03} + \cdots + \Gamma^0_{13}\Gamma^3_{03}\big) + \big(\Gamma^1_{10}\Gamma^0_{13} + \cdots + \Gamma^1_{13}\Gamma^3_{13}\big) + \big(\Gamma^2_{10}\Gamma^0_{23} + \cdots + \Gamma^2_{13}\Gamma^3_{23}\big) + \big(\Gamma^3_{10}\Gamma^0_{33} + \cdots + \Gamma^3_{13}\Gamma^3_{33}\big)\,,$$
$$\big(\Gamma^0_{10}\Gamma^0_{03} + \cdots + \Gamma^0_{13}\Gamma^3_{03}\big) = \frac{1}{c^{16}}\big(q^2 PD_1 PD_3 + c^4 h^2_{123}\,PD_3\,\partial_1 V\big)\,,$$
$$\big(\Gamma^1_{10}\Gamma^0_{13} + \cdots + \Gamma^1_{13}\Gamma^3_{13}\big) = \frac{2}{c^8}\big(c^2-2V\big)^2\partial_1 V\partial_3 V + \frac{1}{c^{12}}\,h_{01}\big(h_{01}\,\partial_3 V + h_{03}\,\partial_1 V\big)PD_1\,,$$
$$\big(\Gamma^2_{10}\Gamma^0_{23} + \cdots + \Gamma^2_{13}\Gamma^3_{23}\big) = \frac{1}{c^8}\big(c^2-2V\big)^2\partial_1 V\partial_3 V + \frac{1}{c^{12}}\,h_{02}\big(h_{02}\,\partial_3 V + h_{03}\,\partial_2 V\big)PD_1\,,$$
$$\big(\Gamma^3_{10}\Gamma^0_{33} + \cdots + \Gamma^3_{13}\Gamma^3_{33}\big) = \frac{2}{c^8}\big(c^2-2V\big)^2\partial_1 V\partial_3 V - \frac{1}{c^{12}}\,h_{03}\big(-h_{03}\,\partial_3 V + h_{02}\,\partial_2 V + h_{01}\,\partial_1 V\big)PD_1\,.$$

Further compacting, it follows,

$$\sum_{i=1}^{3}\frac{\partial\Gamma^{i}_{13}}{\partial x^{i}}=\frac{2}{c^{4}}\Big[\big(c^{2}-2V\big)\,\partial_{13}V-2\,\partial_{1}V\partial_{3}V\Big]\,,$$

$$\sum_{i=1}^{3}\Gamma^{i}_{13}\frac{\partial}{\partial x^{i}}\ln\sqrt{-g}=\frac{2\big(c^{2}-2V\big)\big(2D+P\big)}{c^{4}\big(c^{2}+2V\big)D}\,\partial_{1}V\,\partial_{3}V-\frac{\big(c^{4}-4V^{2}\big)}{c^{4}D}\big(PD_{1}\,\partial_{3}V+PD_{3}\,\partial_{1}V\big)\,,$$

$$\Gamma^{\sigma}_{1\alpha}\Gamma^{\alpha}_{\sigma3}=\frac{1}{c^{16}}\big(q^{2}PD_{1}PD_{3}+c^{4}h^{2}_{123}\,PD_{3}\,\partial_{1}V\big)+\frac{5\big(c^{2}-2V\big)^{2}}{c^{8}}\,\partial_{1}V\partial_{3}V+\frac{1}{c^{12}}\,h^{2}_{123}\,PD_{1}\,\partial_{3}V\,,$$

from which,

$$R_{13}=\frac{2}{c^{4}}\Big[\big(c^{2}-2V\big)\partial_{13}V-2\,\partial_{1}V\partial_{3}V\Big]+\frac{2}{\big(c^{2}+2V\big)^{2}D^{2}}\Big[\big(2D+P\big)\partial_{1}V-\big(c^{2}+2V\big)^{2}PD_{1}\Big]$$
$$\Big[\big(2D+P\big)\partial_{3}V-\big(c^{2}+2V\big)^{2}PD_{3}\Big]-\frac{1}{\big(c^{2}+2V\big)^{2}D}\Big\{2\,\partial_{3}U\big(c^{2}+2V\big)^{2}\Big[\big(c^{2}+2V\big)\partial_{1}U-3\big(c^{2}-2U\big)$$
$$\partial_{1}V\Big]+2\,\partial_{3}V\Big[2\big(D+2P\big)\partial_{1}V-3\big(c^{2}+2V\big)^{2}PD_{1}\Big]-\big(c^{2}+2V\big)\Big[12\big(c^{2}+2V\big)\partial_{1}V\partial_{3}\Phi-\big(2D+P\big)$$
$$\partial_{13}V+\big(c^{2}+2V\big)^{2}PD_{13}\Big]\Big\}+\frac{2\big(c^{2}-2V\big)\big(2D+P\big)}{c^{4}\big(c^{2}+2V\big)D}\,\partial_{1}V\partial_{3}V-\frac{\big(c^{4}-4V^{2}\big)}{c^{4}D}\big(PD_{1}\,\partial_{3}V+PD_{3}\,\partial_{1}V\big)$$
$$-\frac{1}{c^{16}}\,q^{2}PD_{1}PD_{3}-\frac{1}{c^{12}}\,h^{2}_{123}\big(PD_{3}\,\partial_{1}V+PD_{1}\,\partial_{3}V\big)-\frac{5\big(c^{2}-2V\big)^{2}}{c^{8}}\,\partial_{1}V\partial_{3}V\,.$$

D.1.3 R_{2i} components $\quad i=2,3.$

***R_{22}* component**

$$R_{22}=\frac{\partial\Gamma^{\alpha}_{22}}{\partial x^{\alpha}}-\frac{\partial^{2}}{\partial x^{2^{2}}}\ln\sqrt{-g}+\Gamma^{\sigma}_{22}\frac{\partial}{\partial x^{\sigma}}\ln\sqrt{-g}-\Gamma^{\sigma}_{2\alpha}\Gamma^{\alpha}_{\sigma2}\,;$$

- 1-st term

$$\begin{aligned}
&\frac{\partial\Gamma^{\alpha}_{22}}{\partial x^{\alpha}}=\frac{\partial\Gamma^{0}_{22}}{\partial x^{0}}+\frac{\partial\Gamma^{1}_{22}}{\partial x^{1}}+\frac{\partial\Gamma^{2}_{22}}{\partial x^{2}}+\frac{\partial\Gamma^{3}_{22}}{\partial x^{3}}\,,\\
&\frac{\partial\Gamma^{0}_{22}}{\partial x^{0}}=0\,,\\
&\frac{\partial\Gamma^{1}_{22}}{\partial x^{1}}=\frac{1}{c^{4}}\Big[2\big(\partial_{1}V\big)^{2}-\big(c^{2}-2V\big)\partial_{11}V\Big]\,,\\
&\frac{\partial\Gamma^{2}_{22}}{\partial x^{2}}=-\frac{1}{c^{4}}\Big[2\big(\partial_{2}V\big)^{2}-\big(c^{2}-2V\big)\partial_{22}V\Big]\,,\\
&\frac{\partial\Gamma^{3}_{22}}{\partial x^{3}}=\frac{1}{c^{4}}\Big[2\big(\partial_{3}V\big)^{2}-\big(c^{2}-2V\big)\partial_{33}V\Big]\,;
\end{aligned}$$

- 2-nd term

$$\frac{\partial^{2}}{\partial x^{2^{2}}}\ln\sqrt{-g}=-\Big\{2\Big[\big(2D+P\big)\partial_{2}V-\big(c^{2}+2V\big)^{2}PD_{2}\Big]^{2}-D\Big[2\big(c^{2}+2V\big)^{3}\big(\partial_{2}U\big)^{2}$$
$$-12\big(c^{2}+2V\big)^{2}PD_{2}\,\partial_{2}V+4\big(D+2P\big)\big(\partial_{2}V\big)^{2}-\big(c^{2}+2V\big)^{3}PD_{22}+\big(c^{2}+2V\big)$$
$$\big(2D+P\big)\partial_{22}V\Big]\Big\}\Big/\big(c^{2}+2V\big)^{2}D^{2}\,;$$

- 3-rd term

$$\Gamma^{\sigma}_{22}\frac{\partial}{\partial x^{\sigma}}\ln\sqrt{-g}=\Gamma^{0}_{22}\frac{\partial}{\partial x^{0}}\ln\sqrt{-g}+\Gamma^{1}_{22}\frac{\partial}{\partial x^{1}}\ln\sqrt{-g}+\Gamma^{2}_{22}\frac{\partial}{\partial x^{2}}\ln\sqrt{-g}+\Gamma^{3}_{22}\frac{\partial}{\partial x^{3}}\ln\sqrt{-g}\,,$$
$$\Gamma^{0}_{22}\frac{\partial}{\partial x^{0}}\ln\sqrt{-g}=0\,,$$
$$\Gamma^{1}_{22}\frac{\partial}{\partial x^{1}}\ln\sqrt{-g}=-\big(c^2-2V\big)\partial_1 V\Big[\big(2D+P\big)\partial_1 V-\big(c^2+2V\big)^2 PD_1\Big]\Big/c^4\big(c^2+2V\big)D\,,$$
$$\Gamma^{2}_{22}\frac{\partial}{\partial x^{2}}\ln\sqrt{-g}=\big(c^2-2V\big)\partial_2 V\Big[\big(2D+P\big)\partial_2 V-\big(c^2+2V\big)^2 PD_2\Big]\Big/c^4\big(c^2+2V\big)D\,,$$
$$\Gamma^{3}_{22}\frac{\partial}{\partial x^{3}}\ln\sqrt{-g}=-\big(c^2-2V\big)\partial_3 V\Big[\big(2D+P\big)\partial_3 V-\big(c^2+2V\big)^2 PD_3\Big]\Big/c^4\big(c^2+2V\big)D\,;$$

- 4-th term

$$\Gamma^{\sigma}_{2\alpha}\Gamma^{\alpha}_{\sigma 2}=\Big(\Gamma^{0}_{20}\Gamma^{0}_{02}+\cdots+\Gamma^{0}_{23}\Gamma^{3}_{02}\Big)+\Big(\Gamma^{1}_{20}\Gamma^{0}_{12}+\cdots+\Gamma^{1}_{23}\Gamma^{3}_{12}\Big)+\Big(\Gamma^{2}_{20}\Gamma^{0}_{22}+\cdots+\Gamma^{2}_{23}\Gamma^{3}_{22}\Big)$$
$$+\Big(\Gamma^{3}_{20}\Gamma^{0}_{32}+\cdots+\Gamma^{3}_{23}\Gamma^{3}_{32}\Big)\,,$$
$$\Big(\Gamma^{0}_{20}\Gamma^{0}_{02}+\cdots+\Gamma^{0}_{23}\Gamma^{3}_{02}\Big)=\frac{1}{c^{16}}\big(q^2PD_2^2+c^4h_{123}^2\,PD_2\,\partial_2 V\big)\,,$$
$$\Big(\Gamma^{1}_{20}\Gamma^{0}_{12}+\cdots+\Gamma^{1}_{23}\Gamma^{3}_{12}\Big)=\frac{1}{c^{12}}\Big\{c^4\big(c^2-2V\big)^2\Big[\big(\partial_2 V\big)^2-\big(\partial_1 V\big)^2\Big]+h_{01}\big(h_{01}\,\partial_2 V$$
$$+h_{02}\,\partial_1 V\big)PD_2\Big\}\,,$$
$$\Big(\Gamma^{2}_{20}\Gamma^{0}_{22}+\cdots+\Gamma^{2}_{23}\Gamma^{3}_{22}\Big)=-\frac{1}{c^{12}}\Big\{c^4\big(c^2-2V\big)^2\Big[\big(\partial_3 V\big)^2-\big(\partial_2 V\big)^2+\big(\partial_1 V\big)^2\Big]+h_{02}\big(h_{03}\,\partial_3 V$$
$$-h_{02}\,\partial_2 V+h_{01}\,\partial_1 V\big)PD_2\Big\}\,,$$
$$\Big(\Gamma^{3}_{20}\Gamma^{0}_{32}+\cdots+\Gamma^{3}_{23}\Gamma^{3}_{32}\Big)=\frac{1}{c^{12}}\Big\{c^4\big(c^2-2V\big)^2\Big[\big(\partial_2 V\big)^2-\big(\partial_3 V\big)^2\Big]+h_{03}\big(h_{02}\,\partial_3 V+h_{03}\,\partial_2 V\big)$$
$$PD_2\Big\}\,.$$

Further compacting, it follows,

$$\text{1-st terms of }\sum_{i=1}^{3}\frac{\partial\Gamma^{i}_{22}}{\partial x^{i}}=\frac{2}{c^4}\Big[\big(\partial_1 V\big)^2-\big(\partial_2 V\big)^2+\big(\partial_3 V\big)^2\Big]\,,$$
$$\text{1-st terms of }\sum_{i=1}^{3}\Gamma^{i}_{22}\frac{\partial}{\partial x^{i}}\ln\sqrt{-g}=-\frac{\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\Big[\big(\partial_1 V\big)^2-\big(\partial_2 V\big)^2+\big(\partial_3 V\big)^2\Big]\,,$$
$$\text{1-st terms of }\sum_{i=1}^{3}\frac{\partial\Gamma^{i}_{22}}{\partial x^{i}}+\text{1-st terms of }\sum_{i=1}^{3}\Gamma^{i}_{22}\frac{\partial}{\partial x^{i}}\ln\sqrt{-g}$$
$$=\left[-\frac{2}{c^4}+\frac{\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\right]\Big[-\big(\partial_1 V\big)^2+\big(\partial_2 V\big)^2-\big(\partial_3 V\big)^2\Big]\,,$$
$$\text{2-nd terms of }\sum_{i=1}^{3}\frac{\partial\Gamma^{i}_{22}}{\partial x^{i}}=\frac{\big(c^2-2V\big)}{c^4}\big(-\partial_{11}V+\partial_{22}V-\partial_{33}V\big)\,,$$

2-nd terms of $\displaystyle\sum_{i=1}^{3}\Gamma^{i}_{22}\frac{\partial}{\partial x^{i}}\ln\sqrt{-g}=\frac{\left(c^{4}-4V^{2}\right)}{c^{4}D}\left(PD_{1}\,\partial_{1}V-PD_{2}\,\partial_{2}V+PD_{3}\,\partial_{3}V\right),$

4-th terms of $\Gamma^{\sigma}_{2\alpha}\Gamma^{\alpha}_{\sigma 2}=\frac{1}{c^{16}}\left(q^{2}PD_{2}^{2}+c^{4}h_{123}^{2}\,PD_{2}\,\partial_{2}V\right)+\frac{\left(c^{2}-2V\right)^{2}}{c^{8}}\Big[-2\big(\partial_{1}V\big)^{2}+3\big(\partial_{2}V\big)^{2}$
$-2\big(\partial_{3}V\big)^{2}\Big]+\frac{1}{c^{12}}\,h_{123}^{2}\,PD_{2}\,\partial_{2}V\,,$

from which,

$R_{22}=\Big[-\frac{2}{c^{4}}+\frac{\left(c^{2}-2V\right)\left(2D+P\right)}{c^{4}\left(c^{2}+2V\right)D}\Big]\Big[-\big(\partial_{1}V\big)^{2}+\big(\partial_{2}V\big)^{2}-\big(\partial_{3}V\big)^{2}\Big]+\frac{\left(c^{2}-2V\right)}{c^{4}}\big(-\partial_{11}V+\partial_{22}V$
$-\partial_{33}V\big)+\frac{\left(c^{4}-4V^{2}\right)}{c^{4}D}\left(PD_{1}\,\partial_{1}V-PD_{2}\,\partial_{2}V+PD_{3}\,\partial_{3}V\right)+\frac{2}{\left(c^{2}+2V\right)^{2}D^{2}}\Big[\left(2D+P\right)\partial_{2}V$
$-\left(c^{2}+2V\right)^{2}PD_{2}\Big]^{2}-\frac{1}{\left(c^{2}+2V\right)^{2}D}\Big[2\big(c^{2}+2V\big)^{3}\big(\partial_{2}U\big)^{2}-12\big(c^{2}+2V\big)^{2}PD_{2}\,\partial_{2}V+4\big(D+2P\big)$
$\big(\partial_{2}V\big)^{2}-\left(c^{2}+2V\right)^{3}PD_{22}+\left(c^{2}+2V\right)\left(2D+P\right)\partial_{22}V\Big]-\frac{1}{c^{16}}\,q^{2}PD_{2}^{2}-\frac{2}{c^{12}}\,h_{123}^{2}\,PD_{2}\,\partial_{2}V$
$-\frac{\left(c^{2}-2V\right)^{2}}{c^{8}}\Big[-2\big(\partial_{1}V\big)^{2}+3\big(\partial_{2}V\big)^{2}-2\big(\partial_{3}V\big)^{2}\Big].$

R_{23} component

$$R_{23}=\frac{\partial\Gamma^{\alpha}_{23}}{\partial x^{\alpha}}-\frac{\partial^{2}}{\partial x^{2}\partial x^{3}}\ln\sqrt{-g}+\Gamma^{\sigma}_{23}\frac{\partial}{\partial x^{\sigma}}\ln\sqrt{-g}-\Gamma^{\sigma}_{2\alpha}\Gamma^{\alpha}_{\sigma 3}\,;$$

- 1-st term

$$\begin{aligned}
&\frac{\partial\Gamma^{\alpha}_{23}}{\partial x^{\alpha}}=\frac{\partial\Gamma^{0}_{23}}{\partial x^{0}}+\frac{\partial\Gamma^{1}_{23}}{\partial x^{1}}+\frac{\partial\Gamma^{2}_{23}}{\partial x^{2}}+\frac{\partial\Gamma^{3}_{23}}{\partial x^{3}}\,,\\
&\frac{\partial\Gamma^{0}_{23}}{\partial x^{0}}=0\,,\\
&\frac{\partial\Gamma^{1}_{23}}{\partial x^{1}}=0\,,\\
&\frac{\partial\Gamma^{2}_{23}}{\partial x^{2}}=\frac{1}{c^{4}}\Big[\left(c^{2}-2V\right)\partial_{23}V-2\,\partial_{2}V\partial_{3}V\Big]\,,\\
&\frac{\partial\Gamma^{3}_{23}}{\partial x^{3}}=\frac{1}{c^{4}}\Big[\left(c^{2}-2V\right)\partial_{23}V-2\,\partial_{2}V\partial_{3}V\Big]\,;
\end{aligned}$$

- 2-nd term

$$\begin{aligned}
&\frac{\partial^{2}}{\partial x^{2}\partial x^{3}}\ln\sqrt{-g}=-\Bigg\{2\Big[\left(2D+P\right)\partial_{2}V-\left(c^{2}+2V\right)^{2}PD_{2}\Big]\Big[\left(2D+P\right)\partial_{3}V-\left(c^{2}+2V\right)^{2}PD_{3}\Big]\\
&-D\Bigg\{2\,\partial_{3}U\big(c^{2}+2V\big)^{2}\Big[\left(c^{2}+2V\right)\partial_{2}U-3\big(c^{2}-2U\big)\partial_{2}V\Big]+2\,\partial_{3}V\Big[2\big(D+2P\big)\partial_{2}V-3\big(c^{2}\\
&+2V\big)^{2}PD_{2}\Big]-\left(c^{2}+2V\right)\Big[12\big(c^{2}+2V\big)\partial_{2}V\partial_{3}\Phi-\left(2D+P\right)\partial_{23}V+\left(c^{2}+2V\right)^{2}PD_{23}\Big]\Bigg\}\Bigg\}\\
&\Big/\left(c^{2}+2V\right)^{2}D^{2}\,;
\end{aligned}$$

- 3-rd term

$$\Gamma^{\sigma}_{23}\frac{\partial}{\partial x^{\sigma}}\ln\sqrt{-g}=\Gamma^{0}_{23}\frac{\partial}{\partial x^{0}}\ln\sqrt{-g}+\Gamma^{1}_{23}\frac{\partial}{\partial x^{1}}\ln\sqrt{-g}+\Gamma^{2}_{23}\frac{\partial}{\partial x^{2}}\ln\sqrt{-g}+\Gamma^{3}_{23}\frac{\partial}{\partial x^{3}}\ln\sqrt{-g}\,,$$
$$\Gamma^{0}_{23}\frac{\partial}{\partial x^{0}}\ln\sqrt{-g}=0\,,$$
$$\Gamma^{1}_{23}\frac{\partial}{\partial x^{1}}\ln\sqrt{-g}=0\,,$$
$$\Gamma^{2}_{23}\frac{\partial}{\partial x^{2}}\ln\sqrt{-g}=\big(c^2-2V\big)\partial_3V\Big[\big(2D+P\big)\,\partial_2V-\big(c^2+2V\big)^2PD_2\Big]\Big/c^4\big(c^2+2V\big)D\,,$$
$$\Gamma^{3}_{23}\frac{\partial}{\partial x^{3}}\ln\sqrt{-g}=\big(c^2-2V\big)\partial_2V\Big[\big(2D+P\big)\,\partial_3V-\big(c^2+2V\big)^2PD_3\Big]\Big/c^4\big(c^2+2V\big)D\,;$$

- 4-th term

$$\Gamma^{\sigma}_{2\alpha}\Gamma^{\alpha}_{\sigma3}=\big(\Gamma^{0}_{20}\Gamma^{0}_{03}+\cdots+\Gamma^{0}_{23}\Gamma^{3}_{03}\big)+\big(\Gamma^{1}_{20}\Gamma^{0}_{13}+\cdots+\Gamma^{1}_{23}\Gamma^{3}_{13}\big)+\big(\Gamma^{2}_{20}\Gamma^{0}_{23}+\cdots+\Gamma^{2}_{23}\Gamma^{3}_{23}\big)$$
$$+\big(\Gamma^{3}_{20}\Gamma^{0}_{33}+\cdots+\Gamma^{3}_{23}\Gamma^{3}_{33}\big)\,,$$
$$\big(\Gamma^{0}_{20}\Gamma^{0}_{03}+\cdots+\Gamma^{0}_{23}\Gamma^{3}_{03}\big)=\frac{1}{c^{16}}\big(q^2PD_2PD_3+c^4h^2_{123}\,PD_3\,\partial_2V\big)\,,$$
$$\big(\Gamma^{1}_{20}\Gamma^{0}_{13}+\cdots+\Gamma^{1}_{23}\Gamma^{3}_{13}\big)=\frac{1}{c^{8}}\big(c^2-2V\big)^2\partial_2V\partial_3V+\frac{1}{c^{12}}\,h_{01}\big(h_{01}\,\partial_3V+h_{03}\,\partial_1V\big)PD_2\,,$$
$$\big(\Gamma^{2}_{20}\Gamma^{0}_{23}+\cdots+\Gamma^{2}_{23}\Gamma^{3}_{23}\big)=\frac{2}{c^{8}}\big(c^2-2V\big)^2\partial_2V\partial_3V+\frac{1}{c^{12}}\,h_{02}\big(h_{02}\,\partial_3V+h_{03}\,\partial_2V\big)PD_2\,,$$
$$\big(\Gamma^{3}_{20}\Gamma^{0}_{33}+\cdots+\Gamma^{3}_{23}\Gamma^{3}_{33}\big)=\frac{2}{c^{8}}\big(c^2-2V\big)^2\partial_2V\partial_3V-\frac{1}{c^{12}}\,h_{03}\big(-h_{03}\,\partial_3V+h_{02}\,\partial_2V$$
$$+h_{01}\,\partial_1V\big)PD_2\,.$$

Further compacting, it follows,

$$\sum_{i=1}^{3}\frac{\partial\Gamma^{i}_{23}}{\partial x^{i}}=\frac{2}{c^4}\Big[\big(c^2-2V\big)\partial_{23}V-2\,\partial_2V\partial_3V\Big]\,,$$
$$\sum_{i=1}^{3}\Gamma^{i}_{23}\frac{\partial}{\partial x^{i}}\ln\sqrt{-g}=\frac{2\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\,\partial_2V\partial_3V-\frac{\big(c^4-4V^2\big)}{c^4D}\big(PD_2\,\partial_3V+PD_3\,\partial_2V\big)\,,$$
$$\Gamma^{\sigma}_{2\alpha}\Gamma^{\alpha}_{\sigma3}=\frac{1}{c^{16}}\big(q^2PD_2PD_3+c^4h^2_{123}\,PD_3\,\partial_2V\big)+\frac{5\big(c^2-2V\big)^2}{c^8}\,\partial_2V\partial_3V+\frac{1}{c^{12}}\,h^2_{123}\,PD_2\,\partial_3V\,,$$

from which,

$$R_{23}=\frac{2}{c^4}\Big[\big(c^2-2V\big)\partial_{23}V-2\,\partial_2V\partial_3V\Big]+\frac{2}{\big(c^2+2V\big)^2D^2}\Big[\big(2D+P\big)\partial_2V-\big(c^2+2V\big)^2PD_2\Big]$$
$$\Big[\big(2D+P\big)\partial_3V-\big(c^2+2V\big)^2PD_3\Big]-\frac{1}{\big(c^2+2V\big)^2D}\Big\{2\,\partial_3U\big(c^2+2V\big)^2\Big[\big(c^2+2V\big)\partial_2U-3\big(c^2-2U\big)$$
$$\partial_2V\Big]+2\,\partial_3V\Big[2\big(D+2P\big)\partial_2V-3\big(c^2+2V\big)^2PD_2\Big]-\big(c^2+2V\big)\Big[12\big(c^2+2V\big)\partial_2V\partial_3\Phi-\big(2D+P\big)$$
$$\partial_{23}V+\big(c^2+2V\big)^2PD_{23}\Big]\Big\}+\frac{2\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\,\partial_2V\partial_3V-\frac{\big(c^4-4V^2\big)}{c^4D}\big(PD_2\,\partial_3V+PD_3\,\partial_2V\big)$$
$$-\frac{1}{c^{16}}\,q^2PD_2PD_3-\frac{1}{c^{12}}\,h^2_{123}\big(PD_3\,\partial_2V+PD_2\,\partial_3V\big)-\frac{5\big(c^2-2V\big)^2}{c^8}\,\partial_2V\partial_3V\,.$$

D.1.4 R_{33} component

$$R_{33} = \frac{\partial \Gamma^{\alpha}_{33}}{\partial x^{\alpha}} - \frac{\partial^2}{\partial {x^3}^2} \ln\sqrt{-g} + \Gamma^{\sigma}_{33} \frac{\partial}{\partial x^{\sigma}} \ln\sqrt{-g} - \Gamma^{\sigma}_{3\alpha}\Gamma^{\alpha}_{\sigma 3}\,;$$

- 1-st term

$$\begin{aligned}
&\frac{\partial \Gamma^{\alpha}_{33}}{\partial x^{\alpha}} = \frac{\partial \Gamma^{0}_{33}}{\partial x^{0}} + \frac{\partial \Gamma^{1}_{33}}{\partial x^{1}} + \frac{\partial \Gamma^{2}_{33}}{\partial x^{2}} + \frac{\partial \Gamma^{3}_{33}}{\partial x^{3}}\,,\\
&\frac{\partial \Gamma^{0}_{33}}{\partial x^{0}} = 0\,,\\
&\frac{\partial \Gamma^{1}_{33}}{\partial x^{1}} = \frac{1}{c^4}\Big[2\big(\partial_1 V\big)^2 - \big(c^2 - 2V\big)\partial_{11} V\Big]\,,\\
&\frac{\partial \Gamma^{2}_{33}}{\partial x^{2}} = \frac{1}{c^4}\Big[2\big(\partial_2 V\big)^2 - \big(c^2 - 2V\big)\partial_{22} V\Big]\,,\\
&\frac{\partial \Gamma^{3}_{33}}{\partial x^{3}} = -\frac{1}{c^4}\Big[2\big(\partial_3 V\big)^2 - \big(c^2 - 2V\big)\partial_{33} V\Big]\,;
\end{aligned}$$

- 2-nd term

$$\begin{aligned}
&\frac{\partial^2}{\partial {x^3}^2} \ln\sqrt{-g} = -\Big\{2\Big[\big(2D+P\big)\partial_3 V - \big(c^2+2V\big)^2 P D_3\Big]^2 - D\Big[2\big(c^2+2V\big)^3\big(\partial_3 U\big)^2\\
&- 12\big(c^2+2V\big)^2 P D_3\,\partial_3 V + 4\big(D+2P\big)\big(\partial_3 V\big)^2 - \big(c^2+2V\big)^3 P D_{33} + \big(c^2+2V\big)\\
&\big(2D+P\big)\partial_{33} V\Big]\Big\}\Big/\big(c^2+2V\big)^2 D^2\,;
\end{aligned}$$

- 3-rd term

$$\begin{aligned}
&\Gamma^{\sigma}_{33}\frac{\partial}{\partial x^{\sigma}} \ln\sqrt{-g} = \Gamma^{0}_{33}\frac{\partial}{\partial x^{0}} \ln\sqrt{-g} + \Gamma^{1}_{33}\frac{\partial}{\partial x^{1}} \ln\sqrt{-g} + \Gamma^{2}_{33}\frac{\partial}{\partial x^{2}} \ln\sqrt{-g} + \Gamma^{3}_{33}\frac{\partial}{\partial x^{3}} \ln\sqrt{-g}\,,\\
&\Gamma^{0}_{33}\frac{\partial}{\partial x^{0}} \ln\sqrt{-g} = 0\,,\\
&\Gamma^{1}_{33}\frac{\partial}{\partial x^{1}} \ln\sqrt{-g} = -\big(c^2-2V\big)\partial_1 V\Big[\big(2D+P\big)\partial_1 V - \big(c^2+2V\big)^2 P D_1\Big]\Big/ c^4\big(c^2+2V\big)D\,,\\
&\Gamma^{2}_{33}\frac{\partial}{\partial x^{2}} \ln\sqrt{-g} = -\big(c^2-2V\big)\partial_2 V\Big[\big(2D+P\big)\partial_2 V - \big(c^2+2V\big)^2 P D_2\Big]\Big/ c^4\big(c^2+2V\big)D\,,\\
&\Gamma^{3}_{33}\frac{\partial}{\partial x^{3}} \ln\sqrt{-g} = \big(c^2-2V\big)\partial_3 V\Big[\big(2D+P\big)\partial_3 V - \big(c^2+2V\big)^2 P D_3\Big]\Big/ c^4\big(c^2+2V\big)D\,;
\end{aligned}$$

- 4-th term

$$\begin{aligned}
&\Gamma^{\sigma}_{3\alpha}\Gamma^{\alpha}_{\sigma 3} = \Big(\Gamma^{0}_{30}\Gamma^{0}_{03} + \cdots + \Gamma^{0}_{33}\Gamma^{3}_{03}\Big) + \Big(\Gamma^{1}_{30}\Gamma^{0}_{13} + \cdots + \Gamma^{1}_{33}\Gamma^{3}_{13}\Big) + \Big(\Gamma^{2}_{30}\Gamma^{0}_{23} + \cdots + \Gamma^{2}_{33}\Gamma^{3}_{23}\Big)\\
&+ \Big(\Gamma^{3}_{30}\Gamma^{0}_{33} + \cdots + \Gamma^{3}_{33}\Gamma^{3}_{33}\Big)\,,\\
&\Big(\Gamma^{0}_{30}\Gamma^{0}_{03} + \cdots + \Gamma^{0}_{33}\Gamma^{3}_{03}\Big) = \frac{1}{c^{16}}\big(q^2 P D_3^2 + c^4 h_{123}^2\, P D_3\,\partial_3 V\big)\,,
\end{aligned}$$

$$\Big(\Gamma^1_{30}\Gamma^0_{13}+\cdots+\Gamma^1_{33}\Gamma^3_{13}\Big)=\frac{1}{c^{12}}\Big\{c^4\big(c^2-2V\big)^2\Big[\big(\partial_3V\big)^2-\big(\partial_1V\big)^2\Big]+h_{01}\big(h_{01}\,\partial_3V+h_{03}\,\partial_1V\big)$$
$$PD_3\Big\},$$
$$\Big(\Gamma^2_{30}\Gamma^0_{23}+\cdots+\Gamma^2_{33}\Gamma^3_{23}\Big)=\frac{1}{c^{12}}\Big\{c^4\big(c^2-2V\big)^2\Big[\big(\partial_3V\big)^2-\big(\partial_2V\big)^2\Big]+h_{02}\big(h_{02}\,\partial_3V+h_{03}\,\partial_2V\big)$$
$$PD_3\Big\},$$
$$\Big(\Gamma^3_{30}\Gamma^0_{33}+\cdots+\Gamma^3_{33}\Gamma^3_{33}\Big)=-\frac{1}{c^{12}}\Big\{c^4\big(c^2-2V\big)^2\Big[-\big(\partial_3V\big)^2+\big(\partial_2V\big)^2+\big(\partial_1V\big)^2\Big]+h_{03}\big(h_{01}\,\partial_1V$$
$$+h_{02}\,\partial_2V-h_{03}\,\partial_3V\big)PD_3\Big\}.$$

Further compacting, it follows,

$$\text{1-st terms of }\sum_{i=1}^{3}\frac{\partial\Gamma^i_{33}}{\partial x^i}=\frac{2}{c^4}\Big[\big(\partial_1V\big)^2+\big(\partial_2V\big)^2-\big(\partial_3V\big)^2\Big],$$
$$\text{1-st terms of }\sum_{i=1}^{3}\Gamma^i_{33}\frac{\partial}{\partial x^i}\ln\sqrt{-g}=-\frac{\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\Big[\big(\partial_1V\big)^2+\big(\partial_2V\big)^2-\big(\partial_3V\big)^2\Big],$$
$$\text{1-st terms of }\sum_{i=1}^{3}\frac{\partial\Gamma^i_{33}}{\partial x^i}+\text{1-st terms of }\sum_{i=1}^{3}\Gamma^i_{33}\frac{\partial}{\partial x^i}\ln\sqrt{-g}$$
$$=\left[-\frac{2}{c^4}+\frac{\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\right]\Big[-\big(\partial_1V\big)^2-\big(\partial_2V\big)^2+\big(\partial_3V\big)^2\Big],$$
$$\text{2-nd terms of }\sum_{i=1}^{3}\frac{\partial\Gamma^i_{33}}{\partial x^i}=\frac{\big(c^2-2V\big)}{c^4}\big(-\partial_{11}V-\partial_{22}V+\partial_{33}V\big),$$
$$\text{2-nd terms of }\sum_{i=1}^{3}\Gamma^i_{33}\frac{\partial}{\partial x^i}\ln\sqrt{-g}=\frac{\big(c^4-4V^2\big)}{c^4D}\big(PD_1\,\partial_1V+PD_2\,\partial_2V-PD_3\,\partial_3V\big),$$
$$\text{4-th terms of }\Gamma^\sigma_{3\alpha}\Gamma^\alpha_{\sigma3}=\frac{1}{c^{16}}\big(q^2PD_3^2+c^4h_{123}^2\,PD_3\,\partial_3V\big)+\frac{\big(c^2-2V\big)^2}{c^8}\Big[-2\big(\partial_1V\big)^2-2\big(\partial_2V\big)^2$$
$$+3\big(\partial_3V\big)^2\Big]+\frac{1}{c^{12}}\,h_{123}^2\,PD_3\,\partial_3V\,,$$

from which,

$$R_{33}=\left[-\frac{2}{c^4}+\frac{\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\right]\Big[-\big(\partial_1V\big)^2-\big(\partial_2V\big)^2+\big(\partial_3V\big)^2\Big]+\frac{\big(c^2-2V\big)}{c^4}\big(-\partial_{11}V-\partial_{22}V$$
$$+\partial_{33}V\big)+\frac{\big(c^4-4V^2\big)}{c^4D}\big(PD_1\,\partial_1V+PD_2\,\partial_2V-PD_3\,\partial_3V\big)+\frac{2}{\big(c^2+2V\big)^2D^2}\Big[\big(2D+P\big)\partial_3V$$
$$-\big(c^2+2V\big)^2PD_3\Big]^2-\frac{1}{\big(c^2+2V\big)^2D}\Big[2\big(c^2+2V\big)^3\big(\partial_3U\big)^2-12\big(c^2+2V\big)^2PD_3\,\partial_3V+4\big(D+2P\big)$$
$$\big(\partial_3V\big)^2-\big(c^2+2V\big)^3PD_{33}+\big(c^2+2V\big)\big(2D+P\big)\partial_{33}V\Big]-\frac{1}{c^{16}}\,q^2PD_3^2-\frac{2}{c^{12}}\,h_{123}^2\,PD_3\,\partial_3V$$
$$-\frac{\big(c^2-2V\big)^2}{c^8}\Big[-2\big(\partial_1V\big)^2-2\big(\partial_2V\big)^2+3\big(\partial_3V\big)^2\Big].$$

D.2 Final expansions

D.2.1 $R_{0\mu}$ components $\quad \mu = 0, 1, 2, 3.$

***R_{00}* component**

$$
\begin{aligned}
R_{00} = {} & \frac{2}{c^6}\big(PD_1\,\partial_1 V + PD_2\,\partial_2 V + PD_3\,\partial_3 V\big) - \frac{\big(c^2 - 2V\big)}{c^6}\Big\{\Big[PD_{11} - 2\big(\partial_1 U\big)^2\Big] + \Big[PD_{22} \\
& - 2\big(\partial_2 U\big)^2\Big] + \Big[PD_{33} - 2\big(\partial_3 U\big)^2\Big]\Big\} + \frac{1}{c^6\big(c^2 + 2V\big)D}\Big[\big(c^2 - 2V\big)\big(c^2 + 2V\big)^2\big(PD_1^2 + PD_2^2 \\
& + PD_3^2\big) - \big(c^2 - 2V\big)\big(2D + P\big)\big(PD_1\,\partial_1 V + PD_2\,\partial_2 V + PD_3\,\partial_3 V\big)\Big] - \frac{1}{c^{14}}\Big[2\,Q\big(PD_1^2 + PD_2^2 \\
& + PD_3^2\big) + \big(h_{01}PD_1 + h_{02}PD_2 + h_{03}PD_3\big)^2 + 2\,h_{01}h_{02}PD_1PD_2 + 2\,h_{01}h_{03}PD_1PD_3 \\
& + 2\,h_{02}h_{03}PD_2PD_3 + h_{01}^2PD_1^2 + h_{02}^2PD_2^2 + h_{03}^2PD_3^2\Big]\,;
\end{aligned}
$$

- 1-st term

$$
\begin{aligned}
& \big[R_{00}\big]_1 = \frac{2}{c^6}\big(PD_1\,\partial_1 V + PD_2\,\partial_2 V + PD_3\,\partial_3 V\big) \\
& = \frac{2}{c^6}\Big\{\Big[\big(c^2 - 2U\big)\partial_1 U + 2\,\partial_1\Phi\Big]\partial_1 V + \Big[\big(c^2 - 2U\big)\partial_2 U + 2\,\partial_2\Phi\Big]\partial_2 V + \Big[\big(c^2 - 2U\big)\partial_3 U + 2\,\partial_3\Phi\Big] \\
& \partial_3 V\Big\} \\
& = \frac{2}{c^6}\Big[\big(c^2 - 2U\big)\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V + 2\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\Phi\Big] = \frac{2}{c^4}\left(1 - \frac{2U}{c^2}\right)\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V + \frac{4}{c^6}\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\Phi \\
& = \frac{2}{c^4}\Big[1 + O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V + \frac{4}{c^6}\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\Phi \\
& = \frac{2}{c^4}\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
\big[R_{00}\big]_1 &= {}^{(4)}\big[R_{00}\big]_1 + O\big(\epsilon^6\big)\,, \\
{}^{(4)}\big[R_{00}\big]_1 &= \frac{2}{c^4}\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V\,;
\end{aligned}
$$

- 2-nd term

$$
\begin{aligned}
& \big[R_{00}\big]_2 = -\frac{1}{c^6}\big(c^2 - 2V\big)\Big\{\Big[PD_{11} - 2\big(\partial_1 U\big)^2\Big] + \Big[PD_{22} - 2\big(\partial_2 U\big)^2\Big] + \Big[PD_{33} - 2\big(\partial_3 U\big)^2\Big]\Big\} \\
& = -\frac{1}{c^4}\left(1 - \frac{2V}{c^2}\right)\Big[\big(c^2 - 2U\big)\partial_{11}U + 2\,\partial_{11}\Phi - 2\big(\partial_1 U\big)^2 + \big(c^2 - 2U\big)\partial_{22}U + 2\,\partial_{22}\Phi - 2\big(\partial_2 U\big)^2 \\
& + \big(c^2 - 2U\big)\partial_{33}U + 2\,\partial_{33}\Phi - 2\big(\partial_3 U\big)^2\Big] \\
& = -\frac{1}{c^4}\left(1 - \frac{2V}{c^2}\right)\Big[\big(c^2 - 2U\big)\Delta U + 2\,\Delta\Phi - 2\,\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{c^4}\Big(1-\frac{2V}{c^2}\Big)\bigg[c^2\Big(1-\frac{2U}{c^2}\Big)\Delta U + 2\,\Delta\Phi - 2\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\bigg]\\
&= -\frac{1}{c^2}\Big(1-\frac{2V}{c^2}\Big)\Big(1-\frac{2U}{c^2}\Big)\Delta U - \frac{2}{c^4}\Big(1-\frac{2V}{c^2}\Big)\Delta\Phi + \frac{2}{c^4}\Big(1-\frac{2V}{c^2}\Big)\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\\
&= -\frac{1}{c^2}\bigg[1-\frac{2U}{c^2}-\frac{2V}{c^2}+\Big(\frac{2U}{c^2}\Big)\Big(\frac{2V}{c^2}\Big)\bigg]\Delta U - \frac{2}{c^4}\Big(1-\frac{2V}{c^2}\Big)\Delta\Phi + \frac{2}{c^4}\Big(1-\frac{2V}{c^2}\Big)\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\,,
\end{aligned}
$$

$$
\begin{aligned}
\big[R_{00}\big]_2 &= {}^{(2)}\big[R_{00}\big]_2 + {}^{(4)}\big[R_{00}\big]_2 + O\big(\epsilon^6\big)\,,\\
{}^{(2)}\big[R_{00}\big]_2 &= -\frac{1}{c^2}\,\Delta U\,,\\
{}^{(4)}\big[R_{00}\big]_2 &= \frac{2}{c^4}\big(U+V\big)\Delta U - \frac{2}{c^4}\,\Delta\Phi + \frac{2}{c^4}\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\,;
\end{aligned}
$$

- 3-rd term

$$
\big[R_{00}\big]_3 = \frac{1}{c^6}\frac{1}{\big(c^2+2V\big)D}\Big[\big(c^2-2V\big)\big(c^2+2V\big)^2\big(PD_1^2+PD_2^2+PD_3^2\big) - \big(c^2-2V\big)\big(2D+P\big)\big(PD_1\,\partial_1 V + PD_2\,\partial_2 V + PD_3\,\partial_3 V\big)\Big]\,,
$$

where (the details of the f_1 Taylor's expansion are shown in Appendix B on page 85),

$$
\begin{aligned}
\frac{1}{\big(c^2+2V\big)D} &= \frac{1}{c^8}\,f_1\Big(\frac{h_{01}}{c^3},\frac{h_{02}}{c^3},\frac{h_{03}}{c^3},\frac{U}{c^2},\frac{V}{c^2},\frac{\Phi}{c^4}\Big)\\
&= \frac{1}{c^8}\bigg\{\Big(1+\frac{2V}{c^2}\Big)\bigg[\frac{h_{123}^2}{c^6}+\Big(1+\frac{2V}{c^2}\Big)\Big(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\Big)\bigg]\bigg\}^{-1}\\
&= \frac{1}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\,;
\end{aligned}
$$

$$
\begin{aligned}
&\big(c^2-2V\big)\big(c^2+2V\big)^2\big(PD_1^2+PD_2^2+PD_3^2\big) = c^2\Big(1-\frac{2V}{c^2}\Big)\big(c^4+4c^2V+4V^2\big)\big(\dots\big)\\
&= c^6\Big(1-\frac{2V}{c^2}\Big)\Big(1+\frac{4V}{c^2}+\frac{4V^2}{c^4}\Big)\bigg\{\Big[\big(c^2-2U\big)\partial_1 U + 2\,\partial_1\Phi\Big]^2 + \Big[\big(c^2-2U\big)\partial_2 U + 2\,\partial_2\Phi\Big]^2\\
&+ \Big[\big(c^2-2U\big)\partial_3 U + 2\,\partial_3\Phi\Big]^2\bigg\}\\
&= \dots\bigg\{\Big[\big(c^2-2U\big)^2\big(\partial_1 U\big)^2 + 4\big(\partial_1\Phi\big)^2 + 4\big(c^2-2U\big)\partial_1 U\partial_1\Phi\Big] + \dots\bigg\}\\
&= \dots\bigg[\big(c^4-4c^2U+4U^2\big)\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U + 4\,\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi + 4\,c^2\Big(1-\frac{2U}{c^2}\Big)\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\Phi\bigg]\\
&= c^6\bigg[1+\frac{4V}{c^2}-\frac{2V}{c^2}+\frac{4V^2}{c^4}-\frac{8V^2}{c^4}+O\big(\epsilon^6\big)\bigg]\bigg\{c^4\Big(1-\frac{4U}{c^2}+\frac{4U^2}{c^4}\Big)\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U + 4\,\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi\\
&+ 4c^2\Big(1-\frac{2U}{c^2}\Big)\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\Phi\bigg\}
\end{aligned}
$$

$$= c^6\Big[1+\frac{2V}{c^2}-\frac{4V^2}{c^4}+O\big(\epsilon^6\big)\Big]\Big\{c^4\Big(1-\frac{4U}{c^2}+\frac{4U^2}{c^4}\Big)\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U+4\,\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi+4c^2\Big(1-\frac{2U}{c^2}\Big)$$
$$\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\Phi\Big\};$$

multiplying it by $\frac{1}{c^{14}}\,f_1\Big(\frac{h_{01}}{c^3},\ldots,\frac{\Phi}{c^4}\Big)$, it follows,

$$= \frac{1}{c^{14}}\Big[1+O\big(\epsilon^2\big)\Big]\Big\{c^{10}\Big[1+\frac{2V}{c^2}-\frac{4V^2}{c^4}+O\big(\epsilon^6\big)\Big]\Big[1-\frac{4U}{c^2}+\frac{4U^2}{c^4}\Big]\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U+4c^6\Big[1+\frac{2V}{c^2}$$
$$-\frac{4V^2}{c^4}+O\big(\epsilon^6\big)\Big]\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi+4c^8\Big[1+\frac{2V}{c^2}-\frac{4V^2}{c^4}+O\big(\epsilon^6\big)\Big]\Big(1-\frac{2U}{c^2}\Big)\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\Phi\Big\}$$
$$= \frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+\frac{2V}{c^2}-\frac{4V^2}{c^4}+O\big(\epsilon^6\big)\Big]\Big[1-\frac{4U}{c^2}+\frac{4U^2}{c^4}\Big]\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U+\frac{4}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]$$
$$\Big[1+\frac{2V}{c^2}-\frac{4V^2}{c^4}+O\big(\epsilon^6\big)\Big]\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi+\frac{4}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+\frac{2V}{c^2}-\frac{4V^2}{c^4}+O\big(\epsilon^6\big)\Big]\Big(1-\frac{2U}{c^2}\Big)$$
$$\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\Phi$$
$$= \frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U+\frac{4}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi$$
$$+\frac{4}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\Phi$$
$$= \frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U+\frac{4}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi+\frac{4}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\Phi$$
$$= \frac{1}{c^4}\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U+O\big(\epsilon^6\big);$$

$$-\big(c^2-2V\big)\big(2D+P\big)\big(PD_1\,\partial_1V+PD_2\,\partial_2V+PD_3\,\partial_3V\big)=-c^2\Big(1-\frac{2V}{c^2}\Big)\Big[2h_{123}^2$$
$$+3\big(c^2+2V\big)\big(c^4-2c^2U+2U^2-4\Phi\big)\Big]\Big[c^2\Big(1-\frac{2U}{c^2}\Big)\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V+2\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\Phi\Big]$$
$$= -c^2\Big(1-\frac{2V}{c^2}\Big)\Big[2c^6\Big(\frac{h_{123}^2}{c^6}\Big)+3c^6\Big(1+\frac{2V}{c^2}\Big)\Big(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\Big)\Big]\Big[c^2\Big(1-\frac{2U}{c^2}\Big)$$
$$\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V+2\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\Phi\Big]$$
$$= -c^8\Big(1-\frac{2V}{c^2}\Big)\Big[2\Big(\frac{h_{123}^2}{c^6}\Big)+3\Big(1+\frac{2V}{c^2}\Big)\Big(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\Big)\Big]\Big[c^2\Big(1-\frac{2U}{c^2}\Big)\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V$$
$$+2\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\Phi\Big];$$

multiplying it by $\frac{1}{c^{14}}\,f_1\Big(\frac{h_{01}}{c^3},\ldots,\frac{\Phi}{c^4}\Big)$, it follows,

$$= \frac{1}{c^{14}}\Big[1+O\big(\epsilon^2\big)\Big]\Big\{-c^8\Big(1-\frac{2V}{c^2}\Big)\Big[2\Big(\frac{h_{123}^2}{c^6}\Big)+3\Big(1+\frac{2V}{c^2}\Big)\Big(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\Big)\Big]$$

$$\left[c^2\left(1-\frac{2U}{c^2}\right)\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V+2\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\Phi\right]\Bigg\}$$
$$=-\frac{1}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big\{2\,O\big(\epsilon^6\big)+3\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big\}\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V$$
$$+2\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\Phi\Big\}$$
$$=-\frac{1}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big[3+O\big(\epsilon^2\big)\Big]\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}\boldsymbol{U}\cdot\boldsymbol{\nabla}V+2\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\Phi\Big\}$$
$$=-\frac{1}{c^6}\Big[3+O\big(\epsilon^2\big)\Big]\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V+2\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\Phi\Big\}$$
$$=-\frac{1}{c^4}\Big[3+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V-\frac{2}{c^6}\Big[3+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\Phi$$
$$=-\frac{1}{c^4}\Big[3+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V-\frac{2}{c^6}\Big[3+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\Phi$$
$$=-\frac{3}{c^4}\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V+O\big(\epsilon^6\big)\,;$$

therefore,

$$\big[R_{00}\big]_3={}^{(4)}\big[R_{00}\big]_3+O\big(\epsilon^6\big)\,,$$
$${}^{(4)}\big[R_{00}\big]_3=\frac{1}{c^4}\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U-\frac{3}{c^4}\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\,;$$

- 4-th term

$$\big[R_{00}\big]_4=-\frac{1}{c^{14}}\Big[2Q\big(PD_1^2+PD_2^2+PD_3^2\big)+\big(h_{01}PD_1+h_{02}PD_2+h_{03}PD_3\big)^2+2\,h_{01}h_{02}PD_1$$
$$PD_2+2\,h_{01}h_{03}PD_1PD_3+2\,h_{02}h_{03}PD_2PD_3+h_{01}^2PD_1^2+h_{02}^2PD_2^2+h_{03}^2PD_3^2\Big]\,;$$

$$-\frac{1}{c^{14}}\Big[2Q\big(PD_1^2+PD_2^2+PD_3^2\big)\Big]=-\frac{2}{c^{14}}\big(c^2-2V\big)\big(c^4+2c^2U+2U^2+4\Phi\big)\Big[c^4\Big(1-\frac{4U}{c^2}$$
$$+\frac{4U^2}{c^4}\Big)\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U+4\,\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi+4c^2\Big(1-\frac{2U}{c^2}\Big)\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\Phi\Big]$$
$$=-\frac{2}{c^8}\Big(1-\frac{2V}{c^2}\Big)\Big(1+\frac{2U}{c^2}+\frac{2U^2}{c^4}+\frac{4\Phi}{c^4}\Big)\Big[\dots\Big]$$
$$=-\frac{2}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big\{c^4\Big[1+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U+4\,\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi+4c^2\Big[1+O\big(\epsilon^2\big)\Big]$$
$$\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\Phi\Big\}$$
$$=-\frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U-\frac{8}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi$$
$$-\frac{8}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\Phi$$
$$=-\frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U-\frac{8}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi-\frac{8}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\Phi$$

$$= -\frac{2}{c^4}\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U + O\big(\epsilon^6\big)\,;$$

$$-\frac{1}{c^{14}}\big(h_{01}PD_1 + h_{02}PD_2 + h_{03}PD_3\big)^2 = -\frac{1}{c^{14}}\Big(h_{01}^2PD_1^2 + \cdots + h_{03}^2PD_3^2 + 2\,h_{01}h_{02}PD_1PD_2$$
$$+ \cdots + 2\,h_{02}h_{03}PD_2PD_3\Big)$$
$$= -\frac{1}{c^{14}}\Bigg\{c^6\bigg(\frac{h_{01}}{c^3}\bigg)^2\Big[\big(c^2-2U\big)\partial_1U + 2\partial_1\Phi\Big]^2 + \cdots + c^6\bigg(\frac{h_{03}}{c^3}\bigg)^2\Big[\big(c^2-2U\big)\partial_3U + 2\partial_3\Phi\Big]^2 + 2c^6$$
$$\bigg(\frac{h_{01}}{c^3}\bigg)\bigg(\frac{h_{02}}{c^3}\bigg)\Big[\big(c^2-2U\big)\partial_1U + 2\partial_1\Phi\Big]\Big[\big(c^2-2U\big)\partial_2U + 2\partial_2\Phi\Big] + \cdots + 2c^6\bigg(\frac{h_{02}}{c^3}\bigg)\bigg(\frac{h_{03}}{c^3}\bigg)$$
$$\Big[\big(c^2-2U\big)\partial_2U + 2\partial_2\Phi\Big]\Big[\big(c^2-2U\big)\partial_3U + 2\partial_3\Phi\Big]\Bigg\}$$
$$= -\frac{1}{c^8}\bigg(\frac{h_{01}}{c^3}\bigg)^2\Big[\big(c^4-4c^2U+4U^2\big)\big(\partial_1U\big)^2 + 4\big(\partial_1\Phi\big)^2 + 4\big(c^2-2U\big)\partial_1U\partial_1\Phi\Big] - \cdots - \frac{1}{c^8}\bigg(\frac{h_{03}}{c^3}\bigg)^2$$
$$\Big[\big(c^4-4c^2U+4U^2\big)\big(\partial_3U\big)^2 + 4\big(\partial_3\Phi\big)^2 + 4\big(c^2-2U\big)\partial_3U\partial_3\Phi\Big] - \frac{2}{c^8}\bigg(\frac{h_{01}}{c^3}\bigg)\bigg(\frac{h_{02}}{c^3}\bigg)\bigg[c^2\bigg(1-\frac{2U}{c^2}\bigg)$$
$$\partial_1U + 2\partial_1\Phi\bigg]\bigg[c^2\bigg(1-\frac{2U}{c^2}\bigg)\partial_2U + 2\partial_2\Phi\bigg] - \cdots - \frac{2}{c^8}\bigg(\frac{h_{02}}{c^3}\bigg)\bigg(\frac{h_{03}}{c^3}\bigg)\bigg[c^2\bigg(1-\frac{2U}{c^2}\bigg)\partial_2U + 2\partial_2\Phi\bigg]$$
$$\bigg[c^2\bigg(1-\frac{2U}{c^2}\bigg)\partial_3U + 2\partial_3\Phi\bigg]$$
$$= -\frac{1}{c^8}\bigg(\frac{h_{01}}{c^3}\bigg)^2\bigg[c^4\bigg(1-\frac{4U}{c^2}+\frac{4U^2}{c^4}\bigg)\big(\partial_1U\big)^2 + 4\big(\partial_1\Phi\big)^2 + 4c^2\bigg(1-\frac{2U}{c^2}\bigg)\partial_1U\partial_1\Phi\bigg] - \ldots$$
$$-\frac{1}{c^8}\bigg(\frac{h_{03}}{c^3}\bigg)^2\bigg[c^4\bigg(1-\frac{4U}{c^2}+\frac{4U^2}{c^4}\bigg)\big(\partial_3U\big)^2 + 4\big(\partial_3\Phi\big)^2 + 4c^2\bigg(1-\frac{2U}{c^2}\bigg)\partial_3U\partial_3\Phi\bigg] - \frac{2}{c^8}\bigg(\frac{h_{01}}{c^3}\bigg)$$
$$\bigg(\frac{h_{02}}{c^3}\bigg)\bigg[c^4\bigg(1-\frac{2U}{c^2}\bigg)^2\partial_1U\partial_2U + 2c^2\bigg(1-\frac{2U}{c^2}\bigg)\big(\partial_1U\partial_2\Phi + \partial_2U\partial_1\Phi\big) + 4\,\partial_1\Phi\partial_2\Phi\bigg] - \ldots$$
$$-\frac{2}{c^8}\bigg(\frac{h_{02}}{c^3}\bigg)\bigg(\frac{h_{03}}{c^3}\bigg)\bigg[c^4\bigg(1-\frac{2U}{c^2}\bigg)^2\partial_2U\partial_3U + 2c^2\bigg(1-\frac{2U}{c^2}\bigg)\big(\partial_2U\partial_3\Phi + \partial_3U\partial_2\Phi\big) + 4\,\partial_2\Phi\partial_3\Phi\bigg]$$
$$= -\frac{1}{c^4}\bigg(\frac{h_{01}}{c^3}\bigg)^2\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1U\big)^2 - \frac{4}{c^8}\bigg(\frac{h_{01}}{c^3}\bigg)^2\big(\partial_1\Phi\big)^2 - \frac{4}{c^6}\bigg(\frac{h_{01}}{c^3}\bigg)^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_1U\partial_1\Phi - \ldots$$
$$-\frac{1}{c^4}\bigg(\frac{h_{03}}{c^3}\bigg)^2\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_3U\big)^2 - \frac{4}{c^8}\bigg(\frac{h_{03}}{c^3}\bigg)^2\big(\partial_3\Phi\big)^2 - \frac{4}{c^6}\bigg(\frac{h_{03}}{c^3}\bigg)^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_3U\partial_3\Phi - \ldots$$
$$-\frac{2}{c^4}\bigg(\frac{h_{01}}{c^3}\bigg)\bigg(\frac{h_{02}}{c^3}\bigg)\Big[1+O\big(\epsilon^2\big)\Big]\partial_1U\partial_2U - \frac{4}{c^6}\bigg(\frac{h_{01}}{c^3}\bigg)\bigg(\frac{h_{02}}{c^3}\bigg)\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1U\partial_2\Phi + \partial_2U\partial_1\Phi\big)$$
$$-\frac{8}{c^8}\bigg(\frac{h_{01}}{c^3}\bigg)\bigg(\frac{h_{02}}{c^3}\bigg)\partial_1\Phi\partial_2\Phi - \cdots - \frac{2}{c^4}\bigg(\frac{h_{02}}{c^3}\bigg)\bigg(\frac{h_{03}}{c^3}\bigg)\Big[1+O\big(\epsilon^2\big)\Big]\partial_2U\partial_3U - \frac{4}{c^6}\bigg(\frac{h_{02}}{c^3}\bigg)\bigg(\frac{h_{03}}{c^3}\bigg)$$
$$\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_2U\partial_3\Phi + \partial_3U\partial_2\Phi\big) - \frac{8}{c^8}\bigg(\frac{h_{02}}{c^3}\bigg)\bigg(\frac{h_{03}}{c^3}\bigg)\partial_2\Phi\partial_3\Phi \sim O\big(\epsilon^{10}\big)\,;$$

$$-\frac{2}{c^{14}}\big(h_{01}h_{02}PD_1PD_2 + h_{01}h_{03}PD_1PD_3 + h_{02}h_{03}PD_2PD_3\big) \sim O\big(\epsilon^{10}\big)\,,$$

as above;

$$
\begin{aligned}
&-\frac{1}{c^{14}}\Big(h_{01}^2 PD_1^2+h_{02}^2 PD_2^2+h_{03}^2 PD_3^2\Big)=-\frac{1}{c^{14}}\bigg\{c^6\bigg(\frac{h_{01}}{c^3}\bigg)^2\Big[\big(c^2-2U\big)\partial_1 U+2\,\partial_1\Phi\Big]^2\bigg\}+\dots\\
&=-\frac{1}{c^8}\bigg(\frac{h_{01}}{c^3}\bigg)^2\Big[\big(c^4-4c^2U+4U^2\big)\big(\partial_1 U\big)^2+4\big(\partial_1\Phi\big)^2+4\big(c^2-2U\big)\partial_1 U\partial_1\Phi\Big]+\dots\\
&=-\frac{1}{c^8}\bigg(\frac{h_{01}}{c^3}\bigg)^2\bigg[c^4\bigg(1-\frac{4U}{c^2}+\frac{4U^2}{c^4}\bigg)\big(\partial_1 U\big)^2+4\big(\partial_1\Phi\big)^2+4c^2\bigg(1-\frac{2U}{c^2}\bigg)\partial_1 U\partial_1\Phi\bigg]+\dots\\
&=-\frac{1}{c^4}\bigg(\frac{h_{01}}{c^3}\bigg)^2\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1 U\big)^2-\frac{4}{c^8}\bigg(\frac{h_{01}}{c^3}\bigg)^2\big(\partial_1\Phi\big)^2-\frac{4}{c^6}\bigg(\frac{h_{01}}{c^3}\bigg)^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1\Phi+\dots\\
&\sim O\big(\epsilon^{10}\big)\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
\big[R_{00}\big]_4&=\,{}^{(4)}\big[R_{00}\big]_4+O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{00}\big]_4&=-\,\frac{2}{c^4}\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\,.
\end{aligned}
$$

Synoptic list:

$$
\begin{aligned}
\big[R_{00}\big]_1&=\,{}^{(4)}\big[R_{00}\big]_1+O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{00}\big]_1&=\frac{2}{c^4}\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\,;
\end{aligned}
$$

$$
\begin{aligned}
\big[R_{00}\big]_2&=\,{}^{(2)}\big[R_{00}\big]_2+\,{}^{(4)}\big[R_{00}\big]_2+O\big(\epsilon^6\big)\,,\\
{}^{(2)}\big[R_{00}\big]_2&=-\frac{1}{c^2}\,\Delta U\,,\\
{}^{(4)}\big[R_{00}\big]_2&=\frac{2}{c^4}\big(U+V\big)\Delta U-\frac{2}{c^4}\,\Delta\Phi+\frac{2}{c^4}\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\,;
\end{aligned}
$$

$$
\begin{aligned}
\big[R_{00}\big]_3&=\,{}^{(4)}\big[R_{00}\big]_3+O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{00}\big]_3&=\frac{1}{c^4}\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U-\frac{3}{c^4}\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\,;
\end{aligned}
$$

$$
\begin{aligned}
\big[R_{00}\big]_4&=\,{}^{(4)}\big[R_{00}\big]_4+O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{00}\big]_4&=-\,\frac{2}{c^4}\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\,.
\end{aligned}
$$

Summing up,

$$
\begin{aligned}
R_{00}&=\,{}^{(2)}R_{00}+\,{}^{(4)}R_{00}+O\big(\epsilon^6\big)\,,\\
{}^{(2)}R_{00}&=-\frac{1}{c^2}\,\Delta U\,,\\
{}^{(4)}R_{00}&=\frac{1}{c^4}\Big[-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V+2\big(U+V\big)\Delta U-2\,\Delta\Phi+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\Big]\,.
\end{aligned}
$$

R_{01} component

$$R_{01} = \frac{1}{c^7}\Big\{h_{01}\Big[PD_{11} - 2\big(\partial_1 U\big)^2\Big] + h_{02}\big(PD_{12} - 2\,\partial_1 U\,\partial_2 U\big) + h_{03}\big(PD_{13} - 2\,\partial_1 U\,\partial_3 U\big)\Big\}$$
$$+ \frac{1}{c^7\big(c^2+2V\big)D}\Big\{h_{01}\Big[-\big(c^2+2V\big)^2 PD_1^2 + \big(2D+P\big)PD_1\,\partial_1 V\Big] + h_{02}\Big[-\big(c^2+2V\big)^2 PD_1$$
$$PD_2 + \big(2D+P\big)PD_1\,\partial_2 V\Big] + h_{03}\Big[-\big(c^2+2V\big)^2 PD_1 PD_3 + \big(2D+P\big)PD_1\,\partial_3 V\Big]\Big\}$$
$$+ \frac{2\big(c^2-2V\big)}{c^{11}}\Big\{h_{01}\big(PD_2\,\partial_2 V + PD_3\,\partial_3 V\big) - h_{02}PD_1\,\partial_2 V - h_{03}PD_1\,\partial_3 V\Big\};$$

- 1-st term

$$\big[R_{01}\big]_1 = \frac{1}{c^7}\Big\{h_{01}\Big[PD_{11} - 2\big(\partial_1 U\big)^2\Big] + h_{02}\big(PD_{12} - 2\,\partial_1 U\,\partial_2 U\big) + h_{03}\big(PD_{13} - 2\,\partial_1 U\,\partial_3 U\big)\Big\}$$
$$= \frac{1}{c^7}\Bigg\{c^3\bigg(\frac{h_{01}}{c^3}\bigg)\Big[\big(c^2-2U\big)\partial_{11}U + 2\,\partial_{11}\Phi - 2\big(\partial_1 U\big)^2\Big] + c^3\bigg(\frac{h_{02}}{c^3}\bigg)\Big[\big(c^2-2U\big)\partial_{12}U + 2\,\partial_{12}\Phi$$
$$- 2\,\partial_1 U\,\partial_2 U\Big] + c^3\bigg(\frac{h_{03}}{c^3}\bigg)\Big[\big(c^2-2U\big)\partial_{13}U + 2\,\partial_{13}\Phi - 2\,\partial_1 U\,\partial_3 U\Big]\Bigg\}$$
$$= \frac{1}{c^4}\bigg(\frac{h_{01}}{c^3}\bigg)\bigg[c^2\bigg(1-\frac{2U}{c^2}\bigg)\partial_{11}U + 2\,\partial_{11}\Phi - 2\big(\partial_1 U\big)^2\bigg] + \frac{1}{c^4}\bigg(\frac{h_{02}}{c^3}\bigg)\bigg[c^2\bigg(1-\frac{2U}{c^2}\bigg)\partial_{12}U + 2\,\partial_{12}\Phi$$
$$- 2\,\partial_1 U\,\partial_2 U\bigg] + \frac{1}{c^4}\bigg(\frac{h_{03}}{c^3}\bigg)\bigg[c^2\bigg(1-\frac{2U}{c^2}\bigg)\partial_{13}U + 2\,\partial_{13}\Phi - 2\,\partial_1 U\,\partial_3 U\bigg]$$
$$= \bigg(\frac{h_{01}}{c^3}\bigg)\bigg\{\frac{1}{c^2}\Big[1+O(\epsilon^2)\Big]\partial_{11}U + \frac{2}{c^4}\,\partial_{11}\Phi - \frac{2}{c^4}\big(\partial_1 U\big)^2\bigg\} + \bigg(\frac{h_{02}}{c^3}\bigg)\bigg\{\frac{1}{c^2}\Big[1+O(\epsilon^2)\Big]\partial_{12}U$$
$$+ \frac{2}{c^4}\,\partial_{12}\Phi - \frac{2}{c^4}\,\partial_1 U\,\partial_2 U\bigg\} + \bigg(\frac{h_{03}}{c^3}\bigg)\bigg\{\frac{1}{c^2}\Big[1+O(\epsilon^2)\Big]\partial_{13}U + \frac{2}{c^4}\,\partial_{13}\Phi - \frac{2}{c^4}\,\partial_1 U\,\partial_3 U\bigg\}$$
$$= \frac{1}{c^2}\bigg[\bigg(\frac{h_{01}}{c^3}\bigg)\partial_{11}U + \bigg(\frac{h_{02}}{c^3}\bigg)\partial_{12}U + \bigg(\frac{h_{03}}{c^3}\bigg)\partial_{13}U\bigg] + O\big(\epsilon^7\big);$$

therefore,

$$\big[R_{01}\big]_1 = {}^{(5)}\big[R_{01}\big]_1 + O\big(\epsilon^7\big),$$
$${}^{(5)}\big[R_{01}\big]_1 = \frac{1}{c^5}\Big(h_{01}\,\partial_{11}U + h_{02}\,\partial_{12}U + h_{03}\,\partial_{13}U\Big);$$

- 2-nd term

$$\big[R_{01}\big]_2 = \frac{1}{c^7\big(c^2+2V\big)D}\Big\{h_{01}\Big[-\big(c^2+2V\big)^2 PD_1^2 + \big(2D+P\big)PD_1\,\partial_1 V\Big] + h_{02}\Big[-\big(c^2+2V\big)^2 PD_1$$
$$PD_2 + \big(2D+P\big)PD_1\,\partial_2 V\Big] + h_{03}\Big[-\big(c^2+2V\big)^2 PD_1 PD_3 + \big(2D+P\big)PD_1\,\partial_3 V\Big]\Big\},$$

where, as in the previous subsection,

$$\frac{1}{\big(c^2+2V\big)D}=\frac{1}{c^8}\,f_1\Big(\frac{h_{01}}{c^3},\cdots,\frac{\Phi}{c^4}\Big)=\frac{1}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\,,$$

$$h_{01}\Big[-\big(c^2+2V\big)^2PD_1^2+\big(2D+P\big)PD_1\,\partial_1V\Big]=c^3\Big(\frac{h_{01}}{c^3}\Big)\Big\{-\big(c^4+4c^2V+4V^2\big)\Big[\big(c^2-2U\big)\partial_1U$$
$$+2\partial_1\Phi\Big]^2\Big\}+c^3\Big(\frac{h_{01}}{c^3}\Big)\Big[2c^6\Big(\frac{h_{123}^2}{c^6}\Big)+3\big(c^2+2V\big)\big(c^4+2U^2-2c^2U-4\Phi\big)\Big]\Big[\big(c^2-2U\big)\partial_1U$$
$$+2\partial_1\Phi\Big]\partial_1V$$
$$=c^3\Big(\frac{h_{01}}{c^3}\Big)\Big\{-c^4\Big(1+\frac{4V}{c^2}+\frac{4V^2}{c^4}\Big)\Big[\big(c^2-2U\big)^2\big(\partial_1U\big)^2+4\big(\partial_1\Phi\big)^2+4\big(c^2-2U\big)\partial_1U\partial_1\Phi\Big]\Big\}$$
$$+c^3\Big(\frac{h_{01}}{c^3}\Big)\Big[2c^6\Big(\frac{h_{123}^2}{c^6}\Big)+3c^6\Big(1+\frac{2V}{c^2}\Big)\Big(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\Big)\Big]\Big[c^2\Big(1-\frac{2U}{c^2}\Big)\partial_1U$$
$$+2\partial_1\Phi\Big]\partial_1V$$
$$=c^7\Big(\frac{h_{01}}{c^3}\Big)\Big\{-\Big(1+\frac{4V}{c^2}+\frac{4V^2}{c^4}\Big)\Big[\big(c^4+4U^2-4c^2U\big)\big(\partial_1U\big)^2+4\big(\partial_1\Phi\big)^2+4c^2\Big(1-\frac{2U}{c^2}\Big)\partial_1U$$
$$\partial_1\Phi\Big]\Big\}+c^9\Big(\frac{h_{01}}{c^3}\Big)\Big[2\Big(\frac{h_{123}^2}{c^6}\Big)+3\Big(1+\frac{2V}{c^2}\Big)\Big(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\Big)\Big]\Big[c^2\Big(1-\frac{2U}{c^2}\Big)\partial_1U$$
$$+2\partial_1\Phi\Big]\partial_1V$$
$$=c^7\Big(\frac{h_{01}}{c^3}\Big)\Big\{-\Big(1+\frac{4V}{c^2}+\frac{4V^2}{c^4}\Big)\Big[c^4\Big(1+\frac{4U^2}{c^4}-\frac{4U}{c^2}\Big)\big(\partial_1U\big)^2+4\big(\partial_1\Phi\big)^2+4c^2\Big(1-\frac{2U}{c^2}\Big)$$
$$\partial_1U\partial_1\Phi\Big]\Big\}+c^9\Big(\frac{h_{01}}{c^3}\Big)\Big[2\Big(\frac{h_{123}^2}{c^6}\Big)+3\Big(1+\frac{2V}{c^2}\Big)\Big(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\Big)\Big]\Big[c^2\Big(1-\frac{2U}{c^2}\Big)$$
$$\partial_1U+2\partial_1\Phi\Big]\partial_1V$$
$$=-\,c^{11}\Big(\frac{h_{01}}{c^3}\Big)\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1U\big)^2-4\,c^7\Big(\frac{h_{01}}{c^3}\Big)\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1\Phi\big)^2-4\,c^9\Big(\frac{h_{01}}{c^3}\Big)$$
$$\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\partial_1U\partial_1\Phi+c^9\Big(\frac{h_{01}}{c^3}\Big)\Big\{O\big(\epsilon^6\big)+3\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big\}$$
$$\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_1U+2\partial_1\Phi\Big\}\partial_1V\,;$$

multiplying it by $\frac{1}{c^{15}}\,f_1\Big(\frac{h_{01}}{c^3},\ldots,\frac{\Phi}{c^4}\Big)$, it follows,

$$\frac{1}{c^{15}}\Big[1+O\big(\epsilon^2\big)\Big]\Big\{-\,c^{11}\Big(\frac{h_{01}}{c^3}\Big)-\cdots+c^9\Big(\frac{h_{01}}{c^3}\Big)\cdots\Big\}=-\,\frac{1}{c^4}\Big(\frac{h_{01}}{c^3}\Big)\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1U\big)^2-\frac{4}{c^8}$$
$$\Big(\frac{h_{01}}{c^3}\Big)\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1\Phi\big)^2-\frac{4}{c^6}\Big(\frac{h_{01}}{c^3}\Big)\Big[1+O\big(\epsilon^2\big)\Big]\partial_1U\partial_1\Phi+\frac{1}{c^6}\Big(\frac{h_{01}}{c^3}\Big)\Big[1+O\big(\epsilon^2\big)\Big]\Big\{O\big(\epsilon^6\big)$$

$$
\begin{aligned}
&+3\Big[1+O\big(\epsilon^2\big)\Big]\Big\}\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U+2\partial_1\Phi\Big\}\partial_1 V\\
&=\cdots+\frac{1}{c^6}\left(\frac{h_{01}}{c^3}\right)\Big[3+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1 V+2\partial_1 V\partial_1\Phi\Big\}\\
&=\cdots+\frac{1}{c^4}\left(\frac{h_{01}}{c^3}\right)\Big[3+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1 V+\frac{2}{c^6}\left(\frac{h_{01}}{c^3}\right)\Big[3+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_1\Phi\\
&=-\frac{1}{c^4}\left(\frac{h_{01}}{c^3}\right)\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1 U\big)^2-\frac{4}{c^8}\left(\frac{h_{01}}{c^3}\right)\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1\Phi\big)^2-\frac{4}{c^6}\left(\frac{h_{01}}{c^3}\right)\Big[1+O\big(\epsilon^2\big)\Big]\\
&\partial_1 U\partial_1\Phi+\frac{1}{c^4}\left(\frac{h_{01}}{c^3}\right)\Big[3+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1 V+\frac{2}{c^6}\left(\frac{h_{01}}{c^3}\right)\Big[3+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_1\Phi\sim O\big(\epsilon^7\big)\,,\\
&\Big[c^7\big(c^2+2V\big)D\Big]^{-1}\Big\{h_{01}\Big[-\big(c^2+2V\big)^2PD_1^2+\big(2D+P\big)PD_1\,\partial_1 V\Big]\Big\}\sim O\big(\epsilon^7\big)\,;
\end{aligned}
$$

in the same way,

$$
\begin{aligned}
&\Big[c^7\big(c^2+2V\big)D\Big]^{-1}\Big\{h_{02}\Big[-\big(c^2+2V\big)^2PD_1PD_2+\big(2D+P\big)PD_1\,\partial_2 V\Big]\Big\}\sim O\big(\epsilon^7\big)\,,\\
&\Big[c^7\big(c^2+2V\big)D\Big]^{-1}\Big\{h_{03}\Big[-\big(c^2+2V\big)^2PD_1PD_3+\big(2D+P\big)PD_1\,\partial_3 V\Big]\Big\}\sim O\big(\epsilon^7\big)\,;
\end{aligned}
$$

therefore,

$$
\big[R_{01}\big]_2\sim O\big(\epsilon^7\big)\,;
$$

- 3-rd term

$$
\big[R_{01}\big]_3=\frac{2}{c^{11}}\big(c^2-2V\big)\Big\{h_{01}\big(PD_2\,\partial_2 V+PD_3\,\partial_3 V\big)-h_{02}PD_1\,\partial_2 V-h_{03}PD_1\,\partial_3 V\Big\}\,,
$$

$$
\begin{aligned}
&\frac{2}{c^{11}}\big(c^2-2V\big)h_{01}PD_2\,\partial_2 V=\frac{2}{c^{11}}c^5\left(\frac{h_{01}}{c^3}\right)\left(1-\frac{2V}{c^2}\right)\Big[\big(c^2-2U\big)\partial_2 U+2\partial_2\Phi\Big]\partial_2 V\\
&=\frac{2}{c^6}\left(\frac{h_{01}}{c^3}\right)\left(1-\frac{2V}{c^2}\right)\left[c^2\left(1-\frac{2U}{c^2}\right)\partial_2 U+2\partial_2\Phi\right]\partial_2 V\\
&=\frac{2}{c^6}\left(\frac{h_{01}}{c^3}\right)\Big[1+O\big(\epsilon^2\big)\Big]\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_2 U+2\partial_2\Phi\Big\}\partial_2 V\\
&=\frac{2}{c^4}\left(\frac{h_{01}}{c^3}\right)\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\partial_2 U\partial_2 V+\frac{4}{c^6}\left(\frac{h_{01}}{c^3}\right)\Big[1+O\big(\epsilon^2\big)\Big]\partial_2 V\partial_2\Phi\\
&=\frac{2}{c^4}\left(\frac{h_{01}}{c^3}\right)\Big[1+O\big(\epsilon^2\big)\Big]\partial_2 U\partial_2 V+\frac{4}{c^6}\left(\frac{h_{01}}{c^3}\right)\Big[1+O\big(\epsilon^2\big)\Big]\partial_2 V\partial_2\Phi\sim O\big(\epsilon^7\big)\,;
\end{aligned}
$$

in the same way,

$$
\frac{2}{c^{11}}\big(c^2-2V\big)\Big(h_{01}PD_3\,\partial_3 V-h_{02}PD_1\,\partial_2 V-h_{03}PD_1\,\partial_3 V\Big)\sim O\big(\epsilon^7\big)\,;
$$

therefore,

$$
\big[R_{01}\big]_3\sim O\big(\epsilon^7\big)\,.
$$

Synoptic list:

$$
\begin{aligned}
\big[R_{01}\big]_1 &= {}^{(5)}\big[R_{01}\big]_1 + O\big(\epsilon^7\big)\,,\\
{}^{(5)}\big[R_{01}\big]_1 &= \frac{1}{c^5}\Big(h_{01}\,\partial_{11}U + h_{02}\,\partial_{12}U + h_{03}\,\partial_{13}U\Big)\,;
\end{aligned}
$$

$$
\big[R_{01}\big]_2 \sim O\big(\epsilon^7\big)\,;
$$

$$
\big[R_{01}\big]_3 \sim O\big(\epsilon^7\big)\,.
$$

Summing up,

$$
\begin{aligned}
R_{01} &= {}^{(5)}R_{01} + O\big(\epsilon^7\big)\,,\\
{}^{(5)}R_{01} &= \frac{1}{c^5}\Big[h_{01}\,\partial_{11}U + h_{02}\,\partial_{12}U + h_{03}\,\partial_{13}U\Big]\\
&= \frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(\partial_1 U\big)\,.
\end{aligned}
$$

R_{02} component

$$
\begin{aligned}
&R_{02} = \frac{1}{c^7}\Big\{h_{01}\big(PD_{12} - 2\,\partial_1 U\,\partial_2 U\big) + h_{02}\Big[PD_{22} - 2\big(\partial_2 U\big)^2\Big] + h_{03}\big(PD_{23} - 2\,\partial_2 U\,\partial_3 U\big)\Big\}\\
&+ \frac{1}{c^7\big(c^2+2V\big)D}\Big\{h_{01}\Big[-\big(c^2+2V\big)^2 PD_1 PD_2 + \big(2D+P\big)PD_2\,\partial_1 V\Big] + h_{02}\Big[-\big(c^2+2V\big)^2 PD_2^2\\
&+ \big(2D+P\big)PD_2\,\partial_2 V\Big] + h_{03}\Big[-\big(c^2+2V\big)^2 PD_2 PD_3 + \big(2D+P\big)PD_2\,\partial_3 V\Big]\Big\}\\
&- \frac{2\big(c^2-2V\big)}{c^{11}}\Big[h_{01}PD_2\,\partial_1 V - h_{02}\big(PD_1\,\partial_1 V + PD_3\,\partial_3 V\big) + h_{03}PD_2\,\partial_3 V\Big]\,;
\end{aligned}
$$

- 1-st term

$$
\begin{aligned}
&\big[R_{02}\big]_1 = \frac{1}{c^7}\Big\{h_{01}\big(PD_{12} - 2\,\partial_1 U\,\partial_2 U\big) + h_{02}\Big[PD_{22} - 2\big(\partial_2 U\big)^2\Big] + h_{03}\big(PD_{23} - 2\,\partial_2 U\,\partial_3 U\big)\Big\}\\
&= \frac{1}{c^4}\left(\frac{h_{01}}{c^3}\right)\Big[\big(c^2-2U\big)\partial_{12}U + 2\,\partial_{12}\Phi - 2\,\partial_1 U\,\partial_2 U\Big] + \frac{1}{c^4}\left(\frac{h_{02}}{c^3}\right)\Big[\big(c^2-2U\big)\partial_{22}U + 2\,\partial_{22}\Phi\\
&- 2\,\big(\partial_2 U\big)^2\Big] + \frac{1}{c^4}\left(\frac{h_{03}}{c^3}\right)\Big[\big(c^2-2U\big)\partial_{23}U + 2\,\partial_{23}\Phi - 2\,\partial_2 U\,\partial_3 U\Big]\\
&= \frac{1}{c^4}\left(\frac{h_{01}}{c^3}\right)\left[c^2\left(1-\frac{2U}{c^2}\right)\partial_{12}U + 2\,\partial_{12}\Phi - 2\,\partial_1 U\,\partial_2 U\right] + \frac{1}{c^4}\left(\frac{h_{02}}{c^3}\right)\left[c^2\left(1-\frac{2U}{c^2}\right)\partial_{22}U\right.\\
&\left.+ 2\,\partial_{22}\Phi - 2\,\big(\partial_2 U\big)^2\right] + \frac{1}{c^4}\left(\frac{h_{03}}{c^3}\right)\left[c^2\left(1-\frac{2U}{c^2}\right)\partial_{23}U + 2\,\partial_{23}\Phi - 2\,\partial_2 U\,\partial_3 U\right]\\
&= \frac{1}{c^2}\left(\frac{h_{01}}{c^3}\right)\Big[1+O(\epsilon^2)\Big]\partial_{12}U + \frac{2}{c^4}\left(\frac{h_{01}}{c^3}\right)\partial_{12}\Phi - \frac{2}{c^4}\left(\frac{h_{01}}{c^3}\right)\partial_1 U\,\partial_2 U + \frac{1}{c^2}\left(\frac{h_{02}}{c^3}\right)\Big[1+O(\epsilon^2)\Big]\\
&\partial_{22}U + \frac{2}{c^4}\left(\frac{h_{02}}{c^3}\right)\partial_{22}\Phi - \frac{2}{c^4}\left(\frac{h_{02}}{c^3}\right)\big(\partial_2 U\big)^2 + \frac{1}{c^2}\left(\frac{h_{03}}{c^3}\right)\Big[1+O(\epsilon^2)\Big]\partial_{23}U + \frac{2}{c^4}\left(\frac{h_{03}}{c^3}\right)\partial_{23}\Phi
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2}{c^4}\left(\frac{h_{03}}{c^3}\right)\partial_2 U\,\partial_3 U\\
&=\frac{1}{c^2}\left(\frac{h_{01}}{c^3}\right)\partial_{12}U+\frac{1}{c^2}\left(\frac{h_{02}}{c^3}\right)\partial_{22}U+\frac{1}{c^2}\left(\frac{h_{03}}{c^3}\right)\partial_{23}U+O(\epsilon^7)\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
\big[R_{02}\big]_1 &= {}^{(5)}\big[R_{02}\big]_1+O\big(\epsilon^7\big)\,,\\
{}^{(5)}\big[R_{02}\big]_1 &= \frac{1}{c^5}\Big(h_{01}\,\partial_{12}U+h_{02}\,\partial_{22}U+h_{03}\,\partial_{23}U\Big)\,;
\end{aligned}
$$

- 2-nd term

$$
\begin{aligned}
&\big[R_{02}\big]_2=\frac{1}{c^7\big(c^2+2V\big)D}\Big\{h_{01}\Big[-\big(c^2+2V\big)^2PD_1PD_2+\big(2D+P\big)PD_2\,\partial_1V\Big]+h_{02}\Big[-\big(c^2+2V\big)^2\\
&PD_2^2+\big(2D+P\big)PD_2\,\partial_2V\Big]+h_{03}\Big[-\big(c^2+2V\big)^2PD_2PD_3+\big(2D+P\big)PD_2\,\partial_3V\Big]\Big\}\,,
\end{aligned}
$$

$$
\big[R_{02}\big]_2\sim O\big(\epsilon^7\big)\,,
$$

see the previous subsection;

- 3-rd term

$$
\big[R_{02}\big]_3=-\frac{2}{c^{11}}\big(c^2-2V\big)\Big[h_{01}PD_2\,\partial_1V-h_{02}\big(PD_1\,\partial_1V+PD_3\,\partial_3V\big)+h_{03}PD_2\,\partial_3V\Big]\,,
$$

$$
\big[R_{02}\big]_3\sim O\big(\epsilon^7\big)\,,
$$

see the previous subsection.

Synoptic list:

$$
\begin{aligned}
\big[R_{02}\big]_1 &= {}^{(5)}\big[R_{02}\big]_1+O\big(\epsilon^7\big)\,,\\
{}^{(5)}\big[R_{02}\big]_1 &= \frac{1}{c^5}\Big(h_{01}\,\partial_{12}U+h_{02}\,\partial_{22}U+h_{03}\,\partial_{23}U\Big)\,;
\end{aligned}
$$

$$
\big[R_{02}\big]_2\sim O\big(\epsilon^7\big)\,,
$$

$$
\big[R_{02}\big]_3\sim O\big(\epsilon^7\big)\,.
$$

Summing up,

$$
\begin{aligned}
R_{02} &= {}^{(5)}R_{02} + O\big(\epsilon^7\big)\,,\\
{}^{(5)}R_{02} &= \frac{1}{c^5}\Big(h_{01}\,\partial_{12}U + h_{02}\,\partial_{22}U + h_{03}\,\partial_{23}U\Big)\,,\\
&= \frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(\partial_2 U\big)\,.
\end{aligned}
$$

R_{03} component

$$
\begin{aligned}
&R_{03} = \frac{1}{c^7}\Big\{h_{01}\big(PD_{13} - 2\,\partial_1 U\,\partial_3 U\big) + h_{02}\Big(PD_{23} - 2\,\partial_2 U\partial_3 U\Big) + h_{03}\Big[PD_{33} - 2\big(\partial_3 U\big)^2\Big]\Big\}\\
&+ \frac{1}{c^7\big(c^2+2V\big)D}\Big\{h_{01}\Big[-\big(c^2+2V\big)^2 PD_1 PD_3 + \big(2D+P\big)PD_3\,\partial_1 V\Big] + h_{02}\Big[-\big(c^2+2V\big)^2 PD_2\\
&PD_3 + \big(2D+P\big)PD_3\,\partial_2 V\Big] + h_{03}\Big[-\big(c^2+2V\big)^2 PD_3^2 + \big(2D+P\big)PD_3\,\partial_3 V\Big]\Big\}\\
&- \frac{2\big(c^2-2V\big)}{c^{11}}\Big[h_{01}PD_3\,\partial_1 V + h_{02}PD_3\,\partial_2 V - h_{03}\big(PD_1\,\partial_1 V + PD_2\,\partial_2 V\big)\Big]\,;
\end{aligned}
$$

- 1-st term

$$
\begin{aligned}
&\big[R_{03}\big]_1 = \frac{1}{c^7}\Big\{h_{01}\big(PD_{13} - 2\,\partial_1 U\,\partial_3 U\big) + h_{02}\big(PD_{23} - 2\,\partial_2 U\,\partial_3 U\big) + h_{03}\Big[PD_{33} - 2\big(\partial_3 U\big)^2\Big]\Big\}\\
&= \frac{1}{c^4}\left(\frac{h_{01}}{c^3}\right)\Big[\big(c^2-2U\big)\partial_{13}U + 2\,\partial_{13}\Phi - 2\,\partial_1 U\,\partial_3 U\Big] + \frac{1}{c^4}\left(\frac{h_{02}}{c^3}\right)\Big[\big(c^2-2U\big)\partial_{23}U + 2\,\partial_{23}\Phi\\
&- 2\,\partial_2 U\partial_3 U\Big] + \frac{1}{c^4}\left(\frac{h_{03}}{c^3}\right)\Big[\big(c^2-2U\big)\partial_{33}U + 2\,\partial_{33}\Phi - 2\big(\partial_3 U\big)^2\Big]\\
&= \frac{1}{c^4}\left(\frac{h_{01}}{c^3}\right)\left[c^2\left(1-\frac{2U}{c^2}\right)\partial_{13}U + 2\,\partial_{13}\Phi - 2\,\partial_1 U\,\partial_3 U\right] + \frac{1}{c^4}\left(\frac{h_{02}}{c^3}\right)\left[c^2\left(1-\frac{2U}{c^2}\right)\partial_{23}U\right.\\
&\left.+ 2\,\partial_{23}\Phi - 2\,\partial_2 U\partial_3 U\right] + \frac{1}{c^4}\left(\frac{h_{03}}{c^3}\right)\left[c^2\left(1-\frac{2U}{c^2}\right)\partial_{33}U + 2\,\partial_{33}\Phi - 2\big(\partial_3 U\big)^2\right]\\
&= \frac{1}{c^2}\left(\frac{h_{01}}{c^3}\right)\Big[1+O(\epsilon^2)\Big]\partial_{13}U + \frac{2}{c^4}\left(\frac{h_{01}}{c^3}\right)\partial_{13}\Phi - \frac{2}{c^4}\left(\frac{h_{01}}{c^3}\right)\partial_1 U\,\partial_3 U + \frac{1}{c^2}\left(\frac{h_{02}}{c^3}\right)\Big[1+O(\epsilon^2)\Big]\\
&\partial_{23}U + \frac{2}{c^4}\left(\frac{h_{02}}{c^3}\right)\partial_{23}\Phi - \frac{2}{c^4}\left(\frac{h_{02}}{c^3}\right)\partial_2 U\,\partial_3 U + \frac{1}{c^2}\left(\frac{h_{03}}{c^3}\right)\Big[1+O(\epsilon^2)\Big]\partial_{33}U + \frac{2}{c^4}\left(\frac{h_{03}}{c^3}\right)\partial_{33}\Phi\\
&- \frac{2}{c^4}\left(\frac{h_{03}}{c^3}\right)\big(\partial_3 U\big)^2\\
&= \frac{1}{c^2}\left(\frac{h_{01}}{c^3}\right)\partial_{13}U + \frac{1}{c^2}\left(\frac{h_{02}}{c^3}\right)\partial_{23}U + \frac{1}{c^2}\left(\frac{h_{03}}{c^3}\right)\partial_{33}U + O(\epsilon^7)\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
\big[R_{03}\big]_1 &= {}^{(5)}\big[R_{03}\big]_1 + O\big(\epsilon^7\big)\,,\\
{}^{(5)}\big[R_{03}\big]_1 &= \frac{1}{c^5}\Big(h_{01}\,\partial_{13}U + h_{02}\,\partial_{23}U + h_{03}\,\partial_{33}U\Big)\,;
\end{aligned}
$$

- 2-nd term

$$\big[R_{03}\big]_2 = \frac{1}{c^7\big(c^2+2V\big)D}\Big\{h_{01}\Big[-\big(c^2+2V\big)^2 PD_1PD_3 + \big(2D+P\big)PD_3\,\partial_1 V\Big] + h_{02}\Big[-\big(c^2+2V\big)^2$$
$$PD_2PD_3 + \big(2D+P\big)PD_3\,\partial_2 V\Big] + h_{03}\Big[-\big(c^2+2V\big)^2 PD_3^2 + \big(2D+P\big)PD_3\,\partial_3 V\Big]\Big\}\,,$$

$$\big[R_{03}\big]_2 \sim O\big(\epsilon^7\big)\,,$$

see the two previous subsections;

- 3-rd term

$$\big[R_{03}\big]_3 = -\frac{2}{c^{11}}\big(c^2-2V\big)\Big[h_{01}PD_3\,\partial_1 V + h_{02}PD_3\,\partial_2 V - h_{03}\big(PD_1\,\partial_1 V + PD_2\,\partial_2 V\Big]\,,$$

$$\big[R_{03}\big]_3 \sim O\big(\epsilon^7\big)\,,$$

again, see the two previous subsections.

Synoptic list:

$$\begin{aligned}\big[R_{03}\big]_1 &= {}^{(5)}\big[R_{03}\big]_1 + O\big(\epsilon^7\big)\,,\\ {}^{(5)}\big[R_{03}\big]_1 &= \frac{1}{c^5}\Big(h_{01}\,\partial_{13}U + h_{02}\,\partial_{23}U + h_{03}\,\partial_{33}U\Big)\,;\end{aligned}$$

$$\big[R_{03}\big]_2 \sim O\big(\epsilon^7\big)\,;$$

$$\big[R_{03}\big]_3 \sim O\big(\epsilon^7\big)\,.$$

Summing up,

$$\begin{aligned}R_{03} &= {}^{(5)}R_{03} + O\big(\epsilon^7\big)\,,\\ {}^{(5)}R_{03} &= \frac{1}{c^5}\Big(h_{01}\,\partial_{13}U + h_{02}\,\partial_{23}U + h_{03}\,\partial_{33}U\Big)\,,\\ &= \frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(\partial_3 U\big)\,.\end{aligned}$$

D.2.2 R_{1i} components $\quad i = 1, 2, 3.$

***R_{11}* component**

$$R_{11} = \Big[-\frac{2}{c^4} + \frac{\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\Big]\Big[\big(\partial_1 V\big)^2 - \big(\partial_2 V\big)^2 - \big(\partial_3 V\big)^2\Big] + \frac{\big(c^2-2V\big)}{c^4}\big(\partial_{11}V - \partial_{22}V$$
$$-\,\partial_{33}V\big) + \frac{\big(c^4-4V^2\big)}{c^4 D}\big(-\,PD_1\,\partial_1 V + PD_2\,\partial_2 V + PD_3\,\partial_3 V\big) + \frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(2D+P\big)\,\partial_1 V$$

$$-\big(c^2+2V\big)^2 PD_1\Big]^2-\frac{1}{\big(c^2+2V\big)^2 D}\Big[2\big(c^2+2V\big)^3\big(\partial_1 U\big)^2-12\big(c^2+2V\big)^2 PD_1\,\partial_1 V+4\big(D+2P\big)$$
$$\big(\partial_1 V\big)^2-\big(c^2+2V\big)^3 PD_{11}+\big(c^2+2V\big)\big(2D+P\big)\,\partial_{11}V\Big]-\frac{1}{c^{16}}\,q^2 PD_1^2-\frac{2}{c^{12}}\,h_{123}^2\,PD_1\,\partial_1 V$$
$$-\frac{1}{c^8}\big(c^2-2V\big)^2\Big[3\big(\partial_1 V\big)^2-2\big(\partial_2 V\big)^2-2\big(\partial_3 V\big)^2\Big];$$

- 1-st term

$$\big[R_{11}\big]_1=\bigg[-\frac{2}{c^4}+\frac{\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\bigg]\Big[\big(\partial_1 V\big)^2-\big(\partial_2 V\big)^2-\big(\partial_3 V\big)^2\Big]$$
$$=-\frac{2}{c^4}\Big[\big(\partial_1 V\big)^2-\big(\partial_2 V\big)^2-\big(\partial_3 V\big)^2\Big]+\frac{1}{c^4}\frac{\big(c^2-2V\big)\big(2D+P\big)}{\big(c^2+2V\big)D}\Big[\big(\partial_1 V\big)^2-\big(\partial_2 V\big)^2-\big(\partial_3 V\big)^2\Big]$$
$$=\cdots+\frac{1}{c^4}\bigg\{c^2\bigg(1-\frac{2V}{c^2}\bigg)\bigg[2c^6\bigg(\frac{h_{123}^2}{c^6}\bigg)+3c^6\bigg(1+\frac{2V}{c^2}\bigg)\bigg(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\bigg)\bigg]\bigg\}$$
$$\bigg\{c^2\bigg(1+\frac{2V}{c^2}\bigg)\bigg[c^6\bigg(\frac{h_{123}^2}{c^6}\bigg)+c^6\bigg(1+\frac{2V}{c^2}\bigg)\bigg(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\bigg)\bigg]\bigg\}^{-1}\Big[\dots\Big]$$
$$=\cdots+\frac{1}{c^4}\bigg\{\bigg(1-\frac{2V}{c^2}\bigg)\bigg[2\bigg(\frac{h_{123}^2}{c^6}\bigg)+3\bigg(1+\frac{2V}{c^2}\bigg)\bigg(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\bigg)\bigg]\bigg\}\bigg\{\bigg(1+\frac{2V}{c^2}\bigg)$$
$$\bigg[\bigg(\frac{h_{123}^2}{c^6}\bigg)+\bigg(1+\frac{2V}{c^2}\bigg)\bigg(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\bigg)\bigg]\bigg\}^{-1}\Big[\dots\Big],$$

where, as above,

$$\frac{1}{\big(c^2+2V\big)D}=\frac{1}{c^8}\,f_1\bigg(\frac{h_{01}}{c^3},\dots,\frac{\Phi}{c^4}\bigg)=\frac{1}{c^8}\Big[1+O\big(\epsilon^2\big)\Big];$$

$$=\cdots+\frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\bigg\{\bigg(1-\frac{2V}{c^2}\bigg)\bigg[2\bigg(\frac{h_{123}^2}{c^6}\bigg)+3\bigg(1+\frac{2V}{c^2}\bigg)\bigg(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\bigg)\bigg]\bigg\}\Big[\dots\Big]$$
$$=\cdots+\frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big\{O\big(\epsilon^6\big)+3\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big\}\Big[\dots\Big]$$
$$=\cdots+\frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]^2\Big[3+O\big(\epsilon^2\big)\Big]\Big[\dots\Big]$$
$$=\cdots+\frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[3+O\big(\epsilon^2\big)\Big]\Big[\dots\Big]$$
$$=\cdots+\frac{1}{c^4}\Big[3+O\big(\epsilon^2\big)\Big]\Big[\dots\Big]$$
$$=-\frac{2}{c^4}\Big[\big(\partial_1 V\big)^2-\big(\partial_2 V\big)^2-\big(\partial_3 V\big)^2\Big]+\frac{3}{c^4}\Big[\big(\partial_1 V\big)^2-\big(\partial_2 V\big)^2-\big(\partial_3 V\big)^2\Big]+O\big(\epsilon^6\big)$$
$$=\frac{1}{c^4}\Big[\big(\partial_1 V\big)^2-\big(\partial_2 V\big)^2-\big(\partial_3 V\big)^2\Big]+O\big(\epsilon^6\big);$$

therefore,

$$\big[R_{11}\big]_1={}^{(4)}\big[R_{11}\big]_1+O\big(\epsilon^6\big),$$
$${}^{(4)}\big[R_{11}\big]_1=\frac{1}{c^4}\Big[\big(\partial_1 V\big)^2-\big(\partial_2 V\big)^2-\big(\partial_3 V\big)^2\Big];$$

- 2-nd term

$$\begin{aligned}\big[R_{11}\big]_2 &= \frac{\big(c^2-2V\big)}{c^4}\big(\partial_{11}V-\partial_{22}V-\partial_{33}V\big) = \frac{1}{c^2}\bigg(1-\frac{2V}{c^2}\bigg)\big(\partial_{11}V-\partial_{22}V-\partial_{33}V\big)\\ &= \frac{1}{c^2}\big(\partial_{11}V-\partial_{22}V-\partial_{33}V\big)-\frac{2V}{c^4}\big(\partial_{11}V-\partial_{22}V-\partial_{33}V\big)\,;\end{aligned}$$

therefore,

$$\begin{aligned}\big[R_{11}\big]_2 &= {}^{(2)}\big[R_{11}\big]_2+{}^{(4)}\,\big[R_{11}\big]_2\,,\\ {}^{(2)}\big[R_{11}\big]_2 &= \frac{1}{c^2}\big(\partial_{11}V-\partial_{22}V-\partial_{33}V\big)\,,\\ {}^{(4)}\big[R_{11}\big]_2 &= -\frac{2}{c^4}V\big(\partial_{11}V-\partial_{22}V-\partial_{33}V\big)\,;\end{aligned}$$

- 3-rd term

$$\begin{aligned}\big[R_{11}\big]_3 &= \frac{\big(c^4-4V^2\big)}{c^4D}\big(-PD_1\,\partial_1V+PD_2\,\partial_2V+PD_3\,\partial_3V\big) = \frac{1}{c^4}\bigg[c^4\bigg(1-\frac{4V^2}{c^4}\bigg)\bigg]\bigg[c^6\bigg(\frac{h_{123}^2}{c^6}\bigg)\\ &+c^6\bigg(1+\frac{2V}{c^2}\bigg)\bigg(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\bigg)\bigg]^{-1}\bigg\{-\Big[\big(c^2-2U\big)\partial_1U+2\partial_1\Phi\Big]\partial_1V+\Big[\big(c^2-2U\big)\partial_2U\\ &+2\partial_2\Phi\Big]\partial_2V+\Big[\big(c^2-2U\big)\partial_3U+2\partial_3\Phi\Big]\partial_3V\bigg\}\,,\end{aligned}$$

where (the details of f_2 Taylor's expansion are shown in Appendix B on page 86),

$$\begin{aligned}\frac{1}{D} &= \frac{1}{c^6}\,f_2\bigg(\frac{h_{01}}{c^3},\ldots,\frac{\Phi}{c^4}\bigg)\\ &= \frac{1}{c^6}\bigg[\frac{h_{123}^2}{c^6}+\bigg(1+\frac{2V}{c^2}\bigg)\bigg(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\bigg)\bigg]^{-1}\\ &= \frac{1}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\,,\end{aligned}$$

from which,

$$\begin{aligned}&\frac{1}{c^6}\Big[1+O\big(\epsilon^4\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\bigg\{-\bigg[c^2\bigg(1-\frac{2U}{c^2}\bigg)\partial_1U\partial_1V+2\,\partial_1V\partial_1\Phi\bigg]+\bigg[c^2\bigg(1-\frac{2U}{c^2}\bigg)\partial_2U\partial_2V\\ &+2\,\partial_2V\partial_2\Phi\bigg]+\bigg[c^2\bigg(1-\frac{2U}{c^2}\bigg)\partial_3U\partial_3V+2\,\partial_3V\partial_3\Phi\bigg]\bigg\}\\ &\frac{1}{c^6}\Big[1+O\big(\epsilon^4\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\bigg\{-\bigg[c^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_1U\partial_1V+2\,\partial_1V\partial_1\Phi\bigg]+\bigg[c^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_2U\partial_2V\\ &+2\,\partial_2V\partial_2\Phi\bigg]+\bigg[c^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_3U\partial_3V+2\,\partial_3V\partial_3\Phi\bigg]\bigg\}\end{aligned}$$

$$= \frac{1}{c^4}\Big[1+O\big(\epsilon^4\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\big(-\partial_1 U\,\partial_1 V+\partial_2 U\,\partial_2 V+\partial_3 U\,\partial_3 V\big)+\frac{2}{c^6}\Big[1+O\big(\epsilon^4\big)\Big]$$
$$\Big[1+O\big(\epsilon^2\big)\Big]\big(-\partial_1 V\partial_1\Phi+\partial_2 V\partial_2\Phi+\partial_3 V\partial_3\Phi\big)$$
$$= \frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\big(-\partial_1 U\,\partial_1 V+\partial_2 U\,\partial_2 V+\partial_3 U\,\partial_3 V\big)+\frac{2}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\big(-\partial_1 V\partial_1\Phi$$
$$+\partial_2 V\partial_2\Phi+\partial_3 V\partial_3\Phi\big)$$
$$= \frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\big(-\partial_1 U\,\partial_1 V+\partial_2 U\,\partial_2 V+\partial_3 U\,\partial_3 V\big)+\frac{2}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\big(-\partial_1 V\partial_1\Phi+\partial_2 V\partial_2\Phi$$
$$+\partial_3 V\partial_3\Phi\big)\,;$$

therefore,

$$\big[R_{11}\big]_3 = {}^{(4)}\big[R_{11}\big]_3+O\big(\epsilon^6\big)\,,$$
$${}^{(4)}\big[R_{11}\big]_3 = \frac{1}{c^4}\big(-\partial_1 U\,\partial_1 V+\partial_2 U\,\partial_2 V+\partial_3 U\,\partial_3 V\big)\,;$$

- 4-th term

$$\big[R_{11}\big]_4 = \frac{2}{\big(c^2+2V\big)^2D^2}\Big[\big(2D+P\big)\,\partial_1 V-\big(c^2+2V\big)^2PD_1\Big]^2 = \frac{2}{\big(c^2+2V\big)^2D^2}\Big[\big(2D+P\big)^2$$
$$\big(\partial_1 V\big)^2+\big(c^2+2V\big)^4PD_1^2-2\big(c^2+2V\big)^2\big(2D+P\big)PD_1\partial_1 V\Big]\,,$$

where (the details of f_3 Taylor's expansion are shown in Appendix B on page 87),

$$\frac{1}{\big(c^2+2V\big)^2D^2} = \frac{1}{c^{16}}\,f_3\Big(\frac{h_{01}}{c^3},\frac{h_{02}}{c^3},\frac{h_{03}}{c^3},\frac{U}{c^2},\frac{V}{c^2},\frac{\Phi}{c^4}\Big)$$
$$= \frac{1}{c^{16}}\left\{\left(1+\frac{2V}{c^2}\right)^2\left[\frac{h_{123}^2}{c^6}+\left(1+\frac{2V}{c^2}\right)\left(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\right)\right]^2\right\}^{-1}$$
$$= \frac{1}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\,;$$

$$\frac{2}{\big(c^2+2V\big)^2D^2}\Big[\big(2D+P\big)^2\big(\partial_1 V\big)^2\Big] = \frac{2}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(2D+P\big)^2\big(\partial_1 V\big)^2\Big]$$
$$= \frac{2}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\left\{c^{12}\left[2\left(\frac{h_{123}^2}{c^6}\right)+3\left(1+\frac{2V}{c^2}\right)\left(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\right)\right]^2\big(\partial_1 V\big)^2\right.$$
$$= \frac{2}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\left\{c^{12}\left\{O\big(\epsilon^6\big)+3\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\right\}^2\big(\partial_1 V\big)^2\right\}$$
$$= \frac{2}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\left\{c^{12}\Big[3+O\big(\epsilon^2\big)\Big]^2\big(\partial_1 V\big)^2\right\}$$
$$= \frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[3+O\big(\epsilon^2\big)\Big]^2\big(\partial_1 V\big)^2$$
$$= \frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[9+O\big(\epsilon^2\big)\Big]\big(\partial_1 V\big)^2$$

$$= \frac{2}{c^4}\Big[9 + O\big(\epsilon^2\big)\Big]\big(\partial_1 V\big)^2 = \frac{18}{c^4}\big(\partial_1 V\big)^2 + O\big(\epsilon^6\big)\,;$$

$$\frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(c^2+2V\big)^4 P D_1^2\Big] = \frac{2}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(c^2+2V\big)^4 P D_1^2\Big]$$
$$= \frac{2}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\left(1+\frac{2V}{c^2}\right)^4\left[c^4\left(1-\frac{2U}{c^2}\right)^2\big(\partial_1 U\big)^2 + 4\big(\partial_1\Phi\big)^2 + 4c^2\left(1-\frac{2U}{c^2}\right)\partial_1 U\partial_1\Phi\right]$$
$$= \frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\left(1+\frac{2V}{c^2}\right)^4\left(1-\frac{2U}{c^2}\right)^2\big(\partial_1 U\big)^2 + \frac{8}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\left(1+\frac{2V}{c^2}\right)^4\big(\partial_1\Phi\big)^2$$
$$+ \frac{8}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\left(1+\frac{2V}{c^2}\right)^4\left(1-\frac{2U}{c^2}\right)\partial_1 U\partial_1\Phi$$
$$= \frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]^4\Big[1+O\big(\epsilon^2\big)\Big]^2\big(\partial_1 U\big)^2 + \frac{8}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]^4\big(\partial_1\Phi\big)^2$$
$$+ \frac{8}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]^4\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1\Phi$$
$$= \frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1 U\big)^2 + \frac{8}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1\Phi\big)^2$$
$$+ \frac{8}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1\Phi$$
$$= \frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1 U\big)^2 + \frac{8}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1\Phi\big)^2 + \frac{8}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1\Phi$$
$$= \frac{2}{c^4}\big(\partial_1 U\big)^2 + O\big(\epsilon^6\big)\,;$$

$$\frac{2}{\big(c^2+2V\big)^2 D^2}\Big[-2\big(c^2+2V\big)^2(2D+P)P D_1\partial_1 V\Big]$$
$$= \frac{2}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[-2\big(c^2+2V\big)^2(2D+P)P D_1\partial_1 V\Big]$$
$$= \frac{2}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\left\{-2\,c^{10}\left(1+\frac{2V}{c^2}\right)^2\left[2\left(\frac{h_{123}^2}{c^6}\right) + 3\left(1+\frac{2V}{c^2}\right)\left(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\right)\right]\right.$$
$$\left.\left[c^2\left(1-\frac{2U}{c^2}\right)\partial_1 U\partial_1 V + 2\,\partial_1 V\partial_1\Phi\right]\right\}$$
$$= -\frac{4}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\left(1+\frac{2V}{c^2}\right)^2\Big\{O\big(\epsilon^6\big) + 3\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big\}\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1 V$$
$$+ 2\,\partial_1 V\partial_1\Phi\Big\}$$
$$= -\frac{4}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]^2\Big\{O\big(\epsilon^6\big) + \Big[3+O\big(\epsilon^2\big)\Big]\Big\}\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1 V + 2\,\partial_1 V\partial_1\Phi\Big\}$$
$$= -\frac{4}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]^2\Big[3+O\big(\epsilon^2\big)\Big]\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1 V + 2\,\partial_1 V\partial_1\Phi\Big\}$$
$$= -\frac{4}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]^2\Big[3+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1 V - \frac{8}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]^2$$
$$\Big[3+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_1\Phi$$
$$= -\frac{4}{c^4}\Big[3+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1 V - \frac{8}{c^6}\Big[3+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_1\Phi$$

$$= -\frac{12}{c^4}\,\partial_1 U \partial_1 V + O\big(\epsilon^6\big)\,;$$

therefore,

$$\begin{aligned}\big[R_{11}\big]_4 &= {}^{(4)}\big[R_{11}\big]_4 + O\big(\epsilon^6\big)\,,\\ {}^{(4)}\big[R_{11}\big]_4 &= \frac{1}{c^4}\Big[18\big(\partial_1 V\big)^2 + 2\big(\partial_1 U\big)^2 - 12\,\partial_1 U \partial_1 V\Big]\,;\end{aligned}$$

- 5-th term

$$\begin{aligned}\big[R_{11}\big]_5 = &-\frac{1}{\big(c^2+2V\big)^2 D}\Big[2\big(c^2+2V\big)^3\big(\partial_1 U\big)^2 - 12\big(c^2+2V\big)^2 P D_1\,\partial_1 V + 4\big(D+2P\big)\\ &\big(\partial_1 V\big)^2 - \big(c^2+2V\big)^3 P D_{11} + \big(c^2+2V\big)\big(2D+P\big)\,\partial_{11} V\Big]\,,\end{aligned}$$

where (the details of f_4 Taylor's expansion are shown in Appendix B on page 88),

$$\begin{aligned}\frac{1}{\big(c^2+2V\big)^2 D} &= \frac{1}{c^{10}}\,f_4\bigg(\frac{h_{01}}{c^3}, \frac{h_{02}}{c^3}, \frac{h_{03}}{c^3}, \frac{U}{c^2}, \frac{V}{c^2}, \frac{\Phi}{c^4}\bigg)\\ &= \frac{1}{c^{10}}\bigg\{\bigg(1+\frac{2V}{c^2}\bigg)^2\bigg[\frac{h_{123}^2}{c^6} + \bigg(1+\frac{2V}{c^2}\bigg)\bigg(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\bigg)\bigg]\bigg\}^{-1}\\ &= \frac{1}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\,,\end{aligned}$$

$$\begin{aligned}&-\frac{2}{\big(c^2+2V\big)^2 D}\Big[\big(c^2+2V\big)^3\big(\partial_1 U\big)^2\Big] = -\frac{2}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]c^6\bigg(1+\frac{2V}{c^2}\bigg)^3\big(\partial_1 U\big)^2\\ &= -\frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]^3\big(\partial_1 U\big)^2 = -\frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1 U\big)^2\\ &= -\frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1 U\big)^2 = -\frac{2}{c^4}\big(\partial_1 U\big)^2 + O\big(\epsilon^6\big)\,;\end{aligned}$$

$$\begin{aligned}&\frac{12}{\big(c^2+2V\big)^2 D}\Big[\big(c^2+2V\big)^2 P D_1\,\partial_1 V\Big] = \frac{12}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]c^4\bigg(1+\frac{2V}{c^2}\bigg)^2\bigg[c^2\bigg(1-\frac{2U}{c^2}\bigg)\partial_1 U\\ &+ 2\,\partial_1\Phi\bigg]\partial_1 V\\ &= \frac{12}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]^2\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1 V + 2\,\partial_1 V\partial_1\Phi\Big\}\\ &= \frac{12}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1 V + 2\,\partial_1 V\partial_1\Phi\Big\}\\ &= \frac{12}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1 V + \frac{24}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_1\Phi\\ &= \frac{12}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1 V + \frac{24}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_1\Phi = \frac{12}{c^4}\,\partial_1 U\partial_1 V + O\big(\epsilon^6\big)\,;\end{aligned}$$

$$
\begin{aligned}
&-\frac{4}{\big(c^2+2V\big)^2 D}\Big[\big(D+2P\big)\big(\partial_1 V\big)^2\Big] = -\frac{4}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]c^6\Bigg[\frac{h_{123}^2}{c^6}+3\bigg(1+\frac{2V}{c^2}\bigg)\bigg(1-\frac{2U}{c^2}\\
&+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\bigg)\Bigg]\big(\partial_1 V\big)^2\\
&=-\frac{4}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big\{O\big(\epsilon^6\big)+3\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big\}\big(\partial_1 V\big)^2\\
&=-\frac{4}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big\{O\big(\epsilon^6\big)+3\Big[1+O\big(\epsilon^2\big)\Big]\Big\}\big(\partial_1 V\big)^2\\
&=-\frac{4}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[3+O\big(\epsilon^2\big)\Big]\big(\partial_1 V\big)^2 = -\frac{4}{c^4}\Big[3+O\big(\epsilon^2\big)\Big]\big(\partial_1 V\big)^2\\
&=-\frac{12}{c^4}\big(\partial_1 V\big)^2+O\big(\epsilon^6\big)\,;
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{\big(c^2+2V\big)^2 D}\Big[\big(c^2+2V\big)^3 P D_{11}\Big] = \frac{1}{c^{10}}\Big[1+\frac{2U}{c^2}-\frac{6V}{c^2}+\frac{4\Phi}{c^4}+O\big(\epsilon^4\big)\Big]c^6\bigg(1+\frac{2V}{c^2}\bigg)^3\\
&\Bigg[c^2\bigg(1-\frac{2U}{c^2}\bigg)\partial_{11}U+2\partial_{11}\Phi\Bigg]\\
&=\frac{1}{c^4}\Big[1+\frac{2U}{c^2}-\frac{6V}{c^2}+\frac{4\Phi}{c^4}+O\big(\epsilon^4\big)\Big]\bigg(1+\frac{6V}{c^2}+\frac{12V^2}{c^4}+\frac{8V^3}{c^6}\bigg)\Big\{c^2\bigg(1-\frac{2U}{c^2}\bigg)\partial_{11}U+2\partial_{11}\Phi\Big\}\\
&=\frac{1}{c^2}\Big[1+\frac{2U}{c^2}-\frac{6V}{c^2}+\frac{4\Phi}{c^4}+O\big(\epsilon^4\big)\Big]\bigg(1+\frac{6V}{c^2}+\frac{12V^2}{c^4}+\frac{8V^3}{c^6}\bigg)\bigg(1-\frac{2U}{c^2}\bigg)\partial_{11}U+\frac{2}{c^4}\Big[1+\frac{2U}{c^2}\\
&-\frac{6V}{c^2}+\frac{4\Phi}{c^4}+O\big(\epsilon^4\big)\Big]\bigg(1+\frac{6V}{c^2}+\frac{12V^2}{c^4}+\frac{8V^3}{c^6}\bigg)\partial_{11}\Phi\\
&=\frac{1}{c^2}\Big[1+O\big(\epsilon^4\big)\Big]\partial_{11}U+\frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\partial_{11}\Phi\\
&=\frac{1}{c^2}\,\partial_{11}U+\frac{2}{c^4}\,\partial_{11}\Phi+O\big(\epsilon^6\big)\,;
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{\big(c^2+2V\big)^2 D}\Big[\big(c^2+2V\big)\big(2D+P\big)\partial_{11}V\Big] = -\frac{1}{c^{10}}\Big[1+\frac{2U}{c^2}-\frac{6V}{c^2}+\frac{4\Phi}{c^4}+O\big(\epsilon^4\big)\Big]c^2\bigg(1+\frac{2V}{c^2}\bigg)\\
&c^6\Bigg[2\bigg(\frac{h_{123}^2}{c^6}\bigg)+3\bigg(1+\frac{2V}{c^2}\bigg)\bigg(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\bigg)\Bigg]\partial_{11}V\\
&=-\frac{1}{c^2}\Big[1+\frac{2U}{c^2}-\frac{6V}{c^2}+\frac{4\Phi}{c^4}+O\big(\epsilon^4\big)\Big]\bigg(1+\frac{2V}{c^2}\bigg)\Bigg[O\big(\epsilon^6\big)+3\bigg(1+\frac{2V}{c^2}\bigg)\bigg(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}\\
&-\frac{4\Phi}{c^4}\bigg)\Bigg]\partial_{11}V\\
&=-\frac{1}{c^2}\Big[1+\frac{2U}{c^2}-\frac{6V}{c^2}+\frac{4\Phi}{c^4}+O\big(\epsilon^4\big)\Big]\Bigg[3\bigg(1+\frac{2V}{c^2}\bigg)^2\bigg(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\bigg)\Bigg]\partial_{11}V\\
&=-\frac{3}{c^2}\Big[1+\frac{2U}{c^2}-\frac{6V}{c^2}+\frac{4\Phi}{c^4}+O\big(\epsilon^4\big)\Big]\Bigg[\bigg(1+\frac{4V}{c^2}+\frac{4V^2}{c^4}\bigg)\bigg(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\bigg)\Bigg]\partial_{11}V\\
&=-\frac{3}{c^2}\Big[1+\frac{2U}{c^2}-\frac{6V}{c^2}+O\big(\epsilon^4\big)\Big]\Big[1-\frac{2U}{c^2}+\frac{4V}{c^2}+O\big(\epsilon^4\big)\Big]\partial_{11}V
\end{aligned}
$$

$$= -\frac{3}{c^2}\Big[1 - \frac{2V}{c^2} + O\big(\epsilon^4\big)\Big]\partial_{11}V = -\frac{3}{c^2}\,\partial_{11}V + \frac{6}{c^4}V\partial_{11}V + O\big(\epsilon^6\big)\,;$$

therefore,

$$\begin{aligned}
\big[R_{11}\big]_5 &= {}^{(2)}\big[R_{11}\big]_5 + {}^{(4)}\big[R_{11}\big]_5 + O\big(\epsilon^6\big)\,,\\
{}^{(2)}\big[R_{11}\big]_5 &= \frac{1}{c^2}\big(\partial_{11}U - 3\,\partial_{11}V\big)\,,\\
{}^{(4)}\big[R_{11}\big]_5 &= \frac{1}{c^4}\Big[-2\big(\partial_1 U\big)^2 + 12\,\partial_1 U\partial_1 V - 12\big(\partial_1 V\big)^2 + 2\,\partial_{11}\Phi + 6\,V\partial_{11}V\Big]\,;
\end{aligned}$$

- 6-th term

$$\begin{aligned}
&\big[R_{11}\big]_6 = -\frac{1}{c^{16}}\,q^2 P D_1^2\\
&= -\frac{1}{c^{16}}\big(c^4 + 2c^2U + 2U^2 + 4\Phi\big)^2\Big[\big(c^2 - 2U\big)\partial_1 U + 2\partial_1\Phi\Big]^2\\
&= -\frac{1}{c^8}\left(1 + \frac{2U}{c^2} + \frac{2U^2}{c^4} + \frac{4\Phi}{c^4}\right)^2\Big[\big(c^2 - 2U\big)^2\big(\partial_1 U\big)^2 + 4\big(\partial_1\Phi\big)^2 + 4\big(c^2 - 2U\big)\Big]\partial_1 U\partial_1\Phi\Big]\\
&= -\frac{1}{c^8}\Big[1 + \frac{4U}{c^2} + O\big(\epsilon^4\big)\Big]\left[c^4\left(1 - \frac{2U}{c^2}\right)^2\big(\partial_1 U\big)^2 + 4\big(\partial_1\Phi\big)^2 + 4c^2\left(1 - \frac{2U}{c^2}\right)\partial_1 U\partial_1\Phi\right]\\
&= -\frac{1}{c^4}\Big[1 + O\big(\epsilon^2\big)\Big]\Big[1 + O\big(\epsilon^2\big)\Big]^2\big(\partial_1 U\big)^2 - \frac{4}{c^8}\Big[1 + O\big(\epsilon^2\big)\Big]\big(\partial_1\Phi\big)^2 - \frac{4}{c^6}\Big[1 + O\big(\epsilon^2\big)\Big]\Big[1 + O\big(\epsilon^2\big)\Big]\\
&\partial_1 U\partial_1\Phi\\
&= -\frac{1}{c^4}\Big[1 + O\big(\epsilon^2\big)\Big]\Big[1 + O\big(\epsilon^2\big)\Big]\big(\partial_1 U\big)^2 - \frac{4}{c^8}\Big[1 + O\big(\epsilon^2\big)\Big]\big(\partial_1\Phi\big)^2 - \frac{4}{c^6}\Big[1 + O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1\Phi\\
&= -\frac{1}{c^4}\Big[1 + O\big(\epsilon^2\big)\Big]\big(\partial_1 U\big)^2 - \frac{4}{c^8}\Big[1 + O\big(\epsilon^2\big)\Big]\big(\partial_1\Phi\big)^2 - \frac{4}{c^6}\Big[1 + O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1\Phi\,;
\end{aligned}$$

therefore,

$$\begin{aligned}
\big[R_{11}\big]_6 &= {}^{(4)}\big[R_{11}\big]_6 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{11}\big]_6 &= -\frac{1}{c^4}\big(\partial_1 U\big)^2\,;
\end{aligned}$$

- 7-th term

$$\begin{aligned}
&\big[R_{11}\big]_7 = -\frac{2}{c^{12}}\,h_{123}^2\,P D_1\,\partial_1 V\\
&= -\frac{2}{c^6}\left(\frac{h_{123}^2}{c^6}\right)\Big[\big(c^2 - 2U\big)\partial_1 U + 2\,\partial_1\Phi\Big]\partial_1 V\\
&= -\frac{2}{c^6}\,O\big(\epsilon^6\big)\Big[c^2\left(1 - \frac{2U}{c^2}\right)\partial_1 U\partial_1 V + 2\,\partial_1 V\partial_1\Phi\Big]\\
&= -\frac{2}{c^4}\,O\big(\epsilon^6\big)\Big[1 + O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1 V - \frac{4}{c^6}O\big(\epsilon^6\big)\partial_1 V\partial_1\Phi\,;
\end{aligned}$$

therefore,

$$\big[R_{11}\big]_7 \sim O\big(\epsilon^{10}\big)\,;$$

- 8-th term

$$\begin{aligned}\big[R_{11}\big]_8 &= -\frac{1}{c^8}\big(c^2-2V\big)^2\Big[3\big(\partial_1 V\big)^2-2\big(\partial_2 V\big)^2-2\big(\partial_3 V\big)^2\Big]\\ &= -\frac{1}{c^4}\left(1-\frac{2V}{c^2}\right)^2\Big[3\big(\partial_1 V\big)^2-2\big(\partial_2 V\big)^2-2\big(\partial_3 V\big)^2\Big]\\ &= -\frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]^2\Big[3\big(\partial_1 V\big)^2-2\big(\partial_2 V\big)^2-2\big(\partial_3 V\big)^2\Big]\\ &= -\frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[3\big(\partial_1 V\big)^2-2\big(\partial_2 V\big)^2-2\big(\partial_3 V\big)^2\Big];\end{aligned}$$

therefore,

$$\begin{aligned}\big[R_{11}\big]_8 &= {}^{(4)}\big[R_{11}\big]_8+O\big(\epsilon^6\big),\\ {}^{(4)}\big[R_{11}\big]_8 &= -\frac{1}{c^4}\Big[3\big(\partial_1 V\big)^2-2\big(\partial_2 V\big)^2-2\big(\partial_3 V\big)^2\Big].\end{aligned}$$

Synoptic list:

$$\begin{aligned}\big[R_{11}\big]_1 &= {}^{(4)}\big[R_{11}\big]_1+O\big(\epsilon^6\big),\\ {}^{(4)}\big[R_{11}\big]_1 &= \frac{1}{c^4}\Big[\big(\partial_1 V\big)^2-\big(\partial_2 V\big)^2-\big(\partial_3 V\big)^2\Big];\end{aligned}$$

$$\begin{aligned}\big[R_{11}\big]_2 &= {}^{(2)}\big[R_{11}\big]_2+{}^{(4)}\big[R_{11}\big]_2,\\ {}^{(2)}\big[R_{11}\big]_2 &= \frac{1}{c^2}\big(\partial_{11}V-\partial_{22}V-\partial_{33}V\big),\\ {}^{(4)}\big[R_{11}\big]_2 &= -\frac{2}{c^4}V\big(\partial_{11}V-\partial_{22}V-\partial_{33}V\big);\end{aligned}$$

$$\begin{aligned}\big[R_{11}\big]_3 &= {}^{(4)}\big[R_{11}\big]_3+O\big(\epsilon^6\big),\\ {}^{(4)}\big[R_{11}\big]_3 &= \frac{1}{c^4}\big(-\partial_1 U\,\partial_1 V+\partial_2 U\,\partial_2 V+\partial_3 U\,\partial_3 V\big);\end{aligned}$$

$$\begin{aligned}\big[R_{11}\big]_4 &= {}^{(4)}\big[R_{11}\big]_4+O\big(\epsilon^6\big),\\ {}^{(4)}\big[R_{11}\big]_4 &= \frac{1}{c^4}\Big[18\big(\partial_1 V\big)^2+2\big(\partial_1 U\big)^2-12\,\partial_1 U\partial_1 V\Big];\end{aligned}$$

$$\begin{aligned}\big[R_{11}\big]_5 &= {}^{(2)}\big[R_{11}\big]_5+{}^{(4)}\big[R_{11}\big]_5+O\big(\epsilon^6\big),\\ {}^{(2)}\big[R_{11}\big]_5 &= \frac{1}{c^2}\big(\partial_{11}U-3\,\partial_{11}V\big),\\ {}^{(4)}\big[R_{11}\big]_5 &= \frac{1}{c^4}\Big[-2\big(\partial_1 U\big)^2+12\,\partial_1 U\partial_1 V-12\big(\partial_1 V\big)^2+2\,\partial_{11}\Phi+6\,V\partial_{11}V\Big];\end{aligned}$$

$$\begin{aligned}\big[R_{11}\big]_6 &= {}^{(4)}\big[R_{11}\big]_6+O\big(\epsilon^6\big),\\ {}^{(4)}\big[R_{11}\big]_6 &= -\frac{1}{c^4}\big(\partial_1 U\big)^2;\end{aligned}$$

$$\big[R_{11}\big]_7 \sim O\big(\epsilon^{10}\big)\,;$$

$$\begin{aligned}\big[R_{11}\big]_8 &= {}^{(4)}\big[R_{11}\big]_8 + O\big(\epsilon^6\big)\,,\\ {}^{(4)}\big[R_{11}\big]_8 &= -\frac{1}{c^4}\Big[3\big(\partial_1 V\big)^2 - 2\big(\partial_2 V\big)^2 - 2\big(\partial_3 V\big)^2\Big]\,.\end{aligned}$$

Summing up,

$$\begin{aligned}R_{11} &= {}^{(2)}R_{11} + {}^{(4)}R_{11} + O\big(\epsilon^6\big)\,,\\ {}^{(2)}R_{11} &= \frac{1}{c^2}\big(\partial_{11}U - \partial_{11}V - \Delta V\big)\,,\\ {}^{(4)}R_{11} &= \frac{1}{c^4}\Big[3\big(\partial_1 V\big)^2 + \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 2\,V\partial_{11}V + 2\,V\Delta V - 2\,\partial_1 U\partial_1 V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,\partial_{11}\Phi\\ &\quad - \big(\partial_1 U\big)^2\Big]\,.\end{aligned}$$

R_{12} component

$$\begin{aligned}R_{12} &= \frac{2}{c^4}\Big[\big(c^2-2V\big)\partial_{12}V - 2\,\partial_1 V\partial_2 V\Big] + \frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(2D+P\big)\partial_1 V - \big(c^2+2V\big)^2 PD_1\Big]\Big[\big(2D\\ &+P\big)\partial_2 V - \big(c^2+2V\big)^2 PD_2\Big] - \frac{1}{\big(c^2+2V\big)^2 D}\Big\{2\,\partial_2 U\big(c^2+2V\big)^2\Big[\big(c^2+2V\big)\partial_1 U - 3\big(c^2-2U\big)\partial_1 V\Big]\\ &+2\,\partial_2 V\Big[2\big(D+2P\big)\partial_1 V - 3\big(c^2+2V\big)^2 PD_1\Big] - \big(c^2+2V\big)\Big[12\big(c^2+2V\big)\partial_1 V\partial_2\Phi - \big(2D+P\big)\partial_{12}V\\ &+\big(c^2+2V\big)^2 PD_{12}\Big]\Big\} + \frac{2\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\,\partial_1 V\partial_2 V - \frac{\big(c^4-4V^2\big)}{c^4 D}\big(PD_1\partial_2 V + PD_2\partial_1 V\big)\\ &-\frac{1}{c^{16}}q^2 PD_1 PD_2 - \frac{1}{c^{12}}h_{123}^2\big(PD_2\partial_1 V + PD_1\partial_2 V\big) - \frac{5}{c^8}\big(c^2-2V\big)^2\partial_1 V\partial_2 V\,;\end{aligned}$$

- 1-st term

$$\begin{aligned}\big[R_{12}\big]_1 &= \frac{2}{c^4}\Big[\big(c^2-2V\big)\partial_{12}V - 2\,\partial_1 V\partial_2 V\Big]\\ &= \frac{2}{c^4}\left[c^2\left(1-\frac{2V}{c^2}\right)\partial_{12}V - 2\,\partial_1 V\partial_2 V\right]\\ &= \frac{2}{c^2}\left(1-\frac{2V}{c^2}\right)\partial_{12}V - \frac{4}{c^4}\,\partial_1 V\partial_2 V\\ &= \frac{2}{c^2}\,\partial_{12}V - \frac{4}{c^4}\,V\partial_{12}V - \frac{4}{c^4}\,\partial_1 V\partial_2 V\,;\end{aligned}$$

therefore,

$$\begin{aligned}\big[R_{12}\big]_1 &= {}^{(2)}\big[R_{12}\big]_1 + {}^{(4)}\big[R_{12}\big]_1\,,\\ {}^{(2)}\big[R_{12}\big]_1 &= \frac{2}{c^2}\,\partial_{12}V\,,\\ {}^{(4)}\big[R_{12}\big]_1 &= -\frac{4}{c^4}\big(V\partial_{12}V + \partial_1 V\partial_2 V\big)\,;\end{aligned}$$

- 2-nd term

$$\big[R_{12}\big]_2 = \frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(2D+P\big)\partial_1 V - \big(c^2+2V\big)^2 P D_1\Big]\Big[\big(2D+P\big)\partial_2 V - \big(c^2+2V\big)^2 P D_2\Big]$$
$$= \frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(2D+P\big)^2\partial_1 V\partial_2 V - \big(c^2+2V\big)^2\big(2D+P\big)\partial_1 V P D_2 - \big(c^2+2V\big)^2\big(2D+P\big)$$
$$\partial_2 V P D_1 + \big(c^2+2V\big)^4 P D_1 P D_2\Big],$$

where, for details, please refer to the previous subsection ($\big[R_{11}\big]_4$);

$$\frac{1}{\big(c^2+2V\big)^2 D^2} = \frac{1}{c^{16}} f_3\Big(\frac{h_{01}}{c^3},\ldots,\frac{\Phi}{c^4}\Big) = \frac{1}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big];$$

$$\frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(2D+P\big)^2\partial_1 V\partial_2 V\Big] = \frac{2}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(2D+P\big)^2\partial_1 V\partial_2 V\Big]$$
$$= \frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big\{O\big(\epsilon^6\big)+3\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big\}^2\partial_1 V\partial_2 V$$
$$= \frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big\{O\big(\epsilon^6\big)+3\Big[1+O\big(\epsilon^2\big)\Big]\Big\}^2\partial_1 V\partial_2 V$$
$$= \frac{2}{c^4}\Big[1+O\Big(\epsilon^2\Big)\Big]\Big[3+O\big(\epsilon^2\big)\Big]^2\partial_1 V\partial_2 V$$
$$= \frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[9+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2 V$$
$$= \frac{2}{c^4}\Big[9+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2 V = \frac{18}{c^4}\,\partial_1 V\partial_2 V + O\big(\epsilon^6\big);$$

$$\frac{2}{\big(c^2+2V\big)^2 D^2}\Big[-\big(c^2+2V\big)^2\big(2D+P\big)\partial_1 V P D_2\Big] = -\frac{2}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(c^2+2V\big)^2$$
$$\big(2D+P\big)\partial_1 V P D_2\Big]$$
$$= -\frac{2}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]^2\Big\{O\big(\epsilon^6\big)+3\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big\}\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]$$
$$\partial_1 V\partial_2 U + 2\,\partial_1 V\partial_2\Phi\Big\}$$
$$= -\frac{2}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big\{O\big(\epsilon^6\big)+\Big[3+O\big(\epsilon^2\big)\Big]\Big\}\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2 U + 2\,\partial_1 V\partial_2\Phi\Big\}$$
$$= -\frac{2}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[3+O\big(\epsilon^2\big)\Big]\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2 U + 2\,\partial_1 V\partial_2\Phi\Big\}$$
$$= -\frac{2}{c^6}\Big[3+O\big(\epsilon^2\big)\Big]\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2 U + 2\,\partial_1 V\partial_2\Phi\Big\}$$
$$= -\frac{2}{c^4}\Big[3+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2 U - \frac{4}{c^6}\Big[3+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2\Phi$$
$$= -\frac{2}{c^4}\Big[3+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2 U - \frac{4}{c^6}\Big[3+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2\Phi = -\frac{6}{c^4}\,\partial_1 V\partial_2 U + O\big(\epsilon^6\big);$$

$$\frac{2}{\big(c^2+2V\big)^2 D^2}\Big[-\big(c^2+2V\big)^2\big(2D+P\big)\partial_2 V P D_1\Big] = -\frac{6}{c^4}\,\partial_1 U\partial_2 V + O\big(\epsilon^6\big),$$

as above;

$$\frac{2}{\big(c^2+2V\big)^2D^2}\Big[\big(c^2+2V\big)^4PD_1PD_2\Big]=\frac{2}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(c^2+2V\big)^4PD_1PD_2\Big]$$
$$=\frac{2}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]^4\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_1U+2\,\partial_1\Phi\Big\}\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_2U+2\,\partial_2\Phi\Big\}$$
$$=\frac{2}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big\{c^4\Big[1+O\big(\epsilon^2\big)\Big]^2\partial_1U\partial_2U+2c^2\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1U\partial_2\Phi+\partial_2U\partial_1\Phi\big)$$
$$+4\,\partial_1\Phi\partial_2\Phi\Big\}$$
$$=\frac{2}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\Big\{c^4\Big[1+O\big(\epsilon^2\big)\Big]\partial_1U\partial_2U+2c^2\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1U\partial_2\Phi+\partial_2U\partial_1\Phi\big)+4\,\partial_1\Phi\partial_2\Phi\Big\}$$
$$=\frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]^2\partial_1U\partial_2U+\frac{4}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]^2\big(\partial_1U\partial_2\Phi+\partial_2U\partial_1\Phi\big)+\frac{8}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\partial_1\Phi\partial_2\Phi$$
$$=\frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\partial_1U\partial_2U+\frac{4}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1U\partial_2\Phi+\partial_2U\partial_1\Phi\big)+\frac{8}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\partial_1\Phi\partial_2\Phi$$
$$=\frac{2}{c^4}\,\partial_1U\partial_2U+O\big(\epsilon^6\big)\,;$$

therefore,

$$\big[R_{12}\big]_2={}^{(4)}\big[R_{12}\big]_2+O\big(\epsilon^6\big)\,,$$
$${}^{(4)}\big[R_{12}\big]_2=\frac{2}{c^4}\big(9\,\partial_1V\partial_2V-3\,\partial_1V\partial_2U-3\,\partial_1U\partial_2V+\partial_1U\partial_2U\big)\,;$$

- 3-rd term

$$\big[R_{12}\big]_3=-\frac{1}{\big(c^2+2V\big)^2D}\Big\{2\,\partial_2U\big(c^2+2V\big)^2\Big[\big(c^2+2V\big)\partial_1U-3\big(c^2-2U\big)\partial_1V\Big]+2\,\partial_2V$$
$$\Big[2\big(D+2P\big)\partial_1V-3\big(c^2+2V\big)^2PD_1\Big]-\big(c^2+2V\big)\Big[12\big(c^2+2V\big)\partial_1V\partial_2\Phi-\big(2D+P\big)\partial_{12}V$$
$$+\big(c^2+2V\big)^2PD_{12}\Big]\Big\}\,,$$

where, for details, please refer to the previous subsection ($\big[R_{11}\big]_5$),

$$\frac{1}{\big(c^2+2V\big)^2D}=\frac{1}{c^{10}}\,f_4\Big(\frac{h_{01}}{c^3},\ldots,\frac{\Phi}{c^4}\Big)=\frac{1}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\,;$$

$$=-\frac{1}{\big(c^2+2V\big)^2D}\Big\{2\,\partial_2U\big(c^2+2V\big)^2\Big[\big(c^2+2V\big)\partial_1U-3\big(c^2-2U\big)\partial_1V\Big]\Big\}$$
$$=-\frac{1}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\Big\{2\,\partial_2U\big(c^2+2V\big)^2\Big[\big(c^2+2V\big)\partial_1U-3\big(c^2-2U\big)\partial_1V\Big]\Big\}$$
$$=-\frac{2}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]c^4\Big(1+\frac{2V}{c^2}\Big)^2\partial_2U\Big[c^2\Big(1+\frac{2V}{c^2}\Big)\partial_1U-3\,c^2\Big(1-\frac{2U}{c^2}\Big)\partial_1V\Big]$$

$$
\begin{aligned}
&= -\frac{2}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\left[c^2\left(1+\frac{2V}{c^2}\right)^3\partial_1 U\partial_2 U - 3\,c^2\left(1+\frac{2V}{c^2}\right)^2\left(1-\frac{2U}{c^2}\right)\partial_1 V\partial_2 U\right]\\
&= -\frac{2}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\left\{c^2\Big[1+O\big(\epsilon^2\big)\Big]^3\partial_1 U\partial_2 U - 3c^2\Big[1+O\big(\epsilon^2\big)\Big]^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2 U\right\}\\
&= -\frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]^3\partial_1 U\partial_2 U + \frac{6}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2 U\\
&= -\frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_2 U + \frac{6}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2 U\\
&= -\frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_2 U + \frac{6}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2 U\\
&= -\frac{2}{c^4}\,\partial_1 U\partial_2 U + \frac{6}{c^4}\,\partial_1 V\partial_2 U + O\big(\epsilon^6\big)\\
&= -\frac{2}{c^4}\big(\partial_1 U\partial_2 U - 3\,\partial_1 V\partial_2 U\big) + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{\big(c^2+2V\big)^2 D}\left\{2\,\partial_2 V\Big[2\big(D+2P\big)\partial_1 V - 3\big(c^2+2V\big)^2 P D_1\Big]\right\}\\
&= -\frac{2}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[2\big(D+2P\big)\partial_1 V\partial_2 V - 3\big(c^2+2V\big)^2 P D_1\partial_2 V\Big]\\
&= -\frac{4}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\left\{O\big(\epsilon^6\big)+3\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\right\}\partial_1 V\partial_2 V + \frac{6}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\\
&\Big[1+O\big(\epsilon^2\big)\Big]^2\left\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_2 V + 2\,\partial_1\Phi\partial_2 V\right\}\\
&= -\frac{4}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\left\{O\big(\epsilon^6\big)+3\Big[1+O\big(\epsilon^2\big)\Big]\right\}\partial_1 V\partial_2 V + \frac{6}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]^2\\
&\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_2 V + \frac{12}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]^2\partial_1\Phi\partial_2 V\\
&= -\frac{4}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[3+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2 V + \frac{6}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_2 V\\
&+ \frac{12}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\partial_1\Phi\partial_2 V\\
&= -\frac{4}{c^4}\Big[3+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2 V + \frac{6}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_2 V + \frac{12}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\partial_1\Phi\partial_2 V\\
&= -\frac{12}{c^4}\,\partial_1 V\partial_2 V + \frac{6}{c^4}\,\partial_1 U\partial_2 V + O\big(\epsilon^6\big)\\
&= -\frac{6}{c^4}\big(2\,\partial_1 V\partial_2 V - \partial_1 U\partial_2 V\big) + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{\big(c^2+2V\big)^2 D}\Big[-12\big(c^2+2V\big)^2\partial_1 V\partial_2\Phi\Big] = \frac{12}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(c^2+2V\big)^2\partial_1 V\partial_2\Phi\Big]\\
&= \frac{12}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\left\{c^4\left(1+\frac{2V}{c^2}\right)^2\partial_1 V\partial_2\Phi\right\} = \frac{12}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]^2\partial_1 V\partial_2\Phi\\
&= \frac{12}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2\Phi = \frac{12}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2\Phi \sim O\big(\epsilon^6\big)\,;
\end{aligned}
$$

$$
-\frac{1}{\big(c^2+2V\big)^2 D}\Big[\big(c^2+2V\big)\big(2D+P\big)\partial_{12} V\Big] = -\frac{1}{c^{10}}\,f_4\left(\frac{h_{01}}{c^3},\ldots,\frac{\Phi}{c^4}\right)\Big[\big(c^2+2V\big)
$$

$$\big(2D+P\big)\partial_{12}V\Big]\,,$$

for which it has been used the explicit expression of f_4 Taylor's expansion; in this regard, see the Appendix B on page 88;

$$\begin{aligned}
&= -\frac{1}{c^{10}}\left[1+\frac{2U}{c^2}-\frac{6V}{c^2}+O\big(\epsilon^4\big)\right]\left\{c^2\left(1+\frac{2V}{c^2}\right)\left[2c^6\left(\frac{h_{123}^2}{c^6}\right)+3c^6\left(1+\frac{2V}{c^2}\right)\left(1-\frac{2U}{c^2}\right.\right.\right.\\
&\left.\left.\left.+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\right)\right]\right\}\partial_{12}V\\
&= -\frac{1}{c^2}\left[1+\frac{2U}{c^2}-\frac{6V}{c^2}+O\big(\epsilon^4\big)\right]\left(1+\frac{2V}{c^2}\right)\left[2\left(\frac{h_{123}^2}{c^6}\right)+3\left(1+\frac{2V}{c^2}\right)\left(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}\right.\right.\\
&\left.\left.-\frac{4\Phi}{c^4}\right)\right]\partial_{12}V\\
&= -\frac{1}{c^2}\left[1+\frac{2U}{c^2}-\frac{6V}{c^2}+O\big(\epsilon^4\big)\right]\left(1+\frac{2V}{c^2}\right)\left[O\big(\epsilon^6\big)+3\left(1-\frac{2U}{c^2}+\frac{2V}{c^2}+O\big(\epsilon^4\big)\right)\right]\partial_{12}V\\
&= -\frac{1}{c^2}\left[1+\frac{2U}{c^2}-\frac{6V}{c^2}+O\big(\epsilon^4\big)\right]\left(1+\frac{2V}{c^2}\right)\left[3\left(1-\frac{2U}{c^2}+\frac{2V}{c^2}+O\big(\epsilon^4\big)\right)\right]\partial_{12}V\\
&= -\frac{3}{c^2}\left[1+\frac{2U}{c^2}-\frac{6V}{c^2}+O\big(\epsilon^4\big)\right]\left[1-\frac{2U}{c^2}+\frac{2V}{c^2}+\frac{2V}{c^2}+O\big(\epsilon^4\big)\right]\partial_{12}V\\
&= -\frac{3}{c^2}\left[1+\frac{2U}{c^2}-\frac{6V}{c^2}+O\big(\epsilon^4\big)\right]\left[1-\frac{2U}{c^2}+\frac{4V}{c^2}+O\big(\epsilon^4\big)\right]\partial_{12}V\\
&= -\frac{3}{c^2}\left[1+\frac{4V}{c^2}-\frac{6V}{c^2}+O\big(\epsilon^4\big)\right]\partial_{12}V\\
&= -\frac{3}{c^2}\left[1-\frac{2V}{c^2}+O\big(\epsilon^4\big)\right]\partial_{12}V = -\frac{3}{c^2}\,\partial_{12}V+\frac{6}{c^4}\,V\partial_{12}V+O\big(\epsilon^6\big)\,;
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{\big(c^2+2V\big)^2D}\Big[-\big(c^2+2V\big)^3PD_{12}\Big] = \frac{1}{c^{10}}\,f_4\left(\frac{h_{01}}{c^3},\ldots,\frac{\Phi}{c^4}\right)\big(c^2+2V\big)^3PD_{12}\\
&= \frac{1}{c^{10}}\left[1+\frac{2U}{c^2}-\frac{6V}{c^2}+O\big(\epsilon^4\big)\right]\left\{c^6\left(1+\frac{2V}{c^2}\right)^3\left[c^2\left(1-\frac{2U}{c^2}\right)\partial_{12}U+2\partial_{12}\Phi\right]\right\}\\
&= \frac{1}{c^2}\left[1+\frac{2U}{c^2}-\frac{6V}{c^2}+O\big(\epsilon^4\big)\right]\left(1+\frac{2V}{c^2}\right)^3\left(1-\frac{2U}{c^2}\right)\partial_{12}U+\frac{2}{c^4}\left[1+\frac{2U}{c^2}-\frac{6V}{c^2}+O\big(\epsilon^4\big)\right]\\
&\left(1+\frac{2V}{c^2}\right)^3\partial_{12}\Phi\\
&= \frac{1}{c^2}\left[1+\frac{2U}{c^2}-\frac{6V}{c^2}+O\big(\epsilon^4\big)\right]\left(1+\frac{6V}{c^2}+\frac{12V^2}{c^4}+\frac{8V^3}{c^6}\right)\left(1-\frac{2U}{c^2}\right)\partial_{12}U+\frac{2}{c^4}\left[1+\frac{2U}{c^2}\right.\\
&\left.-\frac{6V}{c^2}+O\big(\epsilon^4\big)\right]\left(1+\frac{6V}{c^2}+\frac{12V^2}{c^4}+\frac{8V^3}{c^6}\right)\partial_{12}\Phi\\
&= \frac{1}{c^2}\left[1+\frac{2U}{c^2}-\frac{6V}{c^2}+O\big(\epsilon^4\big)\right]\left[1-\frac{2U}{c^2}+\frac{6V}{c^2}+O\big(\epsilon^4\big)\right]\partial_{12}U+\frac{2}{c^4}\left[1+O\big(\epsilon^2\big)\right]\partial_{12}\Phi
\end{aligned}$$

$$= \frac{1}{c^2}\Big[1+O\big(\epsilon^4\big)\Big]\partial_{12}U + \frac{2}{c^4}\Bigg[1+O\big(\epsilon^2\big)\Bigg]\partial_{12}\Phi$$
$$= \frac{1}{c^2}\,\partial_{12}U + \frac{2}{c^4}\,\partial_{12}\Phi + O\big(\epsilon^6\big)\,;$$

therefore,

$$\big[R_{12}\big]_3 = {}^{(2)}\big[R_{12}\big]_3 + {}^{(4)}\big[R_{12}\big]_3 + O\big(\epsilon^6\big)\,,$$
$${}^{(2)}\big[R_{12}\big]_3 = \frac{1}{c^2}\big(\partial_{12}U - 3\,\partial_{12}V\big)\,,$$
$${}^{(4)}\big[R_{12}\big]_3 = \frac{2}{c^4}\big(-\,\partial_1 U\partial_2 U + 3\,\partial_1 V\partial_2 U - 6\,\partial_1 V\partial_2 V + 3\,\partial_1 U\partial_2 V + 3\,V\partial_{12}V + \partial_{12}\Phi\big)\,;$$

- 4-th term

$$\big[R_{12}\big]_4 = \frac{2\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\,\partial_1 V\partial_2 V = \frac{1}{c^{12}}\,f_1\bigg(\frac{h_{01}}{c^3},\ldots,\frac{\Phi}{c^4}\bigg)\Big[2\big(c^2-2V\big)\big(2D+P\big)\partial_1 V\partial_2 V\Big]$$
$$= \frac{1}{c^{12}}\Big[1+O\big(\epsilon^2\big)\Big]\Bigg\{2c^2\bigg(1-\frac{2V}{c^2}\bigg)\Bigg[2c^6\bigg(\frac{h_{123}^2}{c^6}\bigg)+3\,c^6\bigg(1+\frac{2V}{c^2}\bigg)\bigg(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}$$
$$-\,\frac{4\Phi}{c^4}\bigg)\Bigg]\Bigg\}\partial_1 V\partial_2 V$$
$$= \frac{1}{c^{12}}\Big[1+O\big(\epsilon^2\big)\Big]\Bigg\{2c^8\bigg(1-\frac{2V}{c^2}\bigg)\Bigg[2\bigg(\frac{h_{123}^2}{c^6}\bigg)+3\bigg(1+\frac{2V}{c^2}\bigg)\bigg(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\bigg)\Bigg]\Bigg\}$$
$$\partial_1 V\partial_2 V$$
$$= \frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\bigg(1-\frac{2V}{c^2}\bigg)\Bigg[2\bigg(\frac{h_{123}^2}{c^6}\bigg)+3\bigg(1+\frac{2V}{c^2}\bigg)\bigg(1-\frac{2U}{c^2}+\frac{2U^2}{c^4}-\frac{4\Phi}{c^4}\bigg)\Bigg]\partial_1 V\partial_2 V$$
$$= \frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\bigg(1-\frac{2V}{c^2}\bigg)\Big\{O\big(\epsilon^6\big)+3\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big\}\partial_1 V\partial_2 V$$
$$= \frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big\{O\big(\epsilon^6\big)+3\Big[1+O\big(\epsilon^2\big)\Big]\Big\}\partial_1 V\partial_2 V$$
$$= \frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big[3+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2 V$$
$$= \frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[3+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2 V$$
$$= \frac{2}{c^4}\Big[3+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2 V$$
$$= \frac{6}{c^4}\,\partial_1 V\partial_2 V + O\big(\epsilon^6\big)\,;$$

therefore,

$$\big[R_{12}\big]_4 = {}^{(4)}\big[R_{12}\big]_4 + O\big(\epsilon^6\big)\,,$$
$${}^{(4)}\big[R_{12}\big]_4 = \frac{6}{c^4}\,\partial_1 V\partial_2 V\,;$$

- 5-th term

$$
\begin{aligned}
&\big[R_{12}\big]_5 = -\frac{\big(c^4-4V^2\big)}{c^4 D}\big(PD_1\partial_2 V + PD_2\partial_1 V\big) = -\frac{1}{c^{10}}\, f_2\bigg(\frac{h_{01}}{c^3},\ldots,\frac{\Phi}{c^4}\bigg)\Big[\big(c^4-4V^2\big)\big(PD_1\partial_2 V \\
&+ PD_2\partial_1 V\big)\Big] \\
&= -\frac{1}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]c^4\bigg(1-\frac{4V^2}{c^4}\bigg)\Bigg\{\bigg[c^2\bigg(1-\frac{2U}{c^2}\bigg)\partial_1 U + 2\,\partial_1\Phi\bigg]\partial_2 V + \bigg[c^2\bigg(1-\frac{2U}{c^2}\bigg)\partial_2 U \\
&+ 2\,\partial_2\Phi\bigg]\partial_1 V\Bigg\} \\
&= -\frac{1}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\bigg(1-\frac{4V^2}{c^4}\bigg)\bigg[c^2\bigg(1-\frac{2U}{c^2}\bigg)\big(\partial_1 U\partial_2 V + \partial_1 V\partial_2 U\big) + 2\big(\partial_1\Phi\partial_2 V + \partial_1 V\partial_2\Phi\big)\bigg] \\
&= -\frac{1}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1 U\partial_2 V + \partial_1 V\partial_2 U\big) + 2\big(\partial_1\Phi\partial_2 V + \partial_1 V\partial_2\Phi\big)\Big\} \\
&= -\frac{1}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1 U\partial_2 V + \partial_1 V\partial_2 U\big) + 2\big(\partial_1\Phi\partial_2 V + \partial_1 V\partial_2\Phi\big)\Big\} \\
&= -\frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]^2\big(\partial_1 U\partial_2 V + \partial_1 V\partial_2 U\big) - \frac{2}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1\Phi\partial_2 V + \partial_1 V\partial_2\Phi\big) \\
&= -\frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1 U\partial_2 V + \partial_1 V\partial_2 U\big) - \frac{2}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1\Phi\partial_2 V + \partial_1 V\partial_2\Phi\big) \\
&= -\frac{1}{c^4}\big(\partial_1 U\partial_2 V + \partial_1 V\partial_2 U\big) + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
\big[R_{12}\big]_5 &= {}^{(4)}\big[R_{12}\big]_5 + O\big(\epsilon^6\big)\,, \\
{}^{(4)}\big[R_{12}\big]_5 &= -\frac{1}{c^4}\big(\partial_1 U\partial_2 V + \partial_1 V\partial_2 U\big)\,;
\end{aligned}
$$

- 6-th term

$$
\begin{aligned}
&\big[R_{12}\big]_6 = -\frac{1}{c^{16}}\, q^2\, PD_1PD_2 \\
&= -\frac{1}{c^{16}}\Big(c^4+2c^2U+2U^2+4\Phi\Big)^2\bigg[c^2\bigg(1-\frac{2U}{c^2}\bigg)\partial_1 U + 2\,\partial_1\Phi\bigg]\bigg[c^2\bigg(1-\frac{2U}{c^2}\bigg)\partial_2 U + 2\,\partial_2\Phi\bigg] \\
&= -\frac{1}{c^8}\bigg(1+\frac{2U}{c^2}+\frac{2U^2}{c^4}+\frac{4\Phi}{c^4}\bigg)^2\bigg[c^4\bigg(1-\frac{2U}{c^2}\bigg)^2\partial_1 U\partial_2 U + 2c^2\bigg(1-\frac{2U}{c^2}\bigg)\big(\partial_1 U\partial_2\Phi \\
&+ \partial_2 U\partial_1\Phi\big) + 4\,\partial_1\Phi\partial_2\Phi\bigg] \\
&= -\frac{1}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]^2\Big\{c^4\Big[1+O\big(\epsilon^2\big)\Big]^2\partial_1 U\partial_2 U + 2c^2\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1 U\partial_2\Phi + \partial_2 U\partial_1\Phi\big) \\
&+ 4\,\partial_1\Phi\partial_2\Phi\Big\} \\
&= -\frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]^2\Big[1+O\big(\epsilon^2\big)\Big]^2\partial_1 U\partial_2 U - \frac{2}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]^2\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1 U\partial_2\Phi + \partial_2 U\partial_1\Phi\big)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{4}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]^2 \partial_1\Phi\partial_2\Phi \\
&= -\frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_2 U - \frac{2}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1 U\partial_2\Phi + \partial_2 U\partial_1\Phi\big) \\
&- \frac{4}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\partial_1\Phi\partial_2\Phi \\
&= -\frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_2 U - \frac{2}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1 U\partial_2\Phi + \partial_2 U\partial_1\Phi\big) - \frac{4}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\partial_1\Phi\partial_2\Phi \\
&= -\frac{1}{c^4}\,\partial_1 U\partial_2 U + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
\big[R_{12}\big]_6 &= {}^{(4)}\big[R_{12}\big]_6 + O\big(\epsilon^6\big) \\
{}^{(4)}\big[R_{12}\big]_6 &= -\frac{1}{c^4}\,\partial_1 U\partial_2 U\,;
\end{aligned}
$$

- 7-th term

$$
\begin{aligned}
&\big[R_{12}\big]_7 = -\frac{1}{c^{12}}\,h_{123}^2\big(PD_2\partial_1 V + PD_1\partial_2 V\big) \\
&= -\frac{1}{c^6}\left(\frac{h_{123}^2}{c^6}\right)\left\{\left[c^2\left(1-\frac{2U}{c^2}\right)\partial_2 U + 2\,\partial_2\Phi\right]\partial_1 V + \left[c^2\left(1-\frac{2U}{c^2}\right)\partial_1 U + 2\,\partial_1\Phi\right]\partial_2 V\right\} \\
&= -\frac{1}{c^6}\left(\frac{h_{123}^2}{c^6}\right)\left[c^2\left(1-\frac{2U}{c^2}\right)\big(\partial_2 U\partial_1 V + \partial_1 U\partial_2 V\big) + 2\big(\partial_2\Phi\partial_1 V + \partial_1\Phi\partial_2 V\big)\right] \\
&= -\frac{1}{c^6}\,O\big(\epsilon^6\big)\left\{c^2\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_2 U\partial_1 V + \partial_1 U\partial_2 V\big) + 2\big(\partial_2\Phi\partial_1 V + \partial_1\Phi\partial_2 V\big)\right\} \\
&= -\frac{1}{c^4}\,O\big(\epsilon^6\big)\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_2 U\partial_1 V + \partial_1 U\partial_2 V\big) - \frac{2}{c^6}\,O\big(\epsilon^6\big)\big(\partial_2\Phi\partial_1 V + \partial_1\Phi\partial_2 V\big) \\
&= -\frac{1}{c^4}\,O\big(\epsilon^6\big)\big(\partial_2 U\partial_1 V + \partial_1 U\partial_2 V\big) - \frac{2}{c^6}\,O\big(\epsilon^6\big)\big(\partial_2\Phi\partial_1 V + \partial_1\Phi\partial_2 V\big) \sim O\big(\epsilon^{10}\big)\,;
\end{aligned}
$$

therefore,

$$
\big[R_{12}\big]_7 \sim O\big(\epsilon^{10}\big)\,;
$$

- 8-th term

$$
\begin{aligned}
&\big[R_{12}\big]_8 = -\frac{5}{c^8}\big(c^2-2V\big)^2\partial_1 V\partial_2 V = -\frac{5}{c^4}\left(1-\frac{2V}{c^2}\right)^2\partial_1 V\partial_2 V \\
&= -\frac{5}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]^2\partial_1 V\partial_2 V = -\frac{5}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_2 V \\
&= -\frac{5}{c^4}\,\partial_1 V\partial_2 V + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
\big[R_{12}\big]_8 &= {}^{(4)}\big[R_{12}\big]_8 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{12}\big]_8 &= -\frac{5}{c^4}\,\partial_1 V \partial_2 V\,.
\end{aligned}
$$

Synoptic list:

$$
\begin{aligned}
\big[R_{12}\big]_1 &= {}^{(2)}\big[R_{12}\big]_1 + {}^{(4)}\big[R_{12}\big]_1\,,\\
{}^{(2)}\big[R_{12}\big]_1 &= \frac{2}{c^2}\,\partial_{12} V\,,\\
{}^{(4)}\big[R_{12}\big]_1 &= -\frac{4}{c^4}\big(V\partial_{12}V + \partial_1 V \partial_2 V\big)\,;
\end{aligned}
$$

$$
\begin{aligned}
\big[R_{12}\big]_2 &= {}^{(4)}\big[R_{12}\big]_2 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{12}\big]_2 &= \frac{2}{c^4}\big(9\,\partial_1 V \partial_2 V - 3\,\partial_1 V \partial_2 U - 3\,\partial_1 U \partial_2 V + \partial_1 U \partial_2 U\big)\,;
\end{aligned}
$$

$$
\begin{aligned}
\big[R_{12}\big]_3 &= {}^{(2)}\big[R_{12}\big]_3 + {}^{(4)}\big[R_{12}\big]_3 + O\big(\epsilon^6\big)\,,\\
{}^{(2)}\big[R_{12}\big]_3 &= \frac{1}{c^2}\big(\partial_{12} U - 3\,\partial_{12} V\big)\,,\\
{}^{(4)}\big[R_{12}\big]_3 &= \frac{2}{c^4}\big(-\,\partial_1 U \partial_2 U + 3\,\partial_1 V \partial_2 U - 6\,\partial_1 V \partial_2 V + 3\,\partial_1 U \partial_2 V + 3\,V\partial_{12}V + \partial_{12}\Phi\big)\,;
\end{aligned}
$$

$$
\begin{aligned}
\big[R_{12}\big]_4 &= {}^{(4)}\big[R_{12}\big]_4 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{12}\big]_4 &= \frac{6}{c^4}\,\partial_1 V \partial_2 V\,;
\end{aligned}
$$

$$
\begin{aligned}
\big[R_{12}\big]_5 &= {}^{(4)}\big[R_{12}\big]_5 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{12}\big]_5 &= -\frac{1}{c^4}\big(\partial_1 U \partial_2 V + \partial_2 U \partial_1 V\big)\,;
\end{aligned}
$$

$$
\begin{aligned}
\big[R_{12}\big]_6 &= {}^{(4)}\big[R_{12}\big]_6 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{12}\big]_6 &= -\,\frac{1}{c^4}\,\partial_1 U \partial_2 U\,;
\end{aligned}
$$

$$
\big[R_{12}\big]_7 \sim O\big(\epsilon^{10}\big)\,;
$$

$$
\begin{aligned}
\big[R_{12}\big]_8 &= {}^{(4)}\big[R_{12}\big]_8 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{12}\big]_8 &= -\,\frac{5}{c^4}\,\partial_1 V \partial_2 V\,.
\end{aligned}
$$

Summing up:

$$
\begin{aligned}
R_{12} &= {}^{(2)}R_{12} + {}^{(4)}R_{12} + O\big(\epsilon^6\big)\,,\\
{}^{(2)}R_{12} &= \frac{1}{c^2}\big(\partial_{12} U - \partial_{12} V\big)\,,\\
{}^{(4)}R_{12} &= \frac{1}{c^4}\Big[2\big(V\partial_{12}V + \partial_{12}\Phi\big) + 3\,\partial_1 V \partial_2 V - \partial_1 U \partial_2 U - \big(\partial_1 U \partial_2 V + \partial_1 V \partial_2 U\big)\Big]\,.
\end{aligned}
$$

R_{13} component

The computations concerning R_{13} are the same as those worked out in R_{12}; therefore, i am focusing my attention on the less obvious contributions only. For the rest, i refer to the previous subsection.

$$R_{13} = \frac{2}{c^4}\Big[\big(c^2-2V\big)\partial_{13}V - 2\,\partial_1 V\partial_3 V\Big] + \frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(2D+P\big)\partial_1 V - \big(c^2+2V\big)^2 PD_1\Big]$$
$$\Big[\big(2D+P\big)\partial_3 V - \big(c^2+2V\big)^2 PD_3\Big] - \frac{1}{\big(c^2+2V\big)^2 D}\Big\{2\,\partial_3 U\big(c^2+2V\big)^2\Big[\big(c^2+2V\big)\partial_1 U - 3\big(c^2-2U\big)$$
$$\partial_1 V\Big] + 2\,\partial_3 V\Big[2\big(D+2P\big)\partial_1 V - 3\big(c^2+2V\big)^2 PD_1\Big] - \big(c^2+2V\big)\Big[12\big(c^2+2V\big)\partial_1 V\partial_3\Phi - \big(2D+P\big)$$
$$\partial_{13}V + \big(c^2+2V\big)^2 PD_{13}\Big]\Big\} + \frac{2\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\,\partial_1 V\partial_3 V - \frac{\big(c^4-4V^2\big)}{c^4 D}\big(PD_1\,\partial_3 V + PD_3\,\partial_1 V\big)$$
$$- \frac{1}{c^{16}}\,q^2 PD_1 PD_3 - \frac{1}{c^{12}}\,h_{123}^2\big(PD_3\,\partial_1 V + PD_1\,\partial_3 V\big) - \frac{5}{c^8}\big(c^2-2V\big)^2\,\partial_1 V\partial_3 V\,;$$

- 1-st term

$$\begin{aligned}\big[R_{13}\big]_1 &= \frac{2}{c^4}\Big[\big(c^2-2V\big)\partial_{13}V - 2\,\partial_1 V\partial_3 V\Big]\\ &= \frac{2}{c^4}\left[c^2\left(1-\frac{2V}{c^2}\right)\partial_{13}V - 2\,\partial_1 V\partial_3 V\right];\end{aligned}$$

therefore,

$$\begin{aligned}\big[R_{13}\big]_1 &= {}^{(2)}\big[R_{13}\big]_1 + {}^{(4)}\big[R_{13}\big]_1 + O\big(\epsilon^6\big)\,,\\ {}^{(2)}\big[R_{13}\big]_1 &= \frac{2}{c^2}\,\partial_{13}V\,,\\ {}^{(4)}\big[R_{13}\big]_1 &= -\frac{4}{c^4}\big(V\partial_{13}V + \partial_1 V\partial_3 V\big)\,;\end{aligned}$$

- 2-nd term

$$\big[R_{13}\big]_2 = \frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(2D+P\big)\partial_1 V - \big(c^2+2V\big)^2 PD_1\Big]\Big[\big(2D+P\big)\partial_3 V - \big(c^2+2V\big)^2 PD_3\Big]$$
$$= \frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(2D+P\big)^2\partial_1 V\partial_3 V - \big(c^2+2V\big)^2\big(2D+P\big)PD_3\partial_1 V - \big(c^2+2V\big)^2\big(2D+P\big)$$
$$PD_1\partial_3 V + \big(c^2+2V\big)^4 PD_1 PD_3\Big]\,,$$

where,

$$\frac{1}{\big(c^2+2V\big)^2 D^2} = \frac{1}{c^{16}}\,f_3\left(\frac{h_{01}}{c^3},\ldots,\frac{\Phi}{c^4}\right) = \frac{1}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\,;$$

$$\frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(2D+P\big)^2\partial_1 V\partial_3 V\Big] = \frac{2}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(2D+P\big)^2\partial_1 V\partial_3 V\Big]$$

$$= \frac{2}{c^4}\Big[9 + O\big(\epsilon^2\big)\Big]\partial_1 V \partial_3 V$$
$$= \frac{18}{c^4}\,\partial_1 V \partial_3 V + O\big(\epsilon^6\big)\,;$$

$$\frac{2}{\big(c^2+2V\big)^2 D^2}\Big[-\big(c^2+2V\big)^2\big(2D+P\big)PD_3\partial_1 V\Big] = \frac{2}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[-\big(c^2+2V\big)^2\big(2D+P\big)$$
$$PD_3\partial_1 V\Big]$$
$$= -\frac{6}{c^4}\,\partial_1 V \partial_3 U + O\big(\epsilon^6\big)\,;$$

$$\frac{2}{\big(c^2+2V\big)^2 D^2}\Big[-\big(c^2+2V\big)^2\big(2D+P\big)PD_1\partial_3 V\Big] = \frac{2}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[-\big(c^2+2V\big)^2\big(2D+P\big)$$
$$PD_1\partial_3 V\Big]$$
$$= -\frac{6}{c^4}\,\partial_1 U \partial_3 V + O\big(\epsilon^6\big)\,;$$

$$\frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(c^2+2V\big)^4 PD_1 PD_3\Big] = \frac{2}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(c^2+2V\big)^4 PD_1 PD_3\Big]$$
$$= \frac{2}{c^4}\,\partial_1 U \partial_3 U + O\big(\epsilon^6\big)\,;$$

therefore,

$$\big[R_{13}\big]_2 = {}^{(4)}\big[R_{13}\big]_2 + O\big(\epsilon^6\big)\,,$$
$${}^{(4)}\big[R_{13}\big]_2 = \frac{2}{c^4}\big(9\,\partial_1 V\partial_3 V - 3\,\partial_1 V\partial_3 U - 3\,\partial_1 U\partial_3 V + \partial_1 U\partial_3 U\big)\,;$$

- 3-rd term

$$\big[R_{13}\big]_3 = -\frac{1}{\big(c^2+2V\big)^2 D}\Big\{2\,\partial_3 U\big(c^2+2V\big)^2\Big[\big(c^2+2V\big)\partial_1 U - 3\big(c^2-2U\big)\partial_1 V\Big] + 2\,\partial_3 V$$
$$\Big[2\big(D+2P\big)\partial_1 V - 3\big(c^2+2V\big)^2 PD_1\Big] - \big(c^2+2V\big)\Big[12\big(c^2+2V\big)\partial_1 V\partial_3\Phi - \big(2D+P\big)\partial_{13} V$$
$$+ \big(c^2+2V\big)^2 PD_{13}\Big]\Big\}\,,$$

where,

$$\frac{1}{\big(c^2+2V\big)^2 D} = \frac{1}{c^{10}}\,f_4\Big(\frac{h_{01}}{c^3},\ldots,\frac{\Phi}{c^4}\Big) = \frac{1}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\,;$$

$$-\ \frac{1}{\big(c^2+2V\big)^2 D}\Big\{2\,\partial_3 U\big(c^2+2V\big)^2\Big[\big(c^2+2V\big)\partial_1 U - 3\big(c^2-2U\big)\partial_1 V\Big]\Big\} = -\frac{1}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]$$

$$\Big\{2\,\partial_3 U\big(c^2+2V\big)^2\Big[\big(c^2+2V\big)\partial_1 U-3\big(c^2-2U\big)\partial_1 V\Big]\Big\}$$
$$=-\frac{2}{c^4}\big(\partial_1 U\partial_3 U-3\,\partial_1 V\partial_3 U\big)+O\big(\epsilon^6\big)\,;$$

$$-\frac{1}{\big(c^2+2V\big)^2 D}\Big\{2\,\partial_3 V\Big[2\big(D+2P\big)\partial_1 V-3\big(c^2+2V\big)^2 P D_1\Big]\Big\}=-\frac{1}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]$$
$$\Big\{2\,\partial_3 V\Big[2\big(D+2P\big)\partial_1 V-3\big(c^2+2V\big)^2 P D_1\Big]\Big\}$$
$$=-\frac{6}{c^4}\big(2\,\partial_1 V\partial_3 V-\partial_1 U\partial_3 V\big)+O\big(\epsilon^6\big)\,;$$

$$-\frac{1}{\big(c^2+2V\big)^2 D}\Big[-12\big(c^2+2V\big)^2\partial_1 V\partial_3\Phi\Big]=-\frac{1}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[-12\big(c^2+2V\big)^2\partial_1 V\partial_3\Phi\Big]$$
$$\sim O\big(\epsilon^6\big)\,;$$

$$-\frac{1}{\big(c^2+2V\big)^2 D}\Big[\big(c^2+2V\big)\big(2D+P\big)\partial_{13}V\Big]=-\frac{1}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(c^2+2V\big)\big(2D+P\big)\partial_{13}V\Big]$$
$$=-\frac{3}{c^2}\,\partial_{13}V+\frac{6}{c^4}\,V\partial_{13}V+O\big(\epsilon^6\big)\,;$$

$$\frac{1}{\big(c^2+2V\big)^2 D}\Big[\big(c^2+2V\big)^3 P D_{13}\Big]=\frac{1}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(c^2+2V\big)^3 P D_{13}\Big]$$
$$=\frac{1}{c^2}\,\partial_{13}U+\frac{2}{c^4}\,\partial_{13}\Phi+O\big(\epsilon^6\big)\,;$$

therefore,

$$\big[R_{13}\big]_3=\,{}^{(2)}\big[R_{13}\big]_3+\,{}^{(4)}\big[R_{13}\big]_3+O\big(\epsilon^6\big)\,,$$
$${}^{(2)}\big[R_{13}\big]_3=\frac{1}{c^2}\big(\partial_{13}U-3\,\partial_{13}V\big)\,,$$
$${}^{(4)}\big[R_{13}\big]_3=\frac{2}{c^4}\big(-\partial_1 U\partial_3 U+3\,\partial_1 V\partial_3 U-6\,\partial_1 V\partial_3 V+3\,\partial_1 U\partial_3 V+3\,V\partial_{13}V+\partial_{13}\Phi\big)\,;$$

- 4-th term

$$\big[R_{13}\big]_4=\frac{2\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\,\partial_1 V\partial_3 V\,,$$

where,

$$\frac{1}{\big(c^2+2V\big)D}=\frac{1}{c^8}f_1\Big(\frac{h_{01}}{c^3},\ldots,\frac{\Phi}{c^4}\Big)=\frac{1}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\,;$$

$$\begin{aligned}\big[R_{13}\big]_4 &= \frac{2}{c^{12}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(c^2-2V\big)\big(2D+P\big)\partial_1 V\partial_3 V\Big]\\ &= \frac{2}{c^4}\Big[3+O\big(\epsilon^2\big)\Big]\partial_1 V\partial_3 V\,;\end{aligned}$$

therefore,

$$\begin{aligned}\big[R_{13}\big]_4 &= {}^{(4)}\big[R_{13}\big]_4 + O\big(\epsilon^6\big)\\ {}^{(4)}\big[R_{13}\big]_4 &= \frac{6}{c^4}\,\partial_1 V\partial_3 V\,;\end{aligned}$$

- 5-th term

$$\big[R_{13}\big]_5 = -\frac{\big(c^4-4V^2\big)}{c^4 D}\big(PD_1\,\partial_3 V + PD_3\,\partial_1 V\big)\,,$$

where,

$$\frac{1}{D} = \frac{1}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\,;$$

$$\begin{aligned}\big[R_{13}\big]_5 &= -\frac{1}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\big(c^4-4V^2\big)\big(PD_1\,\partial_3 V + PD_3\,\partial_1 V\big)\\ &= -\frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1 U\partial_3 V+\partial_1 V\partial_3 U\big) - \frac{2}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1\Phi\partial_3 V+\partial_1 V\partial_3\Phi\big)\,;\end{aligned}$$

therefore,

$$\begin{aligned}\big[R_{13}\big]_5 &= {}^{(4)}\big[R_{13}\big]_5 + O\big(\epsilon^6\big)\,,\\ {}^{(4)}\big[R_{13}\big]_5 &= -\frac{1}{c^4}\big(\partial_1 U\partial_3 V+\partial_1 V\partial_3 U\big)\,;\end{aligned}$$

- 6-th term

$$\begin{aligned}\big[R_{13}\big]_6 &= -\frac{1}{c^{16}}\,q^2 PD_1 PD_3\\ &= -\frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\partial_1 U\partial_3 U - \frac{2}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1 U\partial_3\Phi+\partial_1\Phi\partial_3 U\big)\\ &\quad - \frac{4}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\partial_1\Phi\partial_3\Phi\,;\end{aligned}$$

therefore,

$$\begin{aligned}\big[R_{13}\big]_6 &= {}^{(4)}\big[R_{13}\big]_6 + O\big(\epsilon^6\big)\,,\\ {}^{(4)}\big[R_{13}\big]_6 &= -\frac{1}{c^4}\,\partial_1 U\partial_3 U\,;\end{aligned}$$

- 7-th term

$$\begin{aligned}\big[R_{13}\big]_7 &= -\frac{1}{c^{12}}\, h_{123}^2\big(PD_3\,\partial_1 V + PD_1\,\partial_3 V\big)\\ &= -\frac{1}{c^4}\, O\big(\epsilon^6\big)\big(\partial_1 V\partial_3 U + \partial_1 U\partial_3 V\big) - \frac{2}{c^6}\, O\big(\epsilon^6\big)\big(\partial_1 V\partial_3\Phi + \partial_1\Phi\partial_3 V\big)\,;\end{aligned}$$

therefore,

$$\big[R_{13}\big]_7 \sim O\big(\epsilon^{10}\big)\,;$$

- 8-th term

$$\begin{aligned}\big[R_{13}\big]_8 &= -\frac{5}{c^8}\big(c^2 - 2V\big)^2\,\partial_1 V\partial_3 V\\ &= -\frac{5}{c^4}\Big[1 + O\big(\epsilon^2\big)\Big]\partial_1 V\partial_3 V\,;\end{aligned}$$

therefore,

$$\begin{aligned}\big[R_{13}\big]_8 &= {}^{(4)}\big[R_{13}\big]_8 + O\big(\epsilon^6\big)\,,\\ {}^{(4)}\big[R_{13}\big]_8 &= -\frac{5}{c^4}\,\partial_1 V\partial_3 V\,.\end{aligned}$$

Synoptic list:

$$\begin{aligned}\big[R_{13}\big]_1 &= {}^{(2)}\big[R_{13}\big]_1 + {}^{(4)}\big[R_{13}\big]_1 + O\big(\epsilon^6\big)\,,\\ {}^{(2)}\big[R_{13}\big]_1 &= \frac{2}{c^2}\,\partial_{13} V\,,\\ {}^{(4)}\big[R_{13}\big]_1 &= -\frac{4}{c^4}\big(V\partial_{13} V + \partial_1 V\partial_3 V\big)\,;\end{aligned}$$

$$\begin{aligned}\big[R_{13}\big]_2 &= {}^{(4)}\big[R_{13}\big]_2 + O\big(\epsilon^6\big)\,,\\ {}^{(4)}\big[R_{13}\big]_2 &= \frac{2}{c^4}\big(9\,\partial_1 V\partial_3 V - 3\,\partial_1 V\partial_3 U - 3\,\partial_1 U\partial_3 V + \partial_1 U\partial_3 U\big)\,;\end{aligned}$$

$$\begin{aligned}\big[R_{13}\big]_3 &= {}^{(2)}\big[R_{13}\big]_3 + {}^{(4)}\big[R_{13}\big]_3 + O\big(\epsilon^6\big)\,,\\ {}^{(2)}\big[R_{13}\big]_3 &= \frac{1}{c^2}\big(\partial_{13} U - 3\,\partial_{13} V\big)\,,\\ {}^{(4)}\big[R_{13}\big]_3 &= \frac{2}{c^4}\big(-\,\partial_1 U\partial_3 U + 3\,\partial_1 V\partial_3 U - 6\,\partial_1 V\partial_3 V + 3\,\partial_1 U\partial_3 V + 3\,V\partial_{13} V + \partial_{13}\Phi\big)\,;\end{aligned}$$

$$\begin{aligned}\big[R_{13}\big]_4 &= {}^{(4)}\big[R_{13}\big]_4 + O\big(\epsilon^6\big)\\ {}^{(4)}\big[R_{13}\big]_4 &= \frac{6}{c^4}\,\partial_1 V\partial_3 V\,;\end{aligned}$$

$$\big[R_{13}\big]_5 = {}^{(4)}\big[R_{13}\big]_5 + O\big(\epsilon^6\big)\,,$$

$$
\begin{aligned}
{}^{(4)}\big[R_{13}\big]_5 &= -\frac{1}{c^4}\big(\partial_1 U\partial_3 V + \partial_1 V\partial_3 U\big)\,;\\[2ex]
\big[R_{13}\big]_6 &= {}^{(4)}\big[R_{13}\big]_6 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{13}\big]_6 &= -\frac{1}{c^4}\,\partial_1 U\partial_3 U\,;\\[2ex]
\big[R_{13}\big]_7 &\sim O\big(\epsilon^{10}\big)\,;\\[2ex]
\big[R_{13}\big]_8 &= {}^{(4)}\big[R_{13}\big]_8 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{13}\big]_8 &= -\frac{5}{c^4}\,\partial_1 V\partial_3 V\,.
\end{aligned}
$$

Summing up,

$$
\begin{aligned}
R_{13} &= {}^{(2)}R_{13} + {}^{(4)}R_{13} + O\big(\epsilon^6\big)\,,\\
{}^{(2)}R_{13} &= \frac{1}{c^2}\big(\partial_{13}U - \partial_{13}V\big)\,,\\
{}^{(4)}R_{13} &= \frac{1}{c^4}\Big[2\big(V\partial_{13}V + \partial_{13}\Phi\big) + 3\,\partial_1 V\partial_3 V - \partial_1 U\partial_3 U - \big(\partial_1 U\partial_3 V + \partial_1 V\partial_3 U\big)\Big]\,.
\end{aligned}
$$

D.2.3 R_{2i} components $\quad i = 2, 3.$

$\boldsymbol{R_{22}}$ component

The computations concerning R_{22} are the same as those relating to R_{11}; therefore, i am going to outline only the salient developments. For the rest, i refer to the subsection corresponding to R_{11}.

$$
\begin{aligned}
R_{22} &= \Big[-\frac{2}{c^4} + \frac{\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\Big]\Big[-\big(\partial_1 V\big)^2 + \big(\partial_2 V\big)^2 - \big(\partial_3 V\big)^2\Big] + \frac{\big(c^2-2V\big)}{c^4}\big(-\,\partial_{11}V + \partial_{22}V\\
&- \partial_{33}V\big) + \frac{\big(c^4-4V^2\big)}{c^4 D}\big(PD_1\,\partial_1 V - PD_2\,\partial_2 V + PD_3\,\partial_3 V\big) + \frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(2D+P\big)\,\partial_2 V\\
&- \big(c^2+2V\big)^2 PD_2\Big]^2 - \frac{1}{\big(c^2+2V\big)^2 D}\Big[2\big(c^2+2V\big)^3\big(\partial_2 U\big)^2 - 12\big(c^2+2V\big)^2 PD_2\,\partial_2 V + 4\big(D+2P\big)\\
&\big(\partial_2 V\big)^2 - \big(c^2+2V\big)^3 PD_{22} + \big(c^2+2V\big)\big(2D+P\big)\,\partial_{22}V\Big] - \frac{1}{c^{16}}\,q^2 PD_2^2 - \frac{2}{c^{12}}\,h_{123}^2\,PD_2\,\partial_2 V\\
&- \frac{\big(c^2-2V\big)^2}{c^8}\Big[-2\big(\partial_1 V\big)^2 + 3\big(\partial_2 V\big)^2 - 2\big(\partial_3 V\big)^2\Big]\,.
\end{aligned}
$$

- 1-st term

$$
\begin{aligned}
\big[R_{22}\big]_1 &= \Big[-\frac{2}{c^4} + \frac{\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\Big]\Big[-\big(\partial_1 V\big)^2 + \big(\partial_2 V\big)^2 - \big(\partial_3 V\big)^2\Big]\\
&= -\frac{2}{c^4}\Big[\dots\Big] + \frac{1}{c^4}\frac{\big(c^2-2V\big)\big(2D+P\big)}{\big(c^2+2V\big)D}\Big[\dots\Big]\\
&= -\frac{2}{c^4}\Big[\dots\Big] + \frac{1}{c^4}\Big[3 + O\big(\epsilon^2\big)\Big]\Big[\dots\Big]
\end{aligned}
$$

$$= -\frac{2}{c^4}\Big[-\big(\partial_1 V\big)^2 + \big(\partial_2 V\big)^2 - \big(\partial_3 V\big)^2\Big] + \frac{3}{c^4}\Big[-\big(\partial_1 V\big)^2 + \big(\partial_2 V\big)^2 - \big(\partial_3 V\big)^2\Big] + O\big(\epsilon^6\big)$$
$$= \frac{1}{c^4}\Big[-\big(\partial_1 V\big)^2 + \big(\partial_2 V\big)^2 - \big(\partial_3 V\big)^2\Big] + O\big(\epsilon^6\big)\,;$$

therefore,

$$\big[R_{22}\big]_1 = {}^{(4)}\big[R_{22}\big]_1 + O\big(\epsilon^6\big)\,,$$
$${}^{(4)}\big[R_{22}\big]_1 = \frac{1}{c^4}\Big[-\big(\partial_1 V\big)^2 + \big(\partial_2 V\big)^2 - \big(\partial_3 V\big)^2\Big]\,;$$

- 2-nd term

$$\big[R_{22}\big]_2 = \frac{\big(c^2 - 2V\big)}{c^4}\big(-\partial_{11}V + \partial_{22}V - \partial_{33}V\big)$$
$$= \frac{1}{c^2}\left(1 - \frac{2V}{c^2}\right)\big(-\partial_{11}V + \partial_{22}V - \partial_{33}V\big)$$
$$= \frac{1}{c^2}\big(-\partial_{11}V + \partial_{22}V - \partial_{33}V\big) - \frac{2V}{c^4}\big(-\partial_{11}V + \partial_{22}V - \partial_{33}V\big)\,;$$

therefore,

$$\big[R_{22}\big]_2 = {}^{(2)}\big[R_{22}\big]_2 + {}^{(4)}\big[R_{22}\big]_2\,,$$
$${}^{(2)}\big[R_{22}\big]_2 = \frac{1}{c^2}\big(-\partial_{11}V + \partial_{22}V - \partial_{33}V\big)\,,$$
$${}^{(4)}\big[R_{22}\big]_2 = -\frac{2V}{c^4}\big(-\partial_{11}V + \partial_{22}V - \partial_{33}V\big)\,;$$

- 3-rd term

$$\big[R_{22}\big]_3 = \frac{\big(c^4 - 4V^2\big)}{c^4 D}\big(PD_1\,\partial_1 V - PD_2\,\partial_2 V + PD_3\,\partial_3 V\big)$$
$$= \frac{1}{c^{10}}\left[c^4\left(1 - \frac{4V^2}{c^4}\right)\right]\Big[1 + O\big(\epsilon^2\big)\Big]\left\{\left[c^2\left(1 - \frac{2U}{c^2}\right)\partial_1 U + 2\partial_1\Phi\right]\partial_1 V\right.$$
$$\left. - \left[c^2\left(1 - \frac{2U}{c^2}\right)\partial_2 U + 2\partial_2\Phi\right]\partial_2 V + \left[c^2\left(1 - \frac{2U}{c^2}\right)\partial_3 U + 2\partial_3\Phi\right]\partial_3 V\right\}$$
$$= \frac{1}{c^6}\Big[1 + O\big(\epsilon^4\big)\Big]\Big[1 + O\big(\epsilon^2\big)\Big]\left\{\left[c^2\Big[1 + O\big(\epsilon^2\big)\Big]\partial_1 U\partial_1 V + 2\partial_1 V\partial_1\Phi\right] - \left[c^2\Big[1 + O\big(\epsilon^2\big)\Big]\right.\right.$$
$$\left.\left.\partial_2 U\partial_2 V + 2\partial_2 V\partial_2\Phi\right] + \left[c^2\Big[1 + O\big(\epsilon^2\big)\Big]\partial_3 U\partial_3 V + 2\partial_3 V\partial_3\Phi\right]\right\}$$
$$= \frac{1}{c^4}\Big[1 + O\big(\epsilon^2\big)\Big]\Big[1 + O\big(\epsilon^2\big)\Big]\big(\partial_1 U\,\partial_1 V - \partial_2 U\,\partial_2 V + \partial_3 U\,\partial_3 V\big) + \frac{2}{c^6}\Big[1 + O\big(\epsilon^2\big)\Big]$$
$$\big(\partial_1 V\partial_1\Phi - \partial_2 V\partial_2\Phi + \partial_3 V\partial_3\Phi\big)$$
$$= \frac{1}{c^4}\Big[1 + O\big(\epsilon^2\big)\Big]\big(\partial_1 U\,\partial_1 V - \partial_2 U\,\partial_2 V + \partial_3 U\,\partial_3 V\big) + \frac{2}{c^6}\Big[1 + O\big(\epsilon^2\big)\Big]\big(\partial_1 V\partial_1\Phi - \partial_2 V\partial_2\Phi$$

$$+\,\partial_3 V\partial_3\Phi\big)\,;$$

therefore,

$$\begin{aligned}\big[R_{22}\big]_3 &= {}^{(4)}\big[R_{22}\big]_3 + O\big(\epsilon^6\big)\,,\\ {}^{(4)}\big[R_{22}\big]_3 &= \frac{1}{c^4}\big(\partial_1 U\,\partial_1 V - \partial_2 U\,\partial_2 V + \partial_3 U\,\partial_3 V\big)\,;\end{aligned}$$

- 4-th term

$$\begin{aligned}\big[R_{22}\big]_4 &= \frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(2D+P\big)\,\partial_2 V - \big(c^2+2V\big)^2 P D_2\Big]^2\\ &= \frac{2}{c^4}\Big[9+O\big(\epsilon^2\big)\Big]\big(\partial_2 V\big)^2 + \frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_2 U\big)^2 + \frac{8}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_2\Phi\big)^2\\ &+ \frac{8}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\partial_2 U\partial_2\Phi - \frac{4}{c^4}\Big[3+O\big(\epsilon^2\big)\Big]\partial_2 U\partial_2 V - \frac{8}{c^6}\Big[3+O\big(\epsilon^2\big)\Big]\partial_2 V\partial_2\Phi\,;\end{aligned}$$

therefore,

$$\begin{aligned}\big[R_{22}\big]_4 &= {}^{(4)}\big[R_{22}\big]_4 + O\big(\epsilon^6\big)\,,\\ {}^{(4)}\big[R_{22}\big]_4 &= \frac{18}{c^4}\big(\partial_2 V\big)^2 + \frac{2}{c^4}\big(\partial_2 U\big)^2 - \frac{12}{c^4}\,\partial_2 U\partial_2 V\,;\end{aligned}$$

- 5-th term

$$\begin{aligned}\big[R_{22}\big]_5 = &-\frac{1}{\big(c^2+2V\big)^2 D}\Big[2\big(c^2+2V\big)^3\big(\partial_2 U\big)^2 - 12\big(c^2+2V\big)^2 P D_2\,\partial_2 V + 4\big(D+2P\big)\\ &\big(\partial_2 V\big)^2 - \big(c^2+2V\big)^3 P D_{22} + \big(c^2+2V\big)\big(2D+P\big)\,\partial_{22} V\Big]\,,\end{aligned}$$

where,

$$-\frac{2}{\big(c^2+2V\big)^2 D}\Big[\big(c^2+2V\big)^3\big(\partial_2 U\big)^2\Big] = -\frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_2 U\big)^2 = -\frac{2}{c^4}\big(\partial_2 U\big)^2 + O\big(\epsilon^6\big)\,;$$

$$\begin{aligned}\frac{12}{\big(c^2+2V\big)^2 D}\Big[\big(c^2+2V\big)^2 P D_2\,\partial_2 V\Big] &= \frac{12}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\partial_2 U\partial_2 V + \frac{24}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\partial_2 V\partial_2\Phi\\ &= \frac{12}{c^4}\,\partial_2 U\partial_2 V + O\big(\epsilon^6\big)\,;\end{aligned}$$

$$\begin{aligned}-\frac{4}{\big(c^2+2V\big)^2 D}\Big[\big(D+2P\big)\big(\partial_2 V\big)^2\Big] &= -\frac{4}{c^4}\Big[3+O\big(\epsilon^2\big)\Big]\big(\partial_2 V\big)^2\\ &= -\frac{12}{c^4}\big(\partial_2 V\big)^2 + O\big(\epsilon^6\big)\,;\end{aligned}$$

$$\frac{1}{\big(c^2+2V\big)^2 D}\Big[\big(c^2+2V\big)^3 P D_{22}\Big] = \frac{1}{c^2}\Big[1+O\big(\epsilon^4\big)\Big]\partial_{22}U + \frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\partial_{22}\Phi$$
$$= \frac{1}{c^2}\,\partial_{22}U + \frac{2}{c^4}\,\partial_{22}\Phi + O\big(\epsilon^6\big)\,;$$

$$-\frac{1}{\big(c^2+2V\big)^2 D}\Big[\big(c^2+2V\big)\big(2D+P\big)\partial_{22}V\Big] = -\frac{3}{c^2}\Big[1-\frac{2V}{c^2}+O\big(\epsilon^4\big)\Big]\partial_{22}V$$
$$= -\frac{3}{c^2}\,\partial_{22}V + \frac{6V}{c^4}\,\partial_{22}V + O\big(\epsilon^6\big)\,;$$

therefore,

$$\big[R_{22}\big]_5 = {}^{(2)}\big[R_{22}\big]_5 + {}^{(4)}\big[R_{22}\big]_5 + O\big(\epsilon^6\big)\,,$$
$${}^{(2)}\big[R_{22}\big]_5 = \frac{1}{c^2}\,\partial_{22}U - \frac{3}{c^2}\,\partial_{22}V\,,$$
$${}^{(4)}\big[R_{22}\big]_5 = -\frac{2}{c^4}\,\big(\partial_2 U\big)^2 + \frac{12}{c^4}\,\partial_2 U\partial_2 V - \frac{12}{c^4}\big(\partial_2 V\big)^2 + \frac{2}{c^4}\,\partial_{22}\Phi + \frac{6V}{c^4}\,\partial_{22}V\,;$$

- 6-th term

$$\big[R_{22}\big]_6 = -\frac{1}{c^{16}}\,q^2 P D_2^2$$
$$= -\frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_2 U\big)^2 - \frac{4}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_2\Phi\big)^2 - \frac{4}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\partial_2 U\partial_2\Phi$$
$$= -\frac{1}{c^4}\big(\partial_2 U\big)^2 + O\big(\epsilon^6\big)\,;$$

therefore,

$$\big[R_{22}\big]_6 = {}^{(4)}\big[R_{22}\big]_6 + O\big(\epsilon^6\big)\,,$$
$${}^{(4)}\big[R_{22}\big]_6 = -\frac{1}{c^4}\big(\partial_2 U\big)^2\,;$$

- 7-th term

$$\big[R_{22}\big]_7 = -\frac{2}{c^{12}}\,h_{123}^2\,P D_2\,\partial_2 V$$
$$= -\frac{2}{c^4}\,O\big(\epsilon^6\big)\Big[1+O\big(\epsilon^2\big)\Big]\partial_2 U\partial_2 V - \frac{4}{c^6}O\big(\epsilon^6\big)\partial_2 V\partial_2\Phi\,;$$

therefore,

$$\big[R_{22}\big]_7 \sim O\big(\epsilon^{10}\big)\,;$$

- 8-th term

$$
\begin{aligned}
\big[R_{22}\big]_8 &= -\frac{1}{c^8}\big(c^2-2V\big)^2\Big[-2\big(\partial_1 V\big)^2+3\big(\partial_2 V\big)^2-2\big(\partial_3 V\big)^2\Big]\\
&= -\frac{1}{c^4}\left(1-\frac{2V}{c^2}\right)^2\Big[\dots\Big] = -\frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]^2\Big[-2\big(\partial_1 V\big)^2+3\big(\partial_2 V\big)^2-2\big(\partial_3 V\big)^2\Big]\\
&= -\frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\Big[-2\big(\partial_1 V\big)^2+3\big(\partial_2 V\big)^2-2\big(\partial_3 V\big)^2\Big]\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
\big[R_{22}\big]_8 &= {}^{(4)}\big[R_{22}\big]_8 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{22}\big]_8 &= -\frac{1}{c^4}\Big[-2\big(\partial_1 V\big)^2+3\big(\partial_2 V\big)^2-2\big(\partial_3 V\big)^2\Big]\,.
\end{aligned}
$$

Synoptic list:

$$
\begin{aligned}
\big[R_{22}\big]_1 &= {}^{(4)}\big[R_{22}\big]_1 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{22}\big]_1 &= \frac{1}{c^4}\Big[-\big(\partial_1 V\big)^2+\big(\partial_2 V\big)^2-\big(\partial_3 V\big)^2\Big]\,;
\end{aligned}
$$

$$
\begin{aligned}
\big[R_{22}\big]_2 &= {}^{(2)}\big[R_{22}\big]_2 + {}^{(4)}\big[R_{22}\big]_2\,,\\
{}^{(2)}\big[R_{22}\big]_2 &= \frac{1}{c^2}\big(-\partial_{11}V+\partial_{22}V-\partial_{33}V\big)\,,\\
{}^{(4)}\big[R_{22}\big]_2 &= -\frac{2V}{c^4}\big(-\partial_{11}V+\partial_{22}V-\partial_{33}V\big)\,;
\end{aligned}
$$

$$
\begin{aligned}
\big[R_{22}\big]_3 &= {}^{(4)}\big[R_{22}\big]_3 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{22}\big]_3 &= \frac{1}{c^4}\big(\partial_1 U\,\partial_1 V-\partial_2 U\,\partial_2 V+\partial_3 U\,\partial_3 V\big)\,;
\end{aligned}
$$

$$
\begin{aligned}
\big[R_{22}\big]_4 &= {}^{(4)}\big[R_{22}\big]_4 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{22}\big]_4 &= \frac{18}{c^4}\big(\partial_2 V\big)^2+\frac{2}{c^4}\big(\partial_2 U\big)^2-\frac{12}{c^4}\,\partial_2 U\partial_2 V\,;
\end{aligned}
$$

$$
\begin{aligned}
\big[R_{22}\big]_5 &= {}^{(2)}\big[R_{22}\big]_5 + {}^{(4)}\big[R_{22}\big]_5 + O\big(\epsilon^6\big)\,,\\
{}^{(2)}\big[R_{22}\big]_5 &= \frac{1}{c^2}\,\partial_{22}U-\frac{3}{c^2}\,\partial_{22}V\,,\\
{}^{(4)}\big[R_{22}\big]_5 &= -\frac{2}{c^4}\,\big(\partial_2 U\big)^2+\frac{12}{c^4}\,\partial_2 U\partial_2 V-\frac{12}{c^4}\big(\partial_2 V\big)^2+\frac{2}{c^4}\,\partial_{22}\Phi+\frac{6V}{c^4}\,\partial_{22}V\,;
\end{aligned}
$$

$$
\begin{aligned}
\big[R_{22}\big]_6 &= {}^{(4)}\big[R_{22}\big]_6 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{22}\big]_6 &= -\frac{1}{c^4}\big(\partial_2 U\big)^2\,;
\end{aligned}
$$

$$
\big[R_{22}\big]_7 \sim O\big(\epsilon^{10}\big)\,;
$$

$$
\begin{aligned}
\big[R_{22}\big]_8 &= {}^{(4)}\big[R_{22}\big]_8 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{22}\big]_8 &= -\frac{1}{c^4}\Big[-2\big(\partial_1 V\big)^2 + 3\big(\partial_2 V\big)^2 - 2\big(\partial_3 V\big)^2\Big]\,.
\end{aligned}
$$

Summing up,

$$
\begin{aligned}
R_{22} &= {}^{(2)}R_{22} + {}^{(4)}R_{22} + O\big(\epsilon^6\big)\,,\\
{}^{(2)}R_{22} &= \frac{1}{c^2}\big(\partial_{22}U - \partial_{22}V - \Delta V\big)\,,\\
{}^{(4)}R_{22} &= \Big\{\frac{1}{c^4}\Big[3\big(\partial_2 V\big)^2 + \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\Big] + \frac{2V}{c^4}\big(\partial_{22}V + \Delta V\big) + \frac{1}{c^4}\big(-2\,\partial_2 U\partial_2 V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)\\
&\quad + \frac{2}{c^4}\,\partial_{22}\Phi - \frac{1}{c^4}\big(\partial_2 U\big)^2\Big\}\,.
\end{aligned}
$$

$\boldsymbol{R_{23}}$ component

The computations concerning R_{23} are the same as those relating to R_{13}; therefore, you can see the corresponding subsection for the details.

$$
\begin{aligned}
R_{23} &= \frac{2}{c^4}\Big[\big(c^2-2V\big)\partial_{23}V - 2\,\partial_2 V\partial_3 V\Big] + \frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(2D+P\big)\partial_2 V - \big(c^2+2V\big)^2 PD_2\Big]\\
&\Big[\big(2D+P\big)\partial_3 V - \big(c^2+2V\big)^2 PD_3\Big] - \frac{1}{\big(c^2+2V\big)^2 D}\Big\{2\,\partial_3 U\big(c^2+2V\big)^2\Big[\big(c^2+2V\big)\partial_2 U - 3\big(c^2-2U\big)\\
&\partial_2 V\Big] + 2\,\partial_3 V\Big[2\big(D+2P\big)\partial_2 V - 3\big(c^2+2V\big)^2 PD_2\Big] - \big(c^2+2V\big)\Big[12\big(c^2+2V\big)\partial_2 V\partial_3\Phi - \big(2D+P\big)\\
&\partial_{23}V + \big(c^2+2V\big)^2 PD_{23}\Big]\Big\} + \frac{2\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\,\partial_2 V\partial_3 V - \frac{\big(c^4-4V^2\big)}{c^4 D}\big(PD_2\,\partial_3 V + PD_3\,\partial_2 V\big)\\
&- \frac{1}{c^{16}}\,q^2 PD_2 PD_3 - \frac{1}{c^{12}}\,h_{123}^2\big(PD_3\,\partial_2 V + PD_2\,\partial_3 V\big) - \frac{5\big(c^2-2V\big)^2}{c^8}\,\partial_2 V\partial_3 V\,;
\end{aligned}
$$

- 1-st term

$$
\begin{aligned}
\big[R_{23}\big]_1 &= \frac{2}{c^4}\Big[\big(c^2-2V\big)\partial_{23}V - 2\,\partial_2 V\partial_3 V\Big]\\
&= \frac{2}{c^2}\,\partial_{23}V - \frac{4}{c^4}\,V\partial_{23}V - \frac{4}{c^4}\,\partial_2 V\partial_3 V\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
\big[R_{23}\big]_1 &= {}^{(2)}\big[R_{23}\big]_1 + {}^{(4)}\big[R_{23}\big]_1\,,\\
{}^{(2)}\big[R_{23}\big]_1 &= \frac{2}{c^2}\,\partial_{23}V\,,\\
{}^{(4)}\big[R_{23}\big]_1 &= -\frac{4}{c^4}\big(V\partial_{23}V + \partial_2 V\partial_3 V\big)\,;
\end{aligned}
$$

- 2-nd term

$$\big[R_{23}\big]_2 = \frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(2D+P\big)\partial_2 V - \big(c^2+2V\big)^2 P D_2\Big]\Big[\big(2D+P\big)\partial_3 V - \big(c^2+2V\big)^2 P D_3\Big]$$
$$= \frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(2D+P\big)^2\partial_2 V\partial_3 V - \big(c^2+2V\big)^2\big(2D+P\big)PD_3\partial_2 V - \big(c^2+2V\big)^2\big(2D+P\big)$$
$$PD_2\partial_3 V + \big(c^2+2V\big)^4 PD_2 PD_3\Big]\,,$$

where,

$$\frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(2D+P\big)^2\partial_2 V\partial_3 V\Big] = \frac{2}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(2D+P\big)^2\partial_2 V\partial_3 V\Big]$$
$$= \frac{2}{c^4}\Big[9+O\big(\epsilon^2\big)\Big]\partial_2 V\partial_3 V$$
$$= \frac{18}{c^4}\,\partial_2 V\partial_3 V + O\big(\epsilon^6\big)\,;$$

$$\frac{2}{\big(c^2+2V\big)^2 D^2}\Big[-\big(c^2+2V\big)^2\big(2D+P\big)PD_3\partial_2 V\Big] = \frac{2}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[-\big(c^2+2V\big)^2\big(2D+P\big)$$
$$PD_3\partial_2 V\Big]$$
$$= -\frac{6}{c^4}\,\partial_2 V\partial_3 U + O\big(\epsilon^6\big)\,;$$

$$\frac{2}{\big(c^2+2V\big)^2 D^2}\Big[-\big(c^2+2V\big)^2\big(2D+P\big)PD_2\partial_3 V\Big] = \frac{2}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[-\big(c^2+2V\big)^2\big(2D+P\big)$$
$$PD_2\partial_3 V\Big]$$
$$= -\frac{6}{c^4}\,\partial_2 U\partial_3 V + O\big(\epsilon^6\big)\,;$$

$$\frac{2}{\big(c^2+2V\big)^4 D^2}\Big[\big(c^2+2V\big)^4 PD_2 PD_3\Big] = \frac{2}{c^{16}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(c^2+2V\big)^4 PD_2 PD_3\Big]$$
$$= \frac{2}{c^4}\,\partial_2 U\partial_3 U + O\big(\epsilon^6\big)\,;$$

therefore,

$$\big[R_{23}\big]_2 = {}^{(4)}\big[R_{23}\big]_2 + O\big(\epsilon^6\big)\,,$$
$${}^{(4)}\big[R_{23}\big]_2 = \frac{2}{c^4}\big(9\,\partial_2 V\partial_3 V - 3\,\partial_2 V\partial_3 U - 3\,\partial_2 U\partial_3 V + \partial_2 U\partial_3 U\big)\,;$$

- 3-rd term

$$\big[R_{23}\big]_3 = -\frac{1}{\big(c^2+2V\big)^2 D}\Big\{2\,\partial_3 U\big(c^2+2V\big)^2\Big[\big(c^2+2V\big)\partial_2 U - 3\big(c^2-2U\big)\partial_2 V\Big] + 2\,\partial_3 V$$

$$\Big[2\big(D+2P\big)\partial_2 V-3\big(c^2+2V\big)^2 PD_2\Big]-\big(c^2+2V\big)\Big[12\big(c^2+2V\big)\partial_2 V\partial_3\Phi$$
$$-\big(2D+P\big)\partial_{23}V+\big(c^2+2V\big)^2 PD_{23}\Big]\Big\}\,,$$

where,

$$\begin{aligned}
&-\frac{1}{\big(c^2+2V\big)^2 D}\Big\{2\,\partial_3 U\big(c^2+2V\big)^2\Big[\big(c^2+2V\big)\partial_2 U-3\big(c^2-2U\big)\partial_2 V\Big]\Big\}\\
&=-\frac{1}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\Big\{2\,\partial_3 U\big(c^2+2V\big)^2\Big[\big(c^2+2V\big)\partial_2 U-3\big(c^2-2U\big)\partial_2 V\Big]\Big\}\\
&=-\frac{2}{c^4}\big(\partial_2 U\partial_3 U-3\,\partial_2 V\partial_3 U\big)+O\big(\epsilon^6\big)\,;
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{\big(c^2+2V\big)^2 D}\Big\{2\,\partial_3 V\Big[2\big(D+2P\big)\partial_2 V-3\big(c^2+2V\big)^2 PD_2\Big]\Big\}\\
&=-\frac{1}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\Big\{2\,\partial_3 V\Big[2\big(D+2P\big)\partial_2 V-3\big(c^2+2V\big)^2 PD_2\Big]\Big\}\\
&=-\frac{6}{c^4}\big(2\,\partial_2 V\partial_3 V-\partial_2 U\partial_3 V\big)+O\big(\epsilon^6\big)\,;
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\big(c^2+2V\big)^2 D}\Big[12\big(c^2+2V\big)^2\partial_2 V\partial_3\Phi\Big]&=\frac{1}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[12\big(c^2+2V\big)^2\partial_2 V\partial_3\Phi\Big]\\
&\sim O\big(\epsilon^6\big)\,;
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{\big(c^2+2V\big)^2 D}\Big[\big(c^2+2V\big)\big(2D+P\big)\partial_{23}V\Big]&=-\frac{1}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(c^2+2V\big)\big(2D+P\big)\partial_{23}V\Big]\\
&=-\frac{3}{c^2}\,\partial_{23}V+\frac{6}{c^4}\,V\partial_{23}V+O\big(\epsilon^6\big)\,;
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\big(c^2+2V\big)^2 D}\Big[\big(c^2+2V\big)^3 PD_{23}\Big]&=\frac{1}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(c^2+2V\big)^3 PD_{23}\Big]\\
&=\frac{1}{c^2}\,\partial_{23}U+\frac{2}{c^4}\,\partial_{23}\Phi+O\big(\epsilon^6\big)\,;
\end{aligned}$$

therefore,

$$\begin{aligned}
\big[R_{23}\big]_3&=\,{}^{(2)}\big[R_{23}\big]_3+\,{}^{(4)}\big[R_{23}\big]_3+O\big(\epsilon^6\big)\,,\\
{}^{(2)}\big[R_{23}\big]_3&=\frac{1}{c^2}\big(\partial_{23}U-3\,\partial_{23}V\big)\,,\\
{}^{(4)}\big[R_{23}\big]_3&=\frac{2}{c^4}\big(-\,\partial_2 U\partial_3 U+3\,\partial_2 V\partial_3 U-6\,\partial_2 V\partial_3 V+3\,\partial_2 U\partial_3 V+3\,V\partial_{23}V+\partial_{23}\Phi\big)\,;
\end{aligned}$$

- 4-th term

$$\big[R_{23}\big]_4 = \frac{2\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\,\partial_2 V\partial_3 V = \frac{2}{c^{12}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(c^2-2V\big)\big(2D+P\big)\partial_2 V\partial_3 V\Big]$$
$$= \frac{2}{c^4}\Big[3+O\big(\epsilon^2\big)\Big]\partial_2 V\partial_3 V\,;$$

therefore,

$$\big[R_{23}\big]_4 = {}^{(4)}\big[R_{23}\big]_4 + O\big(\epsilon^6\big)\,,$$
$${}^{(4)}\big[R_{23}\big]_4 = \frac{6}{c^4}\,\partial_2 V\partial_3 V\,;$$

- 5-th term

$$\big[R_{23}\big]_5 = -\frac{\big(c^4-4V^2\big)}{c^4 D}\big(PD_2\,\partial_3 V + PD_3\,\partial_2 V\big)$$
$$= -\frac{1}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(c^4-4V^2\big)\big(PD_2\,\partial_3 V + PD_3\,\partial_2 V\big)$$
$$= -\frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_2 U\partial_3 V + \partial_3 U\partial_2 V\big) - \frac{2}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_2\Phi\partial_3 V + \partial_2 V\partial_3\Phi\big)\,;$$

therefore,

$$\big[R_{23}\big]_5 = {}^{(4)}\big[R_{23}\big]_5 + O\big(\epsilon^6\big)\,,$$
$${}^{(4)}\big[R_{23}\big]_5 = -\frac{1}{c^4}\big(\partial_2 U\partial_3 V + \partial_3 U\partial_2 V\big)\,;$$

- 6-th term

$$\big[R_{23}\big]_6 = -\frac{1}{c^{16}}\,q^2 PD_2 PD_3$$
$$= -\frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\partial_2 U\partial_3 U - \frac{2}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_2 U\partial_3\Phi + \partial_3 U\partial_2\Phi\big)$$
$$-\frac{4}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\partial_2\Phi\partial_3\Phi\,;$$

therefore,

$$\big[R_{23}\big]_6 = {}^{(4)}\big[R_{23}\big]_6 + O\big(\epsilon^6\big)\,,$$
$${}^{(4)}\big[R_{23}\big]_6 = -\frac{1}{c^4}\,\partial_2 U\partial_3 U\,;$$

- 7-th term

$$\big[R_{23}\big]_7 = -\frac{1}{c^{12}}\,h_{123}^2\big(PD_3\,\partial_2 V + PD_2\,\partial_3 V\big)$$

$$= -\frac{1}{c^4}\,O\big(\epsilon^6\big)\big(\partial_3 U\partial_2 V+\partial_2 U\partial_3 V\big)-\frac{2}{c^6}\,O\big(\epsilon^6\big)\big(\partial_3\Phi\partial_2 V+\partial_2\Phi\partial_3 V\big)\,;$$

therefore,

$$\big[R_{23}\big]_7\sim O\big(\epsilon^{10}\big)\,;$$

- 8-th term

$$\begin{aligned}\big[R_{23}\big]_8 &= -\frac{5}{c^8}\big(c^2-2V\big)^2\,\partial_2 V\partial_3 V\\ &= -\frac{5}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]^2\partial_2 V\partial_3 V\\ &= -\frac{5}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\partial_2 V\partial_3 V\,;\end{aligned}$$

therefore,

$$\begin{aligned}\big[R_{23}\big]_8 &= {}^{(4)}\big[R_{23}\big]_8+O\big(\epsilon^6\big)\,,\\ {}^{(4)}\big[R_{23}\big]_8 &= -\frac{5}{c^4}\,\partial_2 V\partial_3 V\,.\end{aligned}$$

Synoptic list:

$$\begin{aligned}\big[R_{23}\big]_1 &= {}^{(2)}\big[R_{23}\big]_1+{}^{(4)}\big[R_{23}\big]_1\,,\\ {}^{(2)}\big[R_{23}\big]_1 &= \frac{2}{c^2}\,\partial_{23}V\,,\\ {}^{(4)}\big[R_{23}\big]_1 &= -\frac{4}{c^4}\big(V\partial_{23}V+\partial_2 V\partial_3 V\big)\,;\end{aligned}$$

$$\begin{aligned}\big[R_{23}\big]_2 &= {}^{(4)}\big[R_{23}\big]_2+O\big(\epsilon^6\big)\,,\\ {}^{(4)}\big[R_{23}\big]_2 &= \frac{2}{c^4}\big(9\,\partial_2 V\partial_3 V-3\,\partial_2 V\partial_3 U-3\,\partial_2 U\partial_3 V+\partial_2 U\partial_3 U\big)\,;\end{aligned}$$

$$\begin{aligned}\big[R_{23}\big]_3 &= {}^{(2)}\big[R_{23}\big]_3+{}^{(4)}\big[R_{23}\big]_3+O\big(\epsilon^6\big)\,,\\ {}^{(2)}\big[R_{23}\big]_3 &= \frac{1}{c^2}\big(\partial_{23}U-3\,\partial_{23}V\big)\,,\\ {}^{(4)}\big[R_{23}\big]_3 &= \frac{2}{c^4}\big(-\partial_2 U\partial_3 U+3\,\partial_2 V\partial_3 U-6\,\partial_2 V\partial_3 V+3\,\partial_2 U\partial_3 V+3\,V\partial_{23}V+\partial_{23}\Phi\big)\,;\end{aligned}$$

$$\begin{aligned}\big[R_{23}\big]_4 &= {}^{(4)}\big[R_{23}\big]_4+O\big(\epsilon^6\big)\,,\\ {}^{(4)}\big[R_{23}\big]_4 &= \frac{6}{c^4}\,\partial_2 V\partial_3 V\,;\end{aligned}$$

$$\begin{aligned}\big[R_{23}\big]_5 &= {}^{(4)}\big[R_{23}\big]_5+O\big(\epsilon^6\big)\,,\\ {}^{(4)}\big[R_{23}\big]_5 &= -\frac{1}{c^4}\big(\partial_2 U\partial_3 V+\partial_3 U\partial_2 V\big)\,;\end{aligned}$$

$$
\begin{aligned}
\big[R_{23}\big]_6 &= {}^{(4)}\big[R_{23}\big]_6 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{23}\big]_6 &= -\,\frac{1}{c^4}\,\partial_2 U \partial_3 U\,;
\end{aligned}
$$

$$
\big[R_{23}\big]_7 \sim O\big(\epsilon^{10}\big)\,;
$$

$$
\begin{aligned}
\big[R_{23}\big]_8 &= {}^{(4)}\big[R_{23}\big]_8 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{23}\big]_8 &= -\frac{5}{c^4}\,\partial_2 V \partial_3 V\,.
\end{aligned}
$$

Summing up,

$$
\begin{aligned}
R_{23} &= {}^{(2)}R_{23} + {}^{(4)}R_{23} + O\big(\epsilon^6\big)\,,\\
{}^{(2)}R_{23} &= \frac{1}{c^2}\big(\partial_{23}U - \partial_{23}V\big)\,,\\
{}^{(4)}R_{23} &= \frac{1}{c^4}\Big[2\big(V\partial_{23}V + \partial_{23}\Phi\big) + 3\,\partial_2 V\partial_3 V - \,\partial_2 U\partial_3 U - \big(\partial_2 U\partial_3 V + \partial_3 U\partial_2 V\big)\Big]\,.
\end{aligned}
$$

D.2.4 R_{33} component

The computations concerning R_{33} are the same as those relating to R_{22} and R_{11}; therefore, you can see the corresponding subsections for the details.

$$
\begin{aligned}
R_{33} &= \Big[-\frac{2}{c^4} + \frac{\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\Big]\Big[-\big(\partial_1 V\big)^2 - \big(\partial_2 V\big)^2 + \big(\partial_3 V\big)^2\Big] + \frac{\big(c^2-2V\big)}{c^4}\big(-\,\partial_{11}V - \partial_{22}V\\
&+ \partial_{33}V\big) + \frac{\big(c^4-4V^2\big)}{c^4 D}\big(PD_1\,\partial_1 V + PD_2\,\partial_2 V - PD_3\,\partial_3 V\big) + \frac{2}{\big(c^2+2V\big)^2 D^2}\Big[\big(2D+P\big)\partial_3 V\\
&- \big(c^2+2V\big)^2 PD_3\Big]^2 - \frac{1}{\big(c^2+2V\big)^2 D}\Big[2\big(c^2+2V\big)^3\big(\partial_3 U\big)^2 - 12\big(c^2+2V\big)^2 PD_3\,\partial_3 V + 4\big(D+2P\big)\\
&\big(\partial_3 V\big)^2 - \big(c^2+2V\big)^3 PD_{33} + \big(c^2+2V\big)\big(2D+P\big)\partial_{33}V\Big] - \frac{1}{c^{16}}\,q^2 PD_3^2 - \frac{2}{c^{12}}\,h_{123}^2\,PD_3\,\partial_3 V\\
&- \frac{\big(c^2-2V\big)^2}{c^8}\Big[-2\big(\partial_1 V\big)^2 - 2\big(\partial_2 V\big)^2 + 3\big(\partial_3 V\big)^2\Big]\,.
\end{aligned}
$$

- 1-st term

$$
\begin{aligned}
\big[R_{33}\big]_1 &= \Big[-\frac{2}{c^4} + \frac{\big(c^2-2V\big)\big(2D+P\big)}{c^4\big(c^2+2V\big)D}\Big]\Big[-\big(\partial_1 V\big)^2 - \big(\partial_2 V\big)^2 + \big(\partial_3 V\big)^2\Big]\\
&= \frac{1}{c^4}\Big[-\big(\partial_1 V\big)^2 - \big(\partial_2 V\big)^2 + \big(\partial_3 V\big)^2\Big] + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
\big[R_{33}\big]_1 &= {}^{(4)}\big[R_{33}\big]_1 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{33}\big]_1 &= \frac{1}{c^4}\Big[-\big(\partial_1 V\big)^2 - \big(\partial_2 V\big)^2 + \big(\partial_3 V\big)^2\Big]\,;
\end{aligned}
$$

- 2-nd term

$$\begin{aligned}\big[R_{33}\big]_2 &= \frac{\big(c^2-2V\big)}{c^4}\big(-\partial_{11}V-\partial_{22}V+\partial_{33}V\big)\\ &= \frac{1}{c^2}\bigg(1-\frac{2V}{c^2}\bigg)\big(-\partial_{11}V-\partial_{22}V+\partial_{33}V\big)\\ &= \frac{1}{c^2}\big(-\partial_{11}V-\partial_{22}V+\partial_{33}V\big)-\frac{2V}{c^4}\big(-\partial_{11}V-\partial_{22}V+\partial_{33}V\big)\,;\end{aligned}$$

therefore,

$$\begin{aligned}\big[R_{33}\big]_2 &= {}^{(2)}\big[R_{33}\big]_2+{}^{(4)}\big[R_{33}\big]_2+O\big(\epsilon^6\big)\,,\\ {}^{(2)}\big[R_{33}\big]_2 &= \frac{1}{c^2}\big(-\partial_{11}V-\partial_{22}V+\partial_{33}V\big)\,,\\ {}^{(4)}\big[R_{33}\big]_2 &= -\frac{2V}{c^4}\big(-\partial_{11}V-\partial_{22}V+\partial_{33}V\big)\,;\end{aligned}$$

- 3-rd term

$$\begin{aligned}\big[R_{33}\big]_3 &= \frac{\big(c^4-4V^2\big)}{c^4D}\big(PD_1\,\partial_1V+PD_2\,\partial_2V-PD_3\,\partial_3V\big)\\ &= \frac{1}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1U\partial_1V+\partial_2U\partial_2V-\partial_3U\partial_3V\big)+\frac{2}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_1V\partial_1\Phi+\partial_2V\partial_2\Phi\\ &\quad-\partial_3V\partial_3\Phi\big)\,;\end{aligned}$$

therefore,

$$\begin{aligned}\big[R_{33}\big]_3 &= {}^{(4)}\big[R_{33}\big]_3+O\big(\epsilon^6\big)\,,\\ {}^{(4)}\big[R_{33}\big]_3 &= \frac{1}{c^4}\big(\partial_1U\partial_1V+\partial_2U\partial_2V-\partial_3U\partial_3V\big)\,;\end{aligned}$$

- 4-th term

$$\begin{aligned}\big[R_{33}\big]_4 &= \frac{2}{\big(c^2+2V\big)^2D^2}\Big[\big(2D+P\big)\partial_3V-\big(c^2+2V\big)^2PD_3\Big]^2\\ &= \frac{2}{c^4}\Big[9+O\big(\epsilon^2\big)\Big]\big(\partial_3V\big)^2+\frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_3U\big)^2+\frac{8}{c^8}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_3\Phi\big)^2\\ &\quad+\frac{8}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\partial_3U\partial_3\Phi-\frac{4}{c^4}\Big[3+O\big(\epsilon^2\big)\Big]\partial_3U\partial_3V-\frac{8}{c^6}\Big[3+O\big(\epsilon^2\big)\Big]\partial_3V\partial_3\Phi\,;\end{aligned}$$

therefore,

$$\begin{aligned}\big[R_{33}\big]_4 &= {}^{(4)}\big[R_{33}\big]_4+O\big(\epsilon^6\big)\,,\\ {}^{(4)}\big[R_{33}\big]_4 &= \frac{18}{c^4}\big(\partial_3V\big)^2+\frac{2}{c^4}\big(\partial_3U\big)^2-\frac{12}{c^4}\,\partial_3U\partial_3V\,;\end{aligned}$$

- 5-th term

$$\begin{aligned}\big[R_{33}\big]_5 = -\frac{1}{\big(c^2+2V\big)^2 D}\Big[&2\big(c^2+2V\big)^3\big(\partial_3 U\big)^2 - 12\big(c^2+2V\big)^2 P D_3\,\partial_3 V + 4\big(D+2P\big)\big(\partial_3 V\big)^2 \\ &- \big(c^2+2V\big)^3 P D_{33} + \big(c^2+2V\big)\big(2D+P\big)\partial_{33} V\Big]\,,\end{aligned}$$

where,

$$\begin{aligned}-\frac{1}{\big(c^2+2V\big)^2 D}\Big[2\big(c^2+2V\big)^3\big(\partial_3 U\big)^2\Big] &= -\frac{2}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(c^2+2V\big)^3\big(\partial_3 U\big)^2\Big] \\ &= -\frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\big(\partial_3 U\big)^2 \\ &= -\frac{2}{c^4}\big(\partial_3 U\big)^2 + O\big(\epsilon^6\big)\,;\end{aligned}$$

$$\begin{aligned}-\frac{1}{\big(c^2+2V\big)^2 D}\Big[-12\big(c^2+2V\big)^2 P D_3\,\partial_3 V\Big] &= \frac{12}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(c^2+2V\big)^2 P D_3\,\partial_3 V\Big] \\ &= \frac{12}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\partial_3 U\partial_3 V + \frac{24}{c^6}\Big[1+O\big(\epsilon^2\big)\Big]\partial_3 V\partial_3\Phi \\ &= \frac{12}{c^4}\,\partial_3 U\partial_3 V + O\big(\epsilon^6\big)\,;\end{aligned}$$

$$\begin{aligned}-\frac{1}{\big(c^2+2V\big)^2 D}\Big[4\big(D+2P\big)\big(\partial_3 V\big)^2\Big] &= -\frac{4}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(D+2P\big)\big(\partial_3 V\big)^2\Big] \\ &= -\frac{4}{c^4}\Big[3+O\big(\epsilon^2\big)\Big]\big(\partial_3 V\big)^2 \\ &= -\frac{12}{c^4}\big(\partial_3 V\big)^2 + O\big(\epsilon^6\big)\,;\end{aligned}$$

$$\begin{aligned}-\frac{1}{\big(c^2+2V\big)^2 D}\Big[-\big(c^2+2V\big)^3 P D_{33}\Big] &= \frac{1}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(c^2+2V\big)^3 P D_{33}\Big] \\ &= \frac{1}{c^2}\Big[1+O\big(\epsilon^4\big)\Big]\partial_{33} U + \frac{2}{c^4}\Big[1+O\big(\epsilon^2\big)\Big]\partial_{33}\Phi \\ &= \frac{1}{c^2}\,\partial_{33} U + \frac{2}{c^4}\,\partial_{33}\Phi + O\big(\epsilon^6\big)\,;\end{aligned}$$

$$\begin{aligned}-\frac{1}{\big(c^2+2V\big)^2 D}\Big[\big(c^2+2V\big)\big(2D+P\big)\partial_{33} V\Big] &= -\frac{1}{c^{10}}\Big[1+O\big(\epsilon^2\big)\Big]\Big[\big(c^2+2V\big)\big(2D+P\big)\partial_{33} V\Big] \\ &= -\frac{3}{c^2}\Big[1-\frac{2V}{c^2}+O\big(\epsilon^4\big)\Big]\partial_{33} V \\ &= -\frac{3}{c^2}\,\partial_{33} V + \frac{6V}{c^4}\,\partial_{33} V + O\big(\epsilon^6\big)\,;\end{aligned}$$

therefore,

$$
\begin{aligned}
\big[R_{33}\big]_5 &= {}^{(2)}\big[R_{33}\big]_5 + {}^{(4)}\big[R_{33}\big]_5 + O\big(\epsilon^6\big)\,,\\
{}^{(2)}\big[R_{33}\big]_5 &= \frac{1}{c^2}\big(\partial_{33}U - 3\,\partial_{33}V\big)\,,\\
{}^{(4)}\big[R_{33}\big]_5 &= -\frac{2}{c^4}\big(\partial_3 U\big)^2 + \frac{12}{c^4}\,\partial_3 U\partial_3 V - \frac{12}{c^4}\,\big(\partial_3 V\big)^2 + \frac{2}{c^4}\,\partial_{33}\Phi + \frac{6}{c^4}\,V\partial_{33}V\,;
\end{aligned}
$$

- 6-th term

$$
\begin{aligned}
\big[R_{33}\big]_6 &= -\frac{1}{c^{16}}\,q^2 P D_3^2\\
&= -\frac{1}{c^4}\Big[1 + O\big(\epsilon^2\big)\Big]\big(\partial_3 U\big)^2 - \frac{4}{c^8}\Big[1 + O\big(\epsilon^2\big)\Big]\big(\partial_3\Phi\big)^2 - \frac{4}{c^6}\Big[1 + O\big(\epsilon^2\big)\Big]\partial_3 U\partial_3\Phi\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
\big[R_{33}\big]_6 &= {}^{(4)}\big[R_{33}\big]_6 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{33}\big]_6 &= -\frac{1}{c^4}\big(\partial_3 U\big)^2\,;
\end{aligned}
$$

- 7-th term

$$
\begin{aligned}
\big[R_{33}\big]_7 &= -\frac{2}{c^{12}}\,h_{123}^2\,P D_3\,\partial_3 V = -\frac{2}{c^4}\,O\big(\epsilon^6\big)\,\partial_3 U\partial_3 V - \frac{4}{c^6}\,O\big(\epsilon^6\big)\,\partial_3 V\partial_3\Phi\\
&\sim O\big(\epsilon^{10}\big)\,;
\end{aligned}
$$

therefore,

$$
\big[R_{33}\big]_7 \sim O\big(\epsilon^{10}\big)\,;
$$

- 8-th term

$$
\begin{aligned}
\big[R_{33}\big]_8 &= -\frac{1}{c^8}\big(c^2 - 2V\big)^2\Big[-2\big(\partial_1 V\big)^2 - 2\big(\partial_2 V\big)^2 + 3\big(\partial_3 V\big)^2\Big]\\
&= -\frac{1}{c^4}\Big[1 + O\big(\epsilon^2\big)\Big]\Big[-2\big(\partial_1 V\big)^2 - 2\big(\partial_2 V\big)^2 + 3\big(\partial_3 V\big)^2\Big]\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
\big[R_{33}\big]_8 &= {}^{(4)}\big[R_{33}\big]_8 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{33}\big]_8 &= -\frac{1}{c^4}\Big[-2\big(\partial_1 V\big)^2 - 2\big(\partial_2 V\big)^2 + 3\big(\partial_3 V\big)^2\Big]\,.
\end{aligned}
$$

Synoptic list:

$$
\big[R_{33}\big]_1 = {}^{(4)}\big[R_{33}\big]_1 + O\big(\epsilon^6\big)\,,
$$

$$
\begin{aligned}
{}^{(4)}\big[R_{33}\big]_1 &= \frac{1}{c^4}\Big[-\big(\partial_1 V\big)^2 - \big(\partial_2 V\big)^2 + \big(\partial_3 V\big)^2\Big]\,;\\[2ex]
\big[R_{33}\big]_2 &= {}^{(2)}\big[R_{33}\big]_2 + {}^{(4)}\big[R_{33}\big]_2 + O\big(\epsilon^6\big)\,,\\
{}^{(2)}\big[R_{33}\big]_2 &= \frac{1}{c^2}\big(-\partial_{11}V - \partial_{22}V + \partial_{33}V\big)\,,\\
{}^{(4)}\big[R_{33}\big]_2 &= -\frac{2}{c^4}\,V\big(-\partial_{11}V - \partial_{22}V + \partial_{33}V\big)\,;\\[2ex]
\big[R_{33}\big]_3 &= {}^{(4)}\big[R_{33}\big]_3 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{33}\big]_3 &= \frac{1}{c^4}\big(\partial_1 U\partial_1 V + \partial_2 U\partial_2 V - \partial_3 U\partial_3 V\big)\,;\\[2ex]
\big[R_{33}\big]_4 &= {}^{(4)}\big[R_{33}\big]_4 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{33}\big]_4 &= \frac{18}{c^4}\big(\partial_3 V\big)^2 + \frac{2}{c^4}\big(\partial_3 U\big)^2 - \frac{12}{c^4}\,\partial_3 U\partial_3 V\,;\\[2ex]
\big[R_{33}\big]_5 &= {}^{(2)}\big[R_{33}\big]_5 + {}^{(4)}\big[R_{33}\big]_5 + O\big(\epsilon^6\big)\,,\\
{}^{(2)}\big[R_{33}\big]_5 &= \frac{1}{c^2}\big(\partial_{33}U - 3\,\partial_{33}V\big)\,,\\
{}^{(4)}\big[R_{33}\big]_5 &= -\frac{2}{c^4}\big(\partial_3 U\big)^2 + \frac{12}{c^4}\,\partial_3 U\partial_3 V - \frac{12}{c^4}\,\big(\partial_3 V\big)^2 + \frac{2}{c^4}\,\partial_{33}\Phi + \frac{6}{c^4}\,V\partial_{33}V\,;\\[2ex]
\big[R_{33}\big]_6 &= {}^{(4)}\big[R_{33}\big]_6 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{33}\big]_6 &= -\frac{1}{c^4}\big(\partial_3 U\big)^2\,;\\[2ex]
\big[R_{33}\big]_7 &\sim O\big(\epsilon^{10}\big)\,;\\[2ex]
\big[R_{33}\big]_8 &= {}^{(4)}\big[R_{33}\big]_8 + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{33}\big]_8 &= -\frac{1}{c^4}\Big[-2\big(\partial_1 V\big)^2 - 2\big(\partial_2 V\big)^2 + 3\big(\partial_3 V\big)^2\Big]\,.
\end{aligned}
$$

Summing up,

$$
\begin{aligned}
R_{33} &= {}^{(2)}R_{33} + {}^{(4)}R_{33} + O\big(\epsilon^6\big)\,,\\
{}^{(2)}R_{33} &= \frac{1}{c^2}\big(\partial_{33}U - \partial_{33}V - \Delta V\big)\,,\\
{}^{(4)}R_{33} &= \frac{1}{c^4}\Big\{\Big[3\big(\partial_3 V\big)^2 + \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\Big] + 2\,V\big(\partial_{33}V + \Delta V\big) - 2\,\partial_3 U\partial_3 V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,\partial_{33}\Phi\\
&\quad - \big(\partial_3 U\big)^2\Big\}\,.
\end{aligned}
$$

Appendix E

Curvature scalars of interest

I am going to compute, here, all the curvature scalars of interest, by using the metric tensor in 1PN approximation, referred to (3.30), (3.31), on page 39,

E.1 Ricci's scalar

Through the aforementioned metric tensor, the Ricci's scalar takes the following form,

$$R = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + 2\,g^{0i}R_{0i} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33}\,,$$

from which:

- $g^{00}R_{00}$ term

$$\begin{aligned} g^{00}R_{00} &= \Big[1 + \frac{2U}{c^2} + \frac{1}{c^4}\big(2U^2 + 4\Phi\big) + O\big(\epsilon^6\big)\Big]\Big\{-\frac{1}{c^2}\,\Delta U + \frac{1}{c^4}\Big[-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\big(U+V\big)\Delta U \\ &\quad - 2\,\Delta\Phi + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\Big] + O\big(\epsilon^6\big)\Big\} \\ &= -\frac{1}{c^2}\,\Delta U + \frac{1}{c^4}\Big[-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\big(U+V\big)\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\Big] - \frac{2}{c^4}\,U\Delta U \\ &\quad + O\big(\epsilon^6\big) \\ &= -\frac{1}{c^2}\,\Delta U + \frac{1}{c^4}\Big[-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,V\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\Big] + O\big(\epsilon^6\big)\,; \end{aligned}$$

  therefore

$$\begin{aligned} g^{00}R_{00} &= {}^{(2)}\big[g^{00}R_{00}\big] + {}^{(4)}\big[g^{00}R_{00}\big] + O\big(\epsilon^6\big)\,, \\ {}^{(2)}\big[g^{00}R_{00}\big] &= -\frac{1}{c^2}\,\Delta U\,, \\ {}^{(4)}\big[g^{00}R_{00}\big] &= \frac{1}{c^4}\Big[-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,V\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\Big]\,; \end{aligned}$$

- $2\,g^{0i}R_{0i}$ term

$$\begin{aligned} 2\,g^{0i}R_{0i} &= 2\,\big(g^{01}R_{01} + g^{02}R_{02} + g^{03}R_{03}\big) \\ &= 2\left\{-\left(\frac{h_{01}}{c^3}\right)\frac{1}{c^5}\Big[\boldsymbol{h}\cdot\boldsymbol{\nabla}\big(\partial_1 U\big)\Big] - \left(\frac{h_{02}}{c^3}\right)\frac{1}{c^5}\Big[\boldsymbol{h}\cdot\boldsymbol{\nabla}\big(\partial_2 U\big)\Big] - \left(\frac{h_{03}}{c^3}\right)\frac{1}{c^5}\Big[\boldsymbol{h}\cdot\boldsymbol{\nabla}\big(\partial_3 U\big)\Big]\right\} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{2}{c^8}\Big\{h_{01}\Big[\boldsymbol{h}\cdot\boldsymbol{\nabla}\big(\partial_1 U\big)\Big] + h_{02}\Big[\boldsymbol{h}\cdot\boldsymbol{\nabla}\big(\partial_2 U\big)\Big] + h_{03}\Big[\boldsymbol{h}\cdot\boldsymbol{\nabla}\big(\partial_3 U\big)\Big]\Big\}\\
&= -\frac{2}{c^8}\Big[h_{01}\big(h_{01}\partial_{11}U + h_{02}\partial_{12}U + h_{03}\partial_{13}U\big) + h_{02}\big(h_{01}\partial_{12}U + h_{02}\partial_{22}U + h_{03}\partial_{23}U\big)\\
&+ h_{03}\big(h_{01}\partial_{13}U + h_{02}\partial_{23}U + h_{03}\partial_{33}U\big)\Big]\\
&= -\frac{2}{c^8}\big(h_{01}^2\partial_{11}U + 2\,h_{01}h_{02}\partial_{12}U + 2\,h_{01}h_{03}\partial_{13}U + h_{02}^2\partial_{22}U + 2\,h_{02}h_{03}\partial_{23}U + h_{03}^2\partial_{33}U\big)\\
&= -\frac{2}{c^8}\big(h_{01}^2\partial_{11}U + h_{02}^2\partial_{22}U + h_{03}^2\partial_{33}U\big) - \frac{4}{c^8}\big(h_{01}h_{02}\partial_{12}U + h_{01}h_{03}\partial_{13}U + h_{02}h_{03}\partial_{23}U\big)\\
&= -\frac{2}{c^8}\,h_{0i}h_{0j}\partial_{ij}U \quad i,j = 1,2,3\,;
\end{aligned}
$$

therefore,

$$
2\,g^{0i}R_{0i} \sim O\big(\epsilon^8\big)\,;
$$

- $g^{11}R_{11}$ term

$$
\begin{aligned}
g^{11}R_{11} &= \left[-\left(1-\frac{2V}{c^2}\right) + O\big(\epsilon^4\big)\right]\left\{\frac{1}{c^2}\big(\partial_{11}U - \partial_{11}V - \Delta V\big) + \frac{1}{c^4}\Big[3\big(\partial_1 V\big)^2 + \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 2\,V\right.\\
&\left.\big(\partial_{11}V + \Delta V\big) - 2\,\partial_1 U\partial_1 V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,\partial_{11}\Phi - \big(\partial_1 U\big)^2\Big] + O\big(\epsilon^6\big)\right\}\\
&= -\frac{1}{c^2}\big(\partial_{11}U - \partial_{11}V - \Delta V\big) - \frac{1}{c^4}\Big[3\big(\partial_1 V\big)^2 + \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 2\,V\big(\partial_{11}V + \Delta V\big)\\
&- 2\,\partial_1 U\partial_1 V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,\partial_{11}\Phi - \big(\partial_1 U\big)^2\Big] + \frac{2}{c^4}\,V\big(\partial_{11}U - \partial_{11}V - \Delta V\big) + O\big(\epsilon^6\big)\\
&= -\frac{1}{c^2}\big(\partial_{11}U - \partial_{11}V - \Delta V\big) + \frac{1}{c^4}\Big[-3\big(\partial_1 V\big)^2 - \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V - 2\,V\big(\partial_{11}V + \Delta V\big)\\
&+ 2\,\partial_1 U\partial_1 V - \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V - 2\,\partial_{11}\Phi + \big(\partial_1 U\big)^2 + 2\,V\big(\partial_{11}U - \partial_{11}V\big) - 2\,V\Delta V\Big] + O\big(\epsilon^6\big)\\
&= -\frac{1}{c^2}\big(\partial_{11}U - \partial_{11}V - \Delta V\big) + \frac{1}{c^4}\Big[-3\big(\partial_1 V\big)^2 - \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V - 4\,V\partial_{11}V - 4\,V\Delta V\\
&+ 2\,\partial_1 U\partial_1 V - \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V - 2\,\partial_{11}\Phi + \big(\partial_1 U\big)^2 + 2\,V\partial_{11}U\Big] + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
g^{11}R_{11} &= {}^{(2)}\big[g^{11}R_{11}\big] + {}^{(4)}\big[g^{11}R_{11}\big] + O\big(\epsilon^6\big)\,,\\
{}^{(2)}\big[g^{11}R_{11}\big] &= -\frac{1}{c^2}\big(\partial_{11}U - \partial_{11}V - \Delta V\big)\,,\\
{}^{(4)}\big[g^{11}R_{11}\big] &= \frac{1}{c^4}\Big[-3\big(\partial_1 V\big)^2 - \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V - 4\,V\partial_{11}V - 4\,V\Delta V + 2\,\partial_1 U\partial_1 V - \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\\
&\quad - 2\,\partial_{11}\Phi + \big(\partial_1 U\big)^2 + 2\,V\partial_{11}U\Big]\,;
\end{aligned}
$$

- $g^{22}R_{22}$ term

$$
\begin{aligned}
g^{22}R_{22} = & \left[-\left(1-\frac{2V}{c^2}\right)+O\big(\epsilon^4\big)\right]\left\{\frac{1}{c^2}\big(\partial_{22}U-\partial_{22}V-\Delta V\big)+\frac{1}{c^4}\Big[3\big(\partial_2 V\big)^2+\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V+2\,V\right.\\
& \left.\big(\partial_{22}V+\Delta V\big)-2\,\partial_2 U\partial_2 V+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V+2\,\partial_{22}\Phi-\big(\partial_2 U\big)^2\Big]+O\big(\epsilon^6\big)\right\}\\
= & -\frac{1}{c^2}\big(\partial_{22}U-\partial_{22}V-\Delta V\big)+\frac{1}{c^4}\Big[-3\big(\partial_2 V\big)^2-\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V-4\,V\partial_{22}V-4\,V\Delta V\\
& +2\,\partial_2 U\partial_2 V-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V-2\,\partial_{22}\Phi+\big(\partial_2 U\big)^2+2\,V\partial_{22}U\Big]+O\big(\epsilon^6\big)\,,
\end{aligned}
$$

as above;

therefore,

$$
\begin{aligned}
g^{22}R_{22} = & \ {}^{(2)}\big[g^{22}R_{22}\big]+{}^{(4)}\big[g^{22}R_{22}\big]+O\big(\epsilon^6\big)\,,\\
{}^{(2)}\big[g^{22}R_{22}\big] = & -\frac{1}{c^2}\big(\partial_{22}U-\partial_{22}V-\Delta V\big)\,,\\
{}^{(4)}\big[g^{22}R_{22}\big] = & \frac{1}{c^4}\Big[-3\big(\partial_2 V\big)^2-\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V-4\,V\partial_{22}V-4\,V\Delta V+2\,\partial_2 U\partial_2 V-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\\
& -2\,\partial_{22}\Phi+\big(\partial_2 U\big)^2+2\,V\partial_{22}U\Big]\,;
\end{aligned}
$$

- $g^{33}R_{33}$ term

$$
\begin{aligned}
g^{33}R_{33} = & \left[-\left(1-\frac{2V}{c^2}\right)+O\big(\epsilon^4\big)\right]\left\{\frac{1}{c^2}\big(\partial_{33}U-\partial_{33}V-\Delta V\big)+\frac{1}{c^4}\Big[3\big(\partial_3 V\big)^2+\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V+2\,V\right.\\
& \left.\big(\partial_{33}V+\Delta V\big)-2\,\partial_3 U\partial_3 V+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V+2\,\partial_{33}\Phi-\big(\partial_3 U\big)^2\Big]+O\big(\epsilon^6\big)\right\}\\
= & -\frac{1}{c^2}\big(\partial_{33}U-\partial_{33}V-\Delta V\big)+\frac{1}{c^4}\Big[-3\big(\partial_3 V\big)^2-\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V-4\,V\partial_{33}V-4\,V\Delta V\\
& +2\,\partial_3 U\partial_3 V-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V-2\,\partial_{33}\Phi+\big(\partial_3 U\big)^2+2\,V\partial_{33}U\Big]+O\big(\epsilon^6\big)\,,
\end{aligned}
$$

as above;

therefore,

$$
\begin{aligned}
g^{33}R_{33} = & \ {}^{(2)}\big[g^{33}R_{33}\big]+{}^{(4)}\big[g^{33}R_{33}\big]+O\big(\epsilon^6\big)\,,\\
{}^{(2)}\big[g^{33}R_{33}\big] = & -\frac{1}{c^2}\big(\partial_{33}U-\partial_{33}V-\Delta V\big)\,,\\
{}^{(4)}\big[g^{33}R_{33}\big] = & \frac{1}{c^4}\Big[-3\big(\partial_3 V\big)^2-\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V-4\,V\partial_{33}V-4\,V\Delta V+2\,\partial_3 U\partial_3 V-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\\
& -2\,\partial_{33}\Phi+\big(\partial_3 U\big)^2+2\,V\partial_{33}U\Big]\,.
\end{aligned}
$$

Thus, with all contributions at hand, it is possible to compute the Ricci's scalar:

$$
R=-\frac{1}{c^2}\,\Delta U+\frac{1}{c^4}\big(-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V+2\,V\Delta U-2\,\Delta\Phi+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)-\frac{1}{c^2}\big(\partial_{11}U-\partial_{11}V-\Delta V\big)
$$

$$
\begin{aligned}
&+\frac{1}{c^4}\Big[-3\big(\partial_1 V\big)^2-\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V-4\,V\partial_{11}V-4\,V\Delta V+2\,\partial_1 U\partial_1 V-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V-2\,\partial_{11}\Phi\\
&+\big(\partial_1 U\big)^2+2\,V\partial_{11}U\Big]-\frac{1}{c^2}\big(\partial_{22}U-\partial_{22}V-\Delta V\big)+\frac{1}{c^4}\Big[-3\big(\partial_2 V\big)^2-\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V-4\,V\partial_{22}V\\
&-4\,V\Delta V+2\,\partial_2 U\partial_2 V-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V-2\,\partial_{22}\Phi+\big(\partial_2 U\big)^2+2\,V\partial_{22}U\Big]-\frac{1}{c^2}\big(\partial_{33}U-\partial_{33}V-\Delta V\big)\\
&+\frac{1}{c^4}\Big[-3\big(\partial_3 V\big)^2-\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V-4\,V\partial_{33}V-4\,V\Delta V+2\,\partial_3 U\partial_3 V-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V-2\,\partial_{33}\Phi\\
&+\big(\partial_3 U\big)^2+2\,V\partial_{33}U\Big]+O\big(\epsilon^6\big)\,,
\end{aligned}
$$

from which,

$$
\begin{aligned}
R&={}^{(2)}R+{}^{(4)}R+O\big(\epsilon^6\big)\,,\\
{}^{(2)}R&=-\frac{2}{c^2}\big(\Delta U-2\,\Delta V\big)\,,\\
{}^{(4)}R&=\frac{2}{c^4}\big(-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V+2\,V\Delta U-2\,\Delta\Phi+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U-3\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V-8\,V\Delta V\big)\,.
\end{aligned}
$$

E.2 R^2 scalar

From the here above expression concerning R, immediately follows R^2,

$$
\begin{aligned}
R^2&=\Big\{-\frac{2}{c^2}\big(\Delta U-2\,\Delta V\big)+\frac{2}{c^4}\big(-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V+2\,V\Delta U-2\,\Delta\Phi+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U-3\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\\
&\quad-8\,V\Delta V\big)+O\big(\epsilon^6\big)\Big\}^2\\
&=\Big[-\frac{2}{c^2}\big(\Delta U-2\,\Delta V\big)\Big]^2+O\big(\epsilon^6\big)\,,
\end{aligned}
$$

that is,

$$
\begin{aligned}
R^2&={}^{(4)}R^2+O\big(\epsilon^6\big)\,,\\
{}^{(4)}R^2&=\frac{1}{c^4}\Big[4\big(\Delta U\big)^2+16\big(\Delta V\big)^2-16\,\Delta U\Delta V\Big]\,.
\end{aligned}
$$

E.3 $R_{\mu\nu}R^{\mu\nu}$ scalar

$$
\begin{aligned}
R_{\mu\nu}R^{\mu\nu}&=R_{00}R^{00}+2\big(R_{01}R^{01}+R_{02}R^{02}+R_{03}R^{03}\big)+R_{11}R^{11}+2\big(R_{12}R^{12}+R_{13}R^{13}\big)\\
&\quad+R_{22}R^{22}+2R_{23}R^{23}+R_{33}R^{33}\,,
\end{aligned}
$$

from which:

- $R_{00}R^{00}$ term
  where,

$$
R^{00}=\big(g^{00}\big)^2R_{00}
$$

$$
\begin{aligned}
&= \Big[1 + \frac{2U}{c^2} + \frac{1}{c^4}\big(2U^2 + 4\Phi\big) + O\big(\epsilon^6\big)\Big]^2 \Big\{ - \frac{1}{c^2}\,\Delta U + \frac{1}{c^4}\Big[- \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V + 2\big(U + V\big)\Delta U \\
&- 2\,\Delta\Phi + \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} U\Big] + O\big(\epsilon^6\big)\Big\} \\
&= \Big[1 + \frac{4U}{c^2} + O\big(\epsilon^4\big)\Big]\Big\{ - \frac{1}{c^2}\,\Delta U + \frac{1}{c^4}\Big[- \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V + 2\big(U + V\big)\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} U\Big] \\
&+ O\big(\epsilon^6\big)\Big\} \\
&= - \frac{1}{c^2}\,\Delta U + \frac{1}{c^4}\Big[- \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V + 2\big(U + V\big)\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} U\Big] - \frac{4}{c^4}\,U\Delta U + O\big(\epsilon^6\big)\,, \\
&= - \frac{1}{c^2}\,\Delta U + \frac{1}{c^4}\Big[- \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V + 2\big(V - U\big)\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} U\Big] + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

$$
\begin{aligned}
R_{00}R^{00} &= \Big\{ - \frac{1}{c^2}\,\Delta U + \frac{1}{c^4}\Big[- \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V + 2\big(U + V\big)\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} U\Big] + O\big(\epsilon^6\big)\Big\} \\
&\quad \Big\{ - \frac{1}{c^2}\,\Delta U + \frac{1}{c^4}\Big[- \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V + 2\big(V - U\big)\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} U\Big] + O\big(\epsilon^6\big)\Big\} \\
&= \frac{1}{c^4}\big(\Delta U\big)^2 + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
R_{00}R^{00} &= {}^{(4)}\big[R_{00}R^{00}\big] + O\big(\epsilon^6\big)\,, \\
{}^{(4)}\big[R_{00}R^{00}\big] &= \frac{1}{c^4}\big(\Delta U\big)^2\,;
\end{aligned}
$$

- $R_{01}R^{01}$ term

  where,

$$
\begin{aligned}
R^{01} &= g^{00}g^{11}R_{01} \\
&= \Big[1 + \frac{2U}{c^2} + \frac{1}{c^4}\big(2U^2 + 4\Phi\big) + O\big(\epsilon^6\big)\Big]\Big[- \Big(1 - \frac{2V}{c^2}\Big) + O\big(\epsilon^4\big)\Big]\Big[\frac{1}{c^5}\,\boldsymbol{h} \cdot \boldsymbol{\nabla}\big(\partial_1 U\big) + O\big(\epsilon^7\big)\Big] \\
&= \Big[1 + \frac{2U}{c^2} + \frac{1}{c^4}\big(2U^2 + 4\Phi\big) + O\big(\epsilon^6\big)\Big]\Big[- 1 + \frac{2V}{c^2} + O\big(\epsilon^4\big)\Big]\Big[\frac{1}{c^5}\,\boldsymbol{h} \cdot \boldsymbol{\nabla}\big(\partial_1 U\big) + O\big(\epsilon^7\big)\Big] \\
&= \Big[- 1 + \frac{2}{c^2}\big(V - U\big) + O\big(\epsilon^4\big)\Big]\Big[\frac{1}{c^5}\,\boldsymbol{h} \cdot \boldsymbol{\nabla}\big(\partial_1 U\big) + O\big(\epsilon^7\big)\Big] \\
&= - \frac{1}{c^5}\,\boldsymbol{h} \cdot \boldsymbol{\nabla}\big(\partial_1 U\big) + O\big(\epsilon^7\big)\,;
\end{aligned}
$$

$$
R_{01}R^{01} = \Big[\frac{1}{c^5}\,\boldsymbol{h} \cdot \boldsymbol{\nabla}\big(\partial_1 U\big) + O\big(\epsilon^7\big)\Big]\Big[- \frac{1}{c^5}\,\boldsymbol{h} \cdot \boldsymbol{\nabla}\big(\partial_1 U\big) + O\big(\epsilon^7\big)\Big] \sim O\big(\epsilon^{10}\big)\,;
$$

therefore,

$$
R_{01}R^{01} \sim O\big(\epsilon^{10}\big)\,;
$$

- $R_{02}R^{02}$ term

  where, as above,

$$\begin{aligned} R^{02} &= g^{00}g^{22}R_{02} \\ &= \left[1+\frac{2U}{c^2}+\frac{1}{c^4}\big(2U^2+4\Phi\big)+O\big(\epsilon^6\big)\right]\left[-1+\frac{2V}{c^2}+O\big(\epsilon^4\big)\right]\left[\frac{1}{c^5}\,\boldsymbol{h}\cdot\boldsymbol{\nabla}\big(\partial_2 U\big)+O\big(\epsilon^7\big)\right] \\ &= \left[-1+\frac{2}{c^2}\big(V-U\big)+O\big(\epsilon^4\big)\right]\left[\frac{1}{c^5}\,\boldsymbol{h}\cdot\boldsymbol{\nabla}\big(\partial_2 U\big)+O\big(\epsilon^7\big)\right] \\ &= -\frac{1}{c^5}\,\boldsymbol{h}\cdot\boldsymbol{\nabla}\big(\partial_2 U\big)+O\big(\epsilon^7\big)\,; \end{aligned}$$

$$R_{02}R^{02} = \left[\frac{1}{c^5}\,\boldsymbol{h}\cdot\boldsymbol{\nabla}\big(\partial_2 U\big)+O\big(\epsilon^7\big)\right]\left[-\frac{1}{c^5}\,\boldsymbol{h}\cdot\boldsymbol{\nabla}\big(\partial_2 U\big)+O\big(\epsilon^7\big)\right] \sim O\big(\epsilon^{10}\big)\,;$$

  therefore,

$$R_{02}R^{02} \sim O\big(\epsilon^{10}\big)\,;$$

- $R_{03}R^{03}$ term

  where, as above,

$$\begin{aligned} R^{03} &= g^{00}g^{33}R_{03} \\ &= \left[1+\frac{2U}{c^2}+\frac{1}{c^4}\big(2U^2+4\Phi\big)+O\big(\epsilon^6\big)\right]\left[-1+\frac{2V}{c^2}+O\big(\epsilon^4\big)\right]\left[\frac{1}{c^5}\,\boldsymbol{h}\cdot\boldsymbol{\nabla}\big(\partial_3 U\big)+O\big(\epsilon^7\big)\right] \\ &= \left[-1+\frac{2}{c^2}\big(V-U\big)+O\big(\epsilon^4\big)\right]\left[\frac{1}{c^5}\,\boldsymbol{h}\cdot\boldsymbol{\nabla}\big(\partial_3 U\big)+O\big(\epsilon^7\big)\right] \\ &= -\frac{1}{c^5}\,\boldsymbol{h}\cdot\boldsymbol{\nabla}\big(\partial_3 U\big)+O\big(\epsilon^7\big)\,; \end{aligned}$$

$$R_{03}R^{03} = \left[\frac{1}{c^5}\,\boldsymbol{h}\cdot\boldsymbol{\nabla}\big(\partial_3 U\big)+O\big(\epsilon^7\big)\right]\left[-\frac{1}{c^5}\,\boldsymbol{h}\cdot\boldsymbol{\nabla}\big(\partial_3 U\big)+O\big(\epsilon^7\big)\right] \sim O\big(\epsilon^{10}\big)\,;$$

  therefore,

$$R_{03}R^{03} \sim O\big(\epsilon^{10}\big)\,;$$

- $R_{11}R^{11}$ term

  where,

$$\begin{aligned} R^{11} &= \big(g^{11}\big)^2 R_{11} \\ &= \left[-\left(1-\frac{2V}{c^2}\right)+O\big(\epsilon^4\big)\right]^2 R_{11} \end{aligned}$$

$$
\begin{aligned}
&= \left[1 - \frac{4V}{c^2} + O\big(\epsilon^4\big)\right]\left\{\frac{1}{c^2}\big(\partial_{11}U - \partial_{11}V - \Delta V\big) + \frac{1}{c^4}\Big[3\big(\partial_1 V\big)^2 + \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 2\,V\big(\partial_{11}V\right.\right.\\
&\left.\left.+ \Delta V\big) - 2\,\partial_1 U\partial_1 V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,\partial_{11}\Phi - \big(\partial_1 U\big)^2\Big] + O\big(\epsilon^6\big)\right\}\\
&= \frac{1}{c^2}\big(\partial_{11}U - \partial_{11}V - \Delta V\big) + \frac{1}{c^4}\Big[3\big(\partial_1 V\big)^2 + \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 2\,V\big(\partial_{11}V + \Delta V\big) - 2\,\partial_1 U\partial_1 V\\
&+ \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,\partial_{11}\Phi - \big(\partial_1 U\big)^2\Big] - \frac{4V}{c^4}\big(\partial_{11}U - \partial_{11}V - \Delta V\big) + O\big(\epsilon^6\big)\\
&= \frac{1}{c^2}\big(\partial_{11}U - \partial_{11}V - \Delta V\big) + \frac{1}{c^4}\Big[3\big(\partial_1 V\big)^2 + \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 6\,V\big(\partial_{11}V + \Delta V\big) - 2\,\partial_1 U\partial_1 V\\
&+ \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,\partial_{11}\Phi - \big(\partial_1 U\big)^2 - 4\,V\partial_{11}U\Big] + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

$$
\begin{aligned}
R_{11}R^{11} &= \left\{\frac{1}{c^2}\big(\partial_{11}U - \partial_{11}V - \Delta V\big) + \frac{1}{c^4}\Big[3\big(\partial_1 V\big)^2 + \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 2\,V\big(\partial_{11}V + \Delta V\big)\right.\\
&\quad\left. - 2\,\partial_1 U\partial_1 V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,\partial_{11}\Phi - \big(\partial_1 U\big)^2\Big] + O\big(\epsilon^6\big)\right\}\left\{\frac{1}{c^2}\big(\partial_{11}U - \partial_{11}V - \Delta V\big)\right.\\
&\quad\left. + \frac{1}{c^4}\Big[3\big(\partial_1 V\big)^2 + \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 6\,V\big(\partial_{11}V + \Delta V\big) - 2\,\partial_1 U\partial_1 V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,\partial_{11}\Phi\right.\\
&\quad\left. - \big(\partial_1 U\big)^2 - 4\,V\partial_{11}U\Big] + O\big(\epsilon^6\big)\right\}\\
&= \frac{1}{c^4}\big(\partial_{11}U - \partial_{11}V - \Delta V\big)^2 + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
R_{11}R^{11} &= {}^{(4)}\big[R_{11}R^{11}\big] + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{11}R^{11}\big] &= \frac{1}{c^4}\Big[\big(\partial_{11}U\big)^2 + \big(\partial_{11}V\big)^2 + \big(\Delta V\big)^2 - 2\,\partial_{11}U\partial_{11}V - 2\,\partial_{11}U\Delta V + 2\,\partial_{11}V\Delta V\Big]\,;
\end{aligned}
$$

- $R_{12}R^{12}$ term

  where,

$$
\begin{aligned}
R^{12} &= g^{11}g^{22}R_{12}\\
&= \left[-\left(1 - \frac{2V}{c^2}\right) + O\big(\epsilon^4\big)\right]^2 R_{12}\\
&= \left[1 - \frac{4V}{c^2} + O\big(\epsilon^4\big)\right]\left\{\frac{1}{c^2}\big(\partial_{12}U - \partial_{12}V\big) + \frac{1}{c^4}\Big[2\big(V\partial_{12}V + \partial_{12}\Phi\big) + 3\,\partial_1 V\partial_2 V - \partial_1 U\partial_2 U\right.\\
&\left. - \big(\partial_1 U\partial_2 V + \partial_1 V\partial_2 U\big)\Big] + O\big(\epsilon^6\big)\right\}\\
&= \frac{1}{c^2}\big(\partial_{12}U - \partial_{12}V\big) + \frac{1}{c^4}\Big[2\big(V\partial_{12}V + \partial_{12}\Phi\big) + 3\,\partial_1 V\partial_2 V - \partial_1 U\partial_2 U - \big(\partial_1 U\partial_2 V\\
&+ \partial_1 V\partial_2 U\big)\Big] - \frac{4V}{c^4}\big(\partial_{12}U - \partial_{12}V\big) + O\big(\epsilon^6\big)\\
&= \frac{1}{c^2}\big(\partial_{12}U - \partial_{12}V\big) + \frac{1}{c^4}\Big[2\,V\big(3\,\partial_{12}V - 2\,\partial_{12}U\big) + 2\,\partial_{12}\Phi + 3\,\partial_1 V\partial_2 V - \partial_1 U\partial_2 U
\end{aligned}
$$

$$- \big(\partial_1 U \partial_2 V + \partial_1 V \partial_2 U\big)\Big] + O\big(\epsilon^6\big)\,;$$

$$\begin{aligned} R_{12}R^{12} &= \left\{ \frac{1}{c^2}\big(\partial_{12}U - \partial_{12}V\big) + \frac{1}{c^4}\Big[2\big(V\partial_{12}V + \partial_{12}\Phi\big) + 3\,\partial_1 V\partial_2 V - \partial_1 U\partial_2 U - \big(\partial_1 U\partial_2 V \right. \\ &\quad \left. + \partial_1 V\partial_2 U\big)\Big] + O\big(\epsilon^6\big)\right\}\left\{ \frac{1}{c^2}\big(\partial_{12}U - \partial_{12}V\big) + \frac{1}{c^4}\Big[2\,V\big(3\,\partial_{12}V - 2\,\partial_{12}U\big) + 2\,\partial_{12}\Phi \right. \\ &\quad \left. + 3\,\partial_1 V\partial_2 V - \partial_1 U\partial_2 U - \big(\partial_1 U\partial_2 V + \partial_1 V\partial_2 U\big)\Big] + O\big(\epsilon^6\big)\right\} \\ &= \frac{1}{c^4}\big(\partial_{12}U - \partial_{12}V\big)^2 + O\big(\epsilon^6\big)\,; \end{aligned}$$

therefore,

$$\begin{aligned} R_{12}R^{12} &= {}^{(4)}\big[R_{12}R^{12}\big] + O\big(\epsilon^6\big)\,, \\ {}^{(4)}\big[R_{12}R^{12}\big] &= \frac{1}{c^4}\Big[\big(\partial_{12}U\big)^2 + \big(\partial_{12}V\big)^2 - 2\,\partial_{12}U\partial_{12}V\Big]\,; \end{aligned}$$

- $R_{13}R^{13}$ term
  where,

$$\begin{aligned} R^{13} &= g^{11}g^{33}R_{13} \\ &= \left[-\left(1 - \frac{2V}{c^2}\right) + O\big(\epsilon^4\big)\right]^2 R_{13} \\ &= \left[1 - \frac{4V}{c^2} + O\big(\epsilon^4\big)\right]\left\{ \frac{1}{c^2}\big(\partial_{13}U - \partial_{13}V\big) + \frac{1}{c^4}\Big[2\big(V\partial_{13}V + \partial_{13}\Phi\big) + 3\,\partial_1 V\partial_3 V - \partial_1 U\partial_3 U \right. \\ &\quad \left. - \big(\partial_1 U\partial_3 V + \partial_1 V\partial_3 U\big)\Big] + O\big(\epsilon^6\big)\right\} \\ &= \frac{1}{c^2}\big(\partial_{13}U - \partial_{13}V\big) + \frac{1}{c^4}\Big[2\big(V\partial_{13}V + \partial_{13}\Phi\big) + 3\,\partial_1 V\partial_3 V - \partial_1 U\partial_3 U - \big(\partial_1 U\partial_3 V \\ &\quad + \partial_1 V\partial_3 U\big)\Big] - \frac{4V}{c^4}\big(\partial_{13}U - \partial_{13}V\big) + O\big(\epsilon^6\big) \\ &= \frac{1}{c^2}\big(\partial_{13}U - \partial_{13}V\big) + \frac{1}{c^4}\Big[2\,V\big(3\,\partial_{13}V - 2\,\partial_{13}U\big) + 2\,\partial_{13}\Phi + 3\,\partial_1 V\partial_3 V - \partial_1 U\partial_3 U \\ &\quad - \big(\partial_1 U\partial_3 V + \partial_1 V\partial_3 U\big)\Big] + O\big(\epsilon^6\big)\,; \end{aligned}$$

$$\begin{aligned} R_{13}R^{13} &= \left\{ \frac{1}{c^2}\big(\partial_{13}U - \partial_{13}V\big) + \frac{1}{c^4}\Big[2\big(V\partial_{13}V + \partial_{13}\Phi\big) + 3\,\partial_1 V\partial_3 V - \partial_1 U\partial_3 U - \big(\partial_1 U\partial_3 V \right. \\ &\quad \left. + \partial_1 V\partial_3 U\big)\Big] + O\big(\epsilon^6\big)\right\}\left\{ \frac{1}{c^2}\big(\partial_{13}U - \partial_{13}V\big) + \frac{1}{c^4}\Big[2\,V\big(3\,\partial_{13}V - 2\,\partial_{13}U\big) + 2\,\partial_{13}\Phi \right. \\ &\quad \left. + 3\,\partial_1 V\partial_3 V - \partial_1 U\partial_3 U - \big(\partial_1 U\partial_3 V + \partial_1 V\partial_3 U\big)\Big] + O\big(\epsilon^6\big)\right\} \\ &= \frac{1}{c^4}\big(\partial_{13}U - \partial_{13}V\big)^2 + O\big(\epsilon^6\big)\,; \end{aligned}$$

therefore,

$$R_{13}R^{13} = {}^{(4)}\big[R_{13}R^{13}\big] + O\big(\epsilon^6\big)\,,$$
$${}^{(4)}\big[R_{13}R^{13}\big] = \frac{1}{c^4}\Big[\big(\partial_{13}U\big)^2 + \big(\partial_{13}V\big)^2 - 2\,\partial_{13}U\partial_{13}V\Big]\,;$$

- $R_{22}R^{22}$ term

  where,

$$\begin{aligned}
R^{22} &= \big(g^{22}\big)^2 R_{22} \\
&= \left[-\left(1 - \frac{2V}{c^2}\right) + O\big(\epsilon^4\big)\right]^2 R_{22} \\
&= \left[1 - \frac{4V}{c^2} + O\big(\epsilon^4\big)\right]\left\{\frac{1}{c^2}\big(\partial_{22}U - \partial_{22}V - \Delta V\big) + \frac{1}{c^4}\Big[3\big(\partial_2 V\big)^2 + \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 2\,V\big(\partial_{22}V\right. \right. \\
&\left. \left. + \Delta V\big) - 2\,\partial_2 U\partial_2 V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,\partial_{22}\Phi - \big(\partial_2 U\big)^2\Big] + O\big(\epsilon^6\big)\right\} \\
&= \frac{1}{c^2}\big(\partial_{22}U - \partial_{22}V - \Delta V\big) + \frac{1}{c^4}\Big[3\big(\partial_2 V\big)^2 + \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 2\,V\big(\partial_{22}V + \Delta V\big) - 2\,\partial_2 U\partial_2 V \\
&+ \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,\partial_{22}\Phi - \big(\partial_2 U\big)^2\Big] - \frac{4V}{c^4}\big(\partial_{22}U - \partial_{22}V - \Delta V\big) + O\big(\epsilon^6\big) \\
&= \frac{1}{c^2}\big(\partial_{22}U - \partial_{22}V - \Delta V\big) + \frac{1}{c^4}\Big[3\big(\partial_2 V\big)^2 + \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 6\,V\big(\partial_{22}V + \Delta V\big) - 2\,\partial_2 U\partial_2 V \\
&+ \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,\partial_{22}\Phi - \big(\partial_2 U\big)^2 - 4\,V\partial_{22}U\Big] + O\big(\epsilon^6\big)\,;
\end{aligned}$$

$$\begin{aligned}
R_{22}R^{22} &= \left\{\frac{1}{c^2}\big(\partial_{22}U - \partial_{22}V - \Delta V\big) + \frac{1}{c^4}\Big[3\big(\partial_2 V\big)^2 + \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 2\,V\big(\partial_{22}V + \Delta V\big)\right. \\
&\left. - 2\,\partial_2 U\partial_2 V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,\partial_{22}\Phi - \big(\partial_2 U\big)^2\Big] + O\big(\epsilon^6\big)\right\}\left\{\frac{1}{c^2}\big(\partial_{22}U - \partial_{22}V - \Delta V\big)\right. \\
&\left. + \frac{1}{c^4}\Big[3\big(\partial_2 V\big)^2 + \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 6\,V\big(\partial_{22}V + \Delta V\big) - 2\,\partial_2 U\partial_2 V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,\partial_{22}\Phi\right. \\
&\left. - \big(\partial_2 U\big)^2 - 4\,V\partial_{22}U\Big] + O\big(\epsilon^6\big)\right\} \\
&= \frac{1}{c^4}\big(\partial_{22}U - \partial_{22}V - \Delta V\big)^2 + O\big(\epsilon^6\big)\,;
\end{aligned}$$

therefore,

$$R_{22}R^{22} = {}^{(4)}\big[R_{22}R^{22}\big] + O\big(\epsilon^6\big)\,,$$
$${}^{(4)}\big[R_{22}R^{22}\big] = \frac{1}{c^4}\Big[\big(\partial_{22}U\big)^2 + \big(\partial_{22}V\big)^2 + \big(\Delta V\big)^2 - 2\,\partial_{22}U\partial_{22}V - 2\,\partial_{22}U\Delta V + 2\,\partial_{22}V\Delta V\Big]\,;$$

- $R_{23}R^{23}$ term

  where,

$$
\begin{aligned}
R^{23} &= g^{22}g^{33}R_{23} \\
&= \left[-\left(1-\frac{2V}{c^2}\right)+O\big(\epsilon^4\big)\right]^2 R_{13} \\
&= \left[1-\frac{4V}{c^2}+O\big(\epsilon^4\big)\right]\left\{\frac{1}{c^2}\big(\partial_{23}U-\partial_{23}V\big)+\frac{1}{c^4}\Big[2\big(V\partial_{23}V+\partial_{23}\Phi\big)+3\,\partial_2V\partial_3V-\,\partial_2U\partial_3U\right. \\
&\left.-\,\big(\partial_2U\partial_3V+\partial_2V\partial_3U\big)\Big]+O\big(\epsilon^6\big)\right\} \\
&= \frac{1}{c^2}\big(\partial_{23}U-\partial_{23}V\big)+\frac{1}{c^4}\Big[2\big(V\partial_{23}V+\partial_{23}\Phi\big)+3\,\partial_2V\partial_3V-\,\partial_2U\partial_3U-\big(\partial_2U\partial_3V \\
&+\,\partial_2V\partial_3U\big)\Big]-\frac{4V}{c^4}\big(\partial_{23}U-\partial_{23}V\big)+O\big(\epsilon^6\big) \\
&= \frac{1}{c^2}\big(\partial_{23}U-\partial_{23}V\big)+\frac{1}{c^4}\Big[2\,V\big(3\,\partial_{23}V-2\,\partial_{23}U\big)+2\,\partial_{23}\Phi+3\,\partial_2V\partial_3V-\,\partial_2U\partial_3U \\
&-\,\big(\partial_2U\partial_3V+\partial_2V\partial_3U\big)\Big]+O\big(\epsilon^6\big)\,;
\end{aligned}
$$

$$
\begin{aligned}
R_{23}R^{23} &= \left\{\frac{1}{c^2}\big(\partial_{23}U-\partial_{23}V\big)+\frac{1}{c^4}\Big[2\big(V\partial_{23}V+\partial_{23}\Phi\big)+3\,\partial_2V\partial_3V-\,\partial_2U\partial_3U-\big(\partial_2U\partial_3V\right. \\
&\left.+\,\partial_2V\partial_3U\big)\Big]+O\big(\epsilon^6\big)\right\}\left\{\frac{1}{c^2}\big(\partial_{23}U-\partial_{23}V\big)+\frac{1}{c^4}\Big[2\,V\big(3\,\partial_{23}V-2\,\partial_{23}U\big)+2\,\partial_{23}\Phi\right. \\
&\left.+\,3\,\partial_2V\partial_3V-\,\partial_2U\partial_3U-\big(\partial_2U\partial_3V+\partial_2V\partial_3U\big)\Big]+O\big(\epsilon^6\big)\right\} \\
&= \frac{1}{c^4}\big(\partial_{23}U-\partial_{23}V\big)^2+O\big(\epsilon^6\big)\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
R_{23}R^{23} &= {}^{(4)}\big[R_{23}R^{23}\big]+O\big(\epsilon^6\big)\,, \\
{}^{(4)}\big[R_{23}R^{23}\big] &= \frac{1}{c^4}\Big[\big(\partial_{23}U\big)^2+\big(\partial_{23}V\big)^2-2\,\partial_{23}U\partial_{23}V\Big]\,;
\end{aligned}
$$

- $R_{33}R^{33}$ term
  where,

$$
\begin{aligned}
R^{33} &= \big(g^{33}\big)^2R_{33} \\
&= \left[-\left(1-\frac{2V}{c^2}\right)+O\big(\epsilon^4\big)\right]^2 R_{33} \\
&= \left[1-\frac{4V}{c^2}+O\big(\epsilon^4\big)\right]\left\{\frac{1}{c^2}\big(\partial_{33}U-\partial_{33}V-\Delta V\big)+\frac{1}{c^4}\Big[3\big(\partial_3V\big)^2+\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V+2\,V\big(\partial_{33}V\right. \\
&\left.+\,\Delta V\big)-2\,\partial_3U\partial_3V+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V+2\,\partial_{33}\Phi-\big(\partial_3U\big)^2\Big]+O\big(\epsilon^6\big)\right\} \\
&= \frac{1}{c^2}\big(\partial_{33}U-\partial_{33}V-\Delta V\big)+\frac{1}{c^4}\Big[3\big(\partial_3V\big)^2+\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V+2\,V\big(\partial_{33}V+\Delta V\big)-2\,\partial_3U\partial_3V
\end{aligned}
$$

$$
\begin{aligned}
&+ \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V + 2\,\partial_{33}\Phi - \big(\partial_3 U\big)^2\Big] - \frac{4V}{c^4}\big(\partial_{33}U - \partial_{33}V - \Delta V\big) + O\big(\epsilon^6\big)\\
&= \frac{1}{c^2}\big(\partial_{33}U - \partial_{33}V - \Delta V\big) + \frac{1}{c^4}\Big[3\big(\partial_3 V\big)^2 + \boldsymbol{\nabla} V \cdot \boldsymbol{\nabla} V + 6\,V\big(\partial_{33}V + \Delta V\big) - 2\,\partial_3 U \partial_3 V\\
&+ \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V + 2\,\partial_{33}\Phi - \big(\partial_3 U\big)^2 - 4\,V\partial_{33}U\Big] + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

$$
\begin{aligned}
R_{33}R^{33} = &\left\{\frac{1}{c^2}\big(\partial_{33}U - \partial_{33}V - \Delta V\big) + \frac{1}{c^4}\Big[3\big(\partial_3 V\big)^2 + \boldsymbol{\nabla} V \cdot \boldsymbol{\nabla} V + 2\,V\big(\partial_{33}V + \Delta V\big)\right.\\
&\left. - 2\,\partial_3 U \partial_3 V + \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V + 2\,\partial_{33}\Phi - \big(\partial_3 U\big)^2\Big] + O\big(\epsilon^6\big)\right\}\left\{\frac{1}{c^2}\big(\partial_{33}U - \partial_{33}V - \Delta V\big)\right.\\
&\left. + \frac{1}{c^4}\Big[3\big(\partial_3 V\big)^2 + \boldsymbol{\nabla} V \cdot \boldsymbol{\nabla} V + 6\,V\big(\partial_{33}V + \Delta V\big) - 2\,\partial_3 U \partial_3 V + \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V + 2\,\partial_{33}\Phi\right.\\
&\left. - \big(\partial_3 U\big)^2 - 4\,V\partial_{33}U\Big] + O\big(\epsilon^6\big)\right\}\\
= &\frac{1}{c^4}\big(\partial_{33}U - \partial_{33}V - \Delta V\big)^2 + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
R_{33}R^{33} &= {}^{(4)}\big[R_{33}R^{33}\big] + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{33}R^{33}\big] &= \frac{1}{c^4}\Big[\big(\partial_{33}U\big)^2 + \big(\partial_{33}V\big)^2 + \big(\Delta V\big)^2 - 2\,\partial_{33}U\partial_{33}V - 2\,\partial_{33}U\Delta V + 2\,\partial_{33}V\Delta V\Big]\,.
\end{aligned}
$$

Synoptic list:

$$
\begin{aligned}
R_{00}R^{00} &= {}^{(4)}\big[R_{00}R^{00}\big] + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{00}R^{00}\big] &= \frac{1}{c^4}\big(\Delta U\big)^2\,;
\end{aligned}
$$

$$
R_{01}R^{01} \sim O\big(\epsilon^{10}\big)\,;
$$

$$
R_{02}R^{02} \sim O\big(\epsilon^{10}\big)\,;
$$

$$
R_{03}R^{03} \sim O\big(\epsilon^{10}\big)\,;
$$

$$
\begin{aligned}
R_{11}R^{11} &= {}^{(4)}\big[R_{11}R^{11}\big] + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{11}R^{11}\big] &= \frac{1}{c^4}\Big[\big(\partial_{11}U\big)^2 + \big(\partial_{11}V\big)^2 + \big(\Delta V\big)^2 - 2\,\partial_{11}U\partial_{11}V - 2\,\partial_{11}U\Delta V + 2\,\partial_{11}V\Delta V\Big]\,;
\end{aligned}
$$

$$
\begin{aligned}
R_{12}R^{12} &= {}^{(4)}\big[R_{12}R^{12}\big] + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{12}R^{12}\big] &= \frac{1}{c^4}\Big[\big(\partial_{12}U\big)^2 + \big(\partial_{12}V\big)^2 - 2\,\partial_{12}U\partial_{12}V\Big]\,;
\end{aligned}
$$

$$
R_{13}R^{13} = {}^{(4)}\big[R_{13}R^{13}\big] + O\big(\epsilon^6\big)\,,
$$

$$
{}^{(4)}\big[R_{13}R^{13}\big] = \frac{1}{c^4}\Big[\big(\partial_{13}U\big)^2 + \big(\partial_{13}V\big)^2 - 2\,\partial_{13}U\partial_{13}V\Big]\,;
$$

$$
\begin{aligned}
R_{22}R^{22} &= {}^{(4)}\big[R_{22}R^{22}\big] + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{22}R^{22}\big] &= \frac{1}{c^4}\Big[\big(\partial_{22}U\big)^2 + \big(\partial_{22}V\big)^2 + \big(\Delta V\big)^2 - 2\,\partial_{22}U\partial_{22}V - 2\,\partial_{22}U\Delta V + 2\,\partial_{22}V\Delta V\Big]\,;
\end{aligned}
$$

$$
\begin{aligned}
R_{23}R^{23} &= {}^{(4)}\big[R_{23}R^{23}\big] + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{23}R^{23}\big] &= \frac{1}{c^4}\Big[\big(\partial_{23}U\big)^2 + \big(\partial_{23}V\big)^2 - 2\,\partial_{23}U\partial_{23}V\Big]\,;
\end{aligned}
$$

$$
\begin{aligned}
R_{33}R^{33} &= {}^{(4)}\big[R_{33}R^{33}\big] + O\big(\epsilon^6\big)\,,\\
{}^{(4)}\big[R_{33}R^{33}\big] &= \frac{1}{c^4}\Big[\big(\partial_{33}U\big)^2 + \big(\partial_{33}V\big)^2 + \big(\Delta V\big)^2 - 2\,\partial_{33}U\partial_{33}V - 2\,\partial_{33}U\Delta V + 2\,\partial_{33}V\Delta V\Big]\,.
\end{aligned}
$$

Summing the previous contributions, it follows a preliminar expression of the $R_{\mu\nu}R^{\mu\nu}$ scalar,

$$
\begin{aligned}
&R_{\mu\nu}R^{\mu\nu} = R_{00}R^{00} + 2\big(R_{01}R^{01} + R_{02}R^{02} + R_{03}R^{03}\big) + R_{11}R^{11} + 2\big(R_{12}R^{12} + R_{13}R^{13}\big)\\
&+ R_{22}R^{22} + 2R_{23}R^{23} + R_{33}R^{33}\\
&= \frac{1}{c^4}\big(\Delta U\big)^2 + \frac{1}{c^4}\Big[\big(\partial_{11}U\big)^2 + \big(\partial_{11}V\big)^2 + \big(\Delta V\big)^2 - 2\,\partial_{11}U\partial_{11}V - 2\,\partial_{11}U\Delta V + 2\,\partial_{11}V\Delta V\Big]\\
&+ \frac{2}{c^4}\Big[\big(\partial_{12}U\big)^2 + \big(\partial_{12}V\big)^2 - 2\,\partial_{12}U\partial_{12}V\Big] + \frac{2}{c^4}\Big[\big(\partial_{13}U\big)^2 + \big(\partial_{13}V\big)^2 - 2\,\partial_{13}U\partial_{13}V\Big]\\
&+ \frac{1}{c^4}\Big[\big(\partial_{22}U\big)^2 + \big(\partial_{22}V\big)^2 + \big(\Delta V\big)^2 - 2\,\partial_{22}U\partial_{22}V - 2\,\partial_{22}U\Delta V + 2\,\partial_{22}V\Delta V\Big] + \frac{2}{c^4}\Big[\big(\partial_{23}U\big)^2\\
&+ \big(\partial_{23}V\big)^2 - 2\,\partial_{23}U\partial_{23}V\Big] + \frac{1}{c^4}\Big[\big(\partial_{33}U\big)^2 + \big(\partial_{33}V\big)^2 + \big(\Delta V\big)^2 - 2\,\partial_{33}U\partial_{33}V - 2\,\partial_{33}U\Delta V\\
&+ 2\,\partial_{33}V\Delta V\Big] + O\big(\epsilon^6\big)\\
&= \frac{1}{c^4}\Big[\big(\Delta U\big)^2 + \big(\partial_{11}U\big)^2 + \big(\partial_{11}V\big)^2 + \big(\Delta V\big)^2 - 2\,\partial_{11}U\partial_{11}V - 2\,\partial_{11}U\Delta V + 2\,\partial_{11}V\Delta V\\
&+ 2\big(\partial_{12}U\big)^2 + 2\big(\partial_{12}V\big)^2 - 4\,\partial_{12}U\partial_{12}V + 2\big(\partial_{13}U\big)^2 + 2\big(\partial_{13}V\big)^2 - 4\,\partial_{13}U\partial_{13}V + \big(\partial_{22}U\big)^2\\
&+ \big(\partial_{22}V\big)^2 + \big(\Delta V\big)^2 - 2\,\partial_{22}U\partial_{22}V - 2\,\partial_{22}U\Delta V + 2\,\partial_{22}V\Delta V + 2\big(\partial_{23}U\big)^2 + 2\big(\partial_{23}V\big)^2\\
&- 4\,\partial_{23}U\partial_{23}V + \big(\partial_{33}U\big)^2 + \big(\partial_{33}V\big)^2 + \big(\Delta V\big)^2 - 2\,\partial_{33}U\partial_{33}V - 2\,\partial_{33}U\Delta V + 2\,\partial_{33}V\Delta V\Big]\\
&+ O\big(\epsilon^6\big)\,,
\end{aligned}
$$

from which:

- 1-st contribution

$$
\begin{aligned}
&\big(\partial_{11}U\big)^2 + 2\big(\partial_{12}U\big)^2 + 2\big(\partial_{13}U\big)^2 + \big(\partial_{22}U\big)^2 + 2\big(\partial_{23}U\big)^2 + \big(\partial_{33}U\big)^2\\
&= \Big[\big(\partial_{11}U\big)^2 + \big(\partial_{12}U\big)^2 + \big(\partial_{13}U\big)^2\Big] + \Big[\big(\partial_{12}U\big)^2 + \big(\partial_{22}U\big)^2 + \big(\partial_{23}U\big)^2\Big] + \Big[\big(\partial_{13}U\big)^2 + \big(\partial_{23}U\big)^2\\
&+ \big(\partial_{33}U\big)^2\Big]\\
&= \boldsymbol{\nabla}\big(\partial_1 U\big)\cdot\boldsymbol{\nabla}\big(\partial_1 U\big) + \boldsymbol{\nabla}\big(\partial_2 U\big)\cdot\boldsymbol{\nabla}\big(\partial_2 U\big) + \boldsymbol{\nabla}\big(\partial_3 U\big)\cdot\boldsymbol{\nabla}\big(\partial_3 U\big)\,;
\end{aligned}
$$

- 2-nd contribution

$$
\begin{aligned}
&\big(\partial_{11}V\big)^2 + 2\big(\partial_{12}V\big)^2 + 2\big(\partial_{13}V\big)^2 + \big(\partial_{22}V\big)^2 + 2\big(\partial_{23}V\big)^2 + \big(\partial_{33}V\big)^2 \\
&= \Big[\big(\partial_{11}V\big)^2 + \big(\partial_{12}V\big)^2 + \big(\partial_{13}V\big)^2\Big] + \Big[\big(\partial_{12}V\big)^2 + \big(\partial_{22}V\big)^2 + \big(\partial_{23}V\big)^2\Big] + \Big[\big(\partial_{13}V\big)^2 + \big(\partial_{23}V\big)^2 \\
&+ \big(\partial_{33}V\big)^2\Big] \\
&= \boldsymbol{\nabla}\big(\partial_1 V\big)\cdot\boldsymbol{\nabla}\big(\partial_1 V\big) + \boldsymbol{\nabla}\big(\partial_2 V\big)\cdot\boldsymbol{\nabla}\big(\partial_2 V\big) + \boldsymbol{\nabla}\big(\partial_3 V\big)\cdot\boldsymbol{\nabla}\big(\partial_3 V\big)\,;
\end{aligned}
$$

- 3-rd contribution

$$
\begin{aligned}
&-2\,\partial_{11}U\partial_{11}V - 4\,\partial_{12}U\partial_{12}V - 4\,\partial_{13}U\partial_{13}V - 2\,\partial_{22}U\partial_{22}V - 4\,\partial_{23}U\partial_{23}V - 2\,\partial_{33}U\partial_{33}V \\
&= -2\Big[\partial_{11}U\partial_{11}V + \partial_{12}U\partial_{12}V + \partial_{13}U\partial_{13}V\Big] - 2\Big[\partial_{12}U\partial_{12}V + \partial_{22}U\partial_{22}V + \partial_{23}U\partial_{23}V\Big] \\
&- 2\Big[\partial_{13}U\partial_{13}V + \partial_{23}U\partial_{23}V + \partial_{33}U\partial_{33}V\Big] \\
&= -2\Big[\boldsymbol{\nabla}\big(\partial_1 U\big)\cdot\boldsymbol{\nabla}\big(\partial_1 V\big) + \boldsymbol{\nabla}\big(\partial_2 U\big)\cdot\boldsymbol{\nabla}\big(\partial_2 V\big) + \boldsymbol{\nabla}\big(\partial_3 U\big)\cdot\boldsymbol{\nabla}\big(\partial_3 V\big)\Big]\,;
\end{aligned}
$$

- 4-th contribution

$$
\begin{aligned}
&-2\,\partial_{11}U\Delta V - 2\,\partial_{22}U\Delta V - 2\,\partial_{33}U\Delta V \\
&= -2\big(\partial_{11}U + \partial_{22}U + \partial_{33}U\big)\Delta V \\
&= -2\,\Delta U\Delta V\,;
\end{aligned}
$$

- 5-th contribution

$$
\begin{aligned}
&2\,\partial_{11}V\Delta V + 2\,\partial_{22}V\Delta V + 2\,\partial_{33}V\Delta V \\
&= 2\big(\partial_{11}V + \partial_{22}V + \partial_{33}V\big)\Delta V \\
&= 2\,\big(\Delta V\big)^2\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
R_{\mu\nu}R^{\mu\nu} &= {}^{(4)}R_{\mu\nu}R^{\mu\nu} + O\big(\epsilon^6\big)\,, \\
{}^{(4)}R_{\mu\nu}R^{\mu\nu} &= \frac{1}{c^4}\Big\{\big(\Delta U\big)^2 + 5\big(\Delta V\big)^2 - 2\,\Delta U\Delta V + \Big[\boldsymbol{\nabla}\big(\partial_1 U\big)\cdot\boldsymbol{\nabla}\big(\partial_1 U\big) + \boldsymbol{\nabla}\big(\partial_2 U\big)\cdot\boldsymbol{\nabla}\big(\partial_2 U\big) \\
&\quad + \boldsymbol{\nabla}\big(\partial_3 U\big)\cdot\boldsymbol{\nabla}\big(\partial_3 U\big)\Big] + \Big[\boldsymbol{\nabla}\big(\partial_1 V\big)\cdot\boldsymbol{\nabla}\big(\partial_1 V\big) + \boldsymbol{\nabla}\big(\partial_2 V\big)\cdot\boldsymbol{\nabla}\big(\partial_2 V\big) + \boldsymbol{\nabla}\big(\partial_3 V\big) \\
&\quad \cdot\boldsymbol{\nabla}\big(\partial_3 V\big)\Big] - 2\Big[\boldsymbol{\nabla}\big(\partial_1 U\big)\cdot\boldsymbol{\nabla}\big(\partial_1 V\big) + \boldsymbol{\nabla}\big(\partial_2 U\big)\cdot\boldsymbol{\nabla}\big(\partial_2 V\big) + \boldsymbol{\nabla}\big(\partial_3 U\big)\cdot\boldsymbol{\nabla}\big(\partial_3 V\big)\Big]\Big\}\,.
\end{aligned}
$$

E.4 $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ scalar

From Gauss-Bonnet identity,

$$
R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = 4\,R_{\mu\nu}R^{\mu\nu} - R^2\,,
$$

it follows,

$$
\begin{aligned}
R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = \frac{4}{c^4}\Big\{ &\big(\Delta U\big)^2 + 5\big(\Delta V\big)^2 - 2\,\Delta U\Delta V + \Big[\boldsymbol{\nabla}\big(\partial_1 U\big)\cdot\boldsymbol{\nabla}\big(\partial_1 U\big) + \boldsymbol{\nabla}\big(\partial_2 U\big)\cdot\boldsymbol{\nabla}\big(\partial_2 U\big) \\
&+ \boldsymbol{\nabla}\big(\partial_3 U\big)\cdot\boldsymbol{\nabla}\big(\partial_3 U\big)\Big] + \Big[\boldsymbol{\nabla}\big(\partial_1 V\big)\cdot\boldsymbol{\nabla}\big(\partial_1 V\big) + \boldsymbol{\nabla}\big(\partial_2 V\big)\cdot\boldsymbol{\nabla}\big(\partial_2 V\big) + \boldsymbol{\nabla}\big(\partial_3 V\big) \\
&\cdot\boldsymbol{\nabla}\big(\partial_3 V\big)\Big] - 2\Big[\boldsymbol{\nabla}\big(\partial_1 U\big)\cdot\boldsymbol{\nabla}\big(\partial_1 V\big) + \boldsymbol{\nabla}\big(\partial_2 U\big)\cdot\boldsymbol{\nabla}\big(\partial_2 V\big) + \boldsymbol{\nabla}\big(\partial_3 U\big)\cdot\boldsymbol{\nabla}\big(\partial_3 V\big)\Big]\Big\} \\
&- \frac{4}{c^4}\Big[\big(\Delta U\big)^2 + 4\big(\Delta V\big)^2 - 4\,\Delta U\Delta V\Big] + O\big(\epsilon^6\big)\,,
\end{aligned}
$$

that is,

$$
\begin{aligned}
R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} &= {}^{(4)}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} + O\big(\epsilon^6\big)\,, \\
{}^{(4)}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} &= \frac{4}{c^4}\Big\{\big(\Delta V\big)^2 + 2\,\Delta U\Delta V + \Big[\boldsymbol{\nabla}\big(\partial_1 U\big)\cdot\boldsymbol{\nabla}\big(\partial_1 U\big) + \boldsymbol{\nabla}\big(\partial_2 U\big)\cdot\boldsymbol{\nabla}\big(\partial_2 U\big) \\
&\quad + \boldsymbol{\nabla}\big(\partial_3 U\big)\cdot\boldsymbol{\nabla}\big(\partial_3 U\big)\Big] + \Big[\boldsymbol{\nabla}\big(\partial_1 V\big)\cdot\boldsymbol{\nabla}\big(\partial_1 V\big) + \boldsymbol{\nabla}\big(\partial_2 V\big)\cdot\boldsymbol{\nabla}\big(\partial_2 V\big) \\
&\quad + \boldsymbol{\nabla}\big(\partial_3 V\big)\cdot\boldsymbol{\nabla}\big(\partial_3 V\big)\Big] - 2\Big[\boldsymbol{\nabla}\big(\partial_1 U\big)\cdot\boldsymbol{\nabla}\big(\partial_1 V\big) + \boldsymbol{\nabla}\big(\partial_2 U\big)\cdot\boldsymbol{\nabla}\big(\partial_2 V\big) \\
&\quad + \boldsymbol{\nabla}\big(\partial_3 U\big)\cdot\boldsymbol{\nabla}\big(\partial_3 V\big)\Big]\Big\}\,.
\end{aligned}
$$

Appendix F

Left-hand sides of field equations

F.1 ${}^0H_{\mu\nu}$ - Left-hand sides of the Einstein's field equations

From the well-known field equations,

$$ {}^0H_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = \chi\, T_{\mu\nu}\,, $$

written in presence of matter/energy and by using the expressions of the Ricci's tensor and scalar, obtained in Appendix E, in 1PN approximation of the metric tensor, it is now possible to make explicit the various components of the left-hand side of them.

The expressions concerning the metric tensor components are given by (3.30), (3.31) on page 39, while those one's relating to the Ricci's tensor components are given by (4.1), (4.2), (4.3) on page 44.

F.1.1 Time - Time component

$$
\begin{aligned}
{}^0H_{00} &= R_{00} - \frac{1}{2}\, g_{00}\, R \\
&= -\frac{1}{c^2}\Delta U + \frac{1}{c^4}\Big[-\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V + 2\big(U+V\big)\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\Big] + O\big(\epsilon^6\big) - \frac{1}{2}\Big[1 - \frac{2U}{c^2} \\
&+ \frac{1}{c^4}\big(2U^2 - 4\Phi\big) + O\big(\epsilon^6\big)\Big]\Big[-\frac{2}{c^2}\big(\Delta U - 2\,\Delta V\big) + \frac{2}{c^4}\big(-\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V + 2\,V\Delta U - 2\,\Delta\Phi \\
&+ \boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U - 3\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V - 8\,V\Delta V\big) + O\big(\epsilon^6\big)\Big] \\
&= \cdots - \frac{1}{2}\Big[-\frac{2}{c^2}\big(\Delta U - 2\,\Delta V\big) + \frac{2}{c^4}\big(-\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V + 2\,V\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U \\
&- 3\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V - 8\,V\Delta V\big) + \frac{4U}{c^4}\big(\Delta U - 2\,\Delta V\big) + O\big(\epsilon^6\big)\Big] \\
&= -\frac{1}{c^2}\Delta U + \frac{1}{c^4}\Big[-\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V + 2\big(U+V\big)\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\Big] + \frac{1}{c^2}\big(\Delta U - 2\,\Delta V\big) \\
&- \frac{1}{c^4}\big(-\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V + 2\,V\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U - 3\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V - 8\,V\Delta V\big) - \frac{2U}{c^4}\big(\Delta U \\
&- 2\,\Delta V\big) + O\big(\epsilon^6\big) \\
&= -\frac{2}{c^2}\Delta V + \frac{1}{c^4}\big(3\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V + 8\,V\Delta V + 4\,U\Delta V\big) + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

therefore,

$$ {}^0H_{00} = {}^{(2)\,0}H_{00} + {}^{(4)\,0}H_{00} + O\big(\epsilon^6\big)\,, $$

$$
\begin{aligned}
{}^{(2)\,0}H_{00} &= -\frac{2}{c^2}\Delta V\,,\\
{}^{(4)\,0}H_{00} &= \frac{1}{c^4}\Big[3\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 4\big(2\,V+U\big)\Delta V\Big]\,.
\end{aligned}
$$

F.1.2 Time - Space components

$$
\begin{aligned}
{}^{0}H_{0i} &= R_{0i} - \frac{1}{2}\,g_{0i}\,R \qquad i=1,2,3,\\
&= \frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(\partial_i U\big) + O\big(\epsilon^7\big) - \frac{1}{2}\Big[\frac{h_{0i}}{c^3} + O\big(\epsilon^5\big)\Big]\Big[-\frac{2}{c^2}\big(\Delta U - 2\,\Delta V\big) + \frac{2}{c^4}\big(-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\\
&+ 2\,V\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U - 3\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V - 8\,V\Delta V\big) + O\big(\epsilon^6\big)\Big]\\
&= \frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(\partial_i U\big) - \frac{1}{2}\Big(\frac{h_{0i}}{c^3}\Big)\Big(-\frac{2}{c^2}\Big)\big(\Delta U - 2\,\Delta V\big) + O\big(\epsilon^7\big)\\
&= \frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(\partial_i U\big) + \frac{h_{0i}}{c^5}\big(\Delta U - 2\,\Delta V\big) + O\big(\epsilon^7\big)\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
{}^{0}H_{0i} &= {}^{(5)\,0}H_{0i} + O\big(\epsilon^7\big)\,,\\
{}^{(5)\,0}H_{0i} &= \frac{1}{c^5}\Big[\mathbf{h}\cdot\boldsymbol{\nabla}\big(\partial_i U\big) + h_{0i}\big(\Delta U - 2\,\Delta V\big)\Big]\,.
\end{aligned}
$$

F.1.3 Space - Space components

$$
{}^{0}H_{ij} = R_{ij} - \frac{1}{2}\,g_{ij}\,R \qquad i,j=1,2,3,
$$

from which:

- ${}^{0}H_{11}$ component

$$
\begin{aligned}
{}^{0}H_{11} &= R_{11} - \frac{1}{2}\,g_{11}\,R\\
&= \frac{1}{c^2}\big(\partial_{11}U - \partial_{11}V - \Delta V\big) + \frac{1}{c^4}\Big[3\big(\partial_1 V\big)^2 + \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 2\,V\partial_{11}V + 2\,V\Delta V - 2\,\partial_1 U\partial_1 V\\
&+ \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,\partial_{11}\Phi - \big(\partial_1 U\big)^2\Big] + O\big(\epsilon^6\big) - \frac{1}{2}\Big[-1 - \frac{2V}{c^2} + O\big(\epsilon^4\big)\Big]\Big[-\frac{2}{c^2}\big(\Delta U - 2\,\Delta V\big)\\
&+ \frac{2}{c^4}\big(-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,V\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U - 3\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V - 8\,V\Delta V\big) + O\big(\epsilon^6\big)\Big]\\
&= \cdots - \frac{1}{2}\Big[\frac{2}{c^2}\big(\Delta U - 2\,\Delta V\big) - \frac{2}{c^4}\big(-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,V\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\\
&- 3\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V - 8\,V\Delta V\big) + \frac{4V}{c^4}\big(\Delta U - 2\,\Delta V\big)\Big] + O\big(\epsilon^6\big)\\
&= \frac{1}{c^2}\big(\partial_{11}U - \partial_{11}V - \Delta V\big) + \frac{1}{c^4}\Big[3\big(\partial_1 V\big)^2 + \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 2\,V\partial_{11}V + 2\,V\Delta V - 2\,\partial_1 U\partial_1 V\\
&+ \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,\partial_{11}\Phi - \big(\partial_1 U\big)^2\Big] - \frac{1}{c^2}\big(\Delta U - 2\,\Delta V\big) + \frac{1}{c^4}\big(-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\,V\Delta U - 2\,\Delta\Phi\\
&+ \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U - 3\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V - 8\,V\Delta V - 2V\Delta U + 4V\Delta V\big) + O\big(\epsilon^6\big)
\end{aligned}
$$

$$= \frac{1}{c^2}\big(\partial_{11}U - \partial_{11}V - \Delta U + \Delta V\big) + \frac{1}{c^4}\Big[3\big(\partial_1 V\big)^2 - \big(\partial_1 U\big)^2 - 2\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V + 2\,V\partial_{11}V$$
$$- 2\,V\Delta V - 2\,\partial_1 U\partial_1 V + 2\,\partial_{11}\Phi - 2\,\Delta\Phi + \boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\Big] + O\big(\epsilon^6\big)\,;$$

therefore,

$${}^0H_{11} = \frac{1}{c^2}\Big[\big(\Delta V - \Delta U\big) - \big(V - U\big)_{,11}\Big] + \frac{1}{c^4}\Big\{2\Big[\big(\partial_1 V\big)^2 - \boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V\Big] + \partial_1 V\big(\partial_1 V - 2\,\partial_1 U\big)$$
$$- \Big[\big(\partial_1 U\big)^2 - \boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\Big] + 2\,V\big(\partial_{11}V - \Delta V\big) + 2\big(\partial_{11}\Phi - \Delta\Phi\big)\Big\} + O\big(\epsilon^6\big)\,;$$

- ${}^0H_{12}$ component

$${}^0H_{12} = R_{12} - \frac{1}{2}\,g_{12}\,R$$
$$= \frac{1}{c^2}\big(\partial_{12}U - \partial_{12}V\big) + \frac{1}{c^4}\Big[2\big(V\partial_{12}V + \partial_{12}\Phi\big) + 3\,\partial_1 V\partial_2 V - \partial_1 U\partial_2 U - \big(\partial_1 U\partial_2 V$$
$$+ \partial_1 V\partial_2 U\big)\Big] + O\big(\epsilon^6\big) - \frac{1}{2}\,O\big(\epsilon^4\big)\Big[\,{}^{(2)}R + {}^{(4)}R + O\big(\epsilon^6\big)\Big]\,;$$

therefore,

$${}^0H_{12} = \frac{1}{c^2}\Big[-\big(V - U\big)_{,12}\Big] + \frac{1}{c^4}\Big[2\,\partial_1 V\partial_2 V + \big(\partial_1 V\partial_2 V - \partial_1 V\partial_2 U - \partial_1 U\partial_2 V\big) - \partial_1 U\partial_2 U$$
$$+ 2\,V\partial_{12}V + 2\,\partial_{12}\Phi\Big] + O\big(\epsilon^6\big)\,;$$

- ${}^0H_{13}$ component

$${}^0H_{13} = R_{13} - \frac{1}{2}\,g_{13}\,R$$
$$= \frac{1}{c^2}\big(\partial_{13}U - \partial_{13}V\big) + \frac{1}{c^4}\Big[2\big(V\partial_{13}V + \partial_{13}\Phi\big) + 3\,\partial_1 V\partial_3 V - \partial_1 U\partial_3 U - \big(\partial_1 U\partial_3 V$$
$$+ \partial_1 V\partial_3 U\big)\Big] + O\big(\epsilon^6\big) - \frac{1}{2}\,O\big(\epsilon^4\big)\Big[\,{}^{(2)}R + {}^{(4)}R + O\big(\epsilon^6\big)\Big]\,;$$

therefore,

$${}^0H_{13} = \frac{1}{c^2}\Big[-\big(V - U\big)_{,13}\Big] + \frac{1}{c^4}\Big[2\,\partial_1 V\partial_3 V + \big(\partial_1 V\partial_3 V - \partial_1 V\partial_3 U - \partial_1 U\partial_3 V\big) - \partial_1 U\partial_3 U$$
$$+ 2\,V\partial_{13}V + 2\,\partial_{13}\Phi\Big] + O\big(\epsilon^6\big)\,;$$

- ${}^0H_{22}$ component

$${}^0H_{22} = R_{22} - \frac{1}{2}\,g_{22}\,R$$
$$= \frac{1}{c^2}\big(\partial_{22}U - \partial_{22}V - \Delta V\big) + \frac{1}{c^4}\Big[3\big(\partial_2 V\big)^2 + \boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V + 2V\partial_{22}V + 2V\Delta V - 2\,\partial_2 U\partial_2 V$$

$$
\begin{aligned}
&+ \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V + 2\,\partial_{22}\Phi - \big(\partial_2 U\big)^2\Big] + O\big(\epsilon^6\big) - \frac{1}{2}\Big[-1 - \frac{2V}{c^2} + O\big(\epsilon^4\big)\Big]\Big[-\frac{2}{c^2}\big(\Delta U - 2\,\Delta V\big)\\
&+ \frac{2}{c^4}\big(-\boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V + 2\,V\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} U - 3\,\boldsymbol{\nabla} V \cdot \boldsymbol{\nabla} V - 8\,V\Delta V\big) + O\big(\epsilon^6\big)\Big]\\
&= \cdots - \frac{1}{2}\Big[\frac{2}{c^2}\big(\Delta U - 2\,\Delta V\big) - \frac{2}{c^4}\big(-\boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V + 2\,V\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} U\\
&- 3\,\boldsymbol{\nabla} V \cdot \boldsymbol{\nabla} V - 8\,V\Delta V\big) + \frac{4V}{c^4}\big(\Delta U - 2\,\Delta V\big)\Big] + O\big(\epsilon^6\big)\\
&= \frac{1}{c^2}\big(\partial_{22}U - \partial_{22}V - \Delta V\big) + \frac{1}{c^4}\Big[3\big(\partial_2 V\big)^2 + \boldsymbol{\nabla} V \cdot \boldsymbol{\nabla} V + 2V\partial_{22}V + 2V\Delta V - 2\,\partial_2 U\partial_2 V\\
&+ \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V + 2\,\partial_{22}\Phi - \big(\partial_2 U\big)^2\Big] - \frac{1}{c^2}\big(\Delta U - 2\,\Delta V\big) + \frac{1}{c^4}\big(-\boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V + 2\,V\Delta U - 2\,\Delta\Phi\\
&+ \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} U - 3\,\boldsymbol{\nabla} V \cdot \boldsymbol{\nabla} V - 8\,V\Delta V - 2V\Delta U + 4V\Delta V\big) + O\big(\epsilon^6\big)\\
&= \frac{1}{c^2}\big(\partial_{22}U - \partial_{22}V - \Delta U + \Delta V\big) + \frac{1}{c^4}\Big[3\big(\partial_2 V\big)^2 - \big(\partial_2 U\big)^2 - 2\,\boldsymbol{\nabla} V \cdot \boldsymbol{\nabla} V + 2\,V\partial_{22}V\\
&- 2\,V\Delta V - 2\,\partial_2 U\partial_2 V + 2\,\partial_{22}\Phi - 2\,\Delta\Phi + \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} U\Big] + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
{}^0H_{22} = \frac{1}{c^2}\Big[\big(\Delta V - \Delta U\big) - \big(V - U\big)_{,22}\Big] + \frac{1}{c^4}\Big\{2\Big[\big(\partial_2 V\big)^2 - \boldsymbol{\nabla} V \cdot \boldsymbol{\nabla} V\Big] + \partial_2 V\big(\partial_2 V - 2\,\partial_2 U\big)\\
- \Big[\big(\partial_2 U\big)^2 - \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} U\Big] + 2\,V\big(\partial_{22}V - \Delta V\big) + 2\big(\partial_{22}\Phi - \Delta\Phi\big)\Big\} + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

- ${}^0H_{23}$ component

$$
\begin{aligned}
{}^0H_{23} &= R_{23} - \frac{1}{2}\,g_{23}\,R\\
&= \frac{1}{c^2}\big(\partial_{23}U - \partial_{23}V\big) + \frac{1}{c^4}\Big[2\big(V\partial_{23}V + \partial_{23}\Phi\big) + 3\,\partial_2 V\partial_3 V - \partial_2 U\partial_3 U - \big(\partial_2 U\partial_3 V\\
&+ \partial_2 V\partial_3 U\big)\Big] + O\big(\epsilon^6\big) - \frac{1}{2}\,O\big(\epsilon^4\big)\Big[{}^{(2)}R + {}^{(4)}R + O\big(\epsilon^6\big)\Big]\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
{}^0H_{23} = \frac{1}{c^2}\Big[-\big(V - U\big)_{,23}\Big] + \frac{1}{c^4}\Big[2\,\partial_2 V\partial_3 V + \big(\partial_2 V\partial_3 V - \partial_2 V\partial_3 U - \partial_2 U\partial_3 V\big) - \partial_2 U\partial_3 U\\
+ 2\,V\partial_{23}V + 2\,\partial_{23}\Phi\Big] + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

- ${}^0H_{33}$ component

$$
\begin{aligned}
{}^0H_{33} &= R_{33} - \frac{1}{2}\,g_{33}\,R\\
&= \frac{1}{c^2}\big(\partial_{33}U - \partial_{33}V - \Delta V\big) + \frac{1}{c^4}\Big[3\big(\partial_3 V\big)^2 + \boldsymbol{\nabla} V \cdot \boldsymbol{\nabla} V + 2V\partial_{33}V + 2V\Delta V - 2\,\partial_3 U\partial_3 V\\
&+ \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V + 2\,\partial_{33}\Phi - \big(\partial_3 U\big)^2\Big] + O\big(\epsilon^6\big) - \frac{1}{2}\Big[-1 - \frac{2V}{c^2} + O\big(\epsilon^4\big)\Big]\Big[-\frac{2}{c^2}\big(\Delta U - 2\,\Delta V\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{c^4}\big(-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V+2\,V\Delta U-2\,\Delta\Phi+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U-3\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V-8\,V\Delta V\big)+O\big(\epsilon^6\big)\Big]\\
&=\cdots-\frac{1}{2}\Big[\frac{2}{c^2}\big(\Delta U-2\,\Delta V\big)-\frac{2}{c^4}\big(-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V+2\,V\Delta U-2\,\Delta\Phi+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\\
&-3\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V-8\,V\Delta V\big)+\frac{4V}{c^4}\big(\Delta U-2\,\Delta V\big)\Big]+O\big(\epsilon^6\big)\\
&=\frac{1}{c^2}\big(\partial_{33}U-\partial_{33}V-\Delta V\big)+\frac{1}{c^4}\Big[3\big(\partial_3V\big)^2+\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V+2V\partial_{33}V+2V\Delta V-\,2\,\partial_3U\partial_3V\\
&+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V+2\,\partial_{33}\Phi-\big(\partial_3U\big)^2\Big]-\frac{1}{c^2}\big(\Delta U-2\,\Delta V\big)+\frac{1}{c^4}\big(-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V+2\,V\Delta U-2\,\Delta\Phi\\
&+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U-3\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V-8\,V\Delta V-2V\Delta U+4V\Delta V\big)+O\big(\epsilon^6\big)\\
&=\frac{1}{c^2}\big(\partial_{33}U-\partial_{33}V-\Delta U+\Delta V\big)+\frac{1}{c^4}\Big[3\big(\partial_3V\big)^2-\big(\partial_3U\big)^2-2\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V+2\,V\partial_{33}V\\
&-2\,V\Delta V-2\,\partial_3U\partial_3V+2\,\partial_{33}\Phi-2\,\Delta\Phi+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\Big]+O\big(\epsilon^6\big)\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
{}^0H_{33}=&\frac{1}{c^2}\Big[\big(\Delta V-\Delta U\big)-\big(V-U\big)_{,33}\Big]+\frac{1}{c^4}\Big\{2\Big[\big(\partial_3V\big)^2-\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\Big]+\partial_3V\big(\partial_3V-2\,\partial_3U\big)\\
&-\Big[\big(\partial_3U\big)^2-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\Big]+2\,V\big(\partial_{33}V-\Delta V\big)+2\big(\partial_{33}\Phi-\Delta\Phi\big)\Big\}+O\big(\epsilon^6\big)\,.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
{}^0H_{ij}=&\ {}^{(2)\,0}H_{ij}+{}^{(4)\,0}H_{ij}+O\big(\epsilon^6\big)\,,\\
{}^{(2)\,0}H_{ij}=&\frac{1}{c^2}\Big[\big(\Delta V-\Delta U\big)\delta_{ij}-\big(V-U\big)_{,ij}\Big]\,,\\
{}^{(4)\,0}H_{ij}=&\frac{1}{c^4}\Big\{2\Big[\partial_iV\partial_jV-\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big)\delta_{ij}\Big]+\big(\partial_iV\partial_jV-\partial_iV\partial_jU-\partial_iU\partial_jV\big)-\Big[\partial_iU\partial_jU\\
&-\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)\delta_{ij}\Big]+2\,V\Big[\partial_{ij}V-\big(\Delta V\big)\delta_{ij}\Big]+2\Big[\partial_{ij}\Phi-\big(\Delta\Phi\big)\delta_{ij}\Big\}\,.
\end{aligned}
$$

Summing up,

$$
\begin{aligned}
{}^0H_{00}=&-\frac{2}{c^2}\Delta V+\frac{1}{c^4}\Big[3\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V+4\big(2\,V+U\big)\Delta V\Big]\,,\\
{}^0H_{0i}=&\frac{1}{c^5}\Big[\mathbf{h}\cdot\boldsymbol{\nabla}\big(\partial_iU\big)+h_{0i}\big(\Delta U-2\,\Delta V\big)\Big]\,,\\
{}^0H_{ij}=&\frac{1}{c^2}\Big[\big(\Delta V-\Delta U\big)\delta_{ij}-\big(V-U\big)_{,ij}\Big]+\frac{1}{c^4}\Big\{2\Big[\partial_iV\partial_jV-\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big)\delta_{ij}\Big]+\big(\partial_iV\partial_jV\\
&-\partial_iV\partial_jU-\partial_iU\partial_jV\big)-\Big[\partial_iU\partial_jU-\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)\delta_{ij}\Big]+2\,V\Big[\partial_{ij}V-\big(\Delta V\big)\delta_{ij}\Big]\\
&+2\Big[\partial_{ij}\Phi-\big(\Delta\Phi\big)\delta_{ij}\Big\}\,.
\end{aligned}
$$

F.2 ${}^1H_{\mu\nu}$ - Left-hand sides of the field equations related to R^2 term

Here i am going to develop the components of the left-hand side of the field equations related to R^2 term of the Lagrangian, referred to in (3.11) on page 28,

$$ {}^1H_{\mu\nu} = 2RR_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}R^2 - 2R_{;\mu\nu} + 2\, g_{\mu\nu}\Box R\,, $$

in 1PN approximation of the metric tensor. In this limit of weak fields, namely of approximately flat space-time, the covariant derivative can be substituted with the ordinary derivative and the D'Alembertian operator with the Laplacian operator, according to the following relations,

$$ \begin{gathered} \text{covariant derivative ";"} \rightarrow \text{ordinary derivative ","} \\ \Box_{GR} = \frac{1}{\sqrt{-g}}\,\partial_\alpha\Big(\sqrt{-g}\, g^{\alpha\beta}\partial_\beta\Big) \rightarrow \Box_{SR} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\,, \end{gathered} \tag{F.1} $$

through which the previous equation becomes,

$$ {}^1H_{\mu\nu} = 2RR_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}R^2 - 2R_{,\mu\nu} - 2\, g_{\mu\nu}\Delta R\,, $$

being the Ricci's curvature tensor independent of time and consequently the corresponding scalar too.

F.2.1 Time - Time component

$$ \begin{aligned} {}^1H_{00} &= 2RR_{00} - \frac{1}{2}\, g_{00}R^2 - 2R_{,00} - 2\, g_{00}\Delta R \\ &= 2RR_{00} - \frac{1}{2}\, g_{00}R^2 - 2\, g_{00}\Delta R \\ &= 2\Big\{-\frac{2}{c^2}\big(\Delta U - 2\,\Delta V\big) + \frac{2}{c^4}\Big[\boldsymbol{\nabla} U\cdot\big(\boldsymbol{\nabla} U - \boldsymbol{\nabla} V\big) - 3\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V + 2V\big(\Delta U - 4\,\Delta V\big) \\ &\quad - 2\,\Delta\Phi\Big]\Big\}\Big\{-\frac{1}{c^2}\,\Delta U + \frac{1}{c^4}\Big[-\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V + 2\big(U+V\big)\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\Big]\Big\} - \frac{1}{2}\Big\{1 \\ &\quad - \frac{2U}{c^2} + \frac{2}{c^4}\big(U^2 - 2\Phi\big)\Big\}\Big\{\frac{4}{c^4}\Big[\big(\Delta U\big)^2 + 4\big(\Delta V\big)^2 - 4\,\Delta U\Delta V\Big]\Big\} - 2\Big\{1 - \frac{2U}{c^2} + \frac{2}{c^4}\big(U^2 - 2\Phi\big)\Big\} \\ &\quad \Big\{-\frac{2}{c^2}\big(\Delta^2 U - 2\,\Delta^2 V\big) + \frac{2}{c^4}\Big[\Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\big) - \Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V\big) - 3\,\Delta\big(\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V\big) + 2\,\Delta V\big(\Delta U \\ &\quad - 4\,\Delta V\big) + 4\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\big(\Delta U - 4\,\Delta V\big) + 2V\big(\Delta^2 U - 4\,\Delta^2 V\big) - 2\,\Delta^2\Phi\Big]\Big\} \\ &= 2\Big[\frac{2}{c^4}\big(\Delta U - 2\,\Delta V\big)\Delta U\Big] - \frac{1}{2}\Big\{\frac{4}{c^4}\Big[\big(\Delta U\big)^2 + 4\big(\Delta V\big)^2 - 4\,\Delta U\Delta V\Big]\Big\} - 2\Big\{-\frac{2}{c^2}\big(\Delta^2 U \\ &\quad - 2\,\Delta^2 V\big) + \frac{2}{c^4}\Big[\Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\big) - \Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V\big) - 3\,\Delta\big(\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V\big) + 2\,\Delta V\big(\Delta U - 4\,\Delta V\big) \\ &\quad + 4\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\big(\Delta U - 4\Delta V\big) + 2V\big(\Delta^2 U - 4\,\Delta^2 V\big) - 2\,\Delta^2\Phi\Big] + \frac{4}{c^4}U\big(\Delta^2 U - 2\,\Delta^2 V\big)\Big\} + O\big(\epsilon^6\big)\,, \end{aligned} $$

from which,

$$ \begin{aligned} {}^1H_{00} &= {}^{(2)}{}^1H_{00} + {}^{(4)}{}^1H_{00} + O\big(\epsilon^6\big)\,, \\ {}^{(2)}{}^1H_{00} &= \frac{1}{c^2}\big(4\,\Delta^2 U - 8\,\Delta^2 V\big)\,, \\ {}^{(4)}{}^1H_{00} &= \frac{1}{c^4}\Big\{2\big(\Delta U\big)^2 + 24\big(\Delta V\big)^2 - 8\,\Delta U\Delta V + 8\,\Delta^2\Phi - 8V\big(\Delta^2 U - 4\,\Delta^2 V\big) + 4U\big(\Delta^2 U - 2\,\Delta^2 V\big) \\ &\quad - 4\Big[\Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\big) - \Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V\big) - 3\,\Delta\big(\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V\big) + 4\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\big(\Delta U - 4\,\Delta V\big)\Big]\Big\}\,. \end{aligned} $$

F.2.2 Time - Space components

$$
\begin{aligned}
{}^1H_{0i} &= 2RR_{0i} - \frac{1}{2}\,g_{0i}R^2 - 2R_{,0i} - 2\,g_{0i}\Delta R \\
&= 2RR_{0i} - \frac{1}{2}\,g_{0i}R^2 - 2\,g_{0i}\Delta R \\
&= 2\Big\{-\frac{2}{c^2}\big(\Delta U - 2\,\Delta V\big) + \frac{2}{c^4}\Big[\boldsymbol{\nabla} U\cdot\big(\boldsymbol{\nabla} U - \boldsymbol{\nabla} V\big) - 3\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V + 2V\big(\Delta U - 4\,\Delta V\big) - 2\,\Delta\Phi\Big]\Big\} \\
&\quad \Big\{\frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,i}\big)\Big\} - \frac{1}{2}\Big(\frac{h_{0i}}{c^3}\Big)\Big\{\frac{4}{c^4}\Big[\big(\Delta U\big)^2 + 4\big(\Delta V\big)^2 + 4\,\Delta U\Delta V\Big]\Big\} - 2\Big(\frac{h_{0i}}{c^3}\Big)\Big\{-\frac{2}{c^2}\big(\Delta^2 U \\
&\quad - 2\,\Delta^2 V\big) + \frac{2}{c^4}\Big[\Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\big) - \Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V\big) - 3\,\Delta\big(\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V\big) + 2\,\Delta V\big(\Delta U - 4\,\Delta V\big) \\
&\quad + 4\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\big(\Delta U - 4\,\Delta V\big) + 2V\big(\Delta^2 U - 4\,\Delta^2 V\big) - 2\,\Delta^2\Phi\Big]\Big\} \\
&= -2\Big(\frac{h_{0i}}{c^3}\Big)\Big(-\frac{2}{c^2}\Big)\big(\Delta^2 U - 2\,\Delta^2 V\big) + O\big(\epsilon^7\big) \\
&= 4\Big(\frac{h_{0i}}{c^5}\Big)\big(\Delta^2 U - 2\,\Delta^2 V\big) + O\big(\epsilon^7\big)\,,
\end{aligned}
$$

from which,

$$
\begin{aligned}
{}^1H_{0i} &= {}^{(5)\,1}H_{0i} + O\big(\epsilon^7\big)\,, \\
{}^{(5)\,1}H_{0i} &= \frac{h_{0i}}{c^5}\big(4\,\Delta^2 U - 8\,\Delta^2 V\big)\,.
\end{aligned}
$$

F.2.3 Space - Space components

$$
\begin{aligned}
{}^1H_{ij} &= 2RR_{ij} - \frac{1}{2}\,g_{ij}R^2 - 2R_{,ij} - 2\,g_{ij}\Delta R \\
&= 2\Big\{-\frac{2}{c^2}\big(\Delta U - 2\,\Delta V\big) + \frac{2}{c^4}\Big[\boldsymbol{\nabla} U\cdot\big(\boldsymbol{\nabla} U - \boldsymbol{\nabla} V\big) - 3\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V + 2V\big(\Delta U - 4\,\Delta V\big) - 2\,\Delta\Phi\Big]\Big\} \\
&\quad \Big\{\frac{1}{c^2}\big(U_{,ij} - V_{,ij} - \Delta V\delta_{ij}\big) + \frac{1}{c^4}\Big[3\,V_{,i}V_{,j} + \big(\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V + \boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V + 2V\Delta V\big)\delta_{ij} + 2VV_{,ij} \\
&\quad - \big(U_{,i}V_{,j} + U_{,j}V_{,i}\big) + 2\Phi_{,ij} - U_{,i}U_{,j}\Big]\Big\} + \frac{1}{2}\Big(1 + \frac{2V}{c^2}\Big)\delta_{ij}\Big\{\frac{4}{c^4}\Big[\big(\Delta U\big)^2 + 4\big(\Delta V\big)^2 - 4\,\Delta U\Delta V\Big]\Big\} \\
&\quad - 2\Big\{-\frac{2}{c^2}\big(\Delta U - 2\Delta V\big)_{,ij} + \frac{2}{c^4}\Big[\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\big)_{,ij} - \big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V\big)_{,ij} - 3\big(\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V\big)_{,ij} + 2\,V_{,ij}\big(\Delta U \\
&\quad - 4\,\Delta V\big) + 2V_{,j}\big(\Delta U - 4\,\Delta V\big)_{,i} + 2\,V_{,i}\big(\Delta U - 4\Delta V\big)_{,j} + 2V\big(\Delta U - 4\Delta V\big)_{,ij} - 2\,\Delta\Phi_{,ij}\Big]\Big\} + 2\Big(1 \\
&\quad + \frac{2V}{c^2}\Big)\delta_{ij}\Big\{-\frac{2}{c^2}\big(\Delta^2 U - 2\Delta^2 V\big) + \frac{2}{c^4}\Big[\Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\big) - \Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V\big) - 3\,\Delta\big(\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V\big) \\
&\quad + 2\,\Delta V\big(\Delta U - 4\,\Delta V\big) + 4\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\big(\Delta U - 4\,\Delta V\big) + 2V\big(\Delta^2 U - 4\,\Delta^2 V\big) - 2\,\Delta^2\Phi\Big]\Big\} \\
&= 2\Big[-\frac{2}{c^4}\big(\Delta U - 2\,\Delta V\big)\big(U_{,ij} - V_{,ij} - \Delta V\delta_{ij}\big)\Big] + \frac{2}{c^4}\Big[\big(\Delta U\big)^2 + 4\big(\Delta V\big)^2 - 4\,\Delta U\Delta V\Big]\delta_{ij} + \frac{4}{c^2} \\
&\quad \big(\Delta U - 2\,\Delta V\big)_{,ij} - \frac{4}{c^4}\Big[\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\big)_{,ij} - \big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V\big)_{,ij} - 3\big(\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V\big)_{,ij} + 2\,V_{,ij}\big(\Delta U - 4\,\Delta V\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2V_{,j}\big(\Delta U - 4\Delta V\big)_{,i} + 2V_{,i}\big(\Delta U - 4\,\Delta V\big)_{,j} + 2V\big(\Delta U - 4\,\Delta V\big)_{,ij} - 2\,\Delta\Phi_{,ij}\Big] - \frac{4}{c^2}\big(\Delta^2 U \\
&- 2\,\Delta^2 V\big)\delta_{ij} + \frac{4}{c^4}\Big[\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big) - \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big) - 3\,\Delta\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big) + 2\,\Delta V\big(\Delta U - 4\,\Delta V\big) \\
&+ 4\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U - 4\,\Delta V\big) + 2V\big(\Delta^2 U - 4\,\Delta^2 V\big) - 2\,\Delta^2\Phi\Big]\delta_{ij} - \frac{8}{c^4}V\big(\Delta^2 U - 2\,\Delta^2 V\big)\delta_{ij} \\
&+ O\big(\epsilon^6\big) \\
&= \frac{1}{c^2}\Big[4\big(\Delta U - 2\,\Delta V\big)_{,ij} - 4\big(\Delta^2 U - 2\,\Delta^2 V\big)\delta_{ij}\Big] + \frac{1}{c^4}\Big\{-4\big(\Delta U - 2\,\Delta V\big)\big(U_{,ij} - V_{,ij} - \Delta V\delta_{ij}\big) \\
&+ 2\big(\Delta U\big)^2\delta_{ij} + 8\big(\Delta V\big)^2\delta_{ij} - 8\,\Delta U\Delta V\delta_{ij} - 4\Big[\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)_{,ij} - \big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,ij} - 3\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big)_{,ij} \\
&+ 2V_{,ij}\big(\Delta U - 4\,\Delta V\big) + 2V_{,j}\big(\Delta U - 4\,\Delta V\big)_{,i} + 2V_{,i}\big(\Delta U - 4\,\Delta V\big)_{,j} + 2V\big(\Delta U - 4\,\Delta V\big)_{,ij} \\
&- 2\Delta\Phi_{,ij}\Big] + 4\Big[\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big) - \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big) - 3\,\Delta\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big) + 4\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U - 4\Delta V\big)\Big] \\
&\delta_{ij} + 8\,\Delta V\big(\Delta U - 4\,\Delta V\big)\delta_{ij} + 8V\big(\Delta^2 U - 4\,\Delta^2 V\big)\delta_{ij} - 8\,\Delta^2\Phi\,\delta_{ij} - 8V\big(\Delta^2 U - 2\,\Delta^2 V\big)\delta_{ij}\Big\} \\
&+ O\big(\epsilon^6\big)\,,
\end{aligned}
$$

from which,

$$
\begin{aligned}
{}^{1}H_{ij} &= {}^{(2)\,1}H_{ij} + {}^{(4)\,1}H_{ij} + O\big(\epsilon^6\big)\,, \\
{}^{(2)\,1}H_{ij} &= \frac{1}{c^2}\Big[4\big(\Delta U - 2\,\Delta V\big)_{,ij} - 4\big(\Delta^2 U - 2\,\Delta^2 V\big)\delta_{ij}\Big]\,, \\
{}^{(4)\,1}H_{ij} &= \frac{1}{c^4}\Big\{-4\,\Delta U\big(U+V\big)_{,ij} + 8\,\Delta V\big(U+3V\big)_{,ij} - 4\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)_{,ij} + 4\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,ij} \\
&\quad + 12\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big)_{,ij} - 8V_{,j}\big(\Delta U - 4\,\Delta V\big)_{,i} - 8V_{,i}\big(\Delta U - 4\,\Delta V\big)_{,j} - 8V\big(\Delta U - 4\,\Delta V\big)_{,ij} \\
&\quad + 8\,\Delta\Phi_{,ij} + \Big[4\,\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big) - 4\,\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big) - 12\,\Delta\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big) + 16\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U \\
&\quad - 4\,\Delta V\big) - 16V\Delta^2 V - 8\,\Delta^2\Phi + 4\,\Delta U\Delta V - 32\big(\Delta V\big)^2 + 2\big(\Delta U\big)^2\Big]\delta_{ij}\Big\}\,.
\end{aligned}
$$

Summing up,

$$
\begin{aligned}
{}^{1}H_{00} &= \frac{1}{c^2}\big(4\,\Delta^2 U - 8\,\Delta^2 V\big) + \frac{1}{c^4}\Big\{2\big(\Delta U\big)^2 + 24\big(\Delta V\big)^2 - 8\,\Delta U\Delta V + 8\,\Delta^2\Phi - 8V\big(\Delta^2 U - 4\,\Delta^2 V\big) \\
&\quad + 4U\big(\Delta^2 U - 2\,\Delta^2 V\big) - 4\Big[\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big) - \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big) - 3\,\Delta\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big) + 4\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U \\
&\quad - 4\,\Delta V\big)\Big]\Big\}\,, \\
{}^{1}H_{0i} &= \frac{h_{0i}}{c^5}\big(4\,\Delta^2 U - 8\,\Delta^2 V\big)\,, \\
{}^{1}H_{ij} &= \frac{1}{c^2}\Big[4\big(\Delta U - 2\,\Delta V\big)_{,ij} - 4\big(\Delta^2 U - 2\,\Delta^2 V\big)\delta_{ij}\Big] + \frac{1}{c^4}\Big\{-4\,\Delta U\big(U+V\big)_{,ij} + 8\,\Delta V\big(U+3V\big)_{,ij} \\
&\quad - 4\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)_{,ij} + 4\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,ij} + 12\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big)_{,ij} - 8V_{,j}\big(\Delta U - 4\,\Delta V\big)_{,i} - 8V_{,i}\big(\Delta U \\
&\quad - 4\,\Delta V\big)_{,j} - 8V\big(\Delta U - 4\,\Delta V\big)_{,ij} + 8\,\Delta\Phi_{,ij} + \Big[4\,\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big) - 4\,\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big) - 12\,\Delta\big(\boldsymbol{\nabla}V \\
&\quad \cdot\,\boldsymbol{\nabla}V\big) + 16\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U - 4\,\Delta V\big) - 16V\Delta^2 V - 8\,\Delta^2\Phi + 4\,\Delta U\Delta V - 32\big(\Delta V\big)^2 + 2\big(\Delta U\big)^2\Big]\delta_{ij}\Big\}\,.
\end{aligned}
$$

F.3 ${}^2H_{\mu\nu}$ - Left-hand sides of the field equations related to $R_{\mu\nu}R^{\mu\nu}$ term

The development of the left-hand sides of the field equations associated with $R_{\mu\nu}R^{\mu\nu}$ term of the Lagrangian, referred to in (3.11) on page 28,

$$ {}^2H_{\mu\nu} = 2R_{\mu}{}^{\alpha}R_{\nu\alpha} - \frac{1}{2}\,g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - 2R_{(\mu|}{}^{\alpha}{}_{;|\nu)\,\alpha} + \Box R_{\mu\nu} + g_{\mu\nu}R^{\alpha\beta}{}_{;\alpha\beta}\,, $$

which, in 1PN approximation of the metric tensor, namely in the weak-field limit (see (F.1) on page 186), becomes,

$$ {}^2H_{\mu\nu} = 2R_{\mu}{}^{\alpha}R_{\nu\alpha} - \frac{1}{2}\,g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - 2R_{(\mu|}{}^{\alpha}{}_{,|\nu)\,\alpha} - \Delta R_{\mu\nu} + g_{\mu\nu}R^{\alpha\beta}{}_{,\alpha\beta}\,, $$

has produced the results that will be shown in the next subsections.

F.3.1 Time - Time component

$$ {}^2H_{00} = 2R_{0}{}^{\alpha}R_{0\alpha} - \frac{1}{2}\,g_{00}R_{\alpha\beta}R^{\alpha\beta} - 2R_{(0|}{}^{\alpha}{}_{,|0)\,\alpha} - \Delta R_{00} + g_{00}R^{\alpha\beta}{}_{,\alpha\beta}\,. $$

1-st term

$$ 2R_{0}{}^{\alpha}R_{0\alpha} = 2\big(R_{0}{}^{0}R_{00} + R_{0}{}^{1}R_{01} + R_{0}{}^{2}R_{02} + R_{0}{}^{3}R_{03}\big)\,, $$

where,

- $$ \begin{aligned} R_{0}{}^{0} &= g^{0\alpha}R_{0\alpha} = g^{00}R_{00} + g^{01}R_{01} + g^{02}R_{02} + g^{03}R_{03} \\ &= \left[1 + \frac{2U}{c^2} + \frac{2}{c^4}\big(U^2 + 2\Phi\big)\right]\left\{-\frac{1}{c^2}\Delta U + \frac{1}{c^4}\Big[2\big(U+V\big)\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla}U\cdot\big(\boldsymbol{\nabla}U\right. \right. \\ &\quad \left. \left. - \boldsymbol{\nabla}V\big)\Big]\right\} - \frac{h_{01}}{c^3}\left[\frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,1}\big)\right] - \frac{h_{02}}{c^3}\left[\frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,2}\big)\right] - \frac{h_{03}}{c^3}\left[\frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\right] \\ &= -\frac{1}{c^2}\,\Delta U + \frac{1}{c^4}\Big[2\big(U+V\big)\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla}U\cdot\big(\boldsymbol{\nabla}U - \boldsymbol{\nabla}V\big)\Big] - \frac{2U}{c^4}\,\Delta U + O\big(\epsilon^6\big) \\ &= -\frac{1}{c^2}\,\Delta U + \frac{1}{c^4}\Big[2V\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla}U\cdot\big(\boldsymbol{\nabla}U - \boldsymbol{\nabla}V\big)\Big] + O\big(\epsilon^6\big)\,; \end{aligned} $$

- $$ \begin{aligned} R_{0}{}^{1} &= g^{1\alpha}R_{0\alpha} = g^{10}R_{00} + g^{11}R_{01} + g^{12}R_{02} + g^{13}R_{03} = g^{10}R_{00} + g^{11}R_{01} \\ &= -\frac{h_{01}}{c^3}\left\{-\frac{1}{c^2}\,\Delta U + \frac{1}{c^4}\Big[2\big(U+V\big)\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla}U\cdot\big(\boldsymbol{\nabla}U - \boldsymbol{\nabla}V\big)\Big]\right\} - \left(1 - \frac{2V}{c^2}\right) \\ &\quad \left[\frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,1}\big)\right] \\ &= \frac{h_{01}}{c^5}\,\Delta U - \frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,1}\big) + O\big(\epsilon^7\big) \\ &= \frac{1}{c^5}\Big[h_{01}\,\Delta U - \mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,1}\big)\Big] + O\big(\epsilon^7\big)\,; \end{aligned} $$

- $$R_0{}^2 = g^{2\alpha}R_{0\alpha} = g^{20}R_{00} + g^{21}R_{01} + g^{22}R_{02} + g^{23}R_{03} = g^{20}R_{00} + g^{22}R_{02}$$
$$= -\frac{h_{02}}{c^3}\Big\{-\frac{1}{c^2}\,\Delta U + \frac{1}{c^4}\Big[2\big(U+V\big)\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla}U\cdot\big(\boldsymbol{\nabla}U - \boldsymbol{\nabla}V\big)\Big]\Big\} - \Big(1 - \frac{2V}{c^2}\Big)$$
$$\Big[\frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,2}\big)\Big]$$
$$= \frac{h_{02}}{c^5}\,\Delta U - \frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,2}\big) + O\big(\epsilon^7\big)$$
$$= \frac{1}{c^5}\Big[h_{02}\,\Delta U - \mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,2}\big)\Big] + O\big(\epsilon^7\big)\,;$$

- $$R_0{}^3 = g^{3\alpha}R_{0\alpha} = g^{30}R_{00} + g^{31}R_{01} + g^{32}R_{02} + g^{33}R_{03} = g^{30}R_{00} + g^{33}R_{03}$$
$$= -\frac{h_{03}}{c^3}\Big\{-\frac{1}{c^2}\,\Delta U + \frac{1}{c^4}\Big[2\big(U+V\big)\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla}U\cdot\big(\boldsymbol{\nabla}U - \boldsymbol{\nabla}V\big)\Big]\Big\} - \Big(1 - \frac{2V}{c^2}\Big)$$
$$\Big[\frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big]$$
$$= \frac{h_{03}}{c^5}\,\Delta U - \frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,3}\big) + O\big(\epsilon^7\big)$$
$$= \frac{1}{c^5}\Big[h_{03}\,\Delta U - \mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big] + O\big(\epsilon^7\big)\,;$$

hence,

$$2R_0{}^\alpha R_{0\alpha} = 2\Bigg\{\Big\{-\frac{1}{c^2}\,\Delta U + \frac{1}{c^4}\Big[2V\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla}U\cdot\big(\boldsymbol{\nabla}U - \boldsymbol{\nabla}V\big)\Big]\Big\}\Big\{-\frac{1}{c^2}\,\Delta U + \frac{1}{c^4}$$
$$\Big[2\big(U+V\big)\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla}U\cdot\big(\boldsymbol{\nabla}U - \boldsymbol{\nabla}V\big)\Big]\Big\} + \Big\{\frac{1}{c^5}\Big[h_{01}\,\Delta U - \mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,1}\big)\Big]\Big\}$$
$$\Big[\frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,1}\big)\Big] + \Big\{\frac{1}{c^5}\Big[h_{02}\,\Delta U - \mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,2}\big)\Big]\Big\}\Big[\frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,2}\big)\Big] + \Big\{\frac{1}{c^5}\,\Big[h_{03}\,\Delta U$$
$$- \mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big]\Big\}\Big[\frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big]\Bigg\}$$
$$= \frac{2}{c^4}\big(\Delta U\big)^2 + O\big(\epsilon^6\big)\,,$$

that is,

$$2R_0{}^\alpha R_{0\alpha} = \frac{2}{c^4}\big(\Delta U\big)^2 + O\big(\epsilon^6\big)\,.$$

2-nd term

$$-\frac{1}{2}\,g_{00}R_{\alpha\beta}R^{\alpha\beta} = -\frac{1}{2}\Big[1 - \frac{2U}{c^2} + \frac{2}{c^4}\Big(U^2 - 2\Phi\Big)\Big]\Big\{\frac{1}{c^4}\Big\{\big(\Delta U\big)^2 + 5\big(\Delta V\big)^2 - 2\,\Delta U\Delta V + \Big[\boldsymbol{\nabla}\big(U_{,1}\big)$$
$$\cdot\,\boldsymbol{\nabla}\big(U_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big] + \Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(V_{,2}\big)$$
$$\cdot\,\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] - 2\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)$$

$$\cdot\,\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}\Big\}\,,$$

that is,

$$-\frac{1}{2}\,g_{00}R_{\alpha\beta}R^{\alpha\beta}=\frac{1}{c^4}\Big\{-\frac{1}{2}\big(\Delta U\big)^2-\frac{5}{2}\big(\Delta V\big)^2+\Delta U\Delta V-\frac{1}{2}\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big)+\boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big)$$
$$+\boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big]-\frac{1}{2}\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big)+\boldsymbol{\nabla}\big(V_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big)+\boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]$$
$$+\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big)+\boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big)+\boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}+O\big(\epsilon^6\big).$$

3-rd term

$$-2R_{(0|}{}^{\alpha}{}_{,|0)\,\alpha}=-2\Big[\frac{1}{2}\big(R_0{}^{\alpha}{}_{,0\alpha}+R_0{}^{\alpha}{}_{,0\alpha}\big)\Big]=-2R_0{}^{\alpha}{}_{,0\alpha}=0\,.$$

4-th term

$$-\Delta R_{00}=-\Delta\Big\{-\frac{1}{c^2}\,\Delta U+\frac{1}{c^4}\Big[2\big(U+V\big)\Delta U-2\,\Delta\Phi+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U-\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\Big]+O\big(\epsilon^6\big)\Big\}$$
$$=\frac{1}{c^2}\,\Delta^2U+\frac{1}{c^4}\Big[-2\big(\Delta U+\Delta V\big)\Delta U-4\big(\boldsymbol{\nabla}U+\boldsymbol{\nabla}V\big)\cdot\boldsymbol{\nabla}\big(\Delta U\big)-2\big(U+V\big)\Delta^2U$$
$$+2\,\Delta^2\Phi-\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)+\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)\Big]+O\big(\epsilon^6\big)\,,$$

that is,

$$-\Delta R_{00}=\frac{1}{c^2}\,\Delta^2U+\frac{1}{c^4}\Big[-2\big(\Delta U\big)^2-2\,\Delta U\Delta V-4\big(\boldsymbol{\nabla}U+\boldsymbol{\nabla}V\big)\cdot\boldsymbol{\nabla}\big(\Delta U\big)-2\big(U+V\big)\Delta^2U$$
$$+2\,\Delta^2\Phi-\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)+\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)\Big]+O\big(\epsilon^6\big).$$

5-th term

$$g_{00}R^{\alpha\beta}{}_{,\alpha\beta}=g_{00}\big(R^{00}{}_{,00}+2R^{01}{}_{,01}+2R^{02}{}_{,02}+2R^{03}{}_{,03}+R^{11}{}_{,11}+2R^{12}{}_{,12}+2R^{13}{}_{,13}+R^{22}{}_{,22}$$
$$+2R^{23}{}_{,23}+R^{33}{}_{,33}\big)$$
$$=g_{00}\big(R^{11}{}_{,11}+2R^{12}{}_{,12}+2R^{13}{}_{,13}+R^{22}{}_{,22}+2R^{23}{}_{,23}+R^{33}{}_{,33}\big)\,.$$

First of all, it has to be expanded the sum between brackets, that is,

$$R^{\alpha\beta}{}_{,\alpha\beta}=\big(R^{11}{}_{,11}+2R^{12}{}_{,12}+2R^{13}{}_{,13}+R^{22}{}_{,22}+2R^{23}{}_{,23}+R^{33}{}_{,33}\big)\,;$$

hence,

- $$R^{11}{}_{,11}=\frac{1}{c^2}\big(U_{,11}-V_{,11}-\Delta V\big)_{,11}+\frac{1}{c^4}\Big[3\big(V_{,1}\big)^2+\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V+6VV_{,11}+6V\Delta V-2U_{,1}V_{,1}$$
$$+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V+2\Phi_{,11}-\big(U_{,1}\big)^2-4VU_{,11}\Big]_{,11}+O\big(\epsilon^6\big)$$

$$
\begin{aligned}
&= \frac{1}{c^2}\big(U_{,1111} - V_{,1111} - \Delta V_{,11}\big) + \frac{1}{c^4}\Big[6\big(V_{,11}\big)^2 + 6V_{,1}V_{,111} + \big(\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V\big)_{,11} + 6\big(V_{,11}\big)^2 \\
&+ 12V_{,1}V_{,111} + 6VV_{,1111} + 6V_{,11}\Delta V + 12V_{,1}\Delta V_{,1} + 6V\Delta V_{,11} - 2U_{,111}V_{,1} - 4U_{,11}V_{,11} \\
&- 2U_{,1}V_{,111} + \big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V\big)_{,11} + 2\Phi_{,1111} - 2\big(U_{,11}\big)^2 - 2U_{,1}U_{,111} - 4V_{,11}U_{,11} - 8V_{,1}U_{,111} \\
&- 4VU_{,1111}\Big] + O\big(\epsilon^6\big) \\
&= \frac{1}{c^2}\big(\cdots\big) + \frac{1}{c^4}\Big[12\big(V_{,11}\big)^2 + 18V_{,1}V_{,111} + \big(\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V\big)_{,11} + 6VV_{,1111} + 6V_{,11}\Delta V \\
&+ 12V_{,1}\Delta V_{,1} + 6V\Delta V_{,11} - 10V_{,1}U_{,111} - 8U_{,11}V_{,11} - 2U_{,1}V_{,111} + \big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V\big)_{,11} \\
&+ 2\Phi_{,1111} - 2\big(U_{,11}\big)^2 - 2U_{,1}U_{,111} - 4VU_{,1111}\Big] + O\big(\epsilon^6\big) \\
&= \frac{1}{c^2}\big(\cdots\big) + \frac{1}{c^4}\Big[12\big(V_{,11}\big)^2 + 18V_{,1}V_{,111} + 2\big(V_{,11}\big)^2 + 2V_{,1}V_{,111} + 2\big(V_{,12}\big)^2 + 2V_{,2}V_{,112} \\
&+ 2\big(V_{,13}\big)^2 + 2V_{,3}V_{,113} + 6VV_{,1111} + 6\big(V_{,11}\big)^2 + 6V_{,11}V_{,22} + 6V_{,11}V_{,33} + 12V_{,1}V_{,111} \\
&+ 12V_{,1}V_{,122} + 12V_{,1}V_{,133} + 6VV_{,1111} + 6VV_{,1122} + 6VV_{,1133} - 10V_{,1}U_{,111} - 8U_{,11}V_{,11} \\
&- 2U_{,1}V_{,111} + U_{,111}V_{,1} + 2U_{,11}V_{,11} + U_{,1}V_{,111} + U_{,112}V_{,2} + 2U_{,12}V_{,12} + U_{,2}V_{,112} \\
&+ U_{,113}V_{,3} + 2U_{,13}V_{,13} + U_{,3}V_{,113} + 2\Phi_{,1111} - 2\big(U_{,11}\big)^2 - 2U_{,1}U_{,111} - 4VU_{,1111}\Big] + O\big(\epsilon^6\big)\,,
\end{aligned}
$$

that is,

$$
\begin{aligned}
R^{11}{}_{,11} &= \frac{1}{c^2}\big(U_{,1111} - V_{,1111} - \Delta V_{,11}\big) + \frac{1}{c^4}\Big[20\big(V_{,11}\big)^2 + 32V_{,1}V_{,111} + 2\big(V_{,12}\big)^2 + 2V_{,2}V_{,112} \\
&+ 2\big(V_{,13}\big)^2 + 2V_{,3}V_{,113} + 12VV_{,1111} + 6V_{,11}V_{,22} + 6V_{,11}V_{,33} + 12V_{,1}V_{,122} + 12V_{,1}V_{,133} \\
&+ 6VV_{,1122} + 6VV_{,1133} - 9V_{,1}U_{,111} - 6U_{,11}V_{,11} - U_{,1}V_{,111} + U_{,112}V_{,2} + 2U_{,12}V_{,12} \\
&+ U_{,2}V_{,112} + U_{,113}V_{,3} + 2U_{,13}V_{,13} + U_{,3}V_{,113} + 2\Phi_{,1111} - 2\big(U_{,11}\big)^2 - 2U_{,1}U_{,111} - 4VU_{,1111}\Big] \\
&+ O\big(\epsilon^6\big)\,;
\end{aligned}
$$

-

$$
\begin{aligned}
2R^{12}{}_{,12} &= 2\Big\{\frac{1}{c^2}\big(U_{,12} - V_{,12}\big)_{,12} + \frac{1}{c^4}\big(6VV_{,12} - 4VU_{,12} + 2\Phi_{,12} + 3V_{,1}V_{,2} - U_{,1}U_{,2} - U_{,1}V_{,2} \\
&- V_{,1}U_{,2}\big)_{,12} + O\big(\epsilon^6\big)\Big\} \\
&= \frac{1}{c^2}\big(2U_{,1122} - 2V_{,1122}\big) + \frac{1}{c^4}\Big[12\big(V_{,12}\big)^2 + 12V_{,2}V_{,112} + 12V_{,1}V_{,122} + 12VV_{,1122} \\
&- 8U_{,12}V_{,12} - 8V_{,2}U_{,112} - 8V_{,1}U_{,122} - 8VU_{,1122} + 4\Phi_{,1122} + 6V_{,112}V_{,2} + 6\big(V_{,12}\big)^2 \\
&+ 6V_{,11}V_{,22} + 6V_{,1}V_{,122} - 2U_{,112}U_{,2} - 2\big(U_{,12}\big)^2 - 2U_{,11}U_{,22} - 2U_{,1}U_{,122} - 2U_{,112}V_{,2} \\
&- 2U_{,12}V_{,12} - 2U_{,11}V_{,22} - 2U_{,1}V_{,122} - 2V_{,112}U_{,2} - 2U_{,12}V_{,12} - 2V_{,11}U_{,22} - 2V_{,1}U_{,122}\Big] \\
&+ O\big(\epsilon^6\big)\,,
\end{aligned}
$$

that is,

$$
\begin{aligned}
2R^{12}{}_{,12} &= \frac{1}{c^2}\big(2U_{,1122} - 2V_{,1122}\big) + \frac{1}{c^4}\Big[18\big(V_{,12}\big)^2 + 18V_{,2}V_{,112} + 18V_{,1}V_{,122} + 12VV_{,1122} \\
&- 12U_{,12}V_{,12} - 10V_{,2}U_{,112} - 10V_{,1}U_{,122} - 8VU_{,1122} + 4\Phi_{,1122} + 6V_{,11}V_{,22}
\end{aligned}
$$

$$
\begin{aligned}
&- 2U_{,2}U_{,112} - 2\big(U_{,12}\big)^2 - 2U_{,11}U_{,22} - 2U_{,1}U_{,122} - 2U_{,11}V_{,22} - 2U_{,1}V_{,122} - 2U_{,2}V_{,112} \\
&- 2V_{,11}U_{,22}\Big] + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

- $$
\begin{aligned}
2R^{13}{}_{,13} &= 2\Big\{\frac{1}{c^2}\big(U_{,13} - V_{,13}\big)_{,13} + \frac{1}{c^4}\big(6VV_{,13} - 4VU_{,13} + 2\Phi_{,13} + 3V_{,1}V_{,3} - U_{,1}U_{,3} - U_{,1}V_{,3} \\
&\quad - V_{,1}U_{,3}\big)_{,13} + O\big(\epsilon^6\big)\Big\} \\
&= \frac{1}{c^2}\big(2U_{,1133} - 2V_{,1133}\big) + \frac{1}{c^4}\Big[12\big(V_{,13}\big)^2 + 12V_{,3}V_{,113} + 12V_{,1}V_{,133} + 12VV_{,1133} \\
&\quad - 8U_{,13}V_{,13} - 8V_{,3}U_{,113} - 8V_{,1}U_{,133} - 8VU_{,1133} + 4\Phi_{,1133} + 6V_{,113}V_{,3} + 6\big(V_{,13}\big)^2 \\
&\quad + 6V_{,11}V_{,33} + 6V_{,1}V_{,133} - 2U_{,113}U_{,3} - 2\big(U_{,13}\big)^2 - 2U_{,11}U_{,33} - 2U_{,1}U_{,133} - 2U_{,113}V_{,3} \\
&\quad - 2U_{,13}V_{,13} - 2U_{,11}V_{,33} - 2U_{,1}V_{,133} - 2V_{,113}U_{,3} - 2U_{,13}V_{,13} - 2V_{,11}U_{,33} - 2V_{,1}U_{,133}\Big] \\
&\quad + O\big(\epsilon^6\big)\,,
\end{aligned}
$$

  that is,

$$
\begin{aligned}
2R^{13}{}_{,13} &= \frac{1}{c^2}\big(2U_{,1133} - 2V_{,1133}\big) + \frac{1}{c^4}\Big[18\big(V_{,13}\big)^2 + 18V_{,3}V_{,113} + 18V_{,1}V_{,133} + 12VV_{,1133} \\
&\quad - 12U_{,13}V_{,13} - 10V_{,3}U_{,113} - 10V_{,1}U_{,133} - 8VU_{,1133} + 4\Phi_{,1133} + 6V_{,11}V_{,33} - 2U_{,3}U_{,113} \\
&\quad - 2\big(U_{,13}\big)^2 - 2U_{,11}U_{,33} - 2U_{,1}U_{,133} - 2U_{,11}V_{,33} - 2U_{,1}V_{,133} - 2U_{,3}V_{,113} - 2V_{,11}U_{,33}\Big] \\
&\quad + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

- $$
\begin{aligned}
R^{22}{}_{,22} &= \frac{1}{c^2}\big(U_{,22} - V_{,22} - \Delta V\big)_{,22} + \frac{1}{c^4}\Big[3\big(V_{,2}\big)^2 + \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 6VV_{,22} + 6V\Delta V - 2U_{,2}V_{,2} \\
&\quad + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\Phi_{,22} - \big(U_{,2}\big)^2 - 4VU_{,22}\Big]_{,22} + O\big(\epsilon^6\big) \\
&= \frac{1}{c^2}\big(U_{,2222} - V_{,2222} - \Delta V_{,22}\big) + \frac{1}{c^4}\Big[6\big(V_{,22}\big)^2 + 6V_{,2}V_{,222} + \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big)_{,22} + 6\big(V_{,22}\big)^2 \\
&\quad + 12V_{,2}V_{,222} + 6VV_{,2222} + 6V_{,22}\Delta V + 12V_{,2}\Delta V_{,2} + 6V\Delta V_{,22} - 2U_{,222}V_{,2} - 4U_{,22}V_{,22} \\
&\quad - 2U_{,2}V_{,222} + \big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,22} + 2\Phi_{,2222} - 2\big(U_{,22}\big)^2 - 2U_{,2}U_{,222} - 4V_{,22}U_{,22} - 8V_{,2}U_{,222} \\
&\quad - 4VU_{,2222}\Big] + O\big(\epsilon^6\big) \\
&= \frac{1}{c^2}\big(\cdots\big) + \frac{1}{c^4}\Big[12\big(V_{,22}\big)^2 + 18V_{,2}V_{,222} + \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big)_{,22} + 6VV_{,2222} + 6V_{,22}\Delta V \\
&\quad + 12V_{,2}\Delta V_{,2} + 6V\Delta V_{,22} - 10V_{,2}U_{,222} - 8U_{,22}V_{,22} - 2U_{,2}V_{,222} + \big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,22} \\
&\quad + 2\Phi_{,2222} - 2\big(U_{,22}\big)^2 - 2U_{,2}U_{,222} - 4VU_{,2222}\Big] + O\big(\epsilon^6\big) \\
&= \frac{1}{c^2}\big(\cdots\big) + \frac{1}{c^4}\Big[12\big(V_{,22}\big)^2 + 18V_{,2}V_{,222} + 2\big(V_{,12}\big)^2 + 2V_{,1}V_{,122} + 2\big(V_{,22}\big)^2 + 2V_{,2}V_{,222} \\
&\quad + 2\big(V_{,23}\big)^2 + 2V_{,3}V_{,223} + 6VV_{,2222} + 6\big(V_{,22}\big)^2 + 6V_{,11}V_{,22} + 6V_{,22}V_{,33} + 12V_{,2}V_{,112} \\
&\quad + 12V_{,2}V_{,222} + 12V_{,2}V_{,233} + 6VV_{,2222} + 6VV_{,1122} + 6VV_{,2233} - 10V_{,2}U_{,222} - 8U_{,22}V_{,22} \\
&\quad - 2U_{,2}V_{,222} + U_{,122}V_{,1} + 2U_{,12}V_{,12} + U_{,1}V_{,122} + U_{,222}V_{,2} + 2U_{,22}V_{,22} + U_{,2}V_{,222} \\
&\quad + U_{,223}V_{,3} + 2U_{,23}V_{,23} + U_{,3}V_{,223} + 2\Phi_{,2222} - 2\big(U_{,22}\big)^2 - 2U_{,2}U_{,222} - 4VU_{,2222}\Big] + O\big(\epsilon^6\big)\,,
\end{aligned}
$$

that is,

$$
\begin{aligned}
R^{22}{}_{,22} &= \frac{1}{c^2}\big(U_{,2222} - V_{,2222} - \Delta V_{,22}\big) + \frac{1}{c^4}\Big[20\big(V_{,22}\big)^2 + 32V_{,2}V_{,222} + 2\big(V_{,12}\big)^2 + 2V_{,1}V_{,122} \\
&\quad + 2\big(V_{,23}\big)^2 + 2V_{,3}V_{,223} + 12VV_{,2222} + 6V_{,11}V_{,22} + 6V_{,22}V_{,33} + 12V_{,2}V_{,112} + 12V_{,2}V_{,233} \\
&\quad + 6VV_{,1122} + 6VV_{,2233} - 9V_{,2}U_{,222} - 6U_{,22}V_{,22} - U_{,2}V_{,222} + U_{,122}V_{,1} + 2U_{,12}V_{,12} \\
&\quad + U_{,1}V_{,122} + U_{,223}V_{,3} + 2U_{,23}V_{,23} + U_{,3}V_{,223} + 2\Phi_{,2222} - 2\big(U_{,22}\big)^2 - 2U_{,2}U_{,222} - 4VU_{,2222}\Big] \\
&\quad + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

-

$$
\begin{aligned}
2R^{23}{}_{,23} &= 2\bigg\{\frac{1}{c^2}\big(U_{,23} - V_{,23}\big)_{,23} + \frac{1}{c^4}\big(6VV_{,23} - 4VU_{,23} + 2\Phi_{,23} + 3V_{,2}V_{,3} - U_{,2}U_{,3} - U_{,2}V_{,3} \\
&\quad - V_{,2}U_{,3}\big)_{,23} + O\big(\epsilon^6\big)\bigg\} \\
&= \frac{1}{c^2}\big(2U_{,2233} - 2V_{,2233}\big) + \frac{1}{c^4}\Big[12\big(V_{,23}\big)^2 + 12V_{,3}V_{,223} + 12V_{,2}V_{,233} + 12VV_{,2233} \\
&\quad - 8U_{,23}V_{,23} - 8V_{,3}U_{,223} - 8V_{,2}U_{,233} - 8VU_{,2233} + 4\Phi_{,2233} + 6V_{,223}V_{,3} + 6\big(V_{,23}\big)^2 \\
&\quad + 6V_{,22}V_{,33} + 6V_{,2}V_{,233} - 2U_{,223}U_{,3} - 2\big(U_{,23}\big)^2 - 2U_{,22}U_{,33} - 2U_{,2}U_{,233} - 2U_{,223}V_{,3} \\
&\quad - 2U_{,23}V_{,23} - 2U_{,22}V_{,33} - 2U_{,2}V_{,233} - 2V_{,223}U_{,3} - 2U_{,23}V_{,23} - 2V_{,22}U_{,33} - 2V_{,2}U_{,233}\Big] \\
&\quad + O\big(\epsilon^6\big)\,,
\end{aligned}
$$

that is,

$$
\begin{aligned}
2R^{23}{}_{,23} &= \frac{1}{c^2}\big(2U_{,2233} - 2V_{,2233}\big) + \frac{1}{c^4}\Big[18\big(V_{,23}\big)^2 + 18V_{,3}V_{,223} + 18V_{,2}V_{,233} + 12VV_{,2233} \\
&\quad - 12U_{,23}V_{,23} - 10V_{,3}U_{,223} - 10V_{,2}U_{,233} - 8VU_{,2233} + 4\Phi_{,2233} + 6V_{,22}V_{,33} - 2U_{,3}U_{,223} \\
&\quad - 2\big(U_{,23}\big)^2 - 2U_{,22}U_{,33} - 2U_{,2}U_{,233} - 2U_{,22}V_{,33} - 2U_{,2}V_{,233} - 2U_{,3}V_{,223} - 2V_{,22}U_{,33}\Big] \\
&\quad + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

-

$$
\begin{aligned}
R^{33}{}_{,33} &= \frac{1}{c^2}\big(U_{,33} - V_{,33} - \Delta V\big)_{,33} + \frac{1}{c^4}\Big[3\big(V_{,3}\big)^2 + \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + 6VV_{,33} + 6V\Delta V - 2U_{,3}V_{,3} \\
&\quad + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V + 2\Phi_{,33} - \big(U_{,3}\big)^2 - 4VU_{,33}\Big]_{,33} + O\big(\epsilon^6\big) \\
&= \frac{1}{c^2}\big(U_{,3333} - V_{,3333} - \Delta V_{,33}\big) + \frac{1}{c^4}\Big[6\big(V_{,33}\big)^2 + 6V_{,3}V_{,333} + \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big)_{,33} + 6\big(V_{,33}\big)^2 \\
&\quad + 12V_{,3}V_{,333} + 6VV_{,3333} + 6V_{,33}\Delta V + 12V_{,3}\Delta V_{,3} + 6V\Delta V_{,33} - 2U_{,333}V_{,3} - 4U_{,33}V_{,33} \\
&\quad - 2U_{,3}V_{,333} + \big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,33} + 2\Phi_{,3333} - 2\big(U_{,33}\big)^2 - 2U_{,3}U_{,333} - 4V_{,33}U_{,33} - 8V_{,3}U_{,333} \\
&\quad - 4VU_{,3333}\Big] + O\big(\epsilon^6\big) \\
&= \frac{1}{c^2}\big(\cdots\big) + \frac{1}{c^4}\Big[12\big(V_{,33}\big)^2 + 18V_{,3}V_{,333} + \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big)_{,33} + 6VV_{,3333} + 6V_{,33}\Delta V
\end{aligned}
$$

$$
\begin{aligned}
&+ 12V_{,3}\Delta V_{,3} + 6V\Delta V_{,33} - 10V_{,3}U_{,333} - 8U_{,33}V_{,33} - 2U_{,3}V_{,333} + \big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,33}\\
&+ 2\Phi_{,3333} - 2\big(U_{,33}\big)^2 - 2U_{,3}U_{,333} - 4VU_{,3333}\Big] + O\big(\epsilon^6\big)\\
&= \frac{1}{c^2}\big(\cdots\big) + \frac{1}{c^4}\Big[12\big(V_{,33}\big)^2 + 18V_{,3}V_{,333} + 2\big(V_{,13}\big)^2 + 2V_{,1}V_{,133} + 2\big(V_{,23}\big)^2 + 2V_{,2}V_{,233}\\
&+ 2\big(V_{,33}\big)^2 + 2V_{,3}V_{,333} + 6VV_{,3333} + 6\big(V_{,33}\big)^2 + 6V_{,11}V_{,33} + 6V_{,22}V_{,33} + 12V_{,3}V_{,113}\\
&+ 12V_{,3}V_{,223} + 12V_{,3}V_{,333} + 6VV_{,1133} + 6VV_{,2233} + 6VV_{,3333} - 10V_{,3}U_{,333} - 8U_{,33}V_{,33}\\
&- 2U_{,3}V_{,333} + U_{,133}V_{,1} + 2U_{,13}V_{,13} + U_{,1}V_{,133} + U_{,233}V_{,2} + 2U_{,23}V_{,23} + U_{,2}V_{,233}\\
&+ U_{,333}V_{,3} + 2U_{,33}V_{,33} + U_{,3}V_{,333} + 2\Phi_{,3333} - 2\big(U_{,33}\big)^2 - 2U_{,3}U_{,333} - 4VU_{,3333}\Big] + O\big(\epsilon^6\big)\,,
\end{aligned}
$$

that is,

$$
\begin{aligned}
R^{33}{}_{,33} &= \frac{1}{c^2}\big(U_{,3333} - V_{,3333} - \Delta V_{,33}\big) + \frac{1}{c^4}\Big[20\big(V_{,33}\big)^2 + 32V_{,3}V_{,333} + 2\big(V_{,13}\big)^2 + 2V_{,1}V_{,133}\\
&+ 2\big(V_{,23}\big)^2 + 2V_{,2}V_{,233} + 12VV_{,3333} + 6V_{,11}V_{,33} + 6V_{,22}V_{,33} + 12V_{,3}V_{,113} + 12V_{,3}V_{,223}\\
&+ 6VV_{,1133} + 6VV_{,2233} - 9V_{,3}U_{,333} - 6U_{,33}V_{,33} - U_{,3}V_{,333} + U_{,133}V_{,1} + 2U_{,13}V_{,13}\\
&+ U_{,1}V_{,133} + U_{,233}V_{,2} + 2U_{,23}V_{,23} + U_{,2}V_{,233} + 2\Phi_{,3333} - 2\big(U_{,33}\big)^2 - 2U_{,3}U_{,333} - 4VU_{,3333}\Big]\\
&+ O\big(\epsilon^6\big)\,.
\end{aligned}
$$

Therefore, the second-order term of $R^{\alpha\beta}{}_{,\alpha\beta}$ is given by the following expression,

$$
\begin{aligned}
{}^{(2)}R^{\alpha\beta}{}_{,\alpha\beta} &= \frac{1}{c^2}\big(U_{,1111} - V_{,1111} - \Delta V_{,11} + 2U_{,1122} - 2V_{,1122} + 2U_{,1133} - 2V_{,1133} + U_{,2222} - V_{,2222}\\
&- \Delta V_{,22} + 2U_{,2233} - 2V_{,2233} + U_{,3333} - V_{,3333} - \Delta V_{,33}\big)\\
&= \frac{1}{c^2}\big(U_{,1111} - V_{,1111} - V_{,1111} - V_{,1122} - V_{,1133} + 2U_{,1122} - 2V_{,1122} + 2U_{,1133} - 2V_{,1133}\\
&+ U_{,2222} - V_{,2222} - V_{,1122} - V_{,2222} - V_{,2233} + 2U_{,2233} - 2V_{,2233} + U_{,3333} - V_{,3333} - V_{,1133}\\
&- V_{,2233} - V_{,3333}\big)\,,
\end{aligned}
$$

from which,

- $$
\begin{aligned}
&U_{,1111} + 2U_{,1122} + 2U_{,1133} + U_{,2222} + 2U_{,2233} + U_{,3333}\\
&= \big(U_{,1111} + U_{,1122} + U_{,1133}\big) + \big(U_{,1122} + U_{,2222} + U_{,2233}\big) + \big(U_{,1133} + U_{,2233} + U_{,3333}\big)\\
&= \Delta U_{,11} + \Delta U_{,22} + \Delta U_{,33} = \Delta\big(\Delta U\big) = \Delta^2 U\,;
\end{aligned}
$$

- $$
\begin{aligned}
&- 2V_{,1111} - 4V_{,1122} - 4V_{,1133} - 2V_{,2222} - 4V_{,2233} - 2V_{,3333}\\
&= -2\big(V_{,1111} + 2V_{,1122} + 2V_{,1133} + V_{,2222} + 2V_{,2233} + V_{,3333}\big)\\
&= -2\Big[\big(V_{,1111} + V_{,1122} + V_{,1133}\big) + \big(V_{,1122} + V_{,2222} + V_{,2233}\big) + \big(V_{,1133} + V_{,2233} + V_{,3333}\big)\Big]\\
&= -2\big(\Delta V_{,11} + \Delta V_{,22} + \Delta V_{,33}\big) = -2\,\Delta\big(\Delta V\big) = -2\,\Delta^2 V\,;
\end{aligned}
$$

hence,

$$
{}^{(2)}R^{\alpha\beta}{}_{,\alpha\beta} = \frac{1}{c^2}\big(\Delta^2 U - 2\,\Delta^2 V\big)\,.
$$

The fourth-order term of $R^{\alpha\beta}{}_{,\alpha\beta}$, on the other hand, consists of seven contributions:

- $$
\begin{aligned}
&20\big(V_{,11}\big)^2+2\big(V_{,12}\big)^2+2\big(V_{,13}\big)^2+18\big(V_{,12}\big)^2+18\big(V_{,13}\big)^2+20\big(V_{,22}\big)^2+2\big(V_{,12}\big)^2+2\big(V_{,23}\big)^2\\
&+18\big(V_{,23}\big)^2+20\big(V_{,33}\big)^2+2\big(V_{,13}\big)^2+2\big(V_{,23}\big)^2+6V_{,11}V_{,22}+6V_{,11}V_{,33}+6V_{,11}V_{,22}+6V_{,11}V_{,33}\\
&+6V_{,11}V_{,22}+6V_{,22}V_{,33}+6V_{,22}V_{,33}+6V_{,11}V_{,33}+6V_{,22}V_{,33}\\
&=20\big(V_{,11}\big)^2+22\big(V_{,12}\big)^2+22\big(V_{,13}\big)^2+20\big(V_{,22}\big)^2+22\big(V_{,23}\big)^2+20\big(V_{,33}\big)^2+18V_{,11}V_{,22}\\
&+18V_{,11}V_{,33}+18V_{,22}V_{,33}\\
&=9\Big[\big(V_{,11}\big)^2+\big(V_{,22}\big)^2+\big(V_{,33}\big)^2+2V_{,11}V_{,22}+2V_{,11}V_{,33}+2V_{,22}V_{,33}\Big]+11\big(V_{,11}\big)^2+22\big(V_{,12}\big)^2\\
&+22\big(V_{,13}\big)^2+11\big(V_{,22}\big)^2+22\big(V_{,23}\big)^2+11\big(V_{,33}\big)^2\\
&=9\big(\Delta V\big)^2+11\Big\{\Big[\big(V_{,11}\big)^2+\big(V_{,12}\big)^2+\big(V_{,13}\big)^2\Big]+\Big[\big(V_{,12}\big)^2+\big(V_{,22}\big)^2+\big(V_{,23}\big)^2\Big]+\Big[\big(V_{,13}\big)^2\\
&+\big(V_{,23}\big)^2+\big(V_{,33}\big)^2\Big]\Big\}\\
&=9\big(\Delta V\big)^2+11\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big)+\boldsymbol{\nabla}\big(V_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big)+\boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big];
\end{aligned}
$$

- $$
\begin{aligned}
&-\Big[2\big(U_{,11}\big)^2+2U_{,1}U_{,111}+2U_{,2}U_{,112}+2\big(U_{,12}\big)^2+2U_{,11}U_{,22}+2U_{,1}U_{,122}+2U_{,3}U_{,113}+2\big(U_{,13}\big)^2\\
&+2U_{,11}U_{,33}+2U_{,1}U_{,133}+2\big(U_{,22}\big)^2+2U_{,2}U_{,222}+2U_{,3}U_{,223}+2\big(U_{,23}\big)^2+2U_{,22}U_{,33}+2U_{,2}U_{,233}\\
&+2\big(U_{,33}\big)^2+2U_{,3}U_{,333}\Big]\\
&=-\Big[\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)-2\big(U_{,12}\big)^2-2\big(U_{,13}\big)^2-2\big(U_{,23}\big)^2+2U_{,11}U_{,22}+2U_{,11}U_{,33}+2U_{,22}U_{,33}\Big],
\end{aligned}
$$

for what concerns the explicit expression of the first term between square brackets, see Appendix B, eq. (A.5) on page 84. Furthermore, from

$$
\begin{aligned}
&\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big)+\boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big)+\boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\\
&\qquad=\big(U_{,11}\big)^2+\big(U_{,22}\big)^2+\big(U_{,33}\big)^2+2\big(U_{,12}\big)^2+2\big(U_{,13}\big)^2+2\big(U_{,23}\big)^2,
\end{aligned}
$$

(see Appendix B, eq. (A.9) on page 84) it follows,

$$
\begin{aligned}
&-2\big(U_{,12}\big)^2-2\big(U_{,13}\big)^2-2\big(U_{,23}\big)^2\\
&=\big(U_{,11}\big)^2+\big(U_{,22}\big)^2+\big(U_{,33}\big)^2-\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big)-\boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big)-\boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big),
\end{aligned}
$$

through which the above expression becomes,

$$
\begin{aligned}
&=-\Big[\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)+\big(U_{,11}\big)^2+\big(U_{,22}\big)^2+\big(U_{,33}\big)^2+2U_{,11}U_{,22}+2U_{,11}U_{,33}+2U_{,22}U_{,33}\\
&-\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big)-\boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big)-\boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big]\\
&=-\Big[\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)+\big(\Delta U\big)^2-\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big)-\boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big)-\boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big]
\end{aligned}
$$

that is,

$$
=-\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)-\big(\Delta U\big)^2+\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big)+\boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big)+\boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big];
$$

- $$2\Phi_{,1111} + 4\Phi_{,1122} + 4\Phi_{,1133} + 2\Phi_{,2222} + 4\Phi_{,2233} + 2\Phi_{,3333}$$
$$= 2\Big[\big(\Phi_{,1111} + \Phi_{,1122} + \Phi_{,1133}\big) + \big(\Phi_{,1122} + \Phi_{,2222} + \Phi_{,2233}\big) + \big(\Phi_{,1133} + \Phi_{,2233} + \Phi_{,3333}\big)\Big]$$
$$= 2\Big[\Delta\Phi_{,11} + \Delta\Phi_{,22} + \Delta\Phi_{,33}\Big] = 2\,\Delta\big(\Delta\Phi\big) = 2\,\Delta^2\Phi\,;$$

- $$-4VU_{,1111} - 8VU_{,1122} - 8VU_{,1133} - 4VU_{,2222} - 8VU_{,2233} - 4VU_{,3333}$$
$$= -4V\big(U_{,1111} + 2U_{,1122} + 2U_{,1133} + U_{,2222} + 2U_{,2233} + U_{,3333}\big)$$
$$= -4V\Big[\big(U_{,1111} + U_{,1122} + U_{,1133}\big) + \big(U_{,1122} + U_{,2222} + U_{,2233}\big) + \big(U_{,1133} + U_{,2233} + U_{,3333}\big)\Big]$$
$$= -4V\Big[\Delta U_{,11} + \Delta U_{,22} + \Delta U_{,33}\Big] = -4V\Delta\big(\Delta U\big) = -4V\Delta^2 U\,;$$

- $$12VV_{,1111} + 6VV_{,1122} + 6VV_{,1133} + 12VV_{,1122} + 12VV_{,1133} + 12VV_{,2222} + 6VV_{,1122} + 6VV_{,2233}$$
$$+ 12VV_{,2233} + 12VV_{,3333} + 6VV_{,1133} + 6VV_{,2233}$$
$$= 12VV_{,1111} + 24VV_{,1122} + 24VV_{,1133} + 12VV_{,2222} + 24VV_{,2233} + 12VV_{,3333}$$
$$= 12V\big(V_{,1111} + 2V_{,1122} + 2V_{,1133} + V_{,2222} + 2V_{,2233} + V_{,3333}\big)$$
$$= 12V\Big[\big(V_{,1111} + V_{,1122} + V_{,1133}\big) + \big(V_{,1122} + V_{,2222} + V_{,2233}\big) + \big(V_{,1133} + V_{,2233} + V_{,3333}\big)\Big]$$
$$= 12V\Big[\Delta V_{,11} + \Delta V_{,22} + \Delta V_{,33}\Big] = 12V\Delta\big(\Delta V\big) = 12V\Delta^2 V\,;$$

- $$32V_{,1}V_{,111} + 2V_{,2}V_{,112} + 2V_{,3}V_{,113} + 12V_{,1}V_{,122} + 12V_{,1}V_{,133} + 18V_{,2}V_{,112} + 18V_{,1}V_{,122}$$
$$+ 18V_{,3}V_{,113} + 18V_{,1}V_{,133} + 32V_{,2}V_{,222} + 2V_{,1}V_{,122} + 2V_{,3}V_{,223} + 12V_{,2}V_{,112} + 12V_{,2}V_{,233}$$
$$+ 18V_{,3}V_{,223} + 18V_{,2}V_{,233} + 32V_{,3}V_{,333} + 2V_{,1}V_{,133} + 2V_{,2}V_{,233} + 12V_{,3}V_{,113} + 12V_{,3}V_{,223}$$
$$= 32\big(V_{,1}V_{,111} + V_{,1}V_{,122} + V_{,1}V_{,133} + V_{,2}V_{,112} + V_{,2}V_{,222} + V_{,2}V_{,233} + V_{,3}V_{,113} + V_{,3}V_{,223}$$
$$+ V_{,3}V_{,333}\big) = 32\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta V\big)\,,$$

  otherwise also,

$$= 16\,\Delta\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big) - 32\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(V_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\,,$$

  but for sakes of convenience i prefer to use the first result;

- $$-9V_{,1}U_{,111} - 6U_{,11}V_{,11} - U_{,1}V_{,111} + V_{,2}U_{,112} + 2U_{,12}V_{,12} + U_{,2}V_{,112} + V_{,3}U_{,113} + 2U_{,13}V_{,13}$$
$$+ U_{,3}V_{,113} - 12U_{,12}V_{,12} - 10V_{,2}U_{,112} - 10V_{,1}U_{,122} - 2U_{,11}V_{,22} - 2U_{,1}V_{,122} - 2U_{,2}V_{,112}$$
$$- 2V_{,11}U_{,22} - 12U_{,13}V_{,13} - 10V_{,3}U_{,113} - 10V_{,1}U_{,133} - 2U_{,11}V_{,33} - 2U_{,1}V_{,133} - 2U_{,3}V_{,113}$$
$$- 2V_{,11}U_{,33} - 9V_{,2}U_{,222} - 6U_{,22}V_{,22} - U_{,2}V_{,222} + V_{,1}U_{,122} + 2U_{,12}V_{,12} + U_{,1}V_{,122} + V_{,3}U_{,223}$$
$$+ 2U_{,23}V_{,23} + U_{,3}V_{,223} - 12U_{,23}V_{,23} - 10V_{,3}U_{,223} - 10V_{,2}U_{,233} - 2U_{,22}V_{,33} - 2U_{,2}V_{,233}$$
$$- 2U_{,3}V_{,223} - 2V_{,22}U_{,33} - 9V_{,3}U_{,333} - 6U_{,33}V_{,33} - U_{,3}V_{,333} + V_{,1}U_{,133} + 2U_{,13}V_{,13} + U_{,1}V_{,133}$$

$$
\begin{aligned}
&+ V_{,2}U_{,233} + 2U_{,23}V_{,23} + U_{,2}V_{,233} \\
&= -9V_{,1}U_{,111} - 6U_{,11}V_{,11} - U_{,1}V_{,111} - 9V_{,2}U_{,112} - 8U_{,12}V_{,12} - U_{,2}V_{,112} - 9V_{,3}U_{,113} - 8U_{,13}V_{,13} \\
&- U_{,3}V_{,113} - 9V_{,1}U_{,122} - 2U_{,11}V_{,22} - U_{,1}V_{,122} - 2V_{,11}U_{,22} - 9V_{,1}U_{,133} - 2U_{,11}V_{,33} - U_{,1}V_{,133} \\
&- 2V_{,11}U_{,33} - 9V_{,2}U_{,222} - 6U_{,22}V_{,22} - U_{,2}V_{,222} - 9V_{,3}U_{,223} - 8U_{,23}V_{,23} - 9V_{,2}U_{,233} - 2U_{,22}V_{,33} \\
&- U_{,2}V_{,233} - U_{,3}V_{,223} - 2V_{,22}U_{,33} - 9V_{,3}U_{,333} - 6U_{,33}V_{,33} - U_{,3}V_{,333} \\
&= -9\big(V_{,1}U_{,111} + V_{,2}U_{,112} + V_{,3}U_{,113} + V_{,1}U_{,122} + V_{,1}U_{,133} + V_{,2}U_{,222} + V_{,3}U_{,223} + V_{,2}U_{,233} \\
&+ V_{,3}U_{,333}\big) - \big(U_{,1}V_{,111} + U_{,2}V_{,112} + U_{,3}V_{,113} + U_{,1}V_{,122} + U_{,1}V_{,133} + U_{,2}V_{,222} + U_{,2}V_{,233} \\
&+ U_{,3}V_{,223} + U_{,3}V_{,333}\big) - 4\big(U_{,11}V_{,11} + 2U_{,12}V_{,12} + 2U_{,13}V_{,13} + U_{,22}V_{,22} + 2U_{,23}V_{,23} + U_{,33}V_{,33}\big) \\
&- 2\big(U_{,11}V_{,11} + U_{,11}V_{,22} + V_{,11}U_{,22} + U_{,11}V_{,33} + V_{,11}U_{,33} + U_{,22}V_{,22} + U_{,22}V_{,33} + V_{,22}U_{,33} \\
&+ U_{,33}V_{,33}\big) \\
&= -9\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U\big) - \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(\Delta V\big) - 4\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big) \\
&\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] - 2\,\Delta U\Delta V\,.
\end{aligned}
$$

Summing up,

$$
\begin{aligned}
{}^{(4)}R^{\alpha\beta}{}_{,\alpha\beta} = \frac{1}{c^4}\Big\{&9\big(\Delta V\big)^2 - \big(\Delta U\big)^2 - 2\,\Delta U\Delta V - 4V\big(\Delta^2 U - 3\,\Delta^2 V\big) + 2\,\Delta^2\Phi - \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big) \\
&+ 32\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta V\big) - 9\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U\big) - \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(\Delta V\big) + 11\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(V_{,2}\big) \\
&\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] + \Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big) \\
&\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big] - 4\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\,,
\end{aligned}
$$

from which,

$$
\begin{aligned}
R^{\alpha\beta}{}_{,\alpha\beta} &= {}^{(2)}R^{\alpha\beta}{}_{,\alpha\beta} + {}^{(4)}R^{\alpha\beta}{}_{,\alpha\beta} + O\big(\epsilon^6\big)\,, \\
{}^{(2)}R^{\alpha\beta}{}_{,\alpha\beta} &= \frac{1}{c^2}\big(\Delta^2 U - 2\,\Delta^2 V\big)\,, \\
{}^{(4)}R^{\alpha\beta}{}_{,\alpha\beta} &= \frac{1}{c^4}\Big\{9\big(\Delta V\big)^2 - \big(\Delta U\big)^2 - 2\,\Delta U\Delta V - 4V\big(\Delta^2 U - 3\,\Delta^2 V\big) + 2\,\Delta^2\Phi - \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big) \\
&\quad + 32\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta V\big) - 9\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U\big) - \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(\Delta V\big) + 11\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(V_{,2}\big) \\
&\quad \cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] + \Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big) \\
&\quad \cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big] - 4\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}\,. \qquad \text{(F.2)}
\end{aligned}
$$

At this point, it is possible to compute tha last term of ${}^2H_{\mu\nu}$, that is,

$$
\begin{aligned}
g_{00}R^{\alpha\beta}{}_{,\alpha\beta} &= \Big[1 - \frac{2U}{c^2} + \frac{2}{c^4}\Big(U^2 - 2\Phi\Big) + O\big(\epsilon^6\big)\Big]\Big\{\frac{1}{c^2}\big(\Delta^2 U - 2\,\Delta^2 V\big) + \frac{1}{c^4}\Big\{9\big(\Delta V\big)^2\cdots \\
&\quad \cdots - 4\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \cdots + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}\Big\} \\
&= \frac{1}{c^2}\big(\Delta^2 U - 2\,\Delta^2 V\big) + \frac{1}{c^4}\Big\{\cdots\Big\} - \frac{2U}{c^4}\big(\Delta^2 U - 2\,\Delta^2 V\big) + O\big(\epsilon^6\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{c^2}\big(\Delta^2 U - 2\,\Delta^2 V\big) + \frac{1}{c^4}\Big\{9\big(\Delta V\big)^2 - \big(\Delta U\big)^2 - 2\,\Delta U\Delta V - 4V\big(\Delta^2 U - 3\,\Delta^2 V\big)\\
&+ 2\,\Delta^2\Phi - 2U\big(\Delta^2 U - 2\,\Delta^2 V\big) - \Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\big) + 32\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\big(\Delta V\big) - 9\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\big(\Delta U\big)\\
&- \boldsymbol{\nabla} U\cdot\boldsymbol{\nabla}\big(\Delta V\big) + 11\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(V_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\\
&+ \Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big] - 4\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big)\\
&+ \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\} + O\big(\epsilon^6\big)\,.
\end{aligned}
$$

Therefore, it results that,

$$
\begin{aligned}
{}^{(2)\,2}H_{00} &= \frac{1}{c^2}\,\Delta^2 U + \frac{1}{c^2}\big(\Delta^2 U - 2\,\Delta^2 V\big)\\
&= \frac{2}{c^2}\big(\Delta^2 U - \Delta^2 V\big)\,;
\end{aligned}
$$

$$
\begin{aligned}
{}^{(4)\,2}H_{00} &= \frac{1}{c^4}\Big\{2\big(\Delta U\big)^2 - \frac{1}{2}\big(\Delta U\big)^2 - \frac{5}{2}\big(\Delta V\big)^2 + \Delta U\Delta V - \frac{1}{2}\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\\
&\cdot\boldsymbol{\nabla}\big(U_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big] - \frac{1}{2}\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(V_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(V_{,3}\big)\\
&\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] + \Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] - 2\big(\Delta U\big)^2\\
&- 2\,\Delta U\Delta V - 4\,\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla}\big(\Delta U\big) - 4\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\big(\Delta U\big) - 2U\Delta^2 U - 2V\Delta^2 U + 2\,\Delta^2\Phi\\
&- \Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\big) + \Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V\big) + 9\big(\Delta V\big)^2 - \big(\Delta U\big)^2 - 2\,\Delta U\Delta V - 4V\Delta^2 U\\
&+ 12V\Delta^2 V + 2\,\Delta^2\Phi - 2U\Delta^2 U + 4U\Delta^2 V - \Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\big) + 32\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\big(\Delta V\big)\\
&- 9\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\big(\Delta U\big) - \boldsymbol{\nabla} U\cdot\boldsymbol{\nabla}\big(\Delta V\big) + 11\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(V_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big)\\
&+ \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] + \Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big]\\
&- 4\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\\
&= \frac{1}{c^4}\Big\{-\frac{3}{2}\big(\Delta U\big)^2 + \frac{13}{2}\big(\Delta V\big)^2 - 3\,\Delta U\Delta V - 4\,\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla}\big(\Delta U\big) - 13\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\big(\Delta U\big)\\
&- 4U\Delta^2 U - 6V\Delta^2 U + 4\,\Delta^2\Phi - 2\,\Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\big) + \Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V\big) + 12V\Delta^2 V\\
&+ 4U\Delta^2 V + 32\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\big(\Delta V\big) - \boldsymbol{\nabla} U\cdot\boldsymbol{\nabla}\big(\Delta V\big) + \frac{21}{2}\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(V_{,2}\big)\\
&\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] + \frac{1}{2}\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\\
&\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big] - 3\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}\,.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
{}^{2}H_{00} &= {}^{(2)\,2}H_{00} + {}^{(4)\,2}H_{00} + O\big(\epsilon^6\big)\,,\\
{}^{(2)\,2}H_{00} &= \frac{2}{c^2}\big(\Delta^2 U - \Delta^2 V\big)\,,\\
{}^{(4)\,2}H_{00} &= \frac{1}{c^4}\Big\{-\frac{3}{2}\big(\Delta U\big)^2 + \frac{13}{2}\big(\Delta V\big)^2 - 3\,\Delta U\Delta V - 4\,\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla}\big(\Delta U\big) - 13\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla}\big(\Delta U\big)\\
&- 4U\Delta^2 U - 6V\Delta^2 U + 4\Delta^2\Phi - 2\,\Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} U\big) + \Delta\big(\boldsymbol{\nabla} U\cdot\boldsymbol{\nabla} V\big) + 12V\Delta^2 V
\end{aligned}
$$

$$+ 4U\Delta^2 V + 32\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}(\Delta V) - \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}(\Delta V) + \frac{21}{2}\Big[\boldsymbol{\nabla}(V_{,1})\cdot\boldsymbol{\nabla}(V_{,1}) + \boldsymbol{\nabla}(V_{,2})$$
$$\cdot\,\boldsymbol{\nabla}(V_{,2}) + \boldsymbol{\nabla}(V_{,3})\cdot\boldsymbol{\nabla}(V_{,3})\Big] + \frac{1}{2}\Big[\boldsymbol{\nabla}(U_{,1})\cdot\boldsymbol{\nabla}(U_{,1}) + \boldsymbol{\nabla}(U_{,2})\cdot\boldsymbol{\nabla}(U_{,2}) + \boldsymbol{\nabla}(U_{,3})$$
$$\cdot\,\boldsymbol{\nabla}(U_{,3})\Big] - 3\Big[\boldsymbol{\nabla}(U_{,1})\cdot\boldsymbol{\nabla}(V_{,1}) + \boldsymbol{\nabla}(U_{,2})\cdot\boldsymbol{\nabla}(V_{,2}) + \boldsymbol{\nabla}(U_{,3})\cdot\boldsymbol{\nabla}(V_{,3})\Big]\Big\}.$$

F.3.2 Time - Space components

$$^2H_{0i} = 2R_0{}^{\alpha}R_{i\alpha} - \frac{1}{2}\,g_{0i}R_{\alpha\beta}R^{\alpha\beta} - 2R_{(0|}{}^{\alpha}{}_{,|i)\,\alpha} - \Delta R_{0i} + g_{0i}R^{\alpha\beta}{}_{,\alpha\beta}\,.$$

1-st term

$$2R_0{}^{\alpha}R_{i\alpha} = 2\big(R_0{}^0R_{i0} + R_0{}^1R_{i1} + R_0{}^2R_{i2} + R_0{}^3R_{i3}\big)\,,$$

where:

- $$R_0{}^0R_{i0} = \Big\{-\frac{1}{c^2}\,\Delta U + \frac{1}{c^4}\Big[2V\Delta U - 2\,\Delta\Phi + \boldsymbol{\nabla}U\cdot\big(\boldsymbol{\nabla}U - \boldsymbol{\nabla}V\big)\Big] + O\big(\epsilon^6\big)\Big\}\Big\{\frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,i}\big) + O\big(\epsilon^7\big)\Big\} \sim O\big(\epsilon^7\big)\,;$$

- $$R_0{}^1R_{i1} = \Big\{\frac{1}{c^5}\Big[h_{01}\,\Delta U - \mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,1}\big)\Big] + O\big(\epsilon^7\big)\Big\}\Big\{\frac{1}{c^2}\big(U_{,i1} - V_{,i1} - \Delta V\delta_{i1}\big) + \frac{1}{c^4}\Big[3V_{,i}V_{,1} + \cdots - U_{,i}U_{,1}\Big] + O\big(\epsilon^6\big)\Big\} \sim O\big(\epsilon^7\big)\,;$$

- $$R_0{}^2R_{i2} = \Big\{\frac{1}{c^5}\Big[h_{02}\,\Delta U - \mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,2}\big)\Big] + O\big(\epsilon^7\big)\Big\}\Big\{\frac{1}{c^2}\big(U_{,i2} - V_{,i2} - \Delta V\delta_{i2}\big) + \frac{1}{c^4}\Big[3V_{,i}V_{,2} + \cdots - U_{,i}U_{,2}\Big] + O\big(\epsilon^6\big)\Big\} \sim O\big(\epsilon^7\big)\,;$$

- $$R_0{}^3R_{i3} = \Big\{\frac{1}{c^5}\Big[h_{03}\,\Delta U - \mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big] + O\big(\epsilon^7\big)\Big\}\Big\{\frac{1}{c^2}\big(U_{,i3} - V_{,i3} - \Delta V\delta_{i3}\big) + \frac{1}{c^4}\Big[3V_{,i}V_{,3} + \cdots - U_{,i}U_{,3}\Big] + O\big(\epsilon^6\big)\Big\} \sim O\big(\epsilon^7\big)\,;$$

therefore,

$$2R_0{}^{\alpha}R_{i\alpha} \sim O\big(\epsilon^7\big)\,.$$

2-nd term

$$-\frac{1}{2}\,g_{0i}R_{\alpha\beta}R^{\alpha\beta}=-\frac{1}{2}\Big\{\frac{h_{0i}}{c^3}+O\big(\epsilon^5\big)\Big\}\Big\{\frac{1}{c^4}\Big\{\big(\Delta U\big)^2+5\big(\Delta V\big)^2+\cdots+\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big)+\cdots+\boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big]+\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big)+\cdots+\boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]-2\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big)+\cdots+\boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}\Big\}\sim O\big(\epsilon^7\big)\,,$$

that is,

$$-\frac{1}{2}\,g_{0i}R_{\alpha\beta}R^{\alpha\beta}\sim O\big(\epsilon^7\big)\,.$$

3-rd term

$$\begin{aligned}-2R_{(0|}{}^{\alpha}{}_{,|i)\,\alpha}&=-2\Big[\frac{1}{2}\big(R_0{}^{\alpha}{}_{,i\alpha}+R_i{}^{\alpha}{}_{,0\alpha}\big)\Big]=-R_0{}^{\alpha}{}_{,i\alpha}\\&=-\big(R_0{}^0{}_{,i0}+R_0{}^1{}_{,i1}+R_0{}^2{}_{,i2}+R_0{}^3{}_{,i3}\big)\\&=-\big(R_0{}^1{}_{,i1}+R_0{}^2{}_{,i2}+R_0{}^3{}_{,i3}\big)\,,\end{aligned}$$

where,

- $$R_0{}^1{}_{,i1}=\Big\{\frac{1}{c^5}\Big[h_{01}\,\Delta U-\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,1}\big)\Big]+O\big(\epsilon^7\big)\Big\}_{,i1}=\frac{1}{c^5}\Big\{h_{01}\,\Delta U_{,i1}-\mathbf{h}\cdot\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\Big]_{,i1}\Big\}\sim O\big(\epsilon^5\big)\,;$$

- $$R_0{}^2{}_{,i2}=\Big\{\frac{1}{c^5}\Big[h_{02}\,\Delta U-\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,2}\big)\Big]+O\big(\epsilon^7\big)\Big\}_{,i2}=\frac{1}{c^5}\Big\{h_{02}\,\Delta U_{,i2}-\mathbf{h}\cdot\Big[\boldsymbol{\nabla}\big(U_{,2}\big)\Big]_{,i2}\Big\}\sim O\big(\epsilon^5\big)\,;$$

- $$R_0{}^3{}_{,i3}=\Big\{\frac{1}{c^5}\Big[h_{03}\,\Delta U-\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big]+O\big(\epsilon^7\big)\Big\}_{,i3}=\frac{1}{c^5}\Big\{h_{03}\,\Delta U_{,i3}-\mathbf{h}\cdot\Big[\boldsymbol{\nabla}\big(U_{,3}\big)\Big]_{,i3}\Big\}\sim O\big(\epsilon^5\big)\,;$$

therefore,

$$-2R_{(0|}{}^{\alpha}{}_{,|i)\,\alpha}=-\frac{1}{c^5}\Big\{h_{01}\,\Delta U_{,i1}+h_{02}\,\Delta U_{,i2}+h_{03}\,\Delta U_{,i3}-\mathbf{h}\cdot\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\Big]_{,i1}-\mathbf{h}\cdot\Big[\boldsymbol{\nabla}\big(U_{,2}\big)\Big]_{,i2}-\mathbf{h}\cdot\Big[\boldsymbol{\nabla}\big(U_{,3}\big)\Big]_{,i3}\Big\}\sim O\big(\epsilon^5\big)\,.$$

4-th term

$$-\Delta R_{0i} = -\Delta\Big\{\frac{1}{c^5}\,\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,i}\big) + O\big(\epsilon^7\big)\Big\} = -\frac{1}{c^5}\,\Delta\Big[\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,i}\big)\Big] \sim O\big(\epsilon^5\big)\,.$$

5-th term

$$\begin{aligned} g_{0i}R^{\alpha\beta}{}_{,\alpha\beta} &= \Big\{\frac{h_{0i}}{c^3} + O\big(\epsilon^5\big)\Big\}\Big\{\frac{1}{c^2}\big(\Delta^2 U - 2\,\Delta^2 V\big) + \frac{1}{c^4}\Big\{9\big(\Delta V\big)^2 + \cdots - 4\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) \\ &\quad + \cdots + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\} + O\big(\epsilon^6\big)\Big\} \\ &= \frac{h_{0i}}{c^5}\big(\Delta^2 U - 2\,\Delta^2 V\big) + O\big(\epsilon^7\big) \sim O\big(\epsilon^5\big)\,, \end{aligned}$$

(see (F.2) on page 198). Therefore,

$$\begin{aligned} {}^2H_{0i} &= {}^{(5)}{}^2H_{0i} + O\big(\epsilon^7\big)\,, \\ {}^{(5)}{}^2H_{0i} &= \frac{1}{c^5}\Big\{-h_{01}\,\Delta U_{,i1} - h_{02}\,\Delta U_{,i2} - h_{03}\,\Delta U_{,i3} + \mathbf{h}\cdot\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\Big]_{,i1} + \mathbf{h}\cdot\Big[\boldsymbol{\nabla}\big(U_{,2}\big)\Big]_{,i2} \\ &\quad + \mathbf{h}\cdot\Big[\boldsymbol{\nabla}\big(U_{,3}\big)\Big]_{,i3} - \Delta\Big[\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,i}\big)\Big] + h_{0i}\big(\Delta^2 U - 2\,\Delta^2 V\big)\Big\}\,. \end{aligned}$$

F.3.3 Space - Space components

$${}^2H_{ij} = 2R_i{}^{\alpha}R_{j\alpha} - \frac{1}{2}\,g_{ij}R_{\alpha\beta}R^{\alpha\beta} - 2R_{(i|}{}^{\alpha}{}_{,|j)\,\alpha} - \Delta R_{ij} + g_{ij}R^{\alpha\beta}{}_{,\alpha\beta}\,.$$

1-st term

$$2R_i{}^{\alpha}R_{j\alpha} = 2\big(R_i{}^0R_{j0} + R_i{}^1R_{j1} + R_i{}^2R_{j2} + R_i{}^3R_{j3}\big)\,,$$

where,

- $$R_i{}^0 = g^{0j}R_{ij} = \Big[-\frac{h_{0j}}{c^3} + O\big(\epsilon^5\big)\Big]\Big[\frac{1}{c^2}\big(U_{,ij} - V_{,ij} - \Delta V\delta_{ij}\big) + \frac{1}{c^4}\big(3V_{,i}V_{,j} + \cdots - U_{,i}U_{,j}\big) + O\big(\epsilon^6\big)\Big] \sim O\big(\epsilon^5\big)\,,$$

  from which,

  $$R_i{}^0R_{j0} = O\big(\epsilon^5\big)\Big[\frac{1}{c^5}\;\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,j}\big)\Big] \sim O\big(\epsilon^{10}\big)\,;$$

- $$R_i{}^1 = g^{1j}R_{ij} = \Big[\Big(-1 + \frac{2V}{c^2}\Big)\delta_{1j} + O\big(\epsilon^4\big)\Big]\Big[\frac{1}{c^2}\big(U_{,ij} - V_{,ij} - \Delta V\delta_{ij}\big) + \frac{1}{c^4}\big(3V_{,i}V_{,j}$$

$$
\begin{aligned}
&+\cdots-U_{,i}U_{,j}\big)+O\big(\epsilon^6\big)\Big]\\
&=-\frac{1}{c^2}\big(U_{,ij}-V_{,ij}-\Delta V\delta_{ij}\big)\delta_{1j}-\frac{1}{c^4}\big(3V_{,i}V_{,j}+\cdots-U_{,i}U_{,j}\big)\delta_{1j}+\frac{2V}{c^4}\big(U_{,ij}-V_{,ij}\\
&-\Delta V\delta_{ij}\big)\delta_{1j}+O\big(\epsilon^6\big)\\
&=-\frac{1}{c^2}\big(U_{,i1}-V_{,i1}-\Delta V\delta_{i1}\big)-\frac{1}{c^4}\Big[3V_{,i}V_{,1}+\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V+2V\Delta V\big)\delta_{i1}\\
&+2VV_{,i1}-\big(U_{,i}V_{,1}+U_{,1}V_{,i}\big)+2\Phi_{,i1}-U_{,i}U_{,1}\Big]+\frac{2V}{c^4}\big(U_{,i1}-V_{,i1}-\Delta V\delta_{i1}\big)+O\big(\epsilon^6\big)\\
&=-\frac{1}{c^2}\big(U_{,i1}-V_{,i1}-\Delta V\delta_{i1}\big)+\frac{1}{c^4}\Big[-3V_{,i}V_{,1}-\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)\delta_{i1}\\
&-4V\Delta V\delta_{i1}-4VV_{,i1}+\big(U_{,i}V_{,1}+U_{,1}V_{,i}\big)-2\Phi_{,i1}+U_{,i}U_{,1}+2VU_{,i1}\Big]+O\big(\epsilon^6\big)\,,
\end{aligned}
$$

from which,

$$
\begin{aligned}
R_i{}^1R_{j1}=&\left\{-\frac{1}{c^2}\big(U_{,i1}-V_{,i1}-\Delta V\delta_{i1}\big)+\frac{1}{c^4}\Big[-3V_{,i}V_{,1}-\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)\delta_{i1}\right.\\
&\left.-4V\Delta V\delta_{i1}-4VV_{,i1}+\big(U_{,i}V_{,1}+U_{,1}V_{,i}\big)-2\Phi_{,i1}+U_{,i}U_{,1}+2VU_{,i1}\Big]+O\big(\epsilon^6\big)\right\}\\
&\left[\frac{1}{c^2}\big(U_{,j1}-V_{,j1}-\Delta V\delta_{j1}\big)+\frac{1}{c^4}\big(3V_{,j}V_{,1}+\cdots-U_{,j}U_{,1}\big)+O\big(\epsilon^6\big)\right],
\end{aligned}
$$

that is,

$$
R_i{}^1R_{j1}=-\frac{1}{c^4}\big(U_{,i1}-V_{,i1}-\Delta V\delta_{i1}\big)\big(U_{,j1}-V_{,j1}-\Delta V\delta_{j1}\big)+O\big(\epsilon^6\big)\,;
$$

-

$$
\begin{aligned}
R_i{}^2=g^{2j}R_{ij}=&\left[\left(-1+\frac{2V}{c^2}\right)\delta_{2j}+O\big(\epsilon^4\big)\right]\left[\frac{1}{c^2}\big(U_{,ij}-V_{,ij}-\Delta V\delta_{ij}\big)+\frac{1}{c^4}\big(3V_{,i}V_{,j}\right.\\
&\left.+\cdots-U_{,i}U_{,j}\big)+O\big(\epsilon^6\big)\right]\\
&=-\frac{1}{c^2}\big(U_{,ij}-V_{,ij}-\Delta V\delta_{ij}\big)\delta_{2j}-\frac{1}{c^4}\big(3V_{,i}V_{,j}+\cdots-U_{,i}U_{,j}\big)\delta_{2j}+\frac{2V}{c^4}\big(U_{,ij}-V_{,ij}\\
&-\Delta V\delta_{ij}\big)\delta_{2j}+O\big(\epsilon^6\big)\\
&=-\frac{1}{c^2}\big(U_{,i2}-V_{,i2}-\Delta V\delta_{i2}\big)+\frac{1}{c^4}\Big[-3V_{,i}V_{,2}-\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)\delta_{i2}\\
&-4V\Delta V\delta_{i2}-4VV_{,i2}+\big(U_{,i}V_{,2}+U_{,2}V_{,i}\big)-2\Phi_{,i2}+U_{,i}U_{,2}+2VU_{,i2}\Big]+O\big(\epsilon^6\big)\,,
\end{aligned}
$$

from which,

$$
\begin{aligned}
R_i{}^2R_{j2}=&\left\{-\frac{1}{c^2}\big(U_{,i2}-V_{,i2}-\Delta V\delta_{i2}\big)+\frac{1}{c^4}\Big[-3V_{,i}V_{,2}-\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)\delta_{i2}\right.\\
&\left.-4V\Delta V\delta_{i2}-4VV_{,i2}+\big(U_{,i}V_{,2}+U_{,2}V_{,i}\big)-2\Phi_{,i2}+U_{,i}U_{,2}+2VU_{,i2}\Big]+O\big(\epsilon^6\big)\right\}\\
&\left[\frac{1}{c^2}\big(U_{,j2}-V_{,j2}-\Delta V\delta_{j2}\big)+\frac{1}{c^4}\big(3V_{,j}V_{,2}+\cdots-U_{,j}U_{,2}\big)+O\big(\epsilon^6\big)\right],
\end{aligned}
$$

that is,

$$R_i{}^2 R_{j2} = -\frac{1}{c^4}\big(U_{,i2} - V_{,i2} - \Delta V \delta_{i2}\big)\big(U_{,j2} - V_{,j2} - \Delta V \delta_{j2}\big) + O\big(\epsilon^6\big)\,;$$

•

$$\begin{aligned}
R_i{}^3 = g^{3j} R_{ij} &= \left[\left(-1 + \frac{2V}{c^2}\right)\delta_{3j} + O\big(\epsilon^4\big)\right]\left[\frac{1}{c^2}\big(U_{,ij} - V_{,ij} - \Delta V \delta_{ij}\big) + \frac{1}{c^4}\big(3V_{,i}V_{,j}\right. \\
&\left. + \cdots - U_{,i}U_{,j}\big) + O\big(\epsilon^6\big)\right] \\
&= -\frac{1}{c^2}\big(U_{,ij} - V_{,ij} - \Delta V \delta_{ij}\big)\delta_{3j} - \frac{1}{c^4}\big(3V_{,i}V_{,j} + \cdots - U_{,i}U_{,j}\big)\delta_{3j} + \frac{2V}{c^4}\big(U_{,ij} - V_{,ij} \\
&- \Delta V \delta_{ij}\big)\delta_{3j} + O\big(\epsilon^6\big) \\
&= -\frac{1}{c^2}\big(U_{,i3} - V_{,i3} - \Delta V \delta_{i3}\big) - \frac{1}{c^4}\Big[3V_{,i}V_{,3} + \big(\boldsymbol{\nabla} V \cdot \boldsymbol{\nabla} V + \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V + 2V\Delta V\big)\delta_{i3} \\
&+ 2VV_{,i3} - \big(U_{,i}V_{,3} + U_{,3}V_{,i}\big) + 2\Phi_{,i3} - U_{,i}U_{,3}\Big] + \frac{2V}{c^4}\big(U_{,i3} - V_{,i3} - \Delta V \delta_{i3}\big) + O\big(\epsilon^6\big) \\
&= -\frac{1}{c^2}\big(U_{,i3} - V_{,i3} - \Delta V \delta_{i3}\big) + \frac{1}{c^4}\Big[-3V_{,i}V_{,3} - \big(\boldsymbol{\nabla} V \cdot \boldsymbol{\nabla} V + \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V\big)\delta_{i3} \\
&- 4V\Delta V \delta_{i3} - 4VV_{,i3} + \big(U_{,i}V_{,3} + U_{,3}V_{,i}\big) - 2\Phi_{,i3} + U_{,i}U_{,3} + 2VU_{,i3}\Big] + O\big(\epsilon^6\big)\,,
\end{aligned}$$

from which,

$$\begin{aligned}
R_i{}^3 R_{j3} = \Big\{&-\frac{1}{c^2}\big(U_{,i3} - V_{,i3} - \Delta V \delta_{i3}\big) + \frac{1}{c^4}\Big[-3V_{,i}V_{,3} - \big(\boldsymbol{\nabla} V \cdot \boldsymbol{\nabla} V + \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V\big)\delta_{i3} \\
&- 4V\Delta V \delta_{i3} - 4VV_{,i3} + \big(U_{,i}V_{,3} + U_{,3}V_{,i}\big) - 2\Phi_{,i3} + U_{,i}U_{,3} + 2VU_{,i3}\Big] + O\big(\epsilon^6\big)\Big\} \\
&\left[\frac{1}{c^2}\big(U_{,j3} - V_{,j3} - \Delta V \delta_{j3}\big) + \frac{1}{c^4}\big(3V_{,j}V_{,3} + \cdots - U_{,j}U_{,3}\big) + O\big(\epsilon^6\big)\right],
\end{aligned}$$

that is,

$$R_i{}^3 R_{j3} = -\frac{1}{c^4}\big(U_{,i3} - V_{,i3} - \Delta V \delta_{i3}\big)\big(U_{,j3} - V_{,j3} - \Delta V \delta_{j3}\big) + O\big(\epsilon^6\big)\,.$$

Ultimately,

$$\begin{aligned}
2R_i{}^\alpha R_{j\alpha} = -\frac{2}{c^4}\Big[&\big(U_{,i1} - V_{,i1} - \Delta V \delta_{i1}\big)\big(U_{,j1} - V_{,j1} - \Delta V \delta_{j1}\big) + \big(U_{,i2} - V_{,i2} - \Delta V \delta_{i2}\big)\big(U_{,j2} \\
&- V_{,j2} - \Delta V \delta_{j2}\big) + \big(U_{,i3} - V_{,i3} - \Delta V \delta_{i3}\big)\big(U_{,j3} - V_{,j3} - \Delta V \delta_{j3}\big)\Big] + O\big(\epsilon^6\big)\,,
\end{aligned}$$

that is,

$$\begin{aligned}
2R_i{}^\alpha R_{j\alpha} = \frac{1}{c^4}\Big[&-2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(U_{,j}\big) + 2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + 2\,\boldsymbol{\nabla}\big(V_{,i}\big)\cdot\boldsymbol{\nabla}\big(U_{,j}\big) - 2\,\boldsymbol{\nabla}\big(V_{,i}\big) \\
&\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + 4U_{,ij}\Delta V - 4V_{,ij}\Delta V - 2\big(\Delta V\big)^2\delta_{ij}\Big] + O\big(\epsilon^6\big)\,.
\end{aligned}$$

2-nd term

$$-\frac{1}{2}\,g_{ij}R_{\alpha\beta}R^{\alpha\beta} = -\frac{1}{2}\Big[\Big(-1-\frac{2V}{c^2}\Big)\delta_{ij}+O\big(\epsilon^4\big)\Big]\Big\{\frac{1}{c^4}\Big\{\big(\Delta U\big)^2+5\big(\Delta V\big)^2-2\,\Delta U\Delta V+\cdots$$
$$-\,2\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big)+\cdots+\boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}\Big\}\,,$$

that is,

$$-\frac{1}{2}\,g_{ij}R_{\alpha\beta}R^{\alpha\beta} = \frac{1}{c^4}\Big\{\frac{1}{2}\big(\Delta U\big)^2+\frac{5}{2}\big(\Delta V\big)^2-\Delta U\Delta V+\frac{1}{2}\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big)+\cdots$$
$$+\,\boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big]+\frac{1}{2}\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big)+\cdots+\boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]$$
$$-\,\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big)+\cdots+\boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}+O\big(\epsilon^6\big)\,.$$

3-rd term

$$-2R_{(i|\ \ ,|j)\,\alpha}^{\ \ \ \ \ \ \alpha} = -2\Big[\frac{1}{2}\big(R_i{}^{\alpha}{}_{,j\alpha}+R_j{}^{\alpha}{}_{,i\alpha}\big)\Big] = -\big(R_i{}^{\alpha}{}_{,j\alpha}+R_j{}^{\alpha}{}_{,i\alpha}\big)\,;$$

- $$R_i{}^{\alpha}{}_{,j\alpha} = R_i{}^{0}{}_{,j0}+R_i{}^{1}{}_{,j1}+R_i{}^{2}{}_{,j2}+R_i{}^{3}{}_{,j3}$$
$$= R_i{}^{1}{}_{,j1}+R_i{}^{2}{}_{,j2}+R_i{}^{3}{}_{,j3}\,,$$

  where (see on page 202 and following),

$$R_i{}^{1}{}_{,j1} = -\frac{1}{c^2}\big(U_{,i1}-V_{,i1}-\Delta V\delta_{i1}\big)_{,j1}+\frac{1}{c^4}\Big[-3V_{,i}V_{,1}-\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)\delta_{i1}$$
$$-\,4V\Delta V\delta_{i1}-4VV_{,i1}+\big(U_{,i}V_{,1}+U_{,1}V_{,i}\big)-2\Phi_{,i1}+U_{,i}U_{,1}+2VU_{,i1}\Big]_{,j1}+O\big(\epsilon^6\big)$$
$$= -\frac{1}{c^2}\big(U_{,ij11}-V_{,ij11}-\Delta V_{,j1}\delta_{i1}\big)+\frac{1}{c^4}\Big[-3\big(V_{,ij1}V_{,1}+V_{,i1}V_{,j1}+V_{,ij}V_{,11}+V_{,i}V_{,j11}\big)$$
$$-\,\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,j1}\delta_{i1}-4\big(V\Delta V\big)_{,j1}\delta_{i1}-4\big(V_{,j1}V_{,i1}+V_{,1}V_{,ij1}+V_{,j}V_{,i11}$$
$$+\,VV_{,ij11}\big)+\big(U_{,ij1}V_{,1}+U_{,i1}V_{,j1}+U_{,ij}V_{,11}+U_{,i}V_{,j11}\big)+\big(U_{,j11}V_{,i}+U_{,11}V_{,ij}$$
$$+\,U_{,j1}V_{,i1}+U_{,1}V_{,ij1}\big)-2\Phi_{,ij11}+\big(U_{,ij1}U_{,1}+U_{,i1}U_{,j1}+U_{,ij}U_{,11}+U_{,i}U_{,j11}\big)$$
$$+\,2\big(V_{,j1}U_{,i1}+V_{,1}U_{,ij1}+V_{,j}U_{,i11}+VU_{,ij11}\big)\Big]+O\big(\epsilon^6\big)\,;$$

$$R_i{}^{2}{}_{,j2} = -\frac{1}{c^2}\big(U_{,i2}-V_{,i2}-\Delta V\delta_{i2}\big)_{,j2}+\frac{1}{c^4}\Big[-3V_{,i}V_{,2}-\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)\delta_{i2}$$
$$-\,4V\Delta V\delta_{i2}-4VV_{,i2}+\big(U_{,i}V_{,2}+U_{,2}V_{,i}\big)-2\Phi_{,i2}+U_{,i}U_{,2}+2VU_{,i2}\Big]_{,j2}+O\big(\epsilon^6\big)$$
$$= -\frac{1}{c^2}\big(U_{,ij22}-V_{,ij22}-\Delta V_{,j2}\delta_{i2}\big)+\frac{1}{c^4}\Big[-3\big(V_{,ij2}V_{,2}+V_{,i2}V_{,j2}+V_{,ij}V_{,22}+V_{,i}V_{,j22}\big)$$
$$-\,\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,j2}\delta_{i2}-4\big(V\Delta V\big)_{,j2}\delta_{i2}-4\big(V_{,j2}V_{,i2}+V_{,2}V_{,ij2}+V_{,j}V_{,i22}$$

$$
\begin{aligned}
&+ VV_{,ij22}\big) + \big(U_{,ij2}V_{,2} + U_{,i2}V_{,j2} + U_{,ij}V_{,22} + U_{,i}V_{,j22}\big) + \big(U_{,j22}V_{,i} + U_{,22}V_{,ij}\\
&+ U_{,j2}V_{,i2} + U_{,2}V_{,ij2}\big) - 2\Phi_{,ij22} + \big(U_{,ij2}U_{,2} + U_{,i2}U_{,j2} + U_{,ij}U_{,22} + U_{,i}U_{,j22}\big)\\
&+ 2\big(V_{,j2}U_{,i2} + V_{,2}U_{,ij2} + V_{,j}U_{,i22} + VU_{,ij22}\big)\Big] + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

$$
\begin{aligned}
R_i{}^3{}_{,j3} &= -\frac{1}{c^2}\big(U_{,i3} - V_{,i3} - \Delta V\delta_{i3}\big)_{,j3} + \frac{1}{c^4}\Big[-3V_{,i}V_{,3} - \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)\delta_{i3}\\
&- 4V\Delta V\delta_{i3} - 4VV_{,i3} + \big(U_{,i}V_{,3} + U_{,3}V_{,i}\big) - 2\Phi_{,i3} + U_{,i}U_{,3} + 2VU_{,i3}\Big]_{,j3} + O\big(\epsilon^6\big)\\
&= -\frac{1}{c^2}\big(U_{,ij33} - V_{,ij33} - \Delta V_{,j3}\delta_{i3}\big) + \frac{1}{c^4}\Big[-3\big(V_{,ij3}V_{,3} + V_{,i3}V_{,j3} + V_{,ij}V_{,33} + V_{,i}V_{,j33}\big)\\
&- \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,j3}\delta_{i3} - 4\big(V\Delta V\big)_{,j3}\delta_{i3} - 4\big(V_{,j3}V_{,i3} + V_{,3}V_{,ij3} + V_{,j}V_{,i33}\\
&+ VV_{,ij33}\big) + \big(U_{,ij3}V_{,3} + U_{,i3}V_{,j3} + U_{,ij}V_{,33} + U_{,i}V_{,j33}\big) + \big(U_{,j33}V_{,i} + U_{,33}V_{,ij}\\
&+ U_{,j3}V_{,i3} + U_{,3}V_{,ij3}\big) - 2\Phi_{,ij33} + \big(U_{,ij3}U_{,3} + U_{,i3}U_{,j3} + U_{,ij}U_{,33} + U_{,i}U_{,j33}\big)\\
&+ 2\big(V_{,j3}U_{,i3} + V_{,3}U_{,ij3} + V_{,j}U_{,i33} + VU_{,ij33}\big)\Big] + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
R_i{}^{\alpha}{}_{,j\alpha} &= -\frac{1}{c^2}\Big(\Delta U_{,ij} - \Delta V_{,ij} - \Delta V_{,j1}\delta_{i1} - \Delta V_{,j2}\delta_{i2} - \Delta V_{,j3}\delta_{i3}\Big) + \frac{1}{c^4}\Big[-7\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(V_{,ij}\big)\\
&- 7\,\boldsymbol{\nabla}\big(V_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) - 3V_{,ij}\Delta V - 3V_{,i}\Delta V_{,j} - 4V_{,j}\Delta V_{,i} - 4V\Delta V_{,ij} - \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\\
&+ \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,j1}\delta_{i1} - \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,j2}\delta_{i2} - \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,j3}\delta_{i3}\\
&- 4\big(V\Delta V\big)_{,j1}\delta_{i1} - 4\big(V\Delta V\big)_{,j2}\delta_{i2} - 4\big(V\Delta V\big)_{,j3}\delta_{i3} + 3\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) + 3\,\boldsymbol{\nabla}\big(U_{,i}\big)\\
&\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + U_{,ij}\Delta V + U_{,i}\Delta V_{,j} + V_{,i}\Delta U_{,j} + V_{,ij}\Delta U + \boldsymbol{\nabla}\big(U_{,j}\big)\cdot\boldsymbol{\nabla}\big(V_{,i}\big) + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(V_{,ij}\big)\\
&+ 2V_{,j}\Delta U_{,i} + 2V\Delta U_{,ij} - 2\Delta\Phi_{,ij} + \boldsymbol{\nabla}\big(U_{,ij}\big)\cdot\boldsymbol{\nabla}U + \boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(U_{,j}\big) + U_{,ij}\Delta U\\
&+ U_{,i}\Delta U_{,j}\Big] + O\big(\epsilon^6\big)\,;
\end{aligned}
$$

- by exchanging $i \leftrightarrow j$, immediately, it follows $R_j{}^{\alpha}{}_{,i\alpha}$,

$$
\begin{aligned}
R_j{}^{\alpha}{}_{,i\alpha} &= -\frac{1}{c^2}\Big[\Delta U_{,ji} - \Delta V_{,ji} - \Delta V_{,i1}\delta_{j1} - \Delta V_{,i2}\delta_{j2} - \Delta V_{,i3}\delta_{j3}\Big] + \frac{1}{c^4}\Big[-7\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(V_{,ji}\big)\\
&- 7\,\boldsymbol{\nabla}\big(V_{,j}\big)\cdot\boldsymbol{\nabla}\big(V_{,i}\big) - 3V_{,ji}\Delta V - 3V_{,j}\Delta V_{,i} - 4V_{,i}\Delta V_{,j} - 4V\Delta V_{,ji} - \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\\
&+ \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,i1}\delta_{j1} - \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,i2}\delta_{j2} - \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,i3}\delta_{j3}\\
&- 4\big(V\Delta V\big)_{,i1}\delta_{j1} - 4\big(V\Delta V\big)_{,i2}\delta_{j2} - 4\big(V\Delta V\big)_{,i3}\delta_{j3} + 3\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(U_{,ji}\big) + 3\,\boldsymbol{\nabla}\big(U_{,j}\big)\\
&\cdot\boldsymbol{\nabla}\big(V_{,i}\big) + U_{,ji}\Delta V + U_{,j}\Delta V_{,i} + V_{,j}\Delta U_{,i} + V_{,ji}\Delta U + \boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(V_{,ji}\big)\\
&+ 2V_{,i}\Delta U_{,j} + 2V\Delta U_{,ji} - 2\Delta\Phi_{,ji} + \boldsymbol{\nabla}\big(U_{,ji}\big)\cdot\boldsymbol{\nabla}U + \boldsymbol{\nabla}\big(U_{,j}\big)\cdot\boldsymbol{\nabla}\big(U_{,i}\big) + U_{,ji}\Delta U\\
&+ U_{,j}\Delta U_{,i}\Big] + O\big(\epsilon^6\big)\,.
\end{aligned}
$$

Ultimately,

$$
\begin{aligned}
-2R_{(i|}{}^{\alpha}{}_{,|j)\,\alpha} &= -\big(R_i{}^{\alpha}{}_{,j\alpha} + R_j{}^{\alpha}{}_{,i\alpha}\big)\\
&= \frac{1}{c^2}\big(2\,\Delta U_{,ij} - 2\,\Delta V_{,ij} - \Delta V_{,j1}\delta_{i1} - \Delta V_{,j2}\delta_{i2} - \Delta V_{,j3}\delta_{i3} - \Delta V_{,i1}\delta_{j1} - \Delta V_{,i2}\delta_{j2}
\end{aligned}
$$

$$
\begin{aligned}
&- \Delta V_{,i3}\delta_{j3}\big) + \frac{1}{c^4}\Big[14\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(V_{,ij}\big) + 14\,\boldsymbol{\nabla}\big(V_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + 6V_{,ij}\Delta V + 3V_{,i}\Delta V_{,j} \\
&+ 3V_{,j}\Delta V_{,i} + 4V_{,j}\Delta V_{,i} + 4V_{,i}\Delta V_{,j} + 8V\Delta V_{,ij} + \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,j1}\delta_{i1} \\
&+ \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,j2}\delta_{i2} + \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,j3}\delta_{i3} + \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V \\
&+ \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,i1}\delta_{j1} + \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,i2}\delta_{j2} + \big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,i3} \\
&\delta_{j3} + 4\big(V\Delta V\big)_{,j1}\delta_{i1} + 4\big(V\Delta V\big)_{,j2}\delta_{i2} + 4\big(V\Delta V\big)_{,j3}\delta_{i3} + 4\big(V\Delta V\big)_{,i1}\delta_{j1} \\
&+ 4\big(V\Delta V\big)_{,i2}\delta_{j2} + 4\big(V\Delta V\big)_{,i3}\delta_{j3} - 6\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) - 3\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) \\
&- 3\,\boldsymbol{\nabla}\big(U_{,j}\big)\cdot\boldsymbol{\nabla}\big(V_{,i}\big) - 2U_{,ij}\Delta V - U_{,i}\Delta V_{,j} - U_{,j}\Delta V_{,i} - V_{,i}\Delta U_{,j} - V_{,j}\Delta U_{,i} \\
&- 2V_{,ij}\Delta U - \boldsymbol{\nabla}\big(U_{,j}\big)\cdot\boldsymbol{\nabla}\big(V_{,i}\big) - \boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) - 2\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(V_{,ij}\big) - 2V_{,j}\Delta U_{,i} \\
&- 2V_{,i}\Delta U_{,j} - 4V\Delta U_{,ij} + 4\,\Delta\Phi_{,ij} - 2\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) - \boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(U_{,j}\big) - \boldsymbol{\nabla}\big(U_{,j}\big) \\
&\cdot\boldsymbol{\nabla}\big(U_{,i}\big) - 2U_{,ij}\Delta U - U_{,i}\Delta U_{,j} - U_{,j}\Delta U_{,i}\Big] + O\big(\epsilon^6\big) \\
&= \frac{1}{c^2}\big(2\,\Delta U_{,ij} - 4\,\Delta V_{,ij}\big) + \frac{1}{c^4}\Big[14\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(V_{,ij}\big) + 14\,\boldsymbol{\nabla}\big(V_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + 6V_{,ij}\Delta V \\
&+ 7V_{,i}\Delta V_{,j} + 7V_{,j}\Delta V_{,i} + 8V\Delta V_{,ij} + 2\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,ij} + 8\big(V\Delta V\big)_{,ij} \\
&- 6\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) - 4\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) - 4\,\boldsymbol{\nabla}\big(U_{,j}\big)\cdot\boldsymbol{\nabla}\big(V_{,i}\big) - 2U_{,ij}\Delta V - U_{,i}\Delta V_{,j} \\
&- U_{,j}\Delta V_{,i} - 3V_{,i}\Delta U_{,j} - 3V_{,j}\Delta U_{,i} - 2V_{,ij}\Delta U - 2\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(V_{,ij}\big) - 4V\Delta U_{,ij} \\
&+ 4\,\Delta\Phi_{,ij} - 2\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) - 2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(U_{,j}\big) - 2U_{,ij}\Delta U - U_{,i}\Delta U_{,j} - U_{,j}\Delta U_{,i}\Big] \\
&+ O\big(\epsilon^6\big)\,,
\end{aligned}
$$

and by expanding the 8-th term of the 4-th order expression, it follows,

$$
\begin{aligned}
&= \frac{1}{c^2}\big(2\,\Delta U_{,ij} - 4\,\Delta V_{,ij}\big) + \frac{1}{c^4}\Big[14\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(V_{,ij}\big) + 14\,\boldsymbol{\nabla}\big(V_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + 6V_{,ij}\Delta V \\
&+ 7V_{,i}\Delta V_{,j} + 7V_{,j}\Delta V_{,i} + 8V\Delta V_{,ij} + 2\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,ij} + 8V_{,ij}\Delta V + 8V_{,j}\Delta V_{,i} \\
&+ 8V_{,i}\Delta V_{,j} + 8V\Delta V_{,ij} - 6\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) - 4\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) - 4\,\boldsymbol{\nabla}\big(U_{,j}\big)\cdot\boldsymbol{\nabla}\big(V_{,i}\big) - 2U_{,ij}\Delta V \\
&- U_{,i}\Delta V_{,j} - U_{,j}\Delta V_{,i} - 3V_{,i}\Delta U_{,j} - 3V_{,j}\Delta U_{,i} - 2V_{,ij}\Delta U - 2\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(V_{,ij}\big) - 4V\Delta U_{,ij} + 4\,\Delta\Phi_{,ij} \\
&- 2\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) - 2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(U_{,j}\big) - 2U_{,ij}\Delta U - U_{,i}\Delta U_{,j} - U_{,j}\Delta U_{,i}\Big] + O\big(\epsilon^6\big)\,,
\end{aligned}
$$

that is,

$$
\begin{aligned}
-2R_{(i|\ \ ,|j)\,\alpha}^{\ \ \alpha} &= \frac{1}{c^2}\big(2\,\Delta U_{,ij} - 4\,\Delta V_{,ij}\big) + \frac{1}{c^4}\Big[14\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(V_{,ij}\big) + 14\,\boldsymbol{\nabla}\big(V_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + 14V_{,ij}\Delta V \\
&+ 15V_{,i}\Delta V_{,j} + 15V_{,j}\Delta V_{,i} + 16V\Delta V_{,ij} + 2\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,ij} - 6\,\boldsymbol{\nabla}V \\
&\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) - 4\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) - 4\,\boldsymbol{\nabla}\big(U_{,j}\big)\cdot\boldsymbol{\nabla}\big(V_{,i}\big) - 2U_{,ij}\Delta V - U_{,i}\Delta V_{,j} - U_{,j}\Delta V_{,i} \\
&- 3V_{,i}\Delta U_{,j} - 3V_{,j}\Delta U_{,i} - 2V_{,ij}\Delta U - 2\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(V_{,ij}\big) - 4V\Delta U_{,ij} + 4\,\Delta\Phi_{,ij} - 2\,\boldsymbol{\nabla}U \\
&\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) - 2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(U_{,j}\big) - 2U_{,ij}\Delta U - U_{,i}\Delta U_{,j} - U_{,j}\Delta U_{,i}\Big] + O\big(\epsilon^6\big)\,.
\end{aligned}
$$

4-th term

$$
\begin{aligned}
-\Delta R_{ij} &= -\Delta\bigg[\frac{1}{c^2}\big(U_{,ij} - V_{,ij} - \Delta V\delta_{ij}\big) + \frac{1}{c^4}\big(3V_{,i}V_{,j} + \cdots - U_{,i}U_{,j}\big) + O\big(\epsilon^6\big)\bigg] \\
&= -\frac{1}{c^2}\big(\Delta U_{,ij} - \Delta V_{,ij} - \Delta^2 V\delta_{ij}\big) - \frac{1}{c^4}\Big\{3V_{,j}\Delta V_{,i} + 6\,\boldsymbol{\nabla}\big(V_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + 3V_{,i}\Delta V_{,j}
\end{aligned}
$$

$$
\begin{aligned}
&+\Delta(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V)\delta_{ij}+\Delta(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V)\delta_{ij}+2\Big[(\Delta V)^2+2\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}(\Delta V)+V\Delta^2V\Big]\delta_{ij}\\
&+2\Big[V_{,ij}\Delta V+2\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}(V_{,ij})+V\Delta V_{,ij}\Big]-\Big[V_{,j}\Delta U_{,i}+2\,\boldsymbol{\nabla}(U_{,i})\cdot\boldsymbol{\nabla}(V_{,j})+U_{,i}\Delta V_{,j}\Big]\\
&-\Big[V_{,i}\Delta U_{,j}+2\,\boldsymbol{\nabla}(U_{,j})\cdot\boldsymbol{\nabla}(V_{,i})+U_{,j}\Delta V_{,i}\Big]+2\,\Delta\Phi_{,ij}-\Big[U_{,j}\Delta U_{,i}+2\,\boldsymbol{\nabla}(U_{,i})\cdot\boldsymbol{\nabla}(U_{,j})\\
&+U_{,i}\Delta U_{,j}\Big]\Big\}+O(\epsilon^6)\,,
\end{aligned}
$$

that is,

$$
\begin{aligned}
-\Delta R_{ij}=\frac{1}{c^2}(-\Delta U_{,ij}+\Delta V_{,ij}+\Delta^2V\delta_{ij})+\frac{1}{c^4}\Big\{&-3V_{,j}\Delta V_{,i}-6\,\boldsymbol{\nabla}(V_{,i})\cdot\boldsymbol{\nabla}(V_{,j})-3V_{,i}\Delta V_{,j}\\
&-\Big[\Delta(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V)+\Delta(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V)+2(\Delta V)^2+4\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}(\Delta V)+2V\Delta^2V\Big]\delta_{ij}\\
&-2V_{,ij}\Delta V-4\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}(V_{,ij})-2V\Delta V_{,ij}+V_{,j}\Delta U_{,i}+2\,\boldsymbol{\nabla}(U_{,i})\cdot\boldsymbol{\nabla}(V_{,j})+U_{,i}\Delta V_{,j}\\
&+V_{,i}\Delta U_{,j}+2\,\boldsymbol{\nabla}(U_{,j})\cdot\boldsymbol{\nabla}(V_{,i})+U_{,j}\Delta V_{,i}-2\,\Delta\Phi_{,ij}+U_{,j}\Delta U_{,i}+2\,\boldsymbol{\nabla}(U_{,i})\cdot\boldsymbol{\nabla}(U_{,j})\\
&+U_{,i}\Delta U_{,j}\Big\}+O(\epsilon^6)\,.
\end{aligned}
$$

5-th term

$$
\begin{aligned}
g_{ij}R^{\alpha\beta}{}_{,\alpha\beta}&=\left[-\left(1+\frac{2V}{c^2}\right)\delta_{ij}+O(\epsilon^4)\right]\left\{\frac{1}{c^2}(\Delta^2U-2\,\Delta^2V)+\frac{1}{c^4}\Big\{9(\Delta V)^2+\cdots-4\Big[\boldsymbol{\nabla}(U_{,1})\right.\\
&\left.\cdot\,\boldsymbol{\nabla}(V_{,1})+\cdots+\boldsymbol{\nabla}(U_{,3})\cdot\boldsymbol{\nabla}(V_{,3})\Big]\Big\}+O(\epsilon^6)\right\}\\
&=-\frac{1}{c^2}(\Delta^2U-2\,\Delta^2V)\delta_{ij}-\frac{1}{c^4}\Big\{9(\Delta V)^2+\cdots-4\Big[\boldsymbol{\nabla}(U_{,1})\cdot\boldsymbol{\nabla}(V_{,1})+\cdots+\boldsymbol{\nabla}(U_{,3})\\
&\cdot\,\boldsymbol{\nabla}(V_{,3})\Big]\Big\}\delta_{ij}-\frac{2V}{c^4}(\Delta^2U-2\,\Delta^2V)\delta_{ij}+O(\epsilon^6)\\
&=\frac{1}{c^2}\Big[(-\Delta^2U+2\,\Delta^2V)\delta_{ij}\Big]+\frac{1}{c^4}\Big\{-9(\Delta V)^2+(\Delta U)^2+2\,\Delta U\Delta V+4V(\Delta^2U\\
&-3\,\Delta^2V)-2\,\Delta^2\Phi+\Delta(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U)-32\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}(\Delta V)+9\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}(\Delta U)+\boldsymbol{\nabla}U\\
&\cdot\,\boldsymbol{\nabla}(\Delta V)-11\Big[\boldsymbol{\nabla}(V_{,1})\cdot\boldsymbol{\nabla}(V_{,1})+\cdots+\boldsymbol{\nabla}(V_{,3})\cdot\boldsymbol{\nabla}(V_{,3})\Big]-\Big[\boldsymbol{\nabla}(U_{,1})\cdot\boldsymbol{\nabla}(U_{,1})+\\
&\cdots+\boldsymbol{\nabla}(U_{,3})\cdot\boldsymbol{\nabla}(U_{,3})\Big]+4\Big[\boldsymbol{\nabla}(U_{,1})\cdot\boldsymbol{\nabla}(V_{,1})+\cdots+\boldsymbol{\nabla}(U_{,3})\cdot\boldsymbol{\nabla}(V_{,3})\Big]-2V(\Delta^2U\\
&-2\,\Delta^2V)\Big\}\delta_{ij}+O(\epsilon^6)\,,
\end{aligned}
$$

(see (F.2) on page 198) that is,

$$
\begin{aligned}
g_{ij}R^{\alpha\beta}{}_{,\alpha\beta}=\frac{1}{c^2}\Big[(-\Delta^2U+2\,\Delta^2V)\delta_{ij}\Big]+\frac{1}{c^4}\Big\{&-9(\Delta V)^2+(\Delta U)^2+2\,\Delta U\Delta V+2V\Delta^2U\\
&-8V\Delta^2V-2\,\Delta^2\Phi+\Delta(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U)-32\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}(\Delta V)+9\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}(\Delta U)\\
&+\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}(\Delta V)-11\Big[\boldsymbol{\nabla}(V_{,1})\cdot\boldsymbol{\nabla}(V_{,1})+\cdots+\boldsymbol{\nabla}(V_{,3})\cdot\boldsymbol{\nabla}(V_{,3})\Big]-\Big[\boldsymbol{\nabla}(U_{,1})\cdot\boldsymbol{\nabla}(U_{,1})\\
&+\cdots+\boldsymbol{\nabla}(U_{,3})\cdot\boldsymbol{\nabla}(U_{,3})\Big]+4\Big[\boldsymbol{\nabla}(U_{,1})\cdot\boldsymbol{\nabla}(V_{,1})+\cdots+\boldsymbol{\nabla}(U_{,3})\cdot\boldsymbol{\nabla}(V_{,3})\Big]\Big\}\delta_{ij}+O(\epsilon^6)\,.
\end{aligned}
$$

Therefore, the second-order term of ${}^{2}H_{ij}$ corresponds to the following expression,

$$
\begin{aligned}
{}^{(2)}{}^{2}H_{ij} &= \frac{1}{c^2}\Big[2\,\Delta U_{,ij} - 2\,\Delta V_{,ij} - \Delta V_{,j1}\delta_{i1} - \Delta V_{,j2}\delta_{i2} - \Delta V_{,j3}\delta_{i3} - \Delta V_{,i1}\delta_{j1} - \Delta V_{,i2}\delta_{j2} \\
&\quad - \Delta V_{,i3}\delta_{j3} - \Delta U_{,ij} + \Delta V_{,ij} + \Delta^2 V\delta_{ij} - \Delta^2 U\delta_{ij} + 2\,\Delta^2 V\delta_{ij}\Big] \\
&= \frac{1}{c^2}\Big[\Delta U_{,ij} - 3\,\Delta V_{,ij} - \Delta^2 U\delta_{ij} + 3\,\Delta^2 V\delta_{ij}\Big] \\
&= \frac{1}{c^2}\Big[\big(\Delta U - 3\,\Delta V\big)_{,ij} + \big(3\,\Delta^2 V - \Delta^2 U\big)\delta_{ij}\Big]\,;
\end{aligned}
$$

the fourth-order term, on the other hand, is given by,

$$
\begin{aligned}
{}^{(4)}{}^{2}H_{ij} &= \frac{1}{c^4}\Bigg\{-2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(U_{,j}\big) + 2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + 2\,\boldsymbol{\nabla}\big(V_{,i}\big)\cdot\boldsymbol{\nabla}\big(U_{,j}\big) - 2\,\boldsymbol{\nabla}\big(V_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) \\
&\quad + 4U_{,ij}\Delta V - 4V_{,ij}\Delta V - 2\,\big(\Delta V\big)^2\delta_{ij} + \Bigg\{\frac{1}{2}\big(\Delta U\big)^2 + \frac{5}{2}\big(\Delta V\big)^2 - \Delta U\Delta V + \frac{1}{2}\Big[\boldsymbol{\nabla}\big(U_{,1}\big) \\
&\quad \cdot\boldsymbol{\nabla}\big(U_{,1}\big) + \cdots + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big] + \frac{1}{2}\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \cdots + \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] \\
&\quad - \Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \cdots + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Bigg\}\delta_{ij} + 14\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(V_{,ij}\big) + 14\,\boldsymbol{\nabla}\big(V_{,i}\big) \\
&\quad \cdot\boldsymbol{\nabla}\big(V_{,j}\big) + 14V_{,ij}\Delta V + 15V_{,i}\Delta V_{,j} + 15V_{,j}\Delta V_{,i} + 16V\Delta V_{,ij} + 2\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,ij} \\
&\quad - 6\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) - 4\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) - 4\,\boldsymbol{\nabla}\big(U_{,j}\big)\cdot\boldsymbol{\nabla}\big(V_{,i}\big) - 2U_{,ij}\Delta V - U_{,i}\Delta V_{,j} \\
&\quad - U_{,j}\Delta V_{,i} - 3V_{,i}\Delta U_{,j} - 3V_{,j}\Delta U_{,i} - 2V_{,ij}\Delta U - 2\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(V_{,ij}\big) - 4V\Delta U_{,ij} + 4\,\Delta\Phi_{,ij} \\
&\quad - 2\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) - 2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(U_{,j}\big) - 2U_{,ij}\Delta U - U_{,i}\Delta U_{,j} - U_{,j}\Delta U_{,i} - 3V_{,j}\Delta V_{,i} \\
&\quad - 6\,\boldsymbol{\nabla}\big(V_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) - 3V_{,i}\Delta V_{,j} + \Bigg\{-\Delta\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big) - \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big) - 2\big(\Delta V\big)^2 - 4\,\boldsymbol{\nabla}V \\
&\quad \cdot\boldsymbol{\nabla}\big(\Delta V\big) - 2V\Delta^2 V\Bigg\}\delta_{ij} - 2V_{,ij}\Delta V - 4\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(V_{,ij}\big) - 2V\Delta V_{,ij} + V_{,j}\Delta U_{,i} + 2\,\boldsymbol{\nabla}\big(U_{,i}\big) \\
&\quad \cdot\boldsymbol{\nabla}\big(V_{,j}\big) + U_{,i}\Delta V_{,j} + V_{,i}\Delta U_{,j} + 2\,\boldsymbol{\nabla}\big(U_{,j}\big)\cdot\boldsymbol{\nabla}\big(V_{,i}\big) + U_{,j}\Delta V_{,i} - 2\,\Delta\Phi_{,ij} + U_{,j}\Delta U_{,i} \\
&\quad + 2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(U_{,j}\big) + U_{,i}\Delta U_{,j} + \Bigg\{-9\big(\Delta V\big)^2 + \big(\Delta U\big)^2 + 2\,\Delta U\Delta V + 2V\Delta^2 U \\
&\quad - 8V\Delta^2 V - 2\,\Delta^2\Phi + \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big) - 32\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta V\big) + 9\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U\big) + \boldsymbol{\nabla}U \\
&\quad \cdot\boldsymbol{\nabla}\big(\Delta V\big) - 11\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \cdots + \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] - \Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big) + \cdots \\
&\quad + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big] + 4\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \cdots + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Bigg\}\delta_{ij}\Bigg\} \\
&= \frac{1}{c^4}\Bigg\{-2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(U_{,j}\big) + 6\,\boldsymbol{\nabla}\big(V_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + 2U_{,ij}\Delta V + 8V_{,ij}\Delta V + 10\,\boldsymbol{\nabla}\big(V\big) \\
&\quad \cdot\boldsymbol{\nabla}\big(V_{,ij}\big) + 12V_{,i}\Delta V_{,j} + 12V_{,j}\Delta V_{,i} + 14V\Delta V_{,ij} + 2\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big)_{,ij} - 6\,\boldsymbol{\nabla}V \\
&\quad \cdot\boldsymbol{\nabla}\big(U_{,ij}\big) - 2V_{,i}\Delta U_{,j} - 2V_{,j}\Delta U_{,i} - 2V_{,ij}\Delta U - 2\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(V_{,ij}\big) - 4V\Delta U_{,ij} + 2\,\Delta\Phi_{,ij} \\
&\quad - 2\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) - 2U_{,ij}\Delta U + \Bigg\{-\frac{21}{2}\big(\Delta V\big)^2 + \frac{3}{2}\big(\Delta U\big)^2 + \Delta U\Delta V - \Delta\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big) \\
&\quad - \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big) - 36\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta V\big) - 10V\Delta^2 V + 2V\Delta^2 U - 2\,\Delta^2\Phi + \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big)
\end{aligned}
$$

$$+ 9\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U\big) + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(\Delta V\big) - \frac{1}{2}\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big) + \cdots + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big]$$
$$- \frac{21}{2}\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \cdots + \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] + 3\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \cdots + \boldsymbol{\nabla}\big(U_{,3}\big)$$
$$\cdot\,\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}\delta_{ij}\Big\},$$

that is,

$$^{(4)\,2}H_{ij} = \frac{1}{c^4}\Big\{-2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(U_{,j}\big) + 10\,\boldsymbol{\nabla}\big(V_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + 2U_{,ij}\Delta V + 8V_{,ij}\Delta V + 14\,\boldsymbol{\nabla}\big(V\big)$$
$$\cdot\,\boldsymbol{\nabla}\big(V_{,ij}\big) + 12V_{,i}\Delta V_{,j} + 12V_{,j}\Delta V_{,i} + 14V\Delta V_{,ij} - 4\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) + 2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big)$$
$$+ 2\,\boldsymbol{\nabla}\big(U_{,j}\big)\cdot\boldsymbol{\nabla}\big(V_{,i}\big) - 2V_{,i}\Delta U_{,j} - 2V_{,j}\Delta U_{,i} - 2V_{,ij}\Delta U - 4V\Delta U_{,ij} + 2\,\Delta\Phi_{,ij} - 2\,\boldsymbol{\nabla}U$$
$$\cdot\,\boldsymbol{\nabla}\big(U_{,ij}\big) - 2U_{,ij}\Delta U + \Big\{-\frac{21}{2}\big(\Delta V\big)^2 + \frac{3}{2}\big(\Delta U\big)^2 + \Delta U\Delta V - \Delta\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big) - \Delta\big(\boldsymbol{\nabla}U$$
$$\cdot\,\boldsymbol{\nabla}V\big) - 36\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta V\big) - 10V\Delta^2V + 2V\Delta^2U - 2\,\Delta^2\Phi + \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big) + 9\,\boldsymbol{\nabla}V$$
$$\cdot\,\boldsymbol{\nabla}\big(\Delta U\big) + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(\Delta V\big) - \frac{1}{2}\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big) + \cdots + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big]$$
$$- \frac{21}{2}\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \cdots + \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] + 3\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \cdots + \boldsymbol{\nabla}\big(U_{,3}\big)$$
$$\cdot\,\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}\delta_{ij}\Big\}.$$

Ultimately,

$$^{2}H_{ij} = {}^{(2)\,2}H_{ij} + {}^{(4)\,2}H_{ij} + O\big(\epsilon^6\big),$$
$$^{(2)\,2}H_{ij} = \frac{1}{c^2}\Big[\big(\Delta U - 3\,\Delta V\big)_{,ij} + \big(3\,\Delta^2V - \Delta^2U\big)\delta_{ij}\Big],$$
$$^{(4)\,2}H_{ij} = \frac{1}{c^4}\Big\{-2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(U_{,j}\big) + 10\,\boldsymbol{\nabla}\big(V_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + 2U_{,ij}\Delta V + 8V_{,ij}\Delta V + 14\,\boldsymbol{\nabla}\big(V\big)$$
$$\cdot\,\boldsymbol{\nabla}\big(V_{,ij}\big) + 12V_{,i}\Delta V_{,j} + 12V_{,j}\Delta V_{,i} + 14V\Delta V_{,ij} - 4\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) + 2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big)$$
$$+ 2\,\boldsymbol{\nabla}\big(U_{,j}\big)\cdot\boldsymbol{\nabla}\big(V_{,i}\big) - 2V_{,i}\Delta U_{,j} - 2V_{,j}\Delta U_{,i} - 2V_{,ij}\Delta U - 4V\Delta U_{,ij} + 2\,\Delta\Phi_{,ij} - 2\,\boldsymbol{\nabla}U$$
$$\cdot\,\boldsymbol{\nabla}\big(U_{,ij}\big) - 2U_{,ij}\Delta U + \Big\{-\frac{21}{2}\big(\Delta V\big)^2 + \frac{3}{2}\big(\Delta U\big)^2 + \Delta U\Delta V - \Delta\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big) - \Delta\big(\boldsymbol{\nabla}U$$
$$\cdot\,\boldsymbol{\nabla}V\big) - 36\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta V\big) - 10V\Delta^2V + 2V\Delta^2U - 2\,\Delta^2\Phi + \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big) + 9\,\boldsymbol{\nabla}V$$
$$\cdot\,\boldsymbol{\nabla}\big(\Delta U\big) + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(\Delta V\big) - \frac{1}{2}\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big) + \cdots + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big]$$
$$- \frac{21}{2}\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \cdots + \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] + 3\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \cdots + \boldsymbol{\nabla}\big(U_{,3}\big)$$
$$\cdot\,\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}\delta_{ij}\Big\}.$$

Summing up,

$$^{2}H_{00} = \frac{2}{c^2}\big(\Delta^2U - \Delta^2V\big) + \frac{1}{c^4}\Big\{-\frac{3}{2}\big(\Delta U\big)^2 + \frac{13}{2}\big(\Delta V\big)^2 - 3\,\Delta U\Delta V - 4\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(\Delta U\big) - 13\,\boldsymbol{\nabla}V$$

$$
\begin{aligned}
&\cdot \boldsymbol{\nabla}\big(\Delta U\big) - 4U\Delta^2 U - 6V\Delta^2 U + 4\Delta^2\Phi - 2\,\Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big) + \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big) + 12V\Delta^2 V\\
&+ 4U\Delta^2 V + 32\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta V\big) - \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(\Delta V\big) + \frac{21}{2}\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(V_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big)\\
&+ \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] + \frac{1}{2}\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big]\\
&- 3\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
{}^2H_{0i} = \frac{1}{c^5}\Big\{&-h_{01}\,\Delta U_{,i1} - h_{02}\,\Delta U_{,i2} - h_{03}\,\Delta U_{,i3} + \mathbf{h}\cdot\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\Big]_{,i1} + \mathbf{h}\cdot\Big[\boldsymbol{\nabla}\big(U_{,2}\big)\Big]_{,i2}\\
&+ \mathbf{h}\cdot\Big[\boldsymbol{\nabla}\big(U_{,3}\big)\Big]_{,i3} - \Delta\Big[\mathbf{h}\cdot\boldsymbol{\nabla}\big(U_{,i}\big)\Big] + h_{0i}\big(\Delta^2 U - 2\,\Delta^2 V\big)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
{}^2H_{ij} = \frac{1}{c^2}\Big[\big(\Delta U - 3\,\Delta V\big)_{,ij} + \big(3\,\Delta^2 V - \Delta^2 U\big)\delta_{ij}\Big] + \frac{1}{c^4}\Big\{&-2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(U_{,j}\big) + 10\,\boldsymbol{\nabla}\big(V_{,i}\big)\\
&\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + 2U_{,ij}\Delta V + 8V_{,ij}\Delta V + 14\,\boldsymbol{\nabla}\big(V\big)\cdot\boldsymbol{\nabla}\big(V_{,ij}\big) + 12V_{,i}\Delta V_{,j} + 12V_{,j}\Delta V_{,i} + 14V\Delta V_{,ij}\\
&- 4\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) + 2\,\boldsymbol{\nabla}\big(U_{,i}\big)\cdot\boldsymbol{\nabla}\big(V_{,j}\big) + 2\,\boldsymbol{\nabla}\big(U_{,j}\big)\cdot\boldsymbol{\nabla}\big(V_{,i}\big) - 2V_{,i}\Delta U_{,j} - 2V_{,j}\Delta U_{,i}\\
&- 2V_{,ij}\Delta U - 4V\Delta U_{,ij} + 2\,\Delta\Phi_{,ij} - 2\,\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(U_{,ij}\big) - 2U_{,ij}\Delta U + \Big\{-\frac{21}{2}\big(\Delta V\big)^2 + \frac{3}{2}\big(\Delta U\big)^2\\
&+ \Delta U\Delta V - \Delta\big(\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}V\big) - \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}V\big) - 36\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta V\big) - 10V\Delta^2 V + 2V\Delta^2 U\\
&- 2\,\Delta^2\Phi + \Delta\big(\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U\big) + 9\,\boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\big(\Delta U\big) + \boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}\big(\Delta V\big) - \frac{1}{2}\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(U_{,1}\big)\\
&+ \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(U_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(U_{,3}\big)\Big] - \frac{21}{2}\Big[\boldsymbol{\nabla}\big(V_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(V_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big)\\
&+ \boldsymbol{\nabla}\big(V_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big] + 3\Big[\boldsymbol{\nabla}\big(U_{,1}\big)\cdot\boldsymbol{\nabla}\big(V_{,1}\big) + \boldsymbol{\nabla}\big(U_{,2}\big)\cdot\boldsymbol{\nabla}\big(V_{,2}\big) + \boldsymbol{\nabla}\big(U_{,3}\big)\cdot\boldsymbol{\nabla}\big(V_{,3}\big)\Big]\Big\}\delta_{ij}\Big\}.
\end{aligned}
$$

Appendix G

Field equations as the coupling constants vary

- Case 1

$$a_2 = 0\,,$$
$$b_1 = 0\,,$$

$$\begin{cases} -2a_1\Delta V = 8\pi G_N \rho\,, \\ a_1\big(\Delta V - \Delta U\big)\delta_{ij} + a_1\big(U - V\big)_{,ij} = 0\,, \end{cases}$$

that is,

$$\begin{cases} \Delta V = -\frac{4\pi G_N}{a_1}\rho\,, \\ \big(\Delta V - \Delta U\big)\delta_{ij} - \big(V - U\big)_{,ij} = 0\,; \end{cases}$$

- Case 2

$$a_1 = 0\,,$$
$$b_1 = 0\,,$$

$$\begin{cases} 4a_2\Delta^2 U - 8a_2\Delta^2 V = 8\pi G_N \rho\,, \\ \big(-4a_2\Delta^2 U + 8a_2\Delta^2 V\big)\delta_{ij} + \big(4a_2\Delta U - 8a_2\Delta V\big)_{,ij} = 0\,, \end{cases}$$

$$\begin{cases} 4a_2\big(\Delta^2 U - 2\Delta^2 V\big) = 8\pi G_N \rho\,, \\ 4a_2\big(-\Delta^2 U + 2\Delta^2 V\big)\delta_{ij} + 4a_2\big(\Delta U - 2\Delta V\big)_{,ij} = 0\,, \end{cases}$$

that is,

$$\begin{cases} 2\Delta^2 V - \Delta^2 U = -\frac{2\pi G_N}{a_2}\rho\,, \\ \big(2\Delta^2 V - \Delta^2 U\big)\delta_{ij} - \big(2\Delta V - \Delta U\big)_{,ij} = 0\,; \end{cases}$$

- Case 3

$$a_1 = 0\,,$$
$$a_2 = 0\,,$$

$$\begin{cases} 2b_1\Delta^2 U - 2b_1\Delta^2 V = 8\pi G_N \rho\,, \\ \big(-b_1\Delta^2 U + 3b_1\Delta^2 V\big)\delta_{ij} + \big(b_1\Delta U - 3b_1\Delta V\big)_{,ij} = 0\,, \end{cases}$$

$$\begin{cases} 2b_1\big(\Delta^2 U - \Delta^2 V\big) = 8\pi G_N \rho\,, \\ b_1\big(3\Delta^2 V - \Delta^2 U\big)\delta_{ij} + b_1\big(\Delta U - 3\Delta V\big)_{,ij} = 0\,, \end{cases}$$

that is,

$$\begin{cases} \Delta^2 U - \Delta^2 V = \frac{4\pi G_N}{b_1}\rho\,, \\ -\big(\Delta^2 U - 3\Delta^2 V\big)\delta_{ij} + \big(\Delta U - 3\Delta V\big)_{,ij} = 0\,; \end{cases}$$

- Case 4

$$b_1 = -2a_2\,,$$

$$\begin{cases} -2a_1\Delta V - 4a_2\Delta^2 V = 8\pi G_N \rho\,, \\ \Big[a_1\big(\Delta V - \Delta U\big) - 2a_2\Delta^2 U + 2a_2\Delta^2 V\Big]\delta_{ij} + \Big[a_1\big(U - V\big) + 2a_2\Delta U - 2a_2\Delta V\Big]_{,ij} = 0\,, \end{cases}$$

$$\begin{cases} -2\big(a_1\Delta V + 2a_2\Delta^2 V\big) = 8\pi G_N \rho\,, \\ \Big[a_1\big(\Delta V - \Delta U\big) + 2a_2\big(\Delta^2 V - \Delta^2 U\big)\Big]\delta_{ij} + \Big[-a_1\big(V - U\big) - 2a_2\big(\Delta V - \Delta U\big)\Big]_{,ij} = 0\,, \end{cases}$$

that is,

$$\begin{cases} a_1\Delta V + 2a_2\Delta^2 V = -4\pi G_N \rho\,, \\ \Big[a_1\big(\Delta V - \Delta U\big) + 2a_2\big(\Delta^2 V - \Delta^2 U\big)\Big]\delta_{ij} - \Big[a_1\big(V - U\big) + 2a_2\big(\Delta V - \Delta U\big)\Big]_{,ij} = 0\,; \end{cases}$$

- Case 5

$$a_1 = 0\,,$$
$$b_1 = -4a_2\,,$$

$$\begin{cases} -4a_2\Delta^2 U = 8\pi G_N \rho\,, \\ -4a_2\Delta^2 V\delta_{ij} + 4a_2\Delta V_{,ij} = 0\,, \end{cases}$$

$$\begin{cases} \Delta^2 U = -\frac{2\pi G_N}{a_2}\rho\,, \\ -\Delta^2 V\delta_{ij} + \Delta V_{,ij} = 0\,, \end{cases}$$

that is,

$$\begin{cases} \Delta^2 U = -\frac{2\pi G_N}{a_2}\rho\,, \\ \Delta^2 V\delta_{ij} - \Delta V_{,ij} = 0\,; \end{cases}$$

- Case 6

$$
\begin{aligned}
a_1 &= 0\,, \\
b_1 &= -2a_2\,,
\end{aligned}
$$

$$
\begin{cases}
-4a_2\Delta^2 V = 8\pi G_N \rho\,, \\
\left(-2a_2\Delta^2 U + 2a_2\Delta^2 V\right)\delta_{ij} + \left(2a_2\Delta U - 2a_2\Delta V\right)_{,ij} = 0\,,
\end{cases}
$$

that is,

$$
\begin{cases}
\Delta^2 V = -\frac{2\pi G_N}{a_2}\rho\,, \\
\left(\Delta^2 V - \Delta^2 U\right)\delta_{ij} - \left(\Delta V - \Delta U\right)_{,ij} = 0\,;
\end{cases}
$$

- Case 7

$$
\begin{aligned}
a_1 &= 0\,, \\
b_1 &= -\frac{8a_2}{3}\,,
\end{aligned}
$$

$$
\begin{cases}
-\frac{4}{3}a_2\Delta^2 U - \frac{8}{3}a_2\Delta^2 V = 8\pi G_N \rho\,, \\
-\frac{4}{3}a_2\Delta^2 U\delta_{ij} + \frac{4}{3}a_2\Delta U_{,ij} = 0\,,
\end{cases}
$$

$$
\begin{cases}
-\frac{1}{3}a_2\Delta^2 U - \frac{2}{3}a_2\Delta^2 V = 2\pi G_N \rho\,, \\
-\Delta^2 U\delta_{ij} + \Delta U_{,ij} = 0\,,
\end{cases}
$$

$$
\begin{cases}
-\frac{a_2}{3}(\Delta^2 U + 2\Delta^2 V) = 2\pi G_N \rho\,, \\
\Delta^2 U\delta_{ij} - \Delta U_{,ij} = 0\,,
\end{cases}
$$

$$
\begin{cases}
\Delta^2 U + 2\Delta^2 V = -\frac{6\pi G_N}{a_2}\rho\,, \\
\Delta^2 U\delta_{ij} - \Delta U_{,ij} = 0\,,
\end{cases}
$$

that is,

$$
\begin{cases}
2\Delta^2 V + \Delta^2 U = -\frac{6\pi G_N}{a_2}\rho\,, \\
\Delta^2 U\delta_{ij} - \Delta U_{,ij} = 0\,.
\end{cases}
$$

Appendix H

Gravitational potentials expressed through Green's function

H.1 Study of the equations as the conditions on the coupling constants vary (Table 5.1)

- Case 1

$$\begin{cases} \Delta V = -\frac{4\pi G_N}{a_1}\rho\,, \\ \Delta(V-U)\delta_{ij} - (V-U)_{,ij} = 0 \end{cases} \quad \Rightarrow \quad V(\mathbf{x}) = U(\mathbf{x})\,,$$

where,

$$\Delta G(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')\,,$$

from which,

$$\begin{aligned} U(\mathbf{x}) = V(\mathbf{x}) &= -\frac{4\pi G_N}{a_1}\int G_{(\Delta)}(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}')d^3\mathbf{x}' \\ &= (2\pi)^{\frac{3}{2}}\frac{G_N}{a_1}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}' \\ &= \frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\,. \end{aligned}$$

- Case 2

$$\begin{cases} \Delta^2(2V-U) = -\frac{2\pi G_N}{a_2}\rho\,, \\ \Delta^2(2V-U)\delta_{ij} - \left[\Delta(2V-U)\right]_{,ij} = 0 \end{cases} \quad \Rightarrow \quad \Delta(2V-U) = 0\,,$$

that is,

$$2V(\mathbf{x}) - U(\mathbf{x}) = \frac{C}{|\mathbf{x}|}\,;$$

on the other hand, from the first equation,

$$2V(\mathbf{x}) - U(\mathbf{x}) = -\frac{2\pi G_N}{a_2}\int G_{(\Delta^2)}(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}')d^3\mathbf{x}'\,,$$

thus,

$$\begin{cases} 2V(\mathbf{x}) - U(\mathbf{x}) = -\frac{2\pi G_N}{a_2} \int G_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')d^3\mathbf{x}' \,, \\ 2V(\mathbf{x}) - U(\mathbf{x}) = \frac{C}{|\mathbf{x}|} \,. \end{cases}$$

This system is not solvable, since the $G_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')$ function is divergent. See for details section H.2.

- Case 3

$$\begin{cases} \Delta^2(U - V) = \frac{4\pi G_N}{b_1}\rho \\ -\Delta^2(U - 3V)\delta_{ij} + \big[\Delta(U - 3V)\big]_{,ij} = 0 \quad \Rightarrow \quad \Delta(U - 3V) = 0 \,, \end{cases}$$

that is,

$$U(\mathbf{x}) - 3V(\mathbf{x}) = \frac{C}{|\mathbf{x}|} \,,$$

and from the first equation,

$$\begin{aligned} U(\mathbf{x}) - V(\mathbf{x}) &= \frac{4\pi G_N}{b_1} \int G_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')d^3\mathbf{x}' \,, \\ 3V(\mathbf{x}) + \frac{C}{|\mathbf{x}|} - V(\mathbf{x}) &= \frac{4\pi G_N}{b_1} \int G_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')d^3\mathbf{x}' \,, \\ 2V(\mathbf{x}) &= -\frac{C}{|\mathbf{x}|} + \frac{4\pi G_N}{b_1} \int G_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')d^3\mathbf{x}' \,, \\ V(\mathbf{x}) &= -\frac{C}{2|\mathbf{x}|} + \frac{2\pi G_N}{b_1} \int G_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')d^3\mathbf{x}' \,; \end{aligned}$$

from this last one, it follows,

$$\begin{aligned} U(\mathbf{x}) = 3V(\mathbf{x}) + \frac{C}{|\mathbf{x}|} &= -\frac{3C}{2|\mathbf{x}|} + \frac{6\pi G_N}{b_1} \int G_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')d^3\mathbf{x}' + \frac{C}{|\mathbf{x}|} \\ &= -\frac{C}{2|\mathbf{x}|} + \frac{6\pi G_N}{b_1} \int G_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')d^3\mathbf{x}' \,; \end{aligned}$$

thus,

$$\begin{cases} U(\mathbf{x}) = -\frac{C}{2|\mathbf{x}|} + \frac{6\pi G_N}{b_1} \int G_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')d^3\mathbf{x}' \,, \\ V(\mathbf{x}) = -\frac{C}{2|\mathbf{x}|} + \frac{2\pi G_N}{b_1} \int G_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')d^3\mathbf{x}' \,, \end{cases}$$

solutions which do not exist. See for details section H.2.

- Case 4

$$\begin{cases} \big(a_1\Delta + 2a_2\Delta^2\big)V = -4\pi G_N\rho \,, \\ \big[a_1\Delta(V - U) + 2a_2\Delta^2(V - U)\big]\delta_{ij} - \big[a_1(V - U) + 2a_2\Delta(V - U)\big]_{,ij} = 0 \,, \end{cases}$$

where, from the second equation, it follows immediately,

$$U(\mathbf{x}) = V(\mathbf{x})\,,$$

and from the first one,

$$V(\mathbf{x}) = -4\pi G_N \int G_{(a_1\Delta + 2a_2\Delta^2)}(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}')d^3\mathbf{x}'\,,$$

with the related fourth-order differential equation,

$$(a_1\Delta_{\mathbf{x}} + 2a_2\Delta^2_{\mathbf{x}})G(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')\,;$$

therefore, in terms of Fourier transforms, it reads,

$$\begin{aligned}
&(a_1\Delta_{\mathbf{x}} + 2a_2\Delta^2_{\mathbf{x}})\frac{1}{(2\pi)^{\frac{3}{2}}}\int G(\mathbf{k})e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}d^3\mathbf{k} = \frac{1}{(2\pi)^{\frac{3}{2}}}\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}d^3\mathbf{k}\,,\\
&\frac{1}{(2\pi)^{\frac{3}{2}}}\int G(\mathbf{k})\Big[-a_1k^2 + 2a_2k^4\Big]e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}d^3\mathbf{k} = \frac{1}{(2\pi)^{\frac{3}{2}}}\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}d^3\mathbf{k}\,,\\
&\big(-a_1k^2 + 2a_2k^4\big)G(\mathbf{k}) = 1 \quad \Rightarrow \quad G(\mathbf{k}) = \frac{1}{(2a_2k^4 - a_1k^2)}\,,
\end{aligned}$$

from which,

$$\begin{aligned}
G(\mathbf{x},\mathbf{x}') &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{1}{(2a_2k^4 - a_1k^2)}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}d^3\mathbf{k}\\
&= \frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{2a_2k^2\Big(k^2 - \frac{a_1}{2a_2}\Big)}d^3\mathbf{k} = \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{2a_2}\int \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k^2\Big(k^2 - \frac{a_1}{2a_2}\Big)}d^3\mathbf{k}\\
&= \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{2a_2}\int_0^{+\infty}\frac{k^2dk}{k^2\Big(k^2 - \frac{a_1}{2a_2}\Big)}\int_0^{\pi}e^{ikr\cos\theta}\sin\theta d\theta\int_0^{2\pi}d\phi
\end{aligned}$$

and setting,

$$\begin{cases} t = \cos\theta\,,\\ dt = -\sin\theta d\theta\,, \end{cases}$$

it follows,

$$\begin{aligned}
G(\mathbf{x},\mathbf{x}') &= \frac{1}{\sqrt{2\pi}}\frac{1}{2a_2}\int_0^{+\infty}\frac{dk}{\Big(k^2 - \frac{a_1}{2a_2}\Big)}\Big[-\int_1^{-1}e^{ikrt}dt\Big] = \frac{1}{\sqrt{2\pi}}\frac{1}{2a_2}\int_0^{+\infty}\frac{dk}{\Big(k^2 - \frac{a_1}{2a_2}\Big)}\int_{-1}^{1}e^{ikrt}dt\\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{2a_2}\int_0^{+\infty}\frac{dk}{\Big(k^2 - \frac{a_1}{2a_2}\Big)}\left[\frac{e^{ikrt}}{ikr}\right]_{-1}^{+1} = \frac{1}{\sqrt{2\pi}}\frac{1}{2a_2}\int_0^{+\infty}\frac{dk}{\Big(k^2 - \frac{a_1}{2a_2}\Big)}\left[\frac{e^{ikr} - e^{-ikr}}{ikr}\right]\\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{2a_2}\int_0^{+\infty}\frac{dk}{\Big(k^2 - \frac{a_1}{2a_2}\Big)}\left[\frac{2i\sin(kr)}{ikr}\right] = \frac{1}{\sqrt{2\pi}}\frac{1}{2a_2}\frac{2}{r}\int_0^{+\infty}\frac{\sin(kr)}{k\Big(k^2 - \frac{a_1}{2a_2}\Big)}dk
\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{a_2}\frac{1}{r}\int_0^{+\infty}\frac{\sin(kr)dk}{k\left(k^2-\frac{a_1}{2a_2}\right)} = \frac{1}{\sqrt{2\pi}}\frac{1}{a_2}\frac{1}{r}I \quad \rightarrow \quad I = \int_0^{+\infty}\frac{\sin(kr)dk}{k\left(k^2-\frac{a_1}{2a_2}\right)};$$

such integral has to be computed in the complex plane by Jordan's lemma, that is,

$$I_z = \oint \frac{e^{irz}dz}{z(z^2-\lambda^2)} = 0 \qquad \lambda = \sqrt{\frac{a_1}{2a_2}} > 0\,,$$
$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\left\{\int_{-R}^{-\lambda-\epsilon}\frac{e^{ikr}dk}{k(k^2-\lambda^2)} + \int_{\gamma_1}\frac{e^{irz}dz}{z(z^2-\lambda^2)} + \int_{-\lambda+\epsilon}^{-\epsilon}\frac{e^{ikr}dk}{k(k^2-\lambda^2)}\right.$$
$$+ \int_{\gamma}\frac{e^{irz}dz}{z(z^2-\lambda^2)} + \int_{\epsilon}^{\lambda-\epsilon}\frac{e^{ikr}dk}{k(k^2-\lambda^2)} + \int_{\gamma_2}\frac{e^{irz}dz}{z(z^2-\lambda^2)} + \int_{\lambda+\epsilon}^{R}\frac{e^{ikr}dk}{k(k^2-\lambda^2)}$$
$$\left. + \int_{\Gamma}\frac{e^{irz}dz}{z(z^2-\lambda^2)}\right\} = 0\,,$$
$$\int_{-\infty}^{-\lambda}\frac{e^{ikr}dk}{k(k^2-\lambda^2)} + \lim_{\epsilon\to 0}\int_{\gamma_1}\frac{e^{irz}dz}{z(z^2-\lambda^2)} + \int_{-\lambda}^{0}\frac{e^{ikr}dk}{k(k^2-\lambda^2)}$$
$$+ \lim_{\epsilon\to 0}\int_{\gamma}\frac{e^{irz}dz}{z(z^2-\lambda^2)} + \int_{0}^{\lambda}\frac{e^{ikr}dk}{k(k^2-\lambda^2)} + \lim_{\epsilon\to 0}\int_{\gamma_2}\frac{e^{irz}dz}{z(z^2-\lambda^2)}$$
$$+ \int_{\lambda}^{+\infty}\frac{e^{ikr}dk}{k(k^2-\lambda^2)} + \lim_{R\to\infty}\int_{\Gamma}\frac{e^{irz}dz}{z(z^2-\lambda^2)} = 0\,,$$
$$+ \int_{-\infty}^{+\infty}\frac{e^{ikr}dk}{k(k^2-\lambda^2)} + \lim_{\epsilon\to 0}\int_{\gamma_1}\frac{e^{irz}dz}{z(z^2-\lambda^2)} + \lim_{\epsilon\to 0}\int_{\gamma}\frac{e^{irz}dz}{z(z^2-\lambda^2)}$$
$$+ \lim_{\epsilon\to 0}\int_{\gamma_2}\frac{e^{irz}dz}{z(z^2-\lambda^2)} = 0\,;$$

here,

$$\begin{cases} z = -\lambda + \epsilon e^{i\theta} & \pi \le \theta \le 0\,, \\ dz = i\epsilon e^{i\theta}d\theta\,, \end{cases}$$

$$\lim_{\epsilon\to 0}\int_{\gamma_1}\frac{e^{irz}dz}{z(z^2-\lambda^2)} = \lim_{\epsilon\to 0}\int_{\pi}^{0}\frac{e^{ir(-\lambda+\epsilon e^{i\theta})}i\epsilon e^{i\theta}d\theta}{(-\lambda+\epsilon e^{i\theta})(\lambda^2+\epsilon^2e^{2i\theta}-2\epsilon\lambda e^{i\theta}-\lambda^2)}$$
$$= \lim_{\epsilon\to 0}\int_{\pi}^{0}\frac{e^{ir(-\lambda+\epsilon e^{i\theta})}i\epsilon e^{i\theta}d\theta}{(-\lambda+\epsilon e^{i\theta})\epsilon e^{i\theta}(\epsilon e^{i\theta}-2\lambda)} = i\lim_{\epsilon\to 0}\int_{\pi}^{0}\frac{e^{ir(-\lambda+\epsilon e^{i\theta})}d\theta}{(-\lambda+\epsilon e^{i\theta})(\epsilon e^{i\theta}-2\lambda)}$$
$$= i\int_{\pi}^{0}\frac{e^{-ir\lambda}}{2\lambda^2}d\theta = \frac{i}{2\lambda^2}e^{-ir\lambda}\int_{\pi}^{0}d\theta = -\frac{i\pi}{2}\frac{e^{-ir\lambda}}{\lambda^2};$$

$$\begin{cases} z = \epsilon e^{i\theta} & 0 \le \theta \le \pi\,, \\ dz = i\epsilon e^{i\theta}d\theta\,, \end{cases}$$

$$\lim_{\epsilon\to 0}\int_{\gamma}\frac{e^{irz}dz}{z(z^2-\lambda^2)} = \lim_{\epsilon\to 0}\int_{\pi}^{0}\frac{e^{ir\epsilon e^{i\theta}}i\epsilon e^{i\theta}d\theta}{\epsilon e^{i\theta}(\epsilon^2e^{2i\theta}-\lambda^2)} = i\lim_{\epsilon\to 0}\int_{\pi}^{0}\frac{e^{ir\epsilon e^{i\theta}}d\theta}{(\epsilon^2e^{2i\theta}-\lambda^2)}$$
$$= -\frac{i}{\lambda^2}\int_{\pi}^{0}d\theta = \frac{i\pi}{\lambda^2};$$

$$\begin{cases} z = \lambda + \epsilon e^{i\theta} & 0 \le \theta \le \pi\,, \\ dz = i\epsilon e^{i\theta} d\theta\,, \end{cases}$$

$$\begin{aligned} &\lim_{\epsilon\to 0}\int_{\gamma_2} \frac{e^{irz}dz}{z(z^2-\lambda^2)} = \lim_{\epsilon\to 0}\int_{\pi}^{0} \frac{e^{ir(\lambda+\epsilon e^{i\theta})} i\epsilon e^{i\theta} d\theta}{(\lambda+\epsilon e^{i\theta})(\lambda^2+\epsilon^2 e^{2i\theta}+2\epsilon\lambda e^{i\theta}-\lambda^2)} \\ &= \lim_{\epsilon\to 0}\int_{\pi}^{0} \frac{e^{ir(\lambda+\epsilon e^{i\theta})} i\epsilon e^{i\theta} d\theta}{(\lambda+\epsilon e^{i\theta})\epsilon e^{i\theta}(\epsilon e^{i\theta}+2\lambda)} = i\lim_{\epsilon\to 0}\int_{\pi}^{0} \frac{e^{ir(\lambda+\epsilon e^{i\theta})} d\theta}{(\lambda+\epsilon e^{i\theta})(\epsilon e^{i\theta}+2\lambda)} \\ &= \frac{i}{2}\frac{e^{ir\lambda}}{\lambda^2}\int_{\pi}^{0} d\theta = -\frac{i\pi}{2}\frac{e^{ir\lambda}}{\lambda^2}\,; \end{aligned}$$

therefore,

$$\begin{aligned} &\int_{-\infty}^{+\infty} \frac{e^{ikr}dk}{k(k^2-\lambda^2)} - \frac{i\pi}{2}\frac{e^{-ir\lambda}}{\lambda^2} + \frac{i\pi}{\lambda^2} - \frac{i\pi}{2}\frac{e^{ir\lambda}}{\lambda^2} = 0\,, \\ &\int_{-\infty}^{+\infty} \frac{e^{ikr}dk}{k(k^2-\lambda^2)} = -\frac{i\pi}{\lambda^2} + \frac{i\pi}{\lambda^2}\left(\frac{e^{ir\lambda}+e^{-ir\lambda}}{2}\right) = -\frac{i\pi}{\lambda^2} + \frac{i\pi}{\lambda^2}\cos(\lambda r) = \frac{i\pi}{\lambda^2}\Big[\cos(\lambda r)-1\Big]\,, \\ &\int_{-\infty}^{+\infty} \frac{\cos(kr)dk}{k(k^2-\lambda^2)} + i\int_{-\infty}^{+\infty} \frac{\sin(kr)dk}{k(k^2-\lambda^2)} = \frac{i\pi}{\lambda^2}\Big[\cos(\lambda r)-1\Big]\,, \end{aligned}$$

from which,

$$\begin{cases} \int_{-\infty}^{+\infty} \frac{\cos(kr)dk}{k(k^2-\lambda^2)} = 0\,, \\ \int_{-\infty}^{+\infty} \frac{\sin(kr)dk}{k(k^2-\lambda^2)} = \frac{\pi}{\lambda^2}\Big[\cos(\lambda r)-1\Big]\,. \end{cases}$$

From the second result, it follows the integral looked for,

$$\begin{aligned} &2\int_{0}^{+\infty} \frac{\sin(kr)dk}{k(k^2-\lambda^2)} = \frac{\pi}{\lambda^2}\Big[\cos(\lambda r)-1\Big]\,, \\ &I = \int_{0}^{+\infty} \frac{\sin(kr)dk}{k(k^2-\lambda^2)} = \frac{\pi}{2\lambda^2}\Big[\cos(\lambda r)-1\Big]\,. \end{aligned}$$

Definitely,

$$\begin{aligned} &G(\mathbf{x},\mathbf{x}') = \frac{1}{\sqrt{2\pi}}\frac{1}{a_2}\frac{1}{r}I = \frac{1}{\sqrt{2\pi}}\frac{1}{a_2}\frac{1}{r}\frac{\pi}{2\lambda^2}\Big[\cos(\lambda r)-1\Big] = \sqrt{\frac{\pi}{2}}\frac{1}{r}\frac{1}{a_1}\Big[\cos(\lambda r)-1\Big]\,, \\ &G(\mathbf{x},\mathbf{x}') = \frac{1}{a_1}\sqrt{\frac{\pi}{2}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\left[\cos\left(\sqrt{\frac{a_1}{2a_2}}|\mathbf{x}-\mathbf{x}'|\right)-1\right]. \end{aligned}$$

Thus, the solution is as follows:

$$U(\mathbf{x}) = V(\mathbf{x}) = -4\pi G_N \frac{1}{a_1}\sqrt{\frac{\pi}{2}}\int \frac{\left[\cos\left(\sqrt{\frac{a_1}{2a_2}}|\mathbf{x}-\mathbf{x}'|\right)-1\right]}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}')d^3\mathbf{x}'$$

$$= (2\pi)^{\frac{3}{2}} \frac{G_N}{a_1} \int \frac{\left[1 - \cos\left(\sqrt{\frac{a_1}{2a_2}}|\mathbf{x} - \mathbf{x}'|\right)\right]}{|\mathbf{x} - \mathbf{x}'|} \rho(\mathbf{x}') d^3\mathbf{x}' \,.$$

- Case 5

$$\begin{cases} \Delta^2 U = -\frac{2\pi G_N}{a_2}\rho \,, \\ \Delta^2 V \delta_{ij} - \Delta V_{,ij} = 0 \quad \Rightarrow \quad \Delta V = 0 \quad \Rightarrow \quad V(\mathbf{x}) = \frac{C}{|\mathbf{x}|} \,, \end{cases}$$

that is,

$$\begin{cases} U(\mathbf{x}) = -\frac{2\pi G_N}{a_2} \int G_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}') d^3\mathbf{x}' \,, \\ V(\mathbf{x}) = \frac{C}{|\mathbf{x}|} \,, \end{cases}$$

for which does not exist solution (see section H.2).

- Case 6

$$\begin{cases} \Delta^2 V = -\frac{2\pi G_N}{a_2}\rho \,, \\ \Delta^2 (V - U)\delta_{ij} - \big[\Delta(V - U)\big]_{,ij} = 0, \quad \Rightarrow \quad \Delta(V - U) = 0 \quad \Rightarrow \quad V(\mathbf{x}) - U(\mathbf{x}) = \frac{C}{|\mathbf{x}|} \,, \end{cases}$$

that is,

$$\begin{cases} U(\mathbf{x}) = V(\mathbf{x}) - \frac{C}{|\mathbf{x}|} = -\frac{C}{|\mathbf{x}|} - \frac{2\pi G_N}{a_2} \int G_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}') d^3\mathbf{x}' \,, \\ V(\mathbf{x}) = -\frac{2\pi G_N}{a_2} \int G_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}') d^3\mathbf{x}' \,, \end{cases}$$

which has not solution (see section H.2).

- Case 7

$$\begin{cases} \Delta^2 (2V + U) = -\frac{6\pi GN}{a_2}\rho \,, \\ \Delta^2 U \delta_{ij} - \Delta U_{,ij} = 0 \quad \Rightarrow \quad \Delta U = 0 \quad \Rightarrow \quad U(\mathbf{x}) = \frac{C}{|\mathbf{x}|} \,, \end{cases}$$

from the first one,

$$\begin{aligned} &2V + U = -\frac{6\pi G_N}{a_2} \int G_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}') d^3\mathbf{x}' \,, \\ &2V = -U - \frac{6\pi G_N}{a_2} \int G_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}') d^3\mathbf{x}' = -\frac{C}{|\mathbf{x}|} - \frac{6\pi G_N}{a_2} \int G_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}') d^3\mathbf{x}' \,, \\ &V(\mathbf{x}) = -\frac{1}{2}\frac{C}{|\mathbf{x}|} - \frac{3\pi G_N}{a_2} \int G_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}') d^3\mathbf{x}' \,, \end{aligned}$$

that is,

$$\begin{cases} U(\mathbf{x}) = \frac{C}{|\mathbf{x}|} \,, \\ V(\mathbf{x}) = -\frac{1}{2}\frac{C}{|\mathbf{x}|} - \frac{3\pi G_N}{a_2} \int G_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}') d^3\mathbf{x}' \,, \end{cases}$$

it too has not solution (see section H.2).

H.2 $G_{(\Delta^2)}(\mathbf{x},\mathbf{x}')$ function

From the corresponding fourth-order differential equation,

$$\Delta^2 G(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')\,,$$

it follows,

$$\begin{aligned}
\Delta^2_{\mathbf{x}}\left\{\frac{1}{(2\pi)^{\frac{3}{2}}}\int G(\mathbf{k})e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}d^3\mathbf{k}\right\} &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}d^3\mathbf{k}\,,\\
\frac{1}{(2\pi)^{\frac{3}{2}}}\int G(\mathbf{k})k^4 e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}d^3\mathbf{k} &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}d^3\mathbf{k}\,,\\
k^4 G(\mathbf{k}) &= 1\,,\\
G(\mathbf{k}) &= \frac{1}{k^4}\,,
\end{aligned}$$

from which,

$$\begin{aligned}
G_{(\Delta^2)}(\mathbf{x},\mathbf{x}') &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{1}{k^4}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}d^3\mathbf{k} = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_0^\infty \frac{1}{k^4}\,k^2 dk\int_0^\pi e^{ikr\cos\theta}\sin\theta d\theta\int_0^{2\pi} d\phi\\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{dk}{k^2}\int_0^\pi e^{ikr\cos\theta}\sin\theta d\theta\,,
\end{aligned}$$

and through the substitution,

$$\begin{cases} t = \cos\theta\,,\\ dt = -\sin\theta d\theta\,, \end{cases}$$

it follows,

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{dk}{k^2}\Big[-\int_1^{-1} e^{ikrt}dt\Big]\\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{dk}{k^2}\int_{-1}^{1} e^{ikrt}dt = \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{dk}{k^2}\left[\frac{e^{ikrt}}{ikr}\right]_{-1}^{+1}\\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{dk}{k^2}\left[\frac{e^{ikr}-e^{-ikr}}{ikr}\right] = \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{dk}{k^2}\frac{2i\sin(kr)}{ikr}\\
&= \sqrt{\frac{2}{\pi}}\frac{1}{r}\int_0^\infty \frac{\sin(kr)}{k^3}dk = \sqrt{\frac{2}{\pi}}\frac{1}{r}I \quad\rightarrow\quad I = \int_0^\infty \frac{\sin(kr)}{k^3}dk\,,
\end{aligned}$$

where the integral has to be computed in the complex plane through Jordan's Lemma; the corresponding closed path does not contain residues, thus,

$$\begin{aligned}
&I_z = \oint \frac{e^{irz}}{z^3}dz = 0\,,\\
&\lim_{R\to\infty}\lim_{\epsilon\to 0}\left\{\int_{-R}^{-\epsilon}\frac{e^{ikr}}{k^3}dk + \int_\gamma \frac{e^{izr}}{z^3}dz + \int_\epsilon^R \frac{e^{ikr}}{k^3}dk + \int_\Gamma \frac{e^{izr}}{z^3}dz\right\} = 0\,,\\
&\int_{-\infty}^0 \frac{e^{ikr}}{k^3}dk + \lim_{\epsilon\to o}\int_\gamma \frac{e^{izr}}{z^3}dz + \int_0^{+\infty}\frac{e^{ikr}}{k^3}dk = 0\,,
\end{aligned}$$

from which,

$$\int_{-\infty}^{+\infty} \frac{e^{ikr}}{k^3} dk + \lim_{\epsilon \to o} \int_{\gamma} \frac{e^{izr}}{z^3} dz = 0\,,$$
$$\int_{-\infty}^{+\infty} \frac{e^{ikr}}{k^3} dk = -\lim_{\epsilon \to o} \int_{\gamma} \frac{e^{irz}}{z^3} dz\,,$$
$$\int_{-\infty}^{+\infty} \frac{\cos(kr) + i\sin(kr)}{k^3} dk = -\lim_{\epsilon \to o} \int_{\gamma} \frac{e^{irz}}{z^3} dz\,;$$

here, it is convenient to set,

$$\begin{cases} z = \epsilon e^{i\theta} & \pi \leq \theta \leq 0\,, \\ dz = i\epsilon e^{i\theta} d\theta\,, \end{cases}$$

through which, the integral on the right hand side of the above equation becomes,

$$\lim_{\epsilon \to o} \int_{\pi}^{0} \frac{e^{ir\epsilon e^{i\theta}}}{\epsilon^3 e^{i3\theta}} i\epsilon e^{i\theta} d\theta = i \lim_{\epsilon \to o} \int_{\pi}^{0} \frac{e^{ir\epsilon e^{i\theta}} e^{-i2\theta}}{\epsilon^2} d\theta \to \infty\,.$$

Therefore the $G_{(\Delta^2)}(\mathbf{x}, \mathbf{x}')$ function does not exist.

Appendix I

Test results on scalars of interest, based on point-like sources potentials

The tests here accomplished, are based on point-like sources potentials and their first and higher-order derivatives, treated in Appendix B, section B.5, on page 89 and following. Therefore, it is referred to the aforementioned appendix for all basic tools necessary to the development of computations here below carried out.

I.1 Test result on R scalar

From the expansion of the Ricci's scalar,

$$
\begin{aligned}
R &= {}^{(2)}R + {}^{(4)}R + O(\epsilon^6)\,, \\
{}^{(2)}R &= -\frac{2}{c^2}\big(\Delta U - 2\,\Delta V\big)\,, \\
{}^{(4)}R &= \frac{2}{c^4}\Big[\boldsymbol{\nabla} U\cdot\big(\boldsymbol{\nabla} U - \boldsymbol{\nabla} V\big) - 3\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V + 2\,V\big(\Delta U - 4\,\Delta V\big) - 2\,\Delta\Phi\Big]\,,
\end{aligned}
$$

it follows:

- ${}^{(2)}R$ term

$$
\begin{aligned}
{}^{(2)}R &= -\frac{2}{c^2}\big(\Delta U - 2\,\Delta V\big) \\
&= -\frac{2}{c^2}\left[-\frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right) + \frac{G_N M}{a_1}\frac{2}{|\mathbf{x}|}\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\right] \\
&= -\frac{2}{c^2}\left(\frac{G_N M}{a_1}\right)\lambda_2^2\,\frac{1}{|\mathbf{x}|}\,e^{-\lambda_2|\mathbf{x}|}\,,
\end{aligned}
$$

that is,

$$
{}^{(2)}R = -\left(\frac{2\,G_N\,M\,\lambda_2^2}{a_1\,c^2}\right)\frac{1}{|\mathbf{x}|}\,e^{-\lambda_2|\mathbf{x}|}\,;
$$

- ${}^{(4)}R$ term

$$
{}^{(4)}R = \frac{2}{c^4}\Big[\boldsymbol{\nabla} U\cdot\big(\boldsymbol{\nabla} U - \boldsymbol{\nabla} V\big) - 3\,\boldsymbol{\nabla} V\cdot\boldsymbol{\nabla} V + 2\,V\big(\Delta U - 4\,\Delta V\big) - 2\,\Delta\Phi\Big]\,;
$$

it is not possible to compute this term, being the $\Phi(\boldsymbol{x})$ potential unknown; this is due to the fact that the field equations related to the quadratic Lagrangians referred to in Chapter 3, have not yet been solved in 1PN approximation of the metric tensor.

I.2 Test result on R^2 scalar

From R^2 expansion,

$$
\begin{aligned}
R^2 &= {}^{(4)}R^2 + O(\epsilon^6)\,,\\
{}^{(4)}R^2 &= \frac{4}{c^4}\Big[\big(\Delta U\big)^2 + 4\big(\Delta V\big)^2 - 4\,\Delta U\Delta V\Big]\,,
\end{aligned}
$$

it follows,

$$
\begin{aligned}
&{}^{(4)}R^2 = \frac{4}{c^4}\Big[\big(\Delta U\big)^2 + 4\big(\Delta V\big)^2 - 4\,\Delta U\Delta V\Big]\\
&= \frac{4}{c^4}\Bigg\{\bigg[-\frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\bigg(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\bigg)\bigg]^2 + 4\bigg[-\frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\bigg(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2\\
&e^{-\lambda_2|\mathbf{x}|}\bigg)\bigg]^2 - 4\bigg[-\frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\bigg(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\bigg)\bigg]\bigg[-\frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\bigg(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2\\
&e^{-\lambda_2|\mathbf{x}|}\bigg)\bigg]\\
&= \frac{4}{c^4}\Bigg[\bigg(\frac{G_N M}{a_1}\bigg)^2\frac{1}{|\mathbf{x}|^2}\bigg(\frac{16}{9}\,\lambda_1^4\,e^{-\,2\,\lambda_1|\mathbf{x}|} + \frac{1}{9}\,\lambda_2^4\,e^{-\,2\,\lambda_2|\mathbf{x}|} - \frac{8}{9}\,\lambda_1^2\,\lambda_2^2\,e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}\bigg) + 4\bigg(\frac{G_N M}{a_1}\bigg)^2\\
&\frac{1}{|\mathbf{x}|^2}\bigg(\frac{4}{9}\,\lambda_1^4\,e^{-\,2\,\lambda_1|\mathbf{x}|} + \frac{1}{9}\,\lambda_2^4\,e^{-\,2\,\lambda_2|\mathbf{x}|} + \frac{4}{9}\,\lambda_1^2\,\lambda_2^2\,e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}\bigg) - 4\bigg(\frac{G_N M}{a_1}\bigg)^2\frac{1}{|\mathbf{x}|^2}\bigg(\frac{8}{9}\,\lambda_1^4\,e^{-\,2\,\lambda_1|\mathbf{x}|}\\
&+ \frac{4}{9}\,\lambda_1^2\lambda_2^2\,e^{-(\lambda_1+\lambda_2)|\mathbf{x}|} - \frac{2}{9}\,\lambda_1^2\,\lambda_2^2\,e^{-(\lambda_1+\lambda_2)|\mathbf{x}|} - \frac{1}{9}\,\lambda_2^4\,e^{-\,2\,\lambda_2|\mathbf{x}|}\bigg)\Bigg]\\
&= \frac{4}{c^4}\bigg(\frac{G_N M}{a_1}\bigg)^2\frac{1}{|\mathbf{x}|^2}\,\lambda_2^4\,e^{-\,2\,\lambda_2|\mathbf{x}|}\,,
\end{aligned}
$$

that is,

$$
{}^{(4)}R^2 = \bigg(\frac{4\,G_N^2\,M^2\,\lambda_2^4}{a_1^2 c^4}\bigg)\frac{1}{|\mathbf{x}|^2}\,e^{-\,2\,\lambda_2|\mathbf{x}|}\,.
$$

I.3 Test result on $R_{\mu\nu}R^{\mu\nu}$ scalar

On the basis of $R_{\mu\nu}R^{\mu\nu}$ expansion,

$$
\begin{aligned}
R_{\mu\nu}R^{\mu\nu} &= {}^{(4)}R_{\mu\nu}R^{\mu\nu} + O(\epsilon^6)\,,\\
{}^{(4)}R_{\mu\nu}R^{\mu\nu} &= \frac{1}{c^4}\Big\{\big(\Delta U\big)^2 + 5\big(\Delta V\big)^2 - 2\,\Delta U\Delta V + \boldsymbol{\nabla}\big(\partial_1 U\big)\cdot\boldsymbol{\nabla}\big(\partial_1 U\big) + \boldsymbol{\nabla}\big(\partial_2 U\big)\cdot\boldsymbol{\nabla}\big(\partial_2 U\big)\\
&\quad + \boldsymbol{\nabla}\big(\partial_3 U\big)\cdot\boldsymbol{\nabla}\big(\partial_3 U\big) + \boldsymbol{\nabla}\big(\partial_1 V\big)\cdot\boldsymbol{\nabla}\big(\partial_1 V\big) + \boldsymbol{\nabla}\big(\partial_2 V\big)\cdot\boldsymbol{\nabla}\big(\partial_2 V\big) + \boldsymbol{\nabla}\big(\partial_3 V\big)\cdot\boldsymbol{\nabla}\big(\partial_3 V\big)\\
&\quad - 2\Big[\boldsymbol{\nabla}\big(\partial_1 U\big)\cdot\boldsymbol{\nabla}\big(\partial_1 V\big) + \boldsymbol{\nabla}\big(\partial_2 U\big)\cdot\boldsymbol{\nabla}\big(\partial_2 V\big) + \boldsymbol{\nabla}\big(\partial_3 U\big)\cdot\boldsymbol{\nabla}\big(\partial_3 V\big)\Big]\Big\}\,,
\end{aligned}
\tag{I.1}
$$

i am going to develop the various contributions separately in the next subsections.

I.3.1 $\Big[\big(\Delta U\big)^2 + 5\big(\Delta V\big)^2 - 2\,\Delta U \Delta V\Big]$

$$
\begin{aligned}
&\big(\Delta U\big)^2 + 5\big(\Delta V\big)^2 - 2\,\Delta U\Delta V \\
&= \left[-\frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\right]^2 + 5\left[-\frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\right]^2 \\
&- 2\left[-\frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\right]\left[-\frac{G_N M}{a_1}\frac{1}{|\mathbf{x}|}\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\right] \\
&= \left(\frac{G_N M}{a_1}\right)^2\frac{1}{|\mathbf{x}|^2}\left(\frac{16}{9}\,\lambda_1^4\,e^{-2\,\lambda_1|\mathbf{x}|} + \frac{1}{9}\,\lambda_2^4\,e^{-2\,\lambda_2|\mathbf{x}|} - \frac{8}{9}\,\lambda_1^2\,\lambda_2^2\,e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}\right) + 5\left(\frac{G_N M}{a_1}\right)^2\frac{1}{|\mathbf{x}|^2} \\
&\left(\frac{4}{9}\,\lambda_1^4\,e^{-2\,\lambda_1|\mathbf{x}|} + \frac{1}{9}\,\lambda_2^4\,e^{-2\,\lambda_2|\mathbf{x}|} + \frac{4}{9}\,\lambda_1^2\,\lambda_2^2\,e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}\right) - 2\left(\frac{G_N M}{a_1}\right)^2\frac{1}{|\mathbf{x}|^2}\left(\frac{8}{9}\,\lambda_1^4\,e^{-2\,\lambda_1|\mathbf{x}|}\right. \\
&\left. + \frac{4}{9}\,\lambda_1^2\lambda_2^2\,e^{-(\lambda_1+\lambda_2)|\mathbf{x}|} - \frac{2}{9}\,\lambda_1^2\,\lambda_2^2\,e^{-(\lambda_1+\lambda_2)|\mathbf{x}|} - \frac{1}{9}\,\lambda_2^4\,e^{-2\,\lambda_2|\mathbf{x}|}\right),
\end{aligned}
$$

that is,

$$
= \left(\frac{G_N M}{a_1}\right)^2\frac{1}{|\mathbf{x}|^2}\left(\frac{20}{9}\,\lambda_1^4\,e^{-2\,\lambda_1|\mathbf{x}|} + \frac{8}{9}\,\lambda_2^4\,e^{-2\,\lambda_2|\mathbf{x}|} + \frac{8}{9}\,\lambda_1^2\,\lambda_2^2\,e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}\right).
$$

I.3.2 $\Big[\boldsymbol{\nabla}\big(\partial_1 U\big)\cdot\boldsymbol{\nabla}\big(\partial_1 U\big) + \boldsymbol{\nabla}\big(\partial_2 U\big)\cdot\boldsymbol{\nabla}\big(\partial_2 U\big) + \boldsymbol{\nabla}\big(\partial_3 U\big)\cdot\boldsymbol{\nabla}\big(\partial_3 U\big)\Big]$

- 1-st term

$$
\begin{aligned}
&\boldsymbol{\nabla}\big(\partial_1 U\big)\cdot\boldsymbol{\nabla}\big(\partial_1 U\big) = \big(\partial_{11}U\big)^2 + \big(\partial_{12}U\big)^2 + \big(\partial_{13}U\big)^2 \\
&= \left(\frac{G_N M}{a_1}\right)^2\left\{\left[-\frac{1}{|\mathbf{x}|^3}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right) + \frac{1}{|\mathbf{x}|^2}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\right.\right. \\
&+ \frac{3x^2}{|\mathbf{x}|^5}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right) - \frac{3x^2}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right) - \frac{x^2}{|\mathbf{x}|^3}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}\right. \\
&\left.\left. - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\right]^2 + \left[\frac{3xy}{|\mathbf{x}|^5}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right) - \frac{3xy}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1 e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\right. \\
&\left. - \frac{xy}{|\mathbf{x}|^3}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\right]^2 + \left[\frac{3xz}{|\mathbf{x}|^5}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}e^{-\lambda_2|\mathbf{x}|}\right) - \frac{3xz}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}\right.\right. \\
&\left.\left. - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right) - \frac{xz}{|\mathbf{x}|^3}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\right]^2\Bigg\} \\
&= \left(\frac{G_N M}{a_1}\right)^2\left[\frac{1}{|\mathbf{x}|^6}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{1}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{3x^2}{|\mathbf{x}|^8}\right. \\
&\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{3x^2}{|\mathbf{x}|^6}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{x^2}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\right. \\
&\left.e^{-\lambda_2|\mathbf{x}|}\right)^2 - \frac{2}{|\mathbf{x}|^5}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right) - \frac{6x^2}{|\mathbf{x}|^7}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}\right. \\
&\left. + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right) - \frac{4x^2}{|\mathbf{x}|^6}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}\right.
\end{aligned}
$$

$$- \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big) + \frac{4x^2}{|\mathbf{x}|^5}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big]\,;$$

- 2-nd term

$$\boldsymbol{\nabla}\big(\partial_2 U\big)\cdot\boldsymbol{\nabla}\big(\partial_2 U\big) = \big(\partial_{12}U\big)^2 + \big(\partial_{22}U\big)^2 + \big(\partial_{23}U\big)^2$$
$$= \Big(\frac{G_N M}{a_1}\Big)^2\Bigg\{\Bigg[\frac{3xy}{|\mathbf{x}|^5}\Big(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big) - \frac{3xy}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big) - \frac{xy}{|\mathbf{x}|^3}$$
$$\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Bigg]^2 + \Bigg[-\frac{1}{|\mathbf{x}|^3}\Big(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big) + \frac{1}{|\mathbf{x}|^2}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}$$
$$- \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big) + \frac{3y^2}{|\mathbf{x}|^5}\Big(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big) - \frac{3y^2}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big) - \frac{y^2}{|\mathbf{x}|^3}$$
$$\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Bigg]^2 + \Bigg[\frac{3yz}{|\mathbf{x}|^5}\Big(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big) - \frac{3yz}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}$$
$$- \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big) - \frac{yz}{|\mathbf{x}|^3}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Bigg]^2\Bigg\}$$
$$= \Big(\frac{G_N M}{a_1}\Big)^2\Bigg[\frac{1}{|\mathbf{x}|^6}\Big(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)^2 + \frac{1}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)^2 + \frac{3y^2}{|\mathbf{x}|^8}$$
$$\Big(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)^2 + \frac{3y^2}{|\mathbf{x}|^6}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)^2 + \frac{y^2}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}$$
$$- \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)^2 - \frac{2}{|\mathbf{x}|^5}\Big(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big) - \frac{6y^2}{|\mathbf{x}|^7}\Big(1$$
$$- \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big) - \frac{4y^2}{|\mathbf{x}|^6}\Big(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{4}{3}\,\lambda_1^2$$
$$e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big) + \frac{4y^2}{|\mathbf{x}|^5}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Bigg]\,;$$

- 3-rd term

$$\boldsymbol{\nabla}\big(\partial_3 U\big)\cdot\boldsymbol{\nabla}\big(\partial_3 U\big) = \big(\partial_{13}U\big)^2 + \big(\partial_{23}U\big)^2 + \big(\partial_{33}U\big)^2$$
$$= \Big(\frac{G_N M}{a_1}\Big)^2\Bigg\{\Bigg[\frac{3xz}{|\mathbf{x}|^5}\Big(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big) - \frac{3xz}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big) - \frac{xz}{|\mathbf{x}|^3}$$
$$\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Bigg]^2 + \Bigg[\frac{3yz}{|\mathbf{x}|^5}\Big(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big) - \frac{3yz}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}$$
$$- \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big) - \frac{yz}{|\mathbf{x}|^3}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Bigg]^2 + \Bigg[-\frac{1}{|\mathbf{x}|^3}\Big(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)$$
$$+ \frac{1}{|\mathbf{x}|^2}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big) + \frac{3z^2}{|\mathbf{x}|^5}\Big(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big) - \frac{3z^2}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}$$
$$- \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big) - \frac{z^2}{|\mathbf{x}|^3}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Bigg]^2\Bigg\}$$

$$= \left(\frac{G_N M}{a_1}\right)^2 \Bigg[\frac{1}{|\mathbf{x}|^6}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{1}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{3z^2}{|\mathbf{x}|^8}$$
$$\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{3z^2}{|\mathbf{x}|^6}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{z^2}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}\right.$$
$$\left. - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)^2 - \frac{2}{|\mathbf{x}|^5}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right) - \frac{6z^2}{|\mathbf{x}|^7}\left(1\right.$$
$$\left. - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right) - \frac{4z^2}{|\mathbf{x}|^6}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{4}{3}\,\lambda_1^2\right.$$
$$\left. e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right) + \frac{4z^2}{|\mathbf{x}|^5}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\Bigg].$$

Summing up,

$$\boldsymbol{\nabla}\big(\partial_1 U\big)\cdot\boldsymbol{\nabla}\big(\partial_1 U\big) + \boldsymbol{\nabla}\big(\partial_2 U\big)\cdot\boldsymbol{\nabla}\big(\partial_2 U\big) + \boldsymbol{\nabla}\big(\partial_3 U\big)\cdot\boldsymbol{\nabla}\big(\partial_3 U\big)$$
$$= \left(\frac{G_N M}{a_1}\right)^2 \Bigg[\frac{3}{|\mathbf{x}|^6}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{3}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)^2$$
$$+ \frac{3\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^8}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{3\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^6}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)^2$$
$$+ \frac{\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)^2 - \frac{6}{|\mathbf{x}|^5}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}\right.$$
$$\left. - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right) - \frac{6\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^7}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)$$
$$- \frac{4\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^6}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right) + \frac{4\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^5}$$
$$\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right),$$

that is,

$$= \left(\frac{G_N M}{a_1}\right)^2 \Bigg[\frac{6}{|\mathbf{x}|^6}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{6}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{1}{|\mathbf{x}|^2}$$
$$\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)^2 - \frac{12}{|\mathbf{x}|^5}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)$$
$$- \frac{4}{|\mathbf{x}|^4}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right) + \frac{4}{|\mathbf{x}|^3}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\right.$$
$$\left. e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\Bigg].$$

I.3.3 $\Big[\boldsymbol{\nabla}\big(\partial_1 V\big)\cdot\boldsymbol{\nabla}\big(\partial_1 V\big) + \boldsymbol{\nabla}\big(\partial_2 V\big)\cdot\boldsymbol{\nabla}\big(\partial_2 V\big) + \boldsymbol{\nabla}\big(\partial_3 V\big)\cdot\boldsymbol{\nabla}\big(\partial_3 V\big)\Big]$

- 1-st term

$$\boldsymbol{\nabla}\big(\partial_1 V\big)\cdot\boldsymbol{\nabla}\big(\partial_1 V\big) = \big(\partial_{11} V\big)^2 + \big(\partial_{12} V\big)^2 + \big(\partial_{13} V\big)^2$$

$$= \left(\frac{G_N M}{a_1}\right)^2 \Bigg\{ \Bigg[-\frac{1}{|\mathbf{x}|^3}\left(1 - \frac{2}{3}\, e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\, e^{-\lambda_2|\mathbf{x}|}\right) + \frac{1}{|\mathbf{x}|^2}\left(\frac{2}{3}\,\lambda_1\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2\, e^{-\lambda_2|\mathbf{x}|}\right) + \frac{3x^2}{|\mathbf{x}|^5}$$
$$\left(1 - \frac{2}{3}\, e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\, e^{-\lambda_2|\mathbf{x}|}\right) - \frac{3x^2}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2\, e^{-\lambda_2|\mathbf{x}|}\right) - \frac{x^2}{|\mathbf{x}|^3}\left(\frac{2}{3}\,\lambda_1^2\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2$$
$$e^{-\lambda_2|\mathbf{x}|}\right)\Bigg]^2 + \Bigg[\frac{3xy}{|\mathbf{x}|^5}\left(1 - \frac{2}{3}\, e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\, e^{-\lambda_2|\mathbf{x}|}\right) - \frac{3xy}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2\, e^{-\lambda_2|\mathbf{x}|}\right) - \frac{xy}{|\mathbf{x}|^3}$$
$$\left(\frac{2}{3}\,\lambda_1^2\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2\, e^{-\lambda_2|\mathbf{x}|}\right)\Bigg]^2 + \Bigg[\frac{3xz}{|\mathbf{x}|^5}\left(1 - \frac{2}{3}\, e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\, e^{-\lambda_2|\mathbf{x}|}\right) - \frac{3xz}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1\, e^{-\lambda_1|\mathbf{x}|}\right.$$
$$\left. + \cdot\frac{1}{3}\,\lambda_2\, e^{-\lambda_2|\mathbf{x}|}\right) - \frac{xz}{|\mathbf{x}|^3}\left(\frac{2}{3}\,\lambda_1^2\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2\, e^{-\lambda_2|\mathbf{x}|}\right)\Bigg]^2\Bigg\}$$
$$= \left(\frac{G_N M}{a_1}\right)^2 \Bigg[\frac{1}{|\mathbf{x}|^6}\left(1 - \frac{2}{3}\, e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\, e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{1}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2\, e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{3x^2}{|\mathbf{x}|^8}$$
$$\left(1 - \frac{2}{3}\, e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\, e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{3x^2}{|\mathbf{x}|^6}\left(\frac{2}{3}\,\lambda_1\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2\, e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{x^2}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1^2\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2$$
$$e^{-\lambda_2|\mathbf{x}|}\right)^2 - \frac{2}{|\mathbf{x}|^5}\left(1 - \frac{2}{3}\, e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\, e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{2}{3}\,\lambda_1\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2\, e^{-\lambda_2|\mathbf{x}|}\right) - \frac{6x^2}{|\mathbf{x}|^7}\left(1 - \frac{2}{3}\, e^{-\lambda_1|\mathbf{x}|}\right.$$
$$\left. - \frac{1}{3}\, e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{2}{3}\,\lambda_1\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2\, e^{-\lambda_2|\mathbf{x}|}\right) - \frac{4x^2}{|\mathbf{x}|^6}\left(1 - \frac{2}{3}\, e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\, e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{2}{3}\,\lambda_1^2\, e^{-\lambda_1|\mathbf{x}|}\right.$$
$$\left. + \frac{1}{3}\,\lambda_2^2\, e^{-\lambda_2|\mathbf{x}|}\right) + \frac{4x^2}{|\mathbf{x}|^5}\left(\frac{2}{3}\,\lambda_1\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2\, e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{2}{3}\,\lambda_1^2\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2\, e^{-\lambda_2|\mathbf{x}|}\right)\Bigg];$$

- 2-nd term

$$\boldsymbol{\nabla}\big(\partial_2 V\big)\cdot\boldsymbol{\nabla}\big(\partial_2 V\big) = \big(\partial_{12}V\big)^2 + \big(\partial_{22}V\big)^2 + \big(\partial_{23}V\big)^2$$
$$= \left(\frac{G_N M}{a_1}\right)^2 \Bigg\{ \Bigg[\frac{3xy}{|\mathbf{x}|^5}\left(1 - \frac{2}{3}\, e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\, e^{-\lambda_2|\mathbf{x}|}\right) - \frac{3xy}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2\, e^{-\lambda_2|\mathbf{x}|}\right) - \frac{xy}{|\mathbf{x}|^3}$$
$$\left(\frac{2}{3}\,\lambda_1^2\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2\, e^{-\lambda_2|\mathbf{x}|}\right)\Bigg]^2 + \Bigg[-\frac{1}{|\mathbf{x}|^3}\left(1 - \frac{2}{3}\, e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\, e^{-\lambda_2|\mathbf{x}|}\right) + \frac{1}{|\mathbf{x}|^2}\left(\frac{2}{3}\,\lambda_1\, e^{-\lambda_1|\mathbf{x}|}\right.$$
$$\left. + \frac{1}{3}\,\lambda_2\, e^{-\lambda_2|\mathbf{x}|}\right) + \frac{3y^2}{|\mathbf{x}|^5}\left(1 - \frac{2}{3}\, e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\, e^{-\lambda_2|\mathbf{x}|}\right) - \frac{3y^2}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2\, e^{-\lambda_2|\mathbf{x}|}\right) - \frac{y^2}{|\mathbf{x}|^3}$$
$$\left(\frac{2}{3}\,\lambda_1^2\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2\, e^{-\lambda_2|\mathbf{x}|}\right)\Bigg]^2 + \Bigg[\frac{3yz}{|\mathbf{x}|^5}\left(1 - \frac{2}{3}\, e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\, e^{-\lambda_2|\mathbf{x}|}\right) - \frac{3yz}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1\, e^{-\lambda_1|\mathbf{x}|}\right.$$
$$\left. + \frac{1}{3}\,\lambda_2\, e^{-\lambda_2|\mathbf{x}|}\right) - \frac{yz}{|\mathbf{x}|^3}\left(\frac{2}{3}\,\lambda_1^2\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2\, e^{-\lambda_2|\mathbf{x}|}\right)\Bigg]^2\Bigg\}$$
$$= \left(\frac{G_N M}{a_1}\right)^2 \Bigg[\frac{1}{|\mathbf{x}|^6}\left(1 - \frac{2}{3}\, e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\, e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{1}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2\, e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{3y^2}{|\mathbf{x}|^8}$$
$$\left(1 - \frac{2}{3}\, e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\, e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{3y^2}{|\mathbf{x}|^6}\left(\frac{2}{3}\,\lambda_1\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2\, e^{-\lambda_2|\mathbf{x}|}\right)^2 + \frac{y^2}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1^2\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2$$
$$e^{-\lambda_2|\mathbf{x}|}\right)^2 - \frac{2}{|\mathbf{x}|^5}\left(1 - \frac{2}{3}\, e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\, e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{2}{3}\,\lambda_1\, e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2\, e^{-\lambda_2|\mathbf{x}|}\right) - \frac{6y^2}{|\mathbf{x}|^7}\left(1 - \frac{2}{3}\, e^{-\lambda_1|\mathbf{x}|}\right.$$

$$
- \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{4y^2}{|\mathbf{x}|^6}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}
$$

$$
+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{4y^2}{|\mathbf{x}|^5}\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Bigg];
$$

- 3-rd term

$$
\boldsymbol{\nabla}\big(\partial_3 V\big)\cdot\boldsymbol{\nabla}\big(\partial_3 V\big)=\big(\partial_{13}V\big)^2+\big(\partial_{23}V\big)^2+\big(\partial_{33}V\big)^2
$$

$$
=\Big(\frac{G_N M}{a_1}\Big)^2\Bigg\{\Bigg[\frac{3xz}{|\mathbf{x}|^5}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3xz}{|\mathbf{x}|^4}\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\cdot\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{xz}{|\mathbf{x}|^3}
$$

$$
\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Bigg]^2+\Bigg[\frac{3yz}{|\mathbf{x}|^5}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3yz}{|\mathbf{x}|^4}\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}
$$

$$
+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{yz}{|\mathbf{x}|^3}\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Bigg]^2+\Bigg[-\frac{1}{|\mathbf{x}|^3}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)
$$

$$
+\frac{1}{|\mathbf{x}|^2}\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{3z^2}{|\mathbf{x}|^5}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3z^2}{|\mathbf{x}|^4}\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}
$$

$$
+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{z^2}{|\mathbf{x}|^3}\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Bigg]^2\Bigg\}
$$

$$
=\Big(\frac{G_N M}{a_1}\Big)^2\Bigg[\frac{1}{|\mathbf{x}|^6}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)^2+\frac{1}{|\mathbf{x}|^4}\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)^2+\frac{3z^2}{|\mathbf{x}|^8}
$$

$$
\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)^2+\frac{3z^2}{|\mathbf{x}|^6}\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)^2+\frac{z^2}{|\mathbf{x}|^4}\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2
$$

$$
e^{-\lambda_2|\mathbf{x}|}\Big)^2-\frac{2}{|\mathbf{x}|^5}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{6z^2}{|\mathbf{x}|^7}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}
$$

$$
-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{4z^2}{|\mathbf{x}|^6}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}
$$

$$
+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{4z^2}{|\mathbf{x}|^5}\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Bigg].
$$

Summing up,

$$
\boldsymbol{\nabla}\big(\partial_1 V\big)\cdot\boldsymbol{\nabla}\big(\partial_1 V\big)+\boldsymbol{\nabla}\big(\partial_2 V\big)\cdot\boldsymbol{\nabla}\big(\partial_2 V\big)+\boldsymbol{\nabla}\big(\partial_3 V\big)\cdot\boldsymbol{\nabla}\big(\partial_3 V\big)
$$

$$
=\Big(\frac{G_N M}{a_1}\Big)^2\Bigg[\frac{3}{|\mathbf{x}|^6}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)^2+\frac{3}{|\mathbf{x}|^4}\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)^2
$$

$$
+\frac{3\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^8}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)^2+\frac{3\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^6}\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)^2
$$

$$
+\frac{\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^4}\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)^2-\frac{6}{|\mathbf{x}|^5}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}
$$

$$
+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{6\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^7}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)
$$

$$
-\frac{4\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^6}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{4\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^5}
$$

$$\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\Bigg],$$

that is,

$$=\left(\frac{G_NM}{a_1}\right)^2\Bigg[\frac{6}{|\mathbf{x}|^6}\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)^2+\frac{6}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)^2+\frac{1}{|\mathbf{x}|^2}$$
$$\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)^2-\frac{12}{|\mathbf{x}|^5}\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)$$
$$-\frac{4}{|\mathbf{x}|^4}\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{4}{|\mathbf{x}|^3}\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2$$
$$e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\Bigg].$$

I.3.4 $\Big[\boldsymbol{\nabla}\big(\partial_1 U\big)\cdot\boldsymbol{\nabla}\big(\partial_1 V\big)+\boldsymbol{\nabla}\big(\partial_2 U\big)\cdot\boldsymbol{\nabla}\big(\partial_2 V\big)+\boldsymbol{\nabla}\big(\partial_3 U\big)\cdot\boldsymbol{\nabla}\big(\partial_3 V\big)\Big]$

- 1-st term

$$\boldsymbol{\nabla}\big(\partial_1 U\big)\cdot\boldsymbol{\nabla}\big(\partial_1 V\big)=\partial_{11}U\partial_{11}V+\partial_{12}U\partial_{12}V+\partial_{13}U\partial_{13}V$$
$$=\left(\frac{G_NM}{a_1}\right)^2\Bigg\{\Bigg[-\frac{1}{|\mathbf{x}|^3}\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{1}{|\mathbf{x}|^2}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{3x^2}{|\mathbf{x}|^5}$$
$$\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{3x^2}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{x^2}{|\mathbf{x}|^3}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2$$
$$e^{-\lambda_2|\mathbf{x}|}\right)\Bigg]\Bigg[-\frac{1}{|\mathbf{x}|^3}\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{1}{|\mathbf{x}|^2}\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{3x^2}{|\mathbf{x}|^5}\Bigg(1$$
$$-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Bigg)-\frac{3x^2}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{x^2}{|\mathbf{x}|^3}\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2$$
$$e^{-\lambda_2|\mathbf{x}|}\right)\Bigg]+\Bigg[\frac{3xy}{|\mathbf{x}|^5}\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{3xy}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{xy}{|\mathbf{x}|^3}$$
$$\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\Bigg]\Bigg[\frac{3xy}{|\mathbf{x}|^5}\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{3xy}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2$$
$$e^{-\lambda_2|\mathbf{x}|}\right)-\frac{xy}{|\mathbf{x}|^3}\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\Bigg]+\Bigg[\frac{3xz}{|\mathbf{x}|^5}\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{3xz}{|\mathbf{x}|^4}$$
$$\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{xz}{|\mathbf{x}|^3}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\Bigg]\Bigg[\frac{3xz}{|\mathbf{x}|^5}\Bigg(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}$$
$$-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Bigg)-\frac{3xz}{|\mathbf{x}|^4}\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\cdot\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{xz}{|\mathbf{x}|^3}\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\Bigg]\Bigg\}$$
$$=\left(\frac{G_NM}{a_1}\right)^2\Bigg[\frac{1}{|\mathbf{x}|^6}\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)\left(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)-\frac{1}{|\mathbf{x}|^5}\left(1-\frac{4}{3}\right.$$
$$e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Bigg)\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)+\frac{3x^2}{|\mathbf{x}|^8}\left(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)\left(1-\frac{2}{3}\right.$$

$$e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3x^2}{|\mathbf{x}|^7}\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{2x^2}{|\mathbf{x}|^6}$$
$$\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{1}{|\mathbf{x}|^5}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)$$
$$\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{1}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)$$
$$-\frac{3x^2}{|\mathbf{x}|^7}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{3x^2}{|\mathbf{x}|^6}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2$$
$$e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{2x^2}{|\mathbf{x}|^5}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2$$
$$e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{2x^2}{|\mathbf{x}|^6}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{2x^2}{|\mathbf{x}|^5}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}$$
$$-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{x^2}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}$$
$$+\frac{1}{3}\,\lambda_2^2e^{-\lambda_2|\mathbf{x}|}\Big)\Big]\,;$$

- 2-nd term

$$\boldsymbol{\nabla}\big(\partial_2 U\big)\cdot\boldsymbol{\nabla}\big(\partial_2 V\big)=\partial_{12}U\partial_{12}V+\partial_{22}U\partial_{22}V+\partial_{23}U\partial_{23}V$$
$$=\Big(\frac{G_N M}{a_1}\Big)^2\Bigg\{\Big[\frac{3xy}{|\mathbf{x}|^5}\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3xy}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{xy}{|\mathbf{x}|^3}$$
$$\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big]\Big[\frac{3xy}{|\mathbf{x}|^5}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3xy}{|\mathbf{x}|^4}\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2$$
$$e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{xy}{|\mathbf{x}|^3}\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big]+\Big[-\frac{1}{|\mathbf{x}|^3}\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{1}{|\mathbf{x}|^2}$$
$$\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{3y^2}{|\mathbf{x}|^5}\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3y^2}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2$$
$$e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{y^2}{|\mathbf{x}|^3}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big]\Big[-\frac{1}{|\mathbf{x}|^3}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{1}{|\mathbf{x}|^2}\Big(\frac{2}{3}\,\lambda_1$$
$$e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{3y^2}{|\mathbf{x}|^5}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3y^2}{|\mathbf{x}|^4}\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)$$
$$-\frac{y^2}{|\mathbf{x}|^3}\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big]+\Big[\frac{3yz}{|\mathbf{x}|^5}\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3yz}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}$$
$$-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{yz}{|\mathbf{x}|^3}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big]\Big[\frac{3yz}{|\mathbf{x}|^5}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3yz}{|\mathbf{x}|^4}$$
$$\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{yz}{|\mathbf{x}|^3}\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big]\Bigg\}$$
$$=\Big(\frac{G_N M}{a_1}\Big)^2\Big[\frac{1}{|\mathbf{x}|^6}\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{1}{|\mathbf{x}|^5}\Big(1-\frac{4}{3}$$

$$e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{3y^2}{|\mathbf{x}|^8}\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(1-\frac{2}{3}$$
$$e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3y^2}{|\mathbf{x}|^7}\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2 e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{2y^2}{|\mathbf{x}|^6}$$
$$\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{1}{|\mathbf{x}|^5}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)$$
$$\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{1}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2 e^{-\lambda_2|\mathbf{x}|}\Big)$$
$$-\frac{3y^2}{|\mathbf{x}|^7}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{3y^2}{|\mathbf{x}|^6}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2$$
$$e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2 e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{2y^2}{|\mathbf{x}|^5}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\lambda_2^2$$
$$e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{2y^2}{|\mathbf{x}|^6}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{2y^2}{|\mathbf{x}|^5}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}$$
$$-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{y^2}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}$$
$$+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big]\,;$$

- 3-rd term

$$\boldsymbol{\nabla}\big(\partial_3 U\big)\cdot\boldsymbol{\nabla}\big(\partial_3 V\big)=\partial_{13}U\partial_{13}V+\partial_{23}U\partial_{23}V+\partial_{33}U\partial_{33}V$$
$$=\Big(\frac{G_N M}{a_1}\Big)^2\Bigg\{\Big[\frac{3xz}{|\mathbf{x}|^5}\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3xz}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{xz}{|\mathbf{x}|^3}$$
$$\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big]\Big[\frac{3xz}{|\mathbf{x}|^5}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3xz}{|\mathbf{x}|^4}\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\cdot\frac{1}{3}\,\lambda_2$$
$$e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{xz}{|\mathbf{x}|^3}\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big]+\Big[\frac{3yz}{|\mathbf{x}|^5}\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3yz}{|\mathbf{x}|^4}\Big(\frac{4}{3}$$
$$\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{yz}{|\mathbf{x}|^3}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big]\Big[\frac{3yz}{|\mathbf{x}|^5}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)$$
$$-\frac{3yz}{|\mathbf{x}|^4}\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{yz}{|\mathbf{x}|^3}\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big]+\Big[-\frac{1}{|\mathbf{x}|^3}\Big(1-\frac{4}{3}$$
$$e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{1}{|\mathbf{x}|^2}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{3z^2}{|\mathbf{x}|^5}\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)$$
$$-\frac{3z^2}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{z^2}{|\mathbf{x}|^3}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big]\Big[-\frac{1}{|\mathbf{x}|^3}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}$$
$$-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{1}{|\mathbf{x}|^2}\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{3z^2}{|\mathbf{x}|^5}\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3z^2}{|\mathbf{x}|^4}$$
$$\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{z^2}{|\mathbf{x}|^3}\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big]\Bigg\}$$
$$=\Big(\frac{G_N M}{a_1}\Big)^2\Big[\frac{1}{|\mathbf{x}|^6}\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{1}{|\mathbf{x}|^5}\Big(1-\frac{4}{3}$$

$$e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+.\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{3z^2}{|\mathbf{x}|^8}\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(1-\frac{2}{3}$$
$$e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3z^2}{|\mathbf{x}|^7}\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+.\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{2z^2}{|\mathbf{x}|^6}$$
$$\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{1}{|\mathbf{x}|^5}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)$$
$$\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{1}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+.\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)$$
$$-\frac{3z^2}{|\mathbf{x}|^7}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{3z^2}{|\mathbf{x}|^6}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2$$
$$e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+.\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{2z^2}{|\mathbf{x}|^5}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}$$
$$\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{2z^2}{|\mathbf{x}|^6}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{2z^2}{|\mathbf{x}|^5}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}$$
$$-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+.\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{z^2}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}$$
$$+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Bigg].$$

Summing up,

$$\boldsymbol{\nabla}\big(\partial_1 U\big)\cdot\boldsymbol{\nabla}\big(\partial_1 V\big)+\boldsymbol{\nabla}\big(\partial_2 U\big)\cdot\boldsymbol{\nabla}\big(\partial_2 V\big)+\boldsymbol{\nabla}\big(\partial_3 U\big)\cdot\boldsymbol{\nabla}\big(\partial_3 V\big)$$
$$=\Big(\frac{G_N M}{a_1}\Big)^2\Bigg[\frac{3}{|\mathbf{x}|^6}\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3}{|\mathbf{x}|^5}\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}$$
$$+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{3\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^8}\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(1-\frac{2}{3}$$
$$e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^7}\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)$$
$$-\frac{2\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^6}\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3}{|\mathbf{x}|^5}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}$$
$$-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{3}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}$$
$$+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{3\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^7}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)$$
$$+\frac{3\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^6}\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{2\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^5}$$
$$\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)-\frac{2\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^6}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2$$
$$e^{-\lambda_2|\mathbf{x}|}\Big)\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{2\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^5}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}$$
$$+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)+\frac{\big(x^2+y^2+z^2\big)}{|\mathbf{x}|^4}\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Bigg],$$

that is,

$$= \left(\frac{G_N M}{a_1}\right)^2 \Bigg[\frac{6}{|\mathbf{x}|^6}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)\left(1 - \frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right) - \frac{6}{|\mathbf{x}|^5}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right) - \frac{2}{|\mathbf{x}|^4}\left(1 - \frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right) - \frac{6}{|\mathbf{x}|^5}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\left(1 - \frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right) + \frac{6}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right) + \frac{2}{|\mathbf{x}|^3}\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right) - \frac{2}{|\mathbf{x}|^4}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\left(1 - \frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\right) + \frac{2}{|\mathbf{x}|^3}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right) + \frac{1}{|\mathbf{x}|^2}\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\Bigg].$$

I.3.5 Assembling of the previous triplets

I am going to put together, here, the contributions coming from the above triplets, related to the different overall factors $\frac{1}{|\mathbf{x}|^n}$, where $2 \leq n \leq 6$, on the basis of the expansion (I.1) on page 226.

- contribution relating to the overall factor $\frac{1}{|\mathbf{x}|^2}$

$$\frac{1}{|\mathbf{x}|^2}\Bigg[\frac{20}{9}\,\lambda_1^4\,e^{-2\lambda_1|\mathbf{x}|} + \frac{8}{9}\,\lambda_2^4\,e^{-2\lambda_2|\mathbf{x}|} + \frac{8}{9}\,\lambda_1^2\,\lambda_2^2\,e^{-(\lambda_1+\lambda_2)|\mathbf{x}|} + \left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)^2 + \left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)^2 - 2\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\Bigg]$$
$$= \frac{4}{3}\,\frac{1}{|\mathbf{x}|^2}\Big[2\,\lambda_1^4\,e^{-2\lambda_1|\mathbf{x}|} + \lambda_2^4\,e^{-2\lambda_2|\mathbf{x}|}\Big];$$

- contribution relating to the overall factor $\frac{1}{|\mathbf{x}|^3}$

$$\frac{1}{|\mathbf{x}|^3}\Bigg[4\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right) + 4\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right) - 4\left(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right) - 4\left(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|} - \frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\right)\left(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|} + \frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\right)\Bigg]$$
$$= \frac{16}{9}\,\frac{1}{|\mathbf{x}|^3}\Big[\lambda_1^3\,e^{-2\lambda_1|\mathbf{x}|} - \lambda_1\lambda_2\big(\lambda_1+\lambda_2\big)e^{-(\lambda_1+\lambda_2)|\mathbf{x}|} + \lambda_2^3\,e^{-2\lambda_2|\mathbf{x}|}\Big];$$

- contribution relating to the overall factor $\frac{1}{|\mathbf{x}|^4}$

$$\frac{1}{|\mathbf{x}|^4}\Bigg[6\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)^2-4\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)+6\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)^2-4\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)+4\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)-12\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)+4\Big(\frac{4}{3}\,\lambda_1^2\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2^2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Bigg]$$
$$=\frac{8}{3}\,\frac{1}{|\mathbf{x}|^4}\Big[\frac{5}{3}\Big(\lambda_1^2\,e^{-2\lambda_1|\mathbf{x}|}+\lambda_2^2\,e^{-2\lambda_2|\mathbf{x}|}\Big)-2\Big(\lambda_1\lambda_2+\frac{1}{3}\lambda_1^2+\frac{1}{3}\lambda_2^2\Big)e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}\Big];$$

- contribution relating to the overall factor $\frac{1}{|\mathbf{x}|^5}$

$$\frac{1}{|\mathbf{x}|^5}\Bigg[-12\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)-12\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)+12\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(\frac{2}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)+12\Big(\frac{4}{3}\,\lambda_1\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,\lambda_2\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Bigg]$$
$$=\frac{16}{3}\,\frac{1}{|\mathbf{x}|^5}\Big[\lambda_1\,e^{-2\lambda_1|\mathbf{x}|}+\lambda_2\,e^{-2\lambda_2|\mathbf{x}|}-\big(\lambda_1+\lambda_2\big)e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}\Big];$$

- contribution relating to the overall factor $\frac{1}{|\mathbf{x}|^6}$

$$\frac{1}{|\mathbf{x}|^6}\Bigg[6\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)^2+6\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)^2-12\Big(1-\frac{4}{3}\,e^{-\lambda_1|\mathbf{x}|}+\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Big(1-\frac{2}{3}\,e^{-\lambda_1|\mathbf{x}|}-\frac{1}{3}\,e^{-\lambda_2|\mathbf{x}|}\Big)\Bigg]$$
$$=\frac{8}{3}\,\frac{1}{|\mathbf{x}|^6}\Big[e^{-2\lambda_1|\mathbf{x}|}+e^{-2\lambda_2|\mathbf{x}|}-2\,e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}\Big].$$

Therefore, it follows the final result:

$$R_{\mu\nu}R^{\mu\nu}=\frac{1}{c^4}\Big(\frac{G_N M}{a_1}\Big)^2\Bigg\{\frac{4}{3}\,\frac{1}{|\mathbf{x}|^2}\Big[2\,\lambda_1^4\,e^{-2\lambda_1|\mathbf{x}|}+\lambda_2^4\,e^{-2\lambda_2|\mathbf{x}|}\Big]+\frac{16}{9}\,\frac{1}{|\mathbf{x}|^3}\Big[\lambda_1^3\,e^{-2\lambda_1|\mathbf{x}|}-\lambda_1\lambda_2\big(\lambda_1+\lambda_2\big)e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}+\lambda_2^3\,e^{-2\lambda_2|\mathbf{x}|}\Big]+\frac{8}{3}\,\frac{1}{|\mathbf{x}|^4}\Big[\frac{5}{3}\Big(\lambda_1^2\,e^{-2\lambda_1|\mathbf{x}|}+\lambda_2^2\,e^{-2\lambda_2|\mathbf{x}|}\Big)-2\Big(\lambda_1\lambda_2+\frac{1}{3}\lambda_1^2+\frac{1}{3}\lambda_2^2\Big)e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}\Big]+\frac{16}{3}\,\frac{1}{|\mathbf{x}|^5}\Big[\lambda_1\,e^{-2\lambda_1|\mathbf{x}|}+\lambda_2\,e^{-2\lambda_2|\mathbf{x}|}-\big(\lambda_1+\lambda_2\big)e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}\Big]$$

$$+\frac{8}{3}\frac{1}{|\mathbf{x}|^6}\left[e^{-2\lambda_1|\mathbf{x}|}+e^{-2\lambda_2|\mathbf{x}|}-2\,e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}\right]\Bigg\}.$$

I.4 Test result on $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ scalar

From the well-known Gauss-Bonnet identity and the previous results concerning $R_{\mu\nu}R^{\mu\nu}$ and R^2 scalars, it follows,

$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}=4\,R_{\mu\nu}R^{\mu\nu}-R^2$$
$$=\frac{1}{c^4}\left(\frac{G_NM}{a_1}\right)^2\Bigg\{\frac{16}{3}\frac{1}{|\mathbf{x}|^2}\left[2\,\lambda_1^4\,e^{-2\lambda_1|\mathbf{x}|}+\lambda_2^4\,e^{-2\lambda_2|\mathbf{x}|}\right]+\frac{64}{9}\frac{1}{|\mathbf{x}|^3}\left[\lambda_1^3\,e^{-2\lambda_1|\mathbf{x}|}-\lambda_1\lambda_2\big(\lambda_1+\lambda_2\big)\right.$$
$$\left.e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}+\lambda_2^3\,e^{-2\lambda_2|\mathbf{x}|}\right]+\frac{32}{3}\frac{1}{|\mathbf{x}|^4}\left[\frac{5}{3}\Big(\lambda_1^2\,e^{-2\lambda_1|\mathbf{x}|}+\lambda_2^2\,e^{-2\lambda_2|\mathbf{x}|}\Big)-2\left(\lambda_1\lambda_2+\frac{1}{3}\lambda_1^2+\frac{1}{3}\lambda_2^2\right)\right.$$
$$\left.e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}\right]+\frac{64}{3}\frac{1}{|\mathbf{x}|^5}\left[\lambda_1\,e^{-2\lambda_1|\mathbf{x}|}+\lambda_2\,e^{-2\lambda_2|\mathbf{x}|}-\big(\lambda_1+\lambda_2\big)e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}\right]+\frac{32}{3}\frac{1}{|\mathbf{x}|^6}\left[e^{-2\lambda_1|\mathbf{x}|}\right.$$
$$\left.+e^{-2\lambda_2|\mathbf{x}|}-2\,e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}\right]\Bigg\}-\frac{1}{c^4}\left(\frac{G_NM}{a_1}\right)^2\frac{1}{|\mathbf{x}|^2}\left[4\,\lambda_2^4\,e^{-2\lambda_2|\mathbf{x}|}\right],$$

that is,

$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}=\frac{1}{c^4}\left(\frac{G_NM}{a_1}\right)^2\Bigg\{\frac{4}{3}\frac{1}{|\mathbf{x}|^2}\left[8\,\lambda_1^4\,e^{-2\lambda_1|\mathbf{x}|}+\lambda_2^4\,e^{-2\lambda_2|\mathbf{x}|}\right]+\frac{64}{9}\frac{1}{|\mathbf{x}|^3}\left[\lambda_1^3\,e^{-2\lambda_1|\mathbf{x}|}-\lambda_1\lambda_2\right.$$
$$\left.\big(\lambda_1+\lambda_2\big)e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}+\lambda_2^3\,e^{-2\lambda_2|\mathbf{x}|}\right]+\frac{32}{3}\frac{1}{|\mathbf{x}|^4}\left[\frac{5}{3}\Big(\lambda_1^2\,e^{-2\lambda_1|\mathbf{x}|}+\lambda_2^2\,e^{-2\lambda_2|\mathbf{x}|}\Big)-2\right.$$
$$\left.\left(\lambda_1\lambda_2+\frac{1}{3}\lambda_1^2+\frac{1}{3}\lambda_2^2\right)e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}\right]+\frac{64}{3}\frac{1}{|\mathbf{x}|^5}\left[\lambda_1\,e^{-2\lambda_1|\mathbf{x}|}+\lambda_2\,e^{-2\lambda_2|\mathbf{x}|}-\big(\lambda_1+\lambda_2\big)\right.$$
$$\left.e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}\right]+\frac{32}{3}\frac{1}{|\mathbf{x}|^6}\left[e^{-2\lambda_1|\mathbf{x}|}+e^{-2\lambda_2|\mathbf{x}|}-2\,e^{-(\lambda_1+\lambda_2)|\mathbf{x}|}\right]\Bigg\}.$$

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