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A generalization of derangement
Sulla generalizzazione delle dismutazioni

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Abstract As a natural extension of the concept of *derangement*, we define as *derangement-3*, *derangement-4*,..., *derangement-K*, the triplet, quadruplets,..., *K*-plets of permutations that have no common elements in the same place. In this paper we propose a theoretical conjecture for the asymptotic behaviour of higher order derangements and validate it by computer simulation for significant values of *K*.

Abstract *Come naturale estensione del concetto di dismutazione definiremo dismutazioni di ordine tre, quattro,..K rispettivamente le terne, quaterne e le K-ple di permutazioni che non hanno elementi in comune nello stesso posto. Nellarticolo viene presentata una congettura per valutarne le probabilità nel comportamento asintotico di queste dismutazioni di ordine superiore, al variare del numero di elementi, confortato con simulazioni per valori significativi di K.*

Key words: Derangement, Married Couples Problem, Montmort's matching problem, Permutation, Problème des Rencontres, Subfactorial.

1 Introduction

The problem of derangement, hereafter referred to as *derangement-2* was formulated and solved long time ago by Pierre M. de Montmort respectively in 1708 and 1713. Nicholas Bernoulli also solved the problem using the inclusion-exclusion principle. The results are summarised for convenience in Section 2, where the generalization to higher order derangements is also discussed. In Section 3 the conjecture is presented and verified experimentally, via simulation, in Section 4. The problem

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remains of finding if possible an exact analytical formulations for the more general *derangement-K*[1].

2 Derangement and generalization

Let S_n be the symmetric group of all permutations of n elements $(1,2,3,\ldots,n)$ whose cardinality is $|S_n|$=n!. A derangement is a permutation in which none of the elements appears in its original position. The number of derangements is indicated by $!n$, called a subfactorial of n and is given by:

$$!n = n! \sum_{i=0,1,2,\ldots,n} \frac{(-1)^i}{i!}. \tag{1}$$

The probability that two elements of S_n do not have some element in same place is:

$$P_2[n] = \frac{!n}{n!} = \sum_{i=0,1,2,\ldots,n} \frac{(-1)^i}{i!} = \sum_{i=2,\ldots,n} \frac{(-1)^i}{i!} \qquad \forall\, n \geq 2.$$

The proof for equation (1) is based on the well known inclusion-exclusion principle. As n increases, the probability converges very rapidly ($n \geq 4$) to $1/e$. Figure 1 shows a graphical representation of this probability as a function of n. Let us now consider what is the probability $P_3[n]$ that three elements of S_n have no element in the same place. More generally we define as $P_K[n]$ with $n \geq K$, the probability that K permutations of S_n have no common elements in the same place. At present the exact solution to this problem is not known. By combinatorial analysis, it is possible to calculate some $P_K[n]$ values for small values of n.
For example in case of $K = 3$ we have:

$$P_3[3] = \frac{1}{18}; \quad P_3[4] = \frac{1}{24} \ldots$$

for $K = 4$:

$$P_4[4] = \frac{1}{24^2}; \ldots.$$

Table 1 shows the exact results for *derangement-3*, for different values of n, together with the results from simulation discussed in Section 4.

[1] We have introduced the notation derangement-K to differentiate it from K-derangement, which has been used by other authors with a completely different meaning [3, 2, 1].

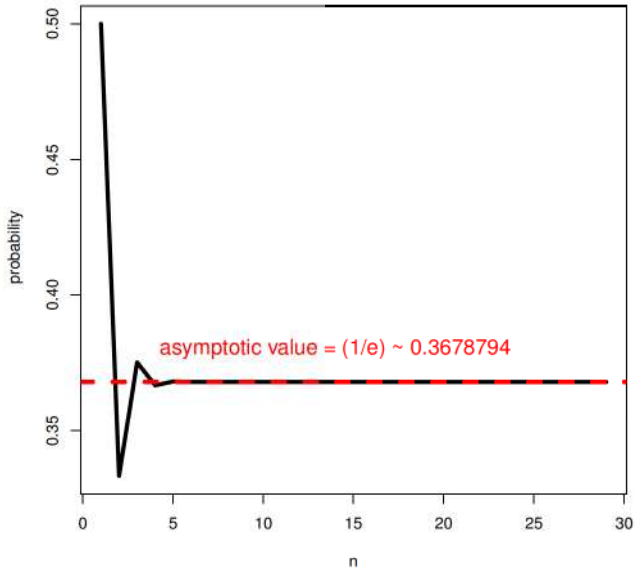

Fig. 1 Derangement

n	N. derangement-3	Probability
3	2	$\frac{2}{3!^2} = \frac{1}{18}$
4	24	$\frac{24}{4!^2} = \frac{1}{24}$
5	552	$\frac{552}{5!^2} = \frac{23}{600}$
6	21280	$\frac{21280}{6!^2} = \frac{133}{3240}$
7	1073760	$\frac{1073760}{7!^2} = \frac{2237}{52920}$
8	70299264	$\frac{70299264}{8!^2} = \frac{26153}{604800}$
9	5792853248	$\frac{5792853248}{9!^2} = \frac{3232619}{73483200}$

Table 1 Number of derangement-3

3 An asymptotic conjecture

Given the difficulty to calculate exactly the probabilities in different cases and by starting from a consideration on *derangement-3*, we attempt to generalize the result previously obtained for *derangement-2*. Taken as X, Y and Z three elements of S_n,

the probability that at two by two don't have elements in the same place is $P_2[n]$. For *derangement-3* we seek the probability of event

$$\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}$$

having indicated with $\mathcal{A}$ the event that X and Y do not have common elements, with $\mathcal{B}$ that X and Z do not have elements in common and with $\mathcal{C}$ that Z and Y have no elements in common. For n high enough and assuming that the events are independent, the probability of $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}$ should be equal $(1/e)^3$. Same consideration for *derangement-4*, but in this case the pairs that have to be independent are $\binom{4}{2}$ and so for *derangement-5* the pairs will be $\binom{5}{2}$. The asymptotic behaviour, as confirmed by simulation (see Section 4), appears to be correct, except, as expected for the initial values of n. This implies that the events are pairwise independent but not three by three. So, by generalizing, we expect they to be not independent for four by four for *derangement-4*, and so on. A simple check to verify this prediction is the case of $P_3[3]$; if they were independent events, one should have as probability $P_3[3] = (P_2[3])^3 = 1/27$ instead, by contrast $P_3[3] = 1/18 \neq 1/27$. In some sense events are only asymptotically independent, i.e. only for $n \to +\infty$. In general it can be conjectured, however, that for *derangement-K* the limit value of the probability is

$$\lim_{n \to +\infty} P_K[n] = \left(\frac{1}{e}\right)^{\binom{K}{2}} \tag{2}$$

This also means that to have practical relevance, it is necessary to have very mall values of K, because these asymptotic probabilities become extremely small, when K increases, as shown in Table 2.

4 Simulation

For each value of K, the fact that the asymptotic value, beyond the first few values of n, is greater than those estimated via simulation, suggests that, if there is an analytic relationship, this might consists of a fixed component plus a variable one with alternating signs. Furthermore the latter should vanish asymptotically, thereby leaving the constant component only. Another consideration emerging from these simulations, in agreement with calculation of Section 2, is that the probability $P_K[K]$ decreases in the next step $P_K[K+1]$ and then tends to the asymptotic value but in an increasingly slow way for increasing K. On the basis of the simulations, it should be noted that the asymptotic trend is reached, for derangement-2 with very small values of $n \geq 4$. Always through simulation it is recognized that even for $K = 5$ the probability is reached asymptotically for $n \approx 500 \div 800$. In Figure 2, it is

K	asymptotic value
2	0.3678794
3	0.04978707
4	0.002478752
5	4.539993e-05
6	3.059023e-07
7	7.58256e-10
8	6.9144e-13
9	2.319523e-16
10	2.862519e-20
11	1.299581e-24
12	2.170522e-29
13	1.333615e-34
14	3.014409e-40
15	2.506567e-46

Table 2 Asymptotic probabilities values

reported graphically the simulation results for $K = 2,...,5$. Note how the Figure 1 corresponds perfectly to Figure 2(a), case of derangement-2.
All simulation are made using software R.

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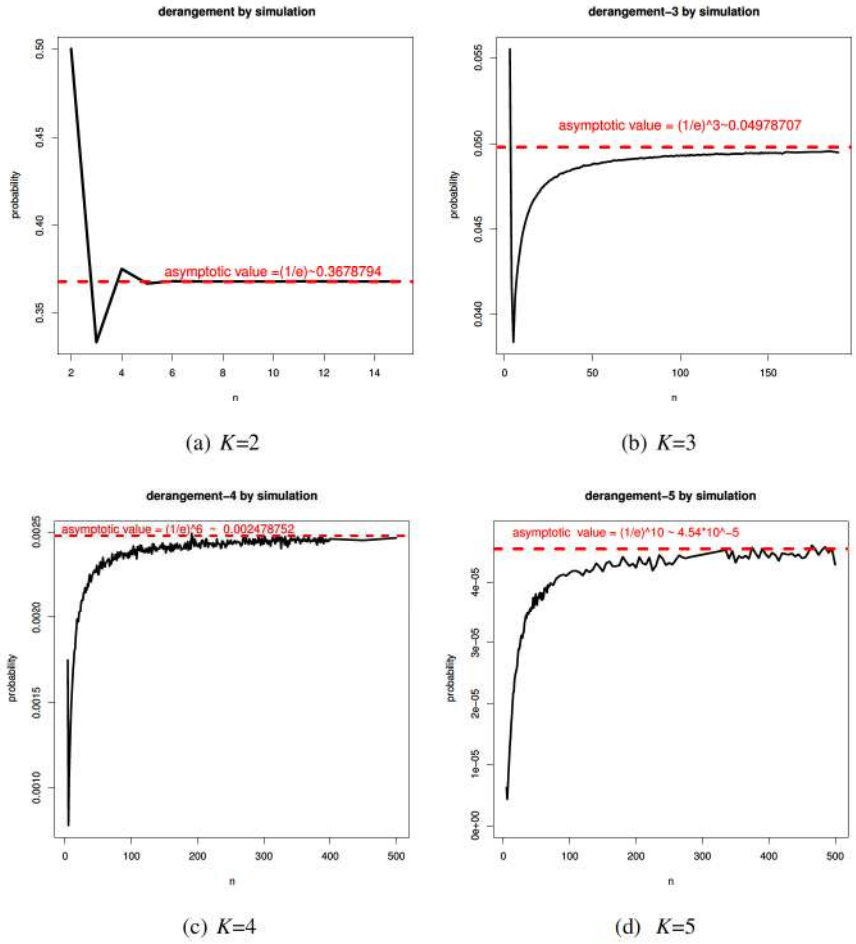

(a) K=2 (b) K=3

(c) K=4 (d) K=5

Fig. 2 Simulations graphics