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Stability Analysis of Discrete-Time Systems with Constrained Delays by Novel Nonlinear Halanay-Type Inequalities

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Contents

Abstract

The research presented in this thesis covers the stability analysis of discrete-time systems with delay where the delay signals are contained to follow a dynamic driven by a delay digraph.

The interest in this class of systems has been motivated traditionally by sampled-data systems in which a process is sampled periodically and then controlled via a machine. This setting leads to relatively cheap control solutions, but requires the discretization of signals which typically introduces time delays. Time-delays often lead to complex behaviors in the dynamics of a system and may cause the loss of stability. Therefore, the investigation of stability is a fundamental problem, which often turns to be highly challenging. More recently the interest in discrete-time systems with delay has been motivated by networked control systems in which the connection between the process and the controller is made through a shared communication network. This communication network increases the flexibility of the control architecture but also introduces effects such as packet dropouts, uncertain time-varying delays and timing jitter.

Motivated by the fact that almost every system in practice is subject to constraints and Lyapunov theory is one of the few methods that can be easily adapted to deal with constraints, all results in the thesis are based on Lyapunov theory.

The delay variation modeled via a delay digraph can have beneficial effect on the stability of a system, and can guarantee stabilization even when instability occurs for one or more values of the delays, taken constant. The delay digraph can describe a large range of cases, including the case of bounded delay variation and the case of arbitrary delay-signals. Some tractable sufficient Lyapunov conditions exploiting a novel nonlinear discrete-time Halanay-type inequality are provided in the stability and the input-to-state stability analysis. The more information is provided by the delays digraph, the more is the reduction of the number of involved inequalities in the provided sufficient Lyapunov conditions. The key role of using the nonlinear Halanay-type inequality is to provide simpler conditions with respect to Lyapunov-Razumikhin and Lyapunov-Krasovskii techniques. Furthermore, the nonlinear Halany-inequality allows to chose simple Lyapunov function candidate in the analysis. In the linear case, by using simple quadratic Lyapunov candidates, the stability properties are obtained by using Linear Matrix Inequality techniques. For nonlinear discrete-time delay systems, the case where the delays are subjected to follow a Markov chain is discussed and some sufficient conditions for the exponential mean square stability are provided.

Sommario

Il progetto di ricerca proposto nella seguente tesi si occupa dell'analisi di stabilitá di sistemi con ritardi a tempo discreto. I segnali di ritardo considerati, sono vincolati a seguire una dinamica indotta da una struttura a grafo orientato, detta grafo dei ritardi. Tipicamente, l'interesse in questa classe di sistemi deriva da sistemi a segnali a campionamento periodico. Questa regolazione richiede la discretizzazione del segnale, la quale introduce ritardi temporali nel sistema. I ritardi temporali derivano dalla propagazione di quantitá fisiche e sono frequentemente usati per ottenere modelli semplici di effetti fisici complessi come la visco-elasticitá, la cristalliazione dei polimeri e la velocitá di reazione. Inoltre, attuatore e sensori connessi ad una rete e controllori digitali posso introdurre ritardi nella dinamica di un sistema. La varietá di effetti che possono essere modellati da un sistema discreto con ritardi è della piú disparata, infatti i modelli soggetti a ritardo temporale sono applicati a diversi campi scientifici come la biologia, la chimica, l'economica e la meccanica [KM13]. Vi sono casi in cui, i ritardi temporali sono introdotti intenzionalmente nella dinamica di un sistema [Ric03], ad esempio, il risonatore ritardato [KGK03] usa un ritardo artificiale per migliorare la performance di assorbimento delle vibrazioni. I sistemi con ritardo sono spesso denominati in letteratura come sistemi ereditari, con memoria oppure a latenza temporale [Nic01]. Questi rappresentano una classe di sistemi infinito-dimensionali ampiamente usati per descrivere fenomeni fisici, come la dinamica delle popolazioni, trasporto, sistemi economici, processi chimici industriali e produzione di energia [NR07]. Nella modellazione fisica, la presenza di ritardi temporali puó essere causata da fenomeni di trasmissione dell'informazione, o dal del trasporto di materiale o puó essere generata dalla complessitá computazionale del problema sotto esamine. Le equazioni differenziali funzionali [Hal77] sono il modello piú comunemente usato per descrivere il comportamento di sistemi dinamici soggetti a ritardo temporale. La principale caratteristica dei sistemi differenziali con ritardo è l'inclusione della storia presente e passata nell'evoluzione del sistema. Quando i ritardi sono piccoli, possono essere ignorati, e questo riduce i problemi del controllo a problemi di tipo standard che possono essere studiati usando la tecnica di Laypunov applicata a sistemi senza ritardi. Esistono due approcci differenti sull'interpretazione della stabilitá di sistemi con ritardo: la prima guarda all'evoluzione del sistema nello spazio delle funzioni (i funzionali di Lyapunov-Krasovskii), la seconda pone attenzione sull'evoluzione del sistema nello spazio euclideo (le funzioni di Lyapunov-Razumikhin). La tecnica di Lyapunov-Krasovskii non considera la grandezza dei ritardi per sviluppare condizioni di stabilitá. Con questo metodo, la derivata (o la differenza nel caso discreto) del funzionale di Lyapunov candidato deve decrescere lungo tutte le traiettorie del sistema affiché si ottenga la stabilitá del sistema. La metodologia di Lyapunov-Razumikhin, la funzione candidata deve avere segno negativo solo per alcuni valori nell'intervallo di definizione dei ritardi e non in tutte le traiettorie. I ritradi temporali possono causare comportamenti complessi alla dinamica di un sistema e conseguentemente

causare perdita di stabilitá. L'intereste nello studio di sistemi con ritardo a tempo discreto è motivato da i sistemi di controllo su reti, in cui, la connessione tra il processore e il controllore è governata da una rete di comunicazione condivisa. Questa rete di comunicazione aumenta la flessibilita' dell'architettura di controllo, ma induce effetti come ritardi variabili nel tempo sogetti ad incertezza e fluttuazioni temporali. La presenza di vincoli è una caratteristica peculiare dei sistemi che modellano un sistema fisico. Tutti i risultati presenti in questa tesi sono basati sulla teoria di Lyapunov, poiché questa è uno dei pochi metodi che puó essere facilmente adattata allo studio dei sistemi vincolati. Lo studio di modelli a tempo discreto con ritardi è motivato tradizionalmente da l'indagine su i sistemi a segnali campionati e piú recentemente da i sistemi di controllo su reti. Oggigiorno, l'efficienza dei calcolatori è aumentata metre il loro costo è diminuito. Come conseguenza, i controllori moderni sono progettato per approssimare a tempo discreto modelli a tempo continuo. In questo caso, il controllore di un modello a tempo discreto stabilizza il modello originale a tempo discreto usando un certo tipo di assunioni [DNS99]. Una conseguenza dell'utilizzo di architetture digitali nel controllo di un sistema è la presenza di ritardi temporali nel ciclo di controllo, dovuto alla disceretizzazione del segnale. Questo produce sistemi a segnali campionati con ingresso ritardato [Fri14]. Un'altra importante applicazione dei sistemi con ritardo a tempo discreto è lo studio di sistemi di controllo su reti. La connessione tra il plant e il controllore è ottenuta attraverso una rete di comunicazone. L'introduzione di una rete di comunicazione porta diversi vantaggi tra cui riduce l'importo di cablaggio. Per questo motivo, i sistemi di controllo su reti sono ampiamente usati nelle applicazioni automobilistiche [CCC13] e nella robotica [CMZ04]. La rete di comunicazione introduce anche la presenza di ritardi variabili non deterministici, vincoli di comunicazione e perdita di pacchetti [AB+10]. Nel caso di sistemi lineari, soggetti ad un controllo su una rete, possono essere modellati da sistemi con ritardi non deterministici a tempo discreto [LHI06]. Un altro approccio alla modellazioni delle reti di comunicazione è l'impiego dei sistemi con ritardo a tempo discreto di tipo switching [ZY08] oppure sistemi con ritardo stocastico a tempo discreto [MDS12]. Inoltre, nei protocolli di comunicazione, possiamo ottenere sistemi a tempo discreto di tipo switching con ritardo [MDS12]. In generale, ritardi temporali divuti a reti di comunicazione sono correlati tra di loro, come conseguenza questi possono essere modellati usando catene di Markov a tempo discreto [DBT18].

Un'altra importante applicazione dei sistemi con ritardo è lo studio delle reti neurali. In questo tipo di sistemi, l'esistenza di ritardi tenporali puo' influenzare la stabilitá del sistema generando oscillazioni e instabilitá. Nell'implementazione della simulazione di reti neurali ricorrenti, è necessario formulare un sistema a tempo discreto a partire dalla sua controparte a tempo continuo, usando uno step di discretizzazione con poche restizioni, questo per mantenere le funzionalitá del sistema discreto similari al sistema continuo da cui ne deriva. Usando un intervallo di campionamento relativamente piccolo, la discretizzazione potrebbe non preservare la dinamica della controparte continua [WB92],[MG00b]. Nella simulazione numerica e nell'implementazione algoritmica di reti neurali continue, è necessario ed essenziale formulare un sistema a tempo discreto che sia analogo al sistema a tempo continuo ([YLL07], [XZD18],[UN13], [ZTJ13],[SZ14],[RYL15],[MJH16],[CS18]) L'ultima classe di modelli che interesserá la nostra analisi sono i sistemi con salto markoviano [MS03]. Questi sono una classe speciale di sistemi con parametri switching, e sono modellati da un insieme di sistemi lineari e nonlineari, in cui la trasizione da un insieme di sistemi all'altro è determinata da una catena di Markov. I sistemi con salto markoviano posso essere interpretati come un caso speciale di sistemi ibridi switching dove i segnali commutano in accordo ad una catena di Markov. Inoltre, i sistemi con salto markoviano posso essere considerati come una classe di sistemi stocastici dove le matrici del sistema variano casualmente su uno spazio discreto governato da una catena di Markov e i cui salti sono invarianti rispetto al tempo. Applicazioni di sistemi con salto markoviano figurano nei sistemi economici [BS75], i sistemi energetici [UP05], e sistemi di controllo su reti [KYX11]. Nell'analisi di sistemi discreti con ritardo, la metodologia di Razumikhin (si veda ad esempio [Tee98],[LM07], [RG13]) è di grande interesse poiche' utilizza condizioni che coinvolgono lo stato del sistema, in opposizione ai segmenti di traiettoria. Ne consegue che, la corrispondente funzione candidata fornisce informazioni sulle traiettorie del sistema e i conti possono essere eseguiti nello spazio degli stati del sistema. La metodologia di Krasovskii utilizza segmenti di traiettoria e non fornisce informazioni riguardante le traiettorie del sistema in sé, questo causa un aumento della complessita' per ritardi grandi. Due fattori importanti sono da tener presente in questi approcci: quale metodologia è la meno conservativa e quale fornisce le condizioni piu' semplici da verificare. Una diretta controparte dell'approccio di Razumikhin nel caso di sistemi con ritardi a tempo continuo è il metodo Razumikhin-backward, la cui complessitá risiede nella verifica delle condizioni imposte sulla funzione candidata. Una variante di queste condizioni è stata proposta in [LM07] ed estesa a sistemi perturbati in [LH09]. In [RG13], è mostrata una variazione del metodo di Razumikhin ed se ne deriva l'esistenza e l'unicitá di condizioni per la stabilitá di sistemi con ritardo a tempo discreto. In [PP17], sono dimostate le proprietá di stabilitá globale asimptotica e la *input-to-state* stabilitá per sistemi con ritardo a tempo discreto con segnale del ritardo variabile vincolato. In [Pep19], sono studiati sistemi con ritardo a tempo discreto con ritardi variabili vincolati e vengono dimostrate condizioni necessarie e sufficienti pe la stabilitá asimptotica globale e la stabilitá *input-to-state*. Funzioni di Lyapunov dipendenti dal ritardo sono impiegate per la trasformazione da un sistema con ritardo a tempo discreto in sistemi a tempo discreto di tipo *switching*. Il vantaggio proposto in [Pep19], consta di condizioni di Lyapunov necessarie e sufficienti che condidera tutti i possibili casi su i vincoli dei ritardi, poiché espressi tramite un grafo orientato. Un'altra metodologia usata nell'analisi di stabilitá di sistemi con ritardo a tempo discreto è la disuguaglianza di Halanay [Hal66]. Alcune generalizzazioni della disuguaglianza di Halanay nel caso discreto sono state formulate in [MG00a], [LF02], [RAS09], [UN09], [YSY13]. In particolare, in [LF02], sono dderivate alcune disuguaglianze a tempo discreto impiegate nello studio della stabilitá asintotica globale per una famiglia di equazioni alle differenze con segnali di ritardo limitati ma non vincolati. In [LM07],[LH09] alcune disuguaglianze lineari di Halanay a tempo discreto sono usate per mostrare la stabilitá *input-to-state* e la stabilitá esponenziale globale per sistemi con ritardi variabili. Uno dei pochi lavori riguardanti la disuguaglianza di Halalany non lineare è [Bak10]. Una generalizazzione al caso multidimensionale è proposto in [WW16]. Sistemi a tempo discreto con switching markoviano sono particolarmente utili nella modellazione di sistemi sogetti a cambiamenti bruschi, come le reti di controllo *wireless*. L'applicabilitá è dovuta alla manifestazione del fenomeno di perdita di pacchietti nel canale *wireless* e questo induce cambiamenti nella dinamica del modello e, conseguentemente, variazione temporale dei ritardi [OC06], [Hua19], [YALB19], [BWT19]. I modelli markoviani consentono di rappresentare l'esplosione della perdita di pacchetti, fenomeno che non è possibile rappresentare con variabili aleatorie bernulliane [PSS08]. Inoltre, in molti protocolli di comunicazione, la perdita di pacchetti puó forzare la ri-trasmissione di un pacchetto, che sará ricevuto con un ritardo dipendente dalla natura stocastica della perdita del pacchetto stesso. Come conseguenza, i sistemi markoviani di tipo *switching* risultano essere una buona approssimazione della caratterizzazione stocastica dei modelli di controllo per reti wireless in presenza di perdita di pacchetti e di ritardi casuali indotti. In [RAW11], [GD19], [YALB19], [YLS20], l'uso di modelli markoviani di tipo *switching* è impiegato nel design e nel controllo di sistemi di controllo su reti *wireless*, e permette di verificare l'instabilita' di un sistema dovuta alla perdita di pacchetti quando i modelli con canale bernoulliano falliscono. Inoltre, la modellazione di canali wireless attraverso sistemi switching markoviano permette di migliorare la stabilizazzione del controllo [YLS20]. La stabilitá di tipo *mean square* per sistemi switching markoviani è stata estensivamente analizzata nel caso lineare [OC06],[YALB19]), pochi lavori, presenti in letteratura, sono dedicati alla studio di questa nozione di stabilitá nell'ambito dei sistemi nonlineari [PPB14], [ATG10]). La disuguaglianza di Halanay a tempo di discreto è esaminata in [LF02], [Bak10], una generalizzazione multidimensionale è fornita in [WW16]. Al meglio delle nostre conoscenze, non vi sono risultati disponibili in letteratura riguardanti disuguaglianze di Halanay nonlineari a tempo discreto [Bak10]. È noto che le condizioni di stabilitá ottenute con le metodologie di Razumikhin e Krasovskii sono, in generale, meno conservative delle condizioni ottenute usando le disuguaglianze di Halanay, sia nel caso continuo che discreto. D'altra parte, le disuguaglianze di Halanay possono garantire condizioni piú semplici da verificare in molti casi. La presente tesi intende di colmare il *gap* nella letteratura riguardante disuguaglianze nonlineari di Halanay a tempo discreto con e senza termine forzante e mostrare condizioni di Lyapunov sufficienti che ricoprano tutti i possibili casi di vincoli sui segnali di ritardo espressi da un grafo orientato.

About

Maria Teresa Grifa received the Master degree in Applied Mathematics from University of Rome La Sapienza. Thereafter, she worked as data scientist in the italian start-up scenario. In 2016, she started working towards a Ph.D. degree in Mathematics and Models at University of L'Aquila. Her research interests include the stability analysis of discrete-time systems with delay and machine learning. She is currently working as data scientist at Bridgestone EMA.

Legalities.

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Introduction

This thesis discusses the stability analysis of discrete-time systems with time-varying delays where the delays dynamic is ruled by a digraph. The peculiarity of a time-delay system lies in its dynamics, which is determined by the past and the present system states. Time delays arise due to the propagation of physical quantities over large distances and are frequently used to obtain relatively simple models of complex physical effects such as visco-elasticity, finite reaction rates and polymer crystallization. Moreover, actuators and sensors connected to networks and digital controllers introduce delays as well. Because of the wide variety of the above effects, models with time delays can be found in many different fields such as biology, chemistry, economics and mechanics, [KM13]. Furthermore, time delays are sometimes introduced intentionally in a dynamical system,e.g., [Ric03], to obtain a response with certain desirable properties, e.g., in [KGK03], a delayed resonator is employed to trigger an artificial delay that enhance a vibration absorber's performance with respect to its sensitivity to the excitation frequency. The main focus of the present thesis is the study of the stability analysis of discrete-time delay systems where delays are subjected to adhere to a delay digraph. The entire exposition will concern nonlinear dynamics and the design problems will be treated in a discrete-time framework.

Time delay systems

Time-delay systems, also called hereditary or with memory, dead-time, or time-lag [Nic01], represent a class of infinite-dimensional systems largely used to describe some physical phenomena, such as population dynamics (reproduction, growing, extinction, etc), transport phenomena, economic systems (investments policy, commodity market evolution), chemical industrial processes and energy production [NR07]. In physical modeling, the presence of time delays may be due to information transmission phenomena, to transport material phenomena or may be generated by the computational complexity of the problem under examination. Functional differential equations [Hal77] are the most common type of models that make use of time delays to describe the behavior of dynamical systems. The main characteristic of the systems described above is that their dynamics can be described by differential equations including the present and the past history of the system's evolution. An illustration of the appearance of time delays can be given by the problem of the water regulation of the shower grip.

Example 1 ([Fri14] Shower mixer) It is common task balancing the blue and red shower mixers in order to obtain suitable water temperature. This is a simple example of a dynamical system with a propagation delay. Adjusting the temperature of the water using red shower-handle can take some time because the water takes a certain amount of time to reach the showerhead. Using our common sense embedded with experience, we might take into account the delay occurrence and hence avoid large temperature deviations. A modeling approach of the Example 1 can be given using a continuous dynamical models with delay.

Example 2([Gie13]) Continuous-time shower mixer model) Let x denote the water temperature at the mixer output and let h denote the time it takes the water from the mixer output to reach the showerhead. Now, let us consider x_d be the desired water temperature to reach. Suppose that the change in water temperature at the mixer output is proportional to the change of the mixer angle α, which influences the ratio of warm and cold water for some constant c. Assume that the rate of rotation of the handle is proportional by a coefficient k to the deviation in water temperature from x_d perceived. At time t, the water temperature at the mixer output is $t-h$, then we can model the rate of the mixer angle as

$$\dot{\alpha} = -k\Big(x(t-h) - x_d(t)\Big),$$

then, we obtain a description of the water temperature as function of the time, for $t \in \mathbb{R}$:

$$\dot{x} = -ck\Big(x(t-h) - x_d(t)\Big).$$

It is worth to mention that when the delays are small, they could be safely ignored, which reduces control problems to more standard problems that can be covered by traditional Lyapunov function techniques for delay free systems. An important research topic on time-delay systems is the analysis based on the Lyapunov's theory. When considering time-delay systems, there are two different ways of interpreting the stability of the considered system: as an evolution in a function space (Lyapunov-Krasovskii functionals) or as an evolution in the Euclidean space (Lyapunov-Razumikhin functions). The Lyapunov-Krasovskii approach does not take into account the delay size information to develop stability conditions, where the derivative (or difference in discrete-time) of the candidate function must decrease along all the system's trajectories to obtain stability. On the other hand, the derivative of Lyapunov-Razumikhin candidate should be negative just for some critical values between the delay interval, and not for all trajectories.

Discrete-time delay systems

The study of discrete-time models with delay is motivated traditionally by sampled-data systems and more recently by networked control systems. Nowadays, the efficiency of computers has

increased while their costs have decreased. As a consequence, modern controllers are designed for a discrete-time approximation of the continuous-time model and then implemented via a computer. In this case, the controller for the discrete-time model also stabilizes the original continuous-time model under some mild assumptions, see, e.g., [DNS99].
A consequence of using a digital architecture to control a system is the presence of time delays in the control loop due to the discretization of signals, and gives rise to sampled-data systems with delayed inputs [Fri14].

Example 3 (Discretization of the shower mixer model [Gie13]) Using a forward Euler technique [CI00], we can achieve a discrete approximation of the continuous-time model with delay in Example 2, for $k \in \mathbb{Z}^+$:

$$x_{k+1} = -ckT_s(x_{k-10} - x_{d,k}) + x_k$$

with sampling time $T_s = 0.1h$ It can be shown that for this particular sampling time, the discrete-time model preserves the stability of the original continuous-time model.

Discrete-time delay systems are largely used in modeling networked control systems. The connection between plant and controller is made through a shared communication network. The introduction of the communication network brings several advantages, most importantly a reduced amount of wiring. For this reason, networked control systems are highly used in automotive applications [CCC13] and robotics [CMZ04]. The communication network also brings the presence of uncertain time-varying delays, communication constraints and packet dropouts, e.g., see [AB+10]. In the case of linear system, controlled over a communication network, can be modeled as an uncertain discrete-time system with delay, see [LHI06]. Another way of modeling a communication network is using a discrete-time switched system with delay [ZY08] or as a stochastic discrete-time system with delay [MDS12] when the stochastic nature of the delay is taken into account. Furthermore, when communication protocols are taken into account a discrete-time switched system with delay [MDS12] is obtained. In general, time delays due to communication networks are correlated each other, and therefore are frequently modeled by discrete-time Markov chains [DBT18].
Another important application of discrete-time delay systems is the study of neural network systems. The implementation of a recurrent neural network requires to derive a discrete-time system from its continuous-time counterpart under little restriction on the discretization step size. However, the discretization may not preserve the dynamics of the continuous-time counterpart even for a small sampling period [WB92],[MG00b]. For numerical simulation and practical implementation of the continuous-time neural networks, it is necessary and essential to formulate a discrete-time system that is an analogue of the continuous-time system, [YLL07], [XZD18],[UN13], [ZTJ13],[SZ14],[RYL15],[MJH16],[CS18]. The last class of model we exploit in our analysis are the Markov jump systems. Markov jump systems are a special class of

parameter-switching systems, and they are modeled by a set of linear or nonlinear systems with the transitions between the models determined by a Markov chain taking values in a finite set [MS03]. Markov jump systems can also be considered as special case of switched hybrid systems with the switching signals governed by a Markov chain. Markov jump systems can be interpreted as a special class of stochastic systems with system matrices changing randomly at discrete-time points governed by a Markov process and remaining time-invariant between random jumps. Applications of Markov jump systems can be found in many real world applications, such as economic systems [BS75], power plant systems [UP05], and networked control systems [KYX11]. To better illustrate when Markov jump linear systems should be used, we consider the solar energy plant example.

Example 4([OC06] Solar energy plant) A solar energy plant consists of a set of adjustable mirrors, capable of focusing sunlight on a tower that contains a boiler, through which flows water. The power transferred to the boiler depends on the atmospheric conditions, more specifically on whether it is a sunny or cloudy day. The attitude of the heliostats is controlled in order to keep sunlight focused onto the boiler. The process dynamics differs with respect to the atmospheric conditions, i.e. with clear skies, the boiler receives more solar energy and so we should operate with a greater flow than on cloudy conditions.

The problem in Example 4 can be modeled via a linear Markov jump system.

Example 5([OC06] Markov jump solar energy model) Given adequate historical data, the atmospheric conditions for boosting the solar energy plant can be modeled as a Markov chain with two states: sunny and cloudy. We can assume that the current state is known, i.e., we do not know how the weather will behave in the future, but that the probabilities of it being sunny or cloudy in the immediate future are known. The stochastic nature of this control problem arises because the system dynamics is heavily dependent on the instantaneous insolation. Cloud movement over the heliostats can cause sudden changes in insolation and can be treated, for practical purposes, as a stochastic process. The thermal receiver is described by the following equations:

$$\begin{aligned} x(k+1) &= A_{r(k)}x(k) + B_{r(k)}u(k) \\ z(k) &= C_{r(k)}x(k) + D_{r(k)}u(k) \end{aligned}$$

for $r(k) \in \{1, 2\}$ and

Sunny	Cloudy
$A_1 = 0.8353$	$A_2 = 0.9646$
$B_1 = 0.0915$	$B_2 = 0.0982$

Stability analysis

As almost every system in practice is subject to constraints and Lyapunov theory is suited for the stability analysis of systems that are subject to constraints, we will essentially restrict our attention to Lyapunov theory. For the stability analysis of discrete-time systems with delay two different approach can be distinguished. The Razumikhin approach (see for instance [Tee98],[LM07], [RG13]) is of interest because it makes use of conditions that involve the system state, as opposed to trajectory segments. As a consequence, the corresponding function provides information about the trajectories of the system directly and the corresponding computations can be executed in the underlying low-dimensional state space of the system dynamics. The Krasovskii approach involves trajectory segments and as such do not provide information about the system trajectories directly, which causes them to become increasingly complex for large delays. Two important factors play a role in these kind of approaches, i.e., which method is the least conservative and which method provides the conditions that are simplest to verify. A direct translation of the Razumikhin approach for continuous-time systems with time-delay to discrete-time delay systems yields a set of so-called backward Razumikhin conditions, which are typically difficult to verify. A practical variant of these conditions was first proposed in [LM07] and extended to systems with disturbances in [LH09]. Even so, for quadratic functions and linear discrete-time delay systems the conditions obtained therein are nonlinear and non-convex and hence, difficult to verify. In [RG13], a modification of the Razumikhin approach is provided, and a necessary and sufficient conditions for stability of discrete-time delay systems is proved. More specifically, the candidate Lyapunov function is required to be less than the maximum over the function values for a number of delayed states. When this number is chosen equal to the size of the delay, Lyapunov functions as introduced in [LH09] are recovered. More specifically, Theorem 4.1 in [RG13], covers the interpretation of the Razumikhin approach that was presented in [LM07]. Moreover, the Lyapunov function in Theorem 4.1 [RG13], provide information about the evolution of the trajectories of the discrete-time delay systems in the original state space $\mathbb{R}^n$, as opposed to $\mathbb{R}^{n(h+1)}$ for the Krasovskii approach. As such, the computations corresponding to Theorem 4.1 can be executed with respect to the original state space, which yields a computational advantage. In [RG13], for exponentially stable discrete-time delay systems, an estimate is constructed for the lower bound on the value of the number of states for which necessity is obtained. Furthermore, for linear discrete-time delay systems and quadratic functions the developed conditions are shown to be equivalent with a Linear Matrix Inequality. In [PP17], both the global asymptotic stability and the input-to-state stability properties (see [Son89] for the definition of input-to-state stability), for discrete-time systems with uncertain and time-varying time delays, have been characterized by the existence of a delay-independent Lyapunov function, which is proved to be not only sufficient for these properties, but also necessary. In [PP17], it is assumed that time delays are bounded and the bound is known. In [Pep19], discrete-time systems with uncertain and time-varying time delays with known common bound and possible constraints in the allowed time-delay signals are considered. Necessary and sufficient conditions for the global asymptotic stability and for the input-to-state stability are provided. Delay-dependent Lyapunov functions are used and the transformation of nonlinear discrete-time time-delay systems into nonlinear discrete-time switching systems is exploited. The new advantage propose in [Pep19], consists in Lyapunov necessary and sufficient conditions which covers all cases with possible constraints on time-delay signals, as expressed by a suitable delays digraph. In this way, the work in [PP17] is covered as a special case in [Pep19]. In [Pep19], in the case of bounded arbitrary time-delay signals and constrained time-delay signals as per a delays digraph, it is shown that the internal and external stability is equivalent to the existence of a delay-dependent Lyapunov function satisfying suitable conditions given by a delays digraph. Another useful technique is based on discrete-time inequalities such as the discrete-time analogue of the continuous-time Halanay inequality [Hal66]. In general, the comparison principle requires finding an additional system, with known stability properties, and then compare that to the original time-delay system. Some linear discrete-time Halanay inequality generalizations and applications can be found in [MG00a], [LF02], [RAS09], [UN09], [YSY13]. In [LF02], the authors derive some discrete-time inequalities and prove the global asymptotic stability of a family of difference equations with unconstrained, bounded delay signals. In [LM07],[LH09] discrete-time linear Halanay-type inequalities are employed to prove the input-to-state stability and the global exponential stability of nonlinear discrete-time delay systems with time-varying delays. One of the few works regarding nonlinear discrete-time Halanay-type inequalities is [Bak10], where most attention is devoted to provide a bound for the involved non-negative functions. Little attention has been paid to the application of this kind of discrete-time nonlinear inequalities to establish related sufficient conditions for the global asymptotic stability of discrete-time delay systems, with or without constrained time delays. A multidimensional generalization is provided in [WW16]. Discrete-time markovian switching systems are particularly useful in the modeling of systems subject to abrupt changes, such as Wireless Control Networks. The applicability is due to the fact that the wireless channel may suffer from packet-losses and re-transmissions, which induce abrupt changes of the dynamical model and possibly time-varying delays [OC06], [Hua19], [YALB19], [BWT19]. Markov models allow representing bursts of packet losses, which is not possible using Bernoulli random variables, [PSS08]. Also, in many communication protocols and standards, packet losses may enforce the re-transmission of a packet, which is thus received with a delay that depends on the stochastic characteristics of bursts of packet losses. As a consequence, markovian switching systems are good approximations of the stochastic characterization of wireless control networks models in presence of packet losses and induced random delays. In [RAW11], [GD19], [YALB19], [YLS20], the use of markovian switching systems handles the challenges in analysis and co-design of wireless networked control systems, and allows to verify instability of a system due to bursts of packet loss when Bernoulli-like channel models fail. Moreover, the Markov modelling of the Wireless channel allows performance improvement in stabilizing control synthesis [YLS20]. The mean square stability of discrete-time markovian switching systems has been extensively analyzed in the linear case [OC06],[YALB19]), only few works presented in the literature investigate this stability notion in the nonlinear framework (see [PPB14],[ATG10]). Motivated by the above discussions, we focus our study on discrete-time systems with markovian delays, linking the methodologies available for Markov jump systems and discrete-time systems with constrained delays. The link between discrete-time systems with delays and switching delay-free systems is provided in [LHI08], [Pep18]. The mean square stability of discrete-time markovian switching systems has been extensively analyzed in the linear case [OC06], [YALB19]. Only few works presented in the literature investigate this stability notion in the nonlinear framework, [PPB14], [ATG10].

Objectives and motivation

As reported in the previous section, times delays in the state of the system in general cannot be neglected. Furthermore, as almost every system in practice is subject to constraints and Lyapunov theory is suited for the stability analysis and control of systems that are also subject to constraints, all results in this thesis will essentially be based on Lyapunov theory. Our main research objective is to fill the gap in the literature regarding the development of nonlinear discrete-time Halanay-type inequality and provide sufficient Lyapunov conditions covering all cases with possible constraints on time-delay signals, as expressed by a suitable delays digraph. The discrete-time Halanay-type inequality is investigated in [LF02], [Bak10], and a multidimensional generalization is provided in [WW16]. To our best knowledge, there are no results available in the literature concerning nonlinear discrete-time Halanay-type inequalities in the case with arbitrarily large, as long as bounded, forcing term (see Theorem 3.12 in [Bak10]). As stated in the previous sections, it is well known that stability conditions obtained by Razumikhin as well as by Krasovskii methodologies are in general less conservative than stability conditions obtained by Halanay's inequalities, in both the continuous-time and the discrete-time cases. On the other hand, Halanay's inequalities may yield tractable and simple conditions to use in many cases and we believe that an insightful exploration of nonlinear Halanay-type inequalities, for novel stability conditions, is worth investigation. Furthermore, in the contest of Lyapunov theory and constrained delays, we present a stability result on a special class of stochastic time-delay systems: nonlinear discrete-time delay systems with delay constrained to vary on a Markov chain. Here, our objective relies in the extension of existing Lyapunov conditions for the global asymptotic stability of discrete-time systems with delay digraph to the study of the mean square stability of discrete-time systems with markovian delays. In the following, the research steps investigated are listed:

1. Definition of the problem statement

2. Definitions for stability and input to state stability

3. Prove of nonlinear Halanay-type inequality with and without forcing term

4. Formulation of Lyapunov sufficient conditions using comparison technique for the stability and the input-to-state stability of the class of systems under consideration

5. Formulation of Lyapunov sufficient conditions for the mean square stability of Markov jump systems.

Outlines of the thesis

In what follows, a summary of results and contributions of this thesis is presented. The work is organized as follows.

Chapter 1

The first chapter contains some recalls of the notions of stability of nonlinear discrete-time systems. We start with the definitions of stability for discrete-time delay free systems. Then we introduce the important concepts of input-to-state stability, which related to the problems of disturbance attenuation and nonlinear systems stabilizability. We illustrate some known techniques for stability analysis of discrete-time delays systems. In this contest, we introduce the Lyapunov-Krasovskii and the Lyapunov-Razumikinh approach. We recall the definition of semi-global stability, which is of sure interest in the context of applications, since the properties introduced take into account that there are bounded set of feasible initial conditions and that the steady-state error could also be "sufficiently small" and not exactly zero. Next, we introduce the problem of stability for linear discrete-time delay systems and the matrix inequality approach on the stability analysis. The last section is devoted on a presentation regarding Markov jump systems. We recall the definition of exponential mean square stability, i.e., stability analysis concerning the behavior of the second moment of the state, which is a related to not deterministic behavior of variation of the time.

Chapter 2

The second chapter contains a brief panoramic on the state-of-the-art of discrete-time Halanay-type inequalities. In this contest, we recall some results on linear discrete version of the Halanay inequality continuous-time counterpart. Halanay-type inequalities may yield tractable and simple conditions to use in the stability analysis of discrete-time systems. We present some linear Halanay-type inequalities for generalized discrete difference equations. Next, we present some established stability results which employ the linear Halanay-type inequality in the Lyapunov-based stability analysis.

Chapter 3

The third chapter is devoted to one of the main result of the work. We begin recalling the delay-digraph assumption for treating the delay signals in a discrete-time system. We specialize the stability analysis in a class of discrete-time delay system. Next, we give the stability definitions related to the system plant introduced, which are strictly dependent on the behavior of the delay digraph associated to the delays of the class of systems considerate. Then, we recall a comparison technique for a class of continuous functions. Next, we present a novel nonlinear Halanay-type inequality involving the asymptotic behavior and the uniform convergence of the sequence considerate. By employing the Halanay technique, we derive Lyapunov sufficient conditions of the global asymptotic and globally exponential stability. In the last part of the chapter, we give some examples for validating our theory.

Chapter 4

The fourth chapter illustrates the results reported in another work form the thesis's author. The system plant is defined as per Chapter 3. We investigate the input-to-state stability analysis of a class of discrete-time delay systems where the delay signals is subject to adhere to a delay digraph. We reformulate the nonlinear Halanay-type inequality reported in Chapter 3, taking into account the effects of adding a forcing term, which it is of interest in the development of the analysis of a system with input source. Next, we develop sufficient Lyapunov conditions guaranteeing the input-to-state stability properties of the plant under consideration. The Lyapunov conditions are established by using the novel discrete-time Halanay-type inequality with forcing term. In the last part of the chapter, we extend the applications given in Chapter 3 to the input-to-state stability analysis.

Chapter 5

In the fifth chapter, the analysis of discrete-time delay systems where delays are modeled via a Markov chain is illustrated. We start defining the modeling framework of discrete-time systems subjected to Markov switching. We perform a first transformation of a class of discrete-time delay system to a switching system where delays are constrained to adhere to a Markov chain. Next, we convert the switching system to a Markov jump system. Next, we perform the analysis of the second moment of the state of the system, considering the exponential mean square stability. The first part of the main findings consists in proving some technical results. Next, we develop sufficient Lyapunov -based conditions for the stability property under consideration. The methodology makes use of multiple Lyapunov functions that depend on the mode of the Markov chain. Finally, we provide a meaningful example and some numerical simulation using the Monte Carlo approach for illustrating our methodology.

Summary of Publications

Grifa M. T., and Pepe P. *"On stability analysis of discrete-time systems with constrained time-delays via nonlinear Halanay-type inequality."*, IEEE Control Systems Letters 5.3 (2020): 869-874.

This is a joint work of the PhD candidate Maria Teresa Grifa with Prof. Pierdomenico Pepe, available online on ieeexplore, which got 2 citations. This work is published in the journal IEEE Control Systems Letters.

The contents of this paper has been presented in

- M.T. Grifa and P. Pepe, On stability analysis of discrete–time systems with constrained time–delays via nonlinear Halanay-type inequality, *IEEE 59th Annual Conference on Decision and Control (CDC 2020)*, Jeju Island, Sounth Korea, 2020.

Grifa,M.T., Pepe P.*"On a Novel Nonlinear Discrete-Time Halanay's Inequality with Forcing Term and Applications to the Input-to-State Stability of Delay Systems"*

This is a joint work of the PhD candidate Maria Teresa Grifa with Professor Pierdomenico Pepe. This work is submitted to *International Journal of Control*.

Impicciatore, A., Grifa, M. T., Pepe, P., D'Innocenzo, A., *"Sufficient Lyapunov conditions for exponential mean square stability of discrete-time systems with markovian delays."*, 2021 29th Mediterranean Conference on Control and Automation (MED). IEEE, 2021. p. 1305-1310.

This is a joint work of the PhD candidate Maria Teresa Grifa with PhD student Anastasia Impicciatore (University of L'Aquila), Prof. Pierdomenico Pepe, Prof. Alessandro D'Innocenzo (University of L'Aquila). This work available online ieeexplore.
The contents of this paper has been presented in

- Impicciatore, A., Grifa, M. T., Pepe, P., D'Innocenzo, A. (2021). Sufficient Lyapunov conditions for exponential mean square stability of discrete-time systems with markovian delays, *29th Mediterranean Conference on Control and Automation, 2021, Bari, Italy, June 2021.*

Chapter 1

Preliminaries

Note 1.0.1. This chapter aims to give to the reader a broad overview about the results used in the thesis. To read the full reference disclaimer, please refer to [0.1].

Chapter Description

This chapter provides technical preliminaries that help to explain the main results and support the presentation of this thesis. We illustrate a broad overview of the classes of systems we exploit in the next chapters and their stability properties which are the objective for our investigation. We report various important results from the literature regarding the stability analysis problem by means of Lyapunov theory.

In Section 1.1, common notation that will be used throughout the thesis are presented. The main purpose of Section 1.2 is to collect stability definitions and Lyapunov related theorems for delay-free discrete-time systems. In Section 1.3, we introduce some important results regarding the stability stability properties of nonlinear discrete-time delay systems. In Section 1.4, definitions and results concerning the stability properties of linear discrete-time delay systems are recalled. In Section 1.5, definitions and results regarding Markov jump systems are presented.

1.1 Notation

Sets

Let D a non-empty set, the cardinality of D is denoted by card(D).

Let $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$ and $\mathbb{Z}^+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively.

Vectors, matrices and norms

For a real number $a \in \mathbb{R}$, $|a|$ denotes its absolute value, $\lfloor a \rfloor$ denote the smallest integer larger that a and $\lceil a \rceil$ denotes the biggest integer smaller that a.

For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm, $\|x\|_\infty$ denotes its infinity-norm, i.e., $\max\{|x_1|, |x_1|, \ldots, |x_n|\}$.

$\mathbf{x}$ denotes the sequence of vectors $\{x_l\}_{l\in\mathbb{Z}^+}$, with $x_l \in \mathbb{R}^n$ for all $l \in \mathbb{Z}^+$, $\|\mathbf{x}\|$ denotes its Euclidean norm, i.e., $\sup\{\|x_l\| : l \in \mathbb{Z}^+\}$.

For a matrix $A \in \mathbb{R}^{n\times m}$, A^T denotes its transpose, A^{-1} denotes its inverse, det(A) denotes its determinant.

For a matrix $A \in \mathbb{R}^{n\times m}$, $\|A\|$ denote its induced Euclidean norm, i.e., $\max\{\|Ax\| : x \in \mathbb{R}^n, \|x\| \leq 1\}$.

For a matrix $A \in \mathbb{R}^{n \times m}$, $A > 0$ denotes that A is positive definite. For all $x \in \mathbb{R}^n - \{0\}$ it holds that $x^T A x > 0$ and $A = A^T$.

I_n denotes the identity matrix of dimension $n \times n$.

diag(D) denotes the diagonal matrix of appropriate dimensions.

For the matrices A and B, diag(A, k) denotes a diagonal matrix with k matrices A on the main diagonal, or diag(A, B) denotes:

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

For a matrix $A \in \mathbb{R}^{n \times m}$, $\lambda(A)$ denotes the eigenvalues of A, called also spectrum of A. Its spectral radius is defined as:

$$\rho(A) := \max_{\lambda \in \lambda(A)} |(\lambda)|.$$

Proposition 1.1.1. ([Boy04] Schur complement). Consider the matrices $Q(x) = Q(x)^T, R(x) = R(x)^T$ and $S(x)$ be affinely dependent on x. The Linear Matrix Inequality (LMI):

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0,$$

is equivalent to the set of nonlinear inequalities:

$$R(x) > 0, \ Q(x) - S(x)R(x)^{-1}S(x)^T > 0,$$

or equivalently:

$$Q(x) > 0, \ R(x) - S(x)^T Q(x)^{-1} S(x) > 0.$$

Proposition 1.1.2. (Cayley-Hamilton Theorem) If $p(\lambda) = \det(\lambda I_n - A)$ is the characteristic polynomial of a matrix $A \in \mathbb{R}^{n \times n}$ then $p(A) = 0$.

Theorem 1.1.1. *(Young inequality) Let a, b be non-negative real numbers and p, q real numbers with $p > 1$ and $q > 1$ such that*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then the following inequality holds

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{1.1.1}$$

if and only if $a^p = b^q$.

Functions

For a positive integer m, for real numbers a, b with $b > a$, a function $g : [a, b) \to \mathbb{R}^m$ is said to be piece-wise continuous if it is bounded, right-continuous, continuous except, at a finite number of points.

For a positive integer m, for real number a, a function $\bar{g} : [a, +\infty) \to \mathbb{R}^m$ is said to be piece-wise continuous if, in any set $[c, d)$, c , d reals with $d > c$, in $[a, +\infty)$, is bounded, right-continuous, continuous except, possibly, at a finite number of points.

For a piece-wise continuous function $\bar{g} : [a, +\infty) \to \mathbb{R}^m$, for a real $T > 0$, with $\bar{g}[0, T] : [a, +\infty) \to \mathbb{R}^m$ is indicated the function defined as

$$u[0, T](t) = \begin{cases} u(t) & \forall t \in [0, T] \\ 0 & \text{elesewhere.} \end{cases}$$

The function sat : $\mathbb{R} \to [-1, 1]$ is defined, for $s \in \mathbb{R}$, as $\text{sat}(s) = \min\{1, \max\{s, -1\}\}$.

The symbol $\circ$ denotes composition of functions.

For a positive integer k, or a function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\gamma^0 = I$ γ^k denotes the k times composition of γ, i.e., $\gamma^k = \gamma \circ \gamma^{k-1}$, $k = 1, 2, \ldots$.

A function $f : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be Lipschitz in $(\bar{k}, \bar{x})$ if

$$\|f(k, x) - f(k, y)\| \leq L\|x - y\|,$$

$\forall (k, x), (k, y)$ in a neighborhood of $(\bar{k}, \bar{x})$. The constant L is called Lipschitz constant.

Class functions

In the following, classes of functions, are recalled (see Definition 4.1, 4.2 in [KG02]).

A continuous function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{P}_0$ if $\gamma(0) = 0$; of class $\mathcal{N}$ if it is of class $\mathcal{P}_0$ and increasing (not necessarily strictly); of class $\mathcal{P}$ if it is of class $\mathcal{P}_0$ and $\alpha(s) > 0$, $s > 0$; of class $\mathcal{K}$ if it is of class $\mathcal{P}$ and strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded; of class $\mathcal{L}$ if it is decreasing and $\lim_{s\to\infty} \gamma(s) = 0$.

A continuous function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{KL}$ if, for each fixed $s \in \mathbb{R}^+$, the function $\beta(\cdot, s)$ is of class $\mathcal{K}$, and, for each fixed $r \in \mathbb{R}^+$, $\beta(r, \cdot)$ is of class $\mathcal{L}$.

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be $\mathcal{K}$-continuous at zero if there exists a function $\alpha \in \mathcal{K}$ such that $\|f(x) - f(0)\| \le \alpha(\|x\|)$, for all $x \in \mathbb{R}^n$.

State space functions

For a positive integer n, for a positive integer Δ, the symbol C denotes the space of functions mapping $\{-\Delta, -\Delta + 1, \dots, 0\}$ into $\mathbb{R}^n$.

For $\phi \in C$, let $\|\phi\|_\infty$ denotes its norm, i.e., $\max\{\|\phi(-j)\| : j = 0, 1, \dots, \Delta\}$.

For a non-negative integer c, for a function $x : \{-\Delta, -\Delta + 1, \dots, c\} \to \mathbb{R}^n$, let x_k denotes the function in C defined as $x_k(\tau) = x(k + \tau)$, $k \in [0, c]$, $\tau \in \{-\Delta, -\Delta + 1, \dots, 0\}$,

For $x = [x_0^T \ x_1^T \cdots x_\Delta^T]^T \in \mathbb{R}^{n(\Delta+1)}$, $x_i \in \mathbb{R}^n$, $i = 0, 1, \dots, \Delta$, the function $\mathcal{I} : \mathbb{R}^{n(\Delta+1)} \to C$ is defined as $\mathcal{I}(x)(-j) = x_j$, $j = 0, 1, \dots, \Delta$.

For $\phi \in C$, the inverse function $\mathcal{I}^{-1} : C \to \mathbb{R}^{n(\Delta+1)}$ is defined as $\mathcal{I}^{-1}(\phi) = [\phi^T(0) \ \phi^T(-1) \ \cdots \ \phi^T(-\Delta)]^T$.

Probability theory

In the following, some probability theory definitions are recalled (see [Ash14]).

A non-empty set $\Omega \subset \mathbb{R}^n$ denotes the sample space.

$\mathcal{F}$ is said to be a σ-algebra on Ω if the following conditions hold:

1. the empty set $\emptyset \in \mathcal{F}$,
2. if $A \in \mathcal{F}$, then $A^C \in \mathcal{F}$ where A^C is the complement of A,
3. if $\{A_i\}_{i\ge1} \in \mathcal{F}$, $\bigcup_{i\ge1} A_i \in \mathcal{F}$

A real valued function $X : \Omega \to \mathbb{R}$ is said to be a random variable or $\mathcal{F}$-measurable if $\{\omega \mid X(\omega) \le x\} \in \mathcal{F}$ for all $x \in \mathbb{R}$.

A function $\mathbb{P} : \mathcal{F} \to [0, 1]$ is said to be a probability measure on the measurable space $(\Omega, \mathcal{F})$ if

1. $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\Omega) = 1$

2. for $\{A_i\}_{i\geq 1} \in \mathcal{F}$ where $A_i \cap A_j = \emptyset$ for all $i \neq j$,

$$\mathbb{P}(\bigcup_{i\geq 1} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Furthermore, the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space. If the σ-algebra is complete; that is, $\mathcal{F} = \mathcal{F}^C$, then the probability space is said to be complete.

The filtered probability space is defined as $(\Omega, \mathcal{F}, \{F_k\}_{k\in\mathbb{N}}, \mathbb{P})$, where $\mathbb{P}$ denotes the unique product probability measure and $\{F_k\}_{k\in\mathbb{N}}$ is the filtration.

Let $\mathbb{E}[\cdot]$ denote the expectation of a random variable with respect to $\mathbb{P}$ and $\mathbb{E}[\cdot \mid \mathcal{F}_k]$ denote the conditional expectation with respect to $\mathcal{F}_k$.

A discrete-time time-homogeneous Markov chain is defined as the sequence $\{r(k)\}_{k\in\mathbb{N}}$, taking values in a finite set $S = \{1, 2, \ldots, N\}$ with N positive integer, initial probability distribution $\nu = \{\nu_1, \nu_2, \ldots, \nu_N\}$ and transition probability matrix $P = (p_{ij})_{i,j\in S} \in \mathbb{R}^{N\times N}$, whose entries are the transition probabilities $p_{ij} := \mathbb{P}(r(k+1) = j \mid r(k) = i)$ for all $k \in \mathbb{N}$, with $\sum_{j=1}^{p} p_{ij} = 1, \forall i \in S$.

List of abbreviations

The following abbreviations are used throughout this thesis:

EMSS:	exponentially mean square stability
GAS:	global asymptotic stability
GES:	global exponential stability
ISS:	input-to-state stability
LMI:	linear matrix inequality
MC:	Markov chain
TPM:	transition probability matrix
UES:	uniform exponential stability
WCNs:	wireless control networks

1.2 Nonlinear discrete-time delay-free systems

Let us consider a nonlinear discrete-time system described by the following equations ([JW01]:

$$x(k+1) = f(k, x(k), u(k)),\ k \geq k_0$$
$$x(k_0) = \xi_0, \tag{1.2.1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector; $f : \mathbb{Z}^+ \times \mathbb{R}^n \to \mathbb{R}^n$; $u : Z^+ \to \mathbb{R}^m$. It is assumed that $f(k, 0, 0) = 0$, i.e., $\xi_0 = 0$ is an equilibrium point.

In the following, well-known definitions and theorems concerning stability properties of the system described by (1.2.1) are recalled. In particular, the definitions of 0-GAS and of ISS and theorems related to such properties for systems described by (1.2.1) are recalled.

1.2.1 Stability properties of nonlinear discrete-time systems

In the following well-known definitions of 0-GAS and 0-GES are reported (see [MWS10]). Furthermore, we introduce the stability definition by means of class $\mathcal{K}$ functions [KG02].

Definition 1.2.1. ([MWS10]) Let us consider the systems described by (1.2.1) with $u(k) \equiv 0$

1. For any given positive integer $k_0 \geq 0$ and a scalar $\epsilon > 0$, if there exists a positive scalar $\delta_1 = \delta(k_0, \epsilon) > 0$ such that

$$\|x(k_0)\| < \delta(k_0, \epsilon) \to \|x(k)\| < \epsilon, \forall k \geq k_0,$$

then the system is stable in the Lyapunov sense at the equilibrium point.

2. If the system is stable at the equilibrium point and there exists a positive scalar $\delta_2 = \delta(k_0, \epsilon)$ such that

$$\|x(k_0)\| < \delta(k_0, \epsilon) \to \lim_{k \to +\infty} x(k) = 0,$$

then the system is asymptotically stable at the equilibrium point.

3. If there exist positive constants δ_3, α, β such that

$$\|x(k_0)\| < \delta_3 \to \|x(k)\| \leq \beta \|x(k_0)\| \exp^{-\alpha(k-k_0)}$$

then the system is exponentially stable at the equilibrium point.

4. If scalars δ_1 or δ_2 can be chosen independently of k_0 , then the system is uniformly stable or uniformly asymptotically stable, respectively, at the equilibrium point.

5. If scalars δ_2 or δ_3 can be an arbitrarily large, finite number, then the system is 0-GAS and 0-GES respectively.

Lemma 1.2.1. ([KG02]) Let us consider the systems described by (1.2.1) with $u(k) \equiv 0$,

1. if there exists a function γ of class $\mathcal{K}$ and a positive constant c, independent of k_0, such that

$$\|x(k)\| \leq \gamma(\|x(k_0)\|), \forall k \geq k_0 \geq 0, \forall \|x(k_0)\| < c$$

   then the systems is uniformly stable at the equilibrium point.

2. If there exists a function ν of class $\mathcal{KL}$ and a positive constant c, independent of k_0, such that

$$\|x(k)\| \leq \nu(\|x(k_0)\|, k - k_0), \forall k \geq k_0 \geq 0, \forall \|x(k_0)\| < c$$

   then the systems is uniformly asymptotically stable at the equilibrium point;

3. if there exists a function ν of class $\mathcal{KL}$ and a positive constant c, independent of k_0, such that

$$\|x(k)\| \leq \nu(\|x(k_0)\|, k - k_0), \forall k \geq k_0 \geq 0, \forall \|x(k_0)\| < c$$

   then the systems is uniformly asymptotically stable at the equilibrium point.

4. If there exists a function ν of class $\mathcal{KL}$ and a positive constant c, independent of k_0, such that

$$\|x(k)\| \leq \nu(\|x(k_0)\|, k - k_0), \forall k \geq k_0 \geq 0, \forall x(k_0)$$

   then the systems is globally uniformly stable at the equilibrium point.

The following theorem, concerning a Lyapunov converse result (see [JW02]), holds for the system described by (1.2.1). In particular, the forthcoming theorem provides necessary and sufficient conditions for the global asymptotic stability of nonlinear discrete-time system described by (1.2.1).

Theorem 1.2.1. *([JW02]) The system described by (1.2.1) with $u(k) \equiv 0$ is 0-GAS if and only if there exist a function $V : \mathbb{R}^n \to \mathbb{R}^+$, α_i $i = 1, 2$ functions of class $\mathcal{K}_\infty$ and α_3 a positive defined function such that the following inequalities hold, for all $\xi \in \mathbb{R}^n$:*

i) $\alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|)$,

ii) $V(f(\xi)) - V(\xi) \leq -\alpha_3(\|\xi\|)$.

1.2.2 Input-to-state stability of nonlinear discrete-time systems

In the following, we illustrate the important concept of ISS for discrete-time systems. The definition of ISS for nonlinear discrete-time systems in [JW01] parallels to the one given for nonlinear continuous-time systems in [Son89]. A system is ISS if every state trajectory corresponding to a bounded control remains bounded and the trajectory eventually becomes small if the input signal is small no matter what the initial state is.

Definition 1.2.2. ([JW01]) The system described by (1.2.1) is said to be ISS if there exists a function γ of class $\mathcal{KL}$ and a function μ of class $\mathcal{K}$ such that, for any initial state $\xi_0 \in \mathbb{R}^n$, for any input signal $u \in l_\infty^m$, the corresponding solution $x(k, \xi_0, u)$ of (1.2.1) satisfies the following inequality, for $k \geq 0$:

$$\|x(k, \xi_0, u)\| \leq \gamma\Big(\|\xi_0\|_\infty, k\Big) + \mu\Big(\max_{j=0,1,\cdots,k-1} |u(j)|\Big), \tag{1.2.2}$$

where the second term of the sum in the right-hand side is taken equal to 0 for $k = 0$.

Definition 1.2.3. ([JW01]) A continuous function $V : \mathbb{R}^n \to R^+$ is called ISS-Lyapunov function for the system (1.2.1) if the following holds:

i) there exist α_1, α_2 functions of class $\mathcal{K}_\infty$ such that

$$\alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|), \quad \forall \xi \in \mathbb{R}^n; \tag{1.2.3}$$

ii) there exist a function α_3 of class $\mathcal{K}_\infty$ and a function σ of class $\mathcal{K}$ such that

$$V(f(\xi, \psi)) - V(\xi) \leq -\alpha_3(\|\xi\|) + \sigma(\|\psi\|), \tag{1.2.4}$$

for all $\xi \in \mathbb{R}^n$, for all $\psi \in \mathbb{R}^m$.

In order to provide converse results for the ISS property of system (1.2.1), we recall the following Lemma. For the reader's convenience, we report here the proof provided in [JW01].

Lemma 1.2.2. ([JW01]) If the system described by (1.2.1) admits a continuous ISS-Lyapunov function, then it is ISS.

Proof. Assume that system (1.2.1) admits an ISS-Lyapunov function V.
Let α_i, $i = 1, 2, 3$ be functions of class $\mathcal{K}_\infty$ and σ a function of class $\mathcal{K}$ such that the conditions $(i) - (ii)$ in Definition 1.2.3 are satisfied. We observe that the inequality (1.2.4) can be rewritten as, for all $\xi \in \mathbb{R}^n$,$\psi \in \mathbb{R}^m$:

$$V(f(\xi, \psi)) - V(\xi) \leq -\alpha_4(V(\xi) + \sigma(\|\psi\|) \tag{1.2.5}$$

where $\alpha_4 = \alpha_3 \circ \alpha^{-1}$. Without loss of generality , we can assume that $(I_d - \alpha_4)$ is a function of class $\mathcal{K}$. Fix a point $\xi \in \mathbb{R}^n$ and pick an input u. Let $x(k)$ denote the corresponding trajectory $x(k, \xi, u)$ of (1.2.1). Let η be a function of class $\mathcal{K}_\infty$ such that $(I_d - \eta)$ is a function of class $\mathcal{K}_\infty$. Let us consider the set defined by:

$$G = \{\xi \mid V(\xi) \leq g\} \tag{1.2.6}$$

where $g = \alpha_4^{-1} \circ \eta^{-1} \circ \sigma(\|u\|)$. We claim that if there is some $k_0 \in \mathbb{N}$, such that $x(k_0) \in G$, then $x(k) \in G$ for all $k \geq k_0$. For proving the claim, we assume that $x(k_0) \in G$. Then $V(x(k_0)) \leq g$, that is $\eta \circ \alpha_4(V(x(k_0))) \leq \sigma(|u|)$. Using the inequality (1.2.5), it follows that:

$$V(x(k_0 + 1)) \leq (I_d - \alpha_4)(V(x(k_0))) + \sigma(\|u\|),$$

and since $(I_d - \alpha_4) \in \mathcal{K}$, we obtain

$$\begin{aligned} V(x(k_0 + 1)) &\leq (I_d - \alpha_4)(g) + \sigma(\|u\|) \\ &= -(I_d - \eta) \circ \alpha_4)(g) + g - \eta \circ \alpha_4(b) + \sigma(\|u\|) \\ &\leq -(I_d - \eta) \circ \alpha_4)(g) + g \leq g. \end{aligned}$$

Using induction, one can show that $V(x(k_0 + j)) \leq g$ for all $j \in \mathbb{N}$, that is, $V(x(k)) \in G$ for all $k \geq k_0$.This concludes the proof of the claim.
Let $j_0 = \min_{k \in \mathbb{N}}\{x(k) \in G\} \leq \infty$, then it follows that, for all $k \geq j_0$:

$$V(x(k)) \leq \bar{\mu}(|u|)$$

where $\bar{\mu}(s) = \bar{\alpha_4}^{-1} \circ \eta^{-1} \circ \sigma(s)$, for $s > 0$. For $k < j_0$, it follows that

$$\eta \circ \bar{\alpha_4}(V(x_k)) > \sigma(\|u\|),$$

and hence,

$$\begin{aligned} V(x(k+1)) - V(x(k)) &\leq -\alpha_4(V(x(k))) + \sigma(\|u\|) \\ &= -(I_d - \eta) \circ \alpha_4)(V(x(k))) - \eta \circ \alpha_4(V(x(k))) + \sigma(\|u\|) \\ &\leq -(I_d - \eta) \circ \alpha_4)(V(x(k))). \end{aligned}$$

By standard comparison lemma, there exists a function $\tilde{\gamma}$ of class $\mathcal{KL}$ such that the following inequality holds, for $0 \leq k \leq j_0 + 1$:

$$V(x(k)) \leq \tilde{\gamma}(V(x(0), k)).$$

Thus, for all $k \geq 0$:

$$V(x(k)) \leq \max\{\tilde{\gamma}(V(\xi, k), \tilde{\mu}(\|u\|))\}. \tag{1.2.7}$$

From (1.2.7), the inequality (1.2.2) is satisfied, for $s, t \in \mathbb{R}^+$, with:

$$\gamma(s,t) = \alpha_1^{-1}\Big(\tilde{\gamma}(\alpha_2(\|\xi\|), t)\Big)$$
$$\mu(s) = \alpha_1^{-1} \circ \tilde{\gamma}(s).$$

□

Finally, we report converse theorem providing the ISS property of the system (1.2.1).

Theorem 1.2.2. *([JW01]) The system described by (1.2.1) is ISS if and only if there exist a function $V : \mathbb{R}^n \to \mathbb{R}^+$, α_i $i = 1, 2, 3$ functions of class $\mathcal{K}_\infty$ and σ a function of class $\mathcal{K}$ such that the following conditions hold, for all $\xi \in \mathbb{R}^n$, $\psi \in \mathbb{R}^m$:*

i) $\alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|x\xi\|)$,

ii) $V(f(\xi, \psi)) - V(\xi) \leq -\alpha_3(\|\xi\|) + \sigma(\|\psi\|)$.

1.3 Nonlinear discrete-time delays systems

Discrete-time delay systems have received much attention in the last few decades due to their importance in many engineering applications. The characterization of stability notions for discrete-time delay systems is instrumental to derive discrete abstractions for large-scale networked control systems [GP16]. Therefore, the investigation of stability is a fundamental problem, which often turns to be highly challenging. In the stability analysis of discrete-time delay systems, commonly used approaches are the Lyapunov-Krasovskii and the Lyapunov-Razumikhin method (see for instance [LM08], [WLN12], [RGT12], [FS03a], [FS05],[OKC13], [RG13], [YLW96],[MM17], [Fri14], [YCL16], [LM07], [LH09], [JSL18], [PP17], [RL19]). In this section, we recall some selected results for illustrating the Lyapunov-Krasovskii and the Lyapunov-Razuminkhin approach for the stability and the input-to-state stability analysis of discrete-time delay systems.

Let us consider the following system plant ([Mah00], [Fri14]):

$$x(k+1) = f(x_k, d(k), u(k)), \; k \in \mathbb{N}$$
$$x(\theta) = \xi_0(\theta), \; \theta \in [-\Delta, 0] \tag{1.3.1}$$

where: $\Delta \in \mathbb{N}$ is the maximum involved time delay which is assumed to be known; $x(k) \in \mathbb{R}^n$, n is a positive integer; $x_k \in C$, $u(k) \in \mathbb{R}^m$ is the input signal, m is a positive integer; $d(k) \in S_r$ is a vector of uncertain, possibly time-varying time delays; r is a positive integer, S_r is a subset of the product set $\{0, 1, \ldots, \Delta\}$; $f : C \times S_r \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz on any bounded set in $C \times \mathbb{R}^m$, uniformly on S_r, and satisfies $f(0, d, 0) = 0$ for any $d \in S_r$; $\xi_0 \in C$

1.3.1 Stability of nonlinear discrete-time time-delay systems

1.3.1.1 Lyapunov-Krasovski approach

This paragraph aims to illustrate the Lyapunov-Krasovskii functional approach in [PP17] for investigating the stability properties of the system described by (1.3.1) with uncertain time-varying delays belonging to a known fine set. Precisely, we report a transformation of the system (1.3.1) into a system where the state is described at time $k + 1$ and k. The number of uncertain time-varying time delays is arbitrary and an arbitrary number of known constant time delays is allowed by the system (1.3.1). All time delays must be not greater than Δ, which can be arbitrarily large as long as it is known. Let $x(k, \phi, d)$ is the solution of (1.3.1). We introduce the following sets:

$$M_{S_r} = \{d : \mathbb{N} \to S_r\} \quad M_u = \{u : \mathbb{N} \to \mathbb{R}^m\}. \tag{1.3.2}$$

Defining the map $F : C \times S_r \times \mathbb{R}^m \to C$, we can rewrite the system (1.3.1) as follows:

$$\begin{aligned} x_{k+1} &= F(x_k, d(k), u(k)) \\ x_0 &= \xi_0 \end{aligned} \tag{1.3.3}$$

where $x_k \in C$, $k \in \mathbb{N}$; $d \in S_r$; $u \in \mathbb{R}^m$ and the map F defined as:

$$F(\phi, d, u)(\theta) = \begin{cases} f(\phi, d, u) & \theta = 0 \\ \phi(\theta + 1) & \theta = -\Delta, -\Delta + 1, \ldots, -1 \end{cases} . \tag{1.3.4}$$

The solution of the system described by (1.3.3) is $x_k(\phi, d, u)$.

Remark 1.3.1. The model (1.3.3) allows to have the state of the system described by (1.3.1) at both time $k + 1$ and time k. For any $k \in \mathbb{N}$, initial state $\xi_0 \in C$, input signal $u \in M_u$, delay signal $d \in M_{S_r}$, the following equality holds:

$$x(k, \xi_0, d, u) = x_k(\xi_0, d, u)(0). \tag{1.3.5}$$

Lemma 1.3.1. ([PP17]) Let there exist a continuous functional $V : C \to \mathbb{R}^+$, functions α_i, $i = 1, 2$

of class $\mathcal{K}_\infty$, a function α_3 of class $\mathcal{K}$ and a function σ of class $\mathcal{P}_0$, such that for any $\phi \in C$, $u \in \mathbb{R}^m$, $d \in S_r$, the following inequalities hold:

i) $\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(\|\phi\|_\infty)$,

ii) $V(F(\phi, d, u)) - V(\phi) \leq -\alpha_3(|\phi(0)|) + \sigma(|u|)$.

where F is the map defined in (1.3.4).
Then, there exist a continuous functional $W : C \to \mathbb{R}^+$, a function $\bar{\alpha}_2$ of class $\mathcal{K}_\infty$ and a function $\bar{\alpha}_3$ of class $\mathcal{K}$, such that for any $\phi \in C$, $u \in \mathbb{R}^m$, $d \in S_r$ the following inequalities hold:

i') $\alpha_1(|\phi(0)|) \leq W(\phi) \leq \bar{\alpha}_2(\|\phi\|_\infty)$,

ii') $W(F(\phi, d, u)) - W(\phi) \leq -\bar{\alpha}_3(\|\phi(0)\|_\infty) + \sigma(|u|)$.

If, moreover, α_3 of class $\mathcal{K}_\infty$, then there exist a continuous functional $W : C \to \mathbb{R}^+$, function $\bar{\alpha}_i$, $i = 1, 2, 3$ of class $\mathcal{K}_\infty$, such that, for any $\phi \in C$, $u \in \mathbb{R}^m$, $d \in S_r$, the following inequalities hold:

i'') $\bar{\alpha}_1(\|\phi\|_\infty) \leq W(\phi) \leq \bar{\alpha}_2(\|\phi\|_\infty)$,

ii'') $W(F(\phi, d, u)) - W(\phi) \leq -\bar{\alpha}_3(\|\phi\|_\infty) + \sigma(|u|)$.

The following theorem provides necessary and sufficient Lyapunov-Krasovskii conditions for the 0-GAS property. Notice that, using the map (1.3.3), the Lyapunov-Krasovskii functionals are defined in the state space.

Theorem 1.3.1. *([PP17]) The system described by (1.3.1), with $u(k) \equiv 0$, for $k \in \mathbb{N}$, is 0-GAS if and only if there exist a continuous functional $V : C \to \mathbb{R}^+$, functions α_1, α_2 of class $\mathcal{K}_\infty$, and a function α_3 of class $\mathcal{K}$ such that the following inequalities hold for all $\phi \in C$ and $d \in S_r$:*

i) $\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(\|\phi\|_\infty)$

ii) $V(F(\phi, d, 0)) - V(\phi) \leq -\alpha_3(|\phi(0)|)$

where F is the map defined in (1.3.4).

In order to study the ISS of nonlinear time-delay systems a Lyapunov-Krasovskii theorem, presented in [PP17], is here recalled. In particular, the following theorem provides necessary and sufficient conditions for the ISS of the system (1.3.1).

Theorem 1.3.2. *([PP17]) The system described by (1.3.1) is ISS if and only if there exist a continuous functional $V : C \to \mathbb{R}^+$, functions α_i, $i = 1, 2, 3$ of class $\mathcal{K}_\infty$, such that the following inequalities hold, for all $\phi \in C$ and $d \in S_r$:*

i) $\alpha_1(|\phi(0)|) \le V(\phi) \le \alpha_2(\|\phi\|_\infty)$

ii) $V(F(\phi, d, 0)) - V(\phi) \le -\alpha_3(|\phi(0)|) + \sigma(|u|)$

where F is the map defined in (1.3.4).

1.3.1.2 Lyapunov-Razumikhin approach

The Lyapunov-Razumikhin approach is of interest because it makes use of conditions that involve the system state, as opposed to trajectory segments used in the Krasovskii approach. In the analysis of discrete-time delay systems, the Krasovskii approach provides conditions that do not provide information about the system trajectories directly, which causes them to become increasingly complex for large delays. The Razumihin approach, relies on a Lyapunov-like function defined in the original, non-augmented state space. As such, the Lyapunov function provides information about the trajectories of the discrete-time delay systems directly and the corresponding computations can be executed in the underlying low-dimensional state space of the discrete-time delay system dynamics. A direct translation of the Razumikhin approach for continuous-time systems with time-delay to discrete-time delay systems yields a set of so-called backward Razumikhin conditions [EZ94], which are typically difficult to verify. A more practical variant of these conditions was first proposed in [LM07], where forward Razumikhin conditions for the uniform asymptotic stability are established.

Theorem 1.3.3. *([LM07] Forward Lyapunov-Razumikhin conditions) Assume that there exist positive functions α_1, $\alpha_2 \in \mathcal{K}$, positive scalar functions $p(\cdot)$, $q(\cdot)$ satisfying $p(s) > s$ and $0 < q(s) < s$ for $s > 0$, and a positive definite function $V(k, x)$ such that the following conditions hold:*

i) $\alpha_1(|x|) \le V(k, x) \le \alpha_2(|x|)$;

ii) for any $\phi \in C$, $j \in \{-\Delta, -\Delta + 1, \dots, 0\}$, *if* $V(k + j, \phi(j)) \le p(V(k + 1, f(k, \phi)))$ *then*

$$V(k + 1, f(k, \phi)) \le V(k, \phi(0)) - q(V(k, \phi(0)));$$

iii) for any $\phi \in C$, $j \in \{-\Delta, -\Delta + 1, \dots, 1\}$, *if* $V(k + j, \phi(j)) \ge V(k, \phi(0))$ *then*

$$p(V(k + 1, f(k, \phi))) \le \max_{j=0,1,\dots,\Delta} V(k - j, \phi(-j));$$

then, the systems described by (1.3.1) with $u(k) \equiv 0$ *is uniform asymptotic stable.*

It is worth to mention that, for quadratic functions and linear discrete-time delay systems the conditions obtained in [LM07] are nonlinear and non-convex and hence difficult to verify.

In [RGT13], a relaxation of the Razumikhin approach is proposed which leads to necessary and sufficient conditions for stability of discrete-time delay system. In order to describe the methodology presented in [RGT13], some definitions are introduced.

Definition 1.3.1. The systems described by (1.3.1) with $u(k) \equiv 0$ is said to be semi-globally asymptotic stable, if for each compact set $\mathbb{X} \subset (\mathbb{R}^n)^{h+1}$ there exists a function $\beta \in \mathcal{KL}$, such that, for all sequence of state $x \in [-\Delta, 0]$ and for all $k \geq 0$

$$\|x_k\| \leq \beta(\|x_{[-\Delta,0]}\|, k).$$

Definition 1.3.2. A sequence of states $x_{[-M,0]} \in (\mathbb{R}^n)^{M+1}$, with $M \in \mathbb{Z}_{\geq\Delta}$ is called a solution to the system described by (1.3.1) with $u(k) \equiv 0$ of length $M+1$ if, for each $M \in \mathbb{Z}_{\geq\Delta+1}$, it holds that $x_{i+1} \in f(x_{[-\Delta+i,i]})$ for all $i \in \mathbb{Z}_{[-M+\Delta,-1]}$. Obviously, any $x_{[-\Delta,0]} \in (\mathbb{R}^n)^{h+1}$ is called a solution to the (1.3.1) with $u(k) \equiv 0$ of length $\Delta + 1$.

A further assumption of the map that generates the dynamic of the system described by (1.3.1) is considerate.

Assumption 1.3.1. The function in (1.3.1) with $u(k) \equiv 0$ is $\mathcal{K}$-continuous at zero and locally Lipschitz continuous at zero.

Theorem 1.3.4. *([RGT13] Necessary and sufficient Lyapunov-Razumikhin conditions) Suppose that Assumption 1.3.1 holds. The following statements are equivalent*

a) There exist a function $V : \mathbb{R}^n \to \mathbb{R}^+$, α_i, $i = 1, 2$ functions of class $\mathcal{K}_\infty$, a constant $c \in \mathbb{R}_{[0,1)}$, and for each compact set $\mathbb{X} \subset (\mathbb{R}^n)^{\Delta+1})$, a finite $M(\mathbb{X}) \in \mathbb{Z}_{\geq\Delta}$ such that the following conditions hold:

  i) $\alpha_1(\|x_0\|) \leq V(x_0) \leq \alpha_2(\|x_0\|)$;

  ii) $V(x_1) \leq c \max_{j=0,1,\ldots,\Delta} V(x(-i))$;

  for all $x_{[-\Delta,0]} \in (\mathbb{R}^n_{M+1})$ and all $x_1 \in f(x_{[-\Delta,0]})$ such that $x_{[-M,0]} \in (\mathbb{R}^n_{M+1})$ is a solution of (1.3.1) with $u(k) \equiv 0$ of length $M+1$ and satisfies $x_{[-M,-M+\Delta]} \in \mathbb{X}$.

b) The system describe by (1.3.1) with $u(k) \equiv 0$ is $\mathcal{KL}$-stable.

Remark 1.3.2. Theorem 1.3.4 recovers the interpretation of the Razumikhin approach that was presented in Theorem 1.3.3 with $M = \Delta$. The sublevel sets of the function V corresponding to (1.3.4) provide information about the evolution of the trajectories of the system (1.3.1) with $u(k) \equiv 0$ in the original state space $\mathbb{R}^n$.

1.4 Linear discrete-time delay systems

In this section, we report some results concerning linear discrete-time delay systems and the application of linear matrix inequalities to the stability problem. For the reader convenience, we report some well-known results that will be object of comparison with the one illustrated in the next chapters (see Chapter 3 and 4).

Consider the following simpler linear discrete-time delay system with time-varying delay, for $k \in \mathbb{N}$, ([FS05], [LHI08]):

$$\begin{aligned} x(k+1) &= A_0 x(k) + A_1 x(k - \dot{}(k)) \\ x(k) &= \phi(k) \end{aligned} \tag{1.4.1}$$

where $A_0, A_1 \in \mathbb{R}^{n \times n}$ and the time varying delay $d(k)$ such that $0 < d(k) \leq -\Delta$, with Δ positive integer.

1.4.1 Linear Matrix Inequality

The next results makes use of the Lyapunov-Razumikihn approach fro the LMI stability analysis of the system described by (1.4.1).

Proposition 1.4.1. ([FS05]) Let $q \in (0,1)$, if there exists a positive defined matrix $P \in \mathbb{R}^{n \times n}$ such that the following linear matrix inequality holds:

$$\Gamma = \begin{bmatrix} A^T P A - (1-q)P & A^T P A_1 \\ \star & A_1^T P A_1 - qP \end{bmatrix} < 0, \tag{1.4.2}$$

then the system described by (1.4.1) is asymptotically stable.

Proof. Let us consider the following Lyapunov function $V(x(k)) = x(k)^T P x(k)$. Let us assume that for some $a > 1$, the following Razumikhin condition holds, for $k \geq 0$ and $-h \leq i \leq -1$:

$$x(k - d(k))^T P x(k - d(k)) \leq a x(k)^T P x(k). \tag{1.4.3}$$

Using the S-procedure, for $q > 0$ we have:

$$\begin{aligned} V(x(k+1)) - V(x(k)) &= [x^T(k)A^T + x^T(k-\tau_k)A_1^T]P[Ax(k) + A_1x(k-d(k))] - x^T(k)Px(k) \\ &\leq [x^T(k)A^T + x^T(k-d(k))A_1^T]P[Ax(k) + A_1x(k-d(k))] - x^T(k)Px(k) \\ &+ q[ax^T(k)Px(k) - x^T(k-d(k))Px(k-d(k))] \\ &\leq [x^T(k)x^T(k-d(k))]\bar{\Gamma}\begin{bmatrix} x(k) \\ x(k-d(k)) \end{bmatrix}, \end{aligned}$$

where

$$\bar{\Gamma} = \begin{bmatrix} A^TPA - (1-qa)P & A^TPA_1 \\ \star & A_1^TPA_1 - qP \end{bmatrix}. \tag{1.4.4}$$

Notice that $\bar{\Gamma} < 0$ is feasible with $a = 1 + \epsilon$ and for $\epsilon > 0$ if $\Gamma < 0$. From the fact that $\Gamma < 0$, it follows that $a < 1$. Using the Razuminkin condition, for $b > 0$ and $k \geq 0$:

$$V(x(k+1)) - V(x(k)) \leq -b|x(k)|^2 \tag{1.4.5}$$

we obtain the asymptotic stability of the system describe by (1.4.1). □

In what follows, we present a necessary and sufficient LMI conditions for the existence of delay dependent Lyapunov-Krasovskii functionals [LHI08]. This approach is based on a theoretical link between the Lyapunov-Krasovskii approach and the switched system transformation in the context of discrete-time systems with time varying delays. Consider the notation:

$$\Gamma(d) = \begin{bmatrix} A_0 & \Xi_1(d) & \Xi_2 & \dots & \Xi_\Delta \\ I & 0 & \dots & \dots & 0 \\ 0 & I & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & I & 0 \end{bmatrix} \tag{1.4.6}$$

with for $i = 1, \dots, \Delta$, for all $d(k) \in \{1, 2, \dots, \Delta\}$

$$\Xi_i = \begin{cases} A_1, & i = d \\ 0, & i \neq d \end{cases} \tag{1.4.7}$$

Theorem 1.4.1. *([LHI08]) The following statement are equivalent.*

a) *There exist* $\Delta(\Delta+1)^2$ *matrices* $P_d^{i,j}$, $d = 1, 2, \ldots, \Delta$, $i, j = 0, \ldots, \Delta$ *such that the block matrix*

$$\Phi(d) = \begin{bmatrix} P_d^{0,0} & P_d^{0,1} & \ldots & P_d^{0,\Delta} \\ P_d^{0,1} & P_d^{1,1} & \ldots & \vdots \\ \vdots & \ddots & \ldots & \vdots \\ P_d^{0,\Delta} & \ldots & \ldots & P_d^{\Delta,\Delta} \end{bmatrix} \tag{1.4.8}$$

satisfies the LMI

$$\begin{bmatrix} \Phi(d_1) & \Gamma(d_1)\Phi(d_2) \\ \Phi(d_2)\Gamma(d_1) & \Phi(d_2) \end{bmatrix} > 0 \tag{1.4.9}$$

for all $d_1, d_2 \in \{1, 2, \ldots, \Delta\}$.

b) *There exists a delay dependent Lyapunov–Krasovskii functionals* $V(k)$ *defines as*

$$V(k) = V(k, x(k), \ldots, x(k-\Delta), d(k)) = \sum_{i=0}^{\Delta}\sum_{j=0}^{\Delta} x^T(k-i)P_{d(k)}^{i,j}x(k-j) \tag{1.4.10}$$

whose difference is strictly negative definite along system (1.4.1) solutions.

Remark 1.4.1. Notice that the Lyapunov matrices $P_{d(k)}^{i,j}(k)$ are not necessarily positive definite. Only the block matrix $\Phi(d)$ is required to be a symmetric positive definite matrix. The proposed functional generalizes the classical functionals from the discrete delay systems in [FS03b] .It represents a discrete-time equivalent of the functional presented in [KR99] for continuous-time.

1.5 Nonlinear discrete-time Markov jump systems

In this section, we introduce another class of systems that is of interest in the further steps of the present manuscript. Switching systems with markovian switching, also known as Markov jump systems, consist of a family of nonlinear subsystems (usually called modes) and a Markov chain that orchestrates the switching among them. These type of systems are particularly useful in the modeling of systems subject to abrupt changes, such as WCNs. Since their introduction [KL61], they have found numerous applications due to their ability to model dynamical systems with random abrupt dynamic changes (failures and repairs) and random time-delays. The applicability is due to the fact that the wireless channel may suffer from packet-losses and re-transmissions, which induce abrupt changes of the dynamical model and possibly time-varying delays [OC06], [Hua19], [YALB19], [BWT19]. Markov models allow representing bursts of packet losses, which is not possible using Bernoulli random variables [PSS08]. Also, in many communication protocols and standards, packet losses may enforce the re-transmission of a packet, which is thus received with a delay that depends on the stochastic characteristics of (bursts of) packet losses. As a consequence, markovian switching systems are good approximations of the stochastic characterization of WCNs models in presence of packet losses and induced random delays. In [RAW11], [GD19], [YALB19], [YLS20], the use of markovian switching systems handles the challenges in analysis and co-design of wireless networked control systems, and allows to verify instability of a system due to bursts of packet loss when Bernoulli-like channel models fail. Moreover, the Markov modelling of the Wireless channel allows performance improvement in stabilizing control synthesis [YLS20].

Consider an autonomous nonlinear discrete-time Markovian switching system, described by the following equations [PPB14]:

$$\begin{aligned} x(k+1) &= f_{\eta(k)}(x(k)) \\ x(0) &= x_0 \end{aligned} \tag{1.5.1}$$

where $x(k) \in \mathbb{R}^n$ is the state of the system, $\{\eta(k)\}_{k\in\mathbb{N}}$ is a discrete-time time-homogeneous Markov chain, taking values in a finite set $\mathcal{S} = \{1, 2, \dots, s\}$, $s \in \mathbb{N}$, initial probability distribution $\nu = \{\nu_1, \dots, \nu_s\}$ and TPM $\mathbb{P} = (p_{ij})$, $i, j \in \mathcal{S}$. For any $i \in \mathcal{S}$, $f_i : \mathbb{R}^n \to \mathbb{R}^n$ are a continuous functions satisfying $f_i(0) = 0$.

1.5.1 Exponential mean square stability

In order to provide the EMSS property for the system described by (1.5.1), we exploit some notation and definitions (see [PPB14], [AIP20]).

Definition 1.5.1. (Admissible Transitions set) The set of all admissible transitions of the MC $\{\eta(k)\}_{k\in\mathbb{N}}$, denoted by $\mathcal{E} \subset \mathcal{S} \times \mathcal{S}$, is defined as

$$\mathcal{E} := \{(i, j) \in \mathcal{S} \times \mathcal{S} \mid p_{ij} > 0\}.$$

Definition 1.5.2. (Admissible switching path) An admissible switching path for (1.5.1), of length $s \in \mathbb{N}$, defined as $\bar{\eta} = (\eta(0), \dots, \eta(s))$, is a switching path, for which $(\eta(k), \eta(k+1)) \in \mathcal{E}$, $\forall k \in \mathbb{N}_{[0,s-1]}$.
Let $\mathcal{B}_s(i)$ denote the set of all admissible switching paths for (1.5.1), of length $s \in \mathbb{N}$, emanating from the state $\eta(0) = i$ of the MC.

Definition 1.5.3. A set X is said to be uniformly positive invariant for for (1.5.1) if $x(k, x_0, i, \bar{\eta}) \in X_{\eta_k}$ $\forall x \in X_i$, $\bar{\eta} \in \mathcal{B}(i)$, $i \in \mathcal{S}$.

Definition 1.5.4. Let X be a uniformly positive invariant set for (1.5.1). The origin is EMSS in X if there exist $M, \zeta \in \mathbb{R}^+$ with $M \geq 1$ and $0 < \zeta \leq 1$, such that for any $\xi_0 \in X_i$, the following inequality holds for any $k \in \mathbb{N}$,

$$\mathbb{E}[\|x(k, x_0, i, \bar{\eta})\|^2] \leq M\zeta^k \Big(\|\xi_0\|_\infty\Big)^2. \tag{1.5.2}$$

We illustrate the established sufficient conditions for exponential mean-square stability of a Markov jump system as reported in [PPB14].

Theorem 1.5.1. *([PPB14]) Consider the autonomous system described by (1.5.1). Let X be a uniformly positive invariant set for (1.5.1). Assume that there exists a positive function $V : \mathbb{R}^n \times \mathcal{S} \to \bar{\mathbb{R}}$, such that $V(0, i) = 0$ for $i \in \mathcal{S}$, and positive scalars a, b and c such that the following inequalities hold:*

i) $a\|x\|^2 \leq V(x, i) \leq b\|x\|^2$

ii) $\mathbb{E}[V(x_{k+1}, \eta_{k+1}) - V(x_k, \eta_k) \mid \mathcal{F}_k] \leq -c\|x\|^2$

for all $x \in X_i$ for $i \in \mathcal{S}$. Then the origin is EMSS in X for (1.5.1).

Chapter 2

Halanay Inequality: Literature Overview

Note 2.0.1. This chapter entails results from several academic sources. To read the full reference disclaimer, please refer to [0.1].

Chapter Description

The main aim of this chapter is to provide a brief overview of several results concerning the discrete-time Halanay-type inequalities and their applications on the stability analysis. They are inspired by the elegant ideas developed by Halanay [Hah67] to prove a stability result for delay differential equations based on differential inequalities involving the max functional. The results contained in the present chapter are the starting point of our research investigation. Furthermore, we put a special emphasis on these ideas, showing how to generalize the known results in a nonlinear framework, including their application to the stability of discrete-time delay systems by means of Lyapunov theory.

In Section 2.1, an overview on some established linear Halanay-type inequalities is presented. In Section 2.2, we illustrate some results on a nonlinear Halanay-type inequalities. In Section 2.3, some stability results concerning the application of linear Halanay-type inequalities in the contest of Lyapunov theory are showed. In Section 2.3, some input-to-state stability results employing linear Halanay's type inequalities in the contest of Lyapunov theory are recalled.

2.1 Discrete-time Halanay inequalities

Discrete-time dynamical systems often arise in the numerical solutions of continuous-time systems. While there are a plethora of methods by which discrete-time analogues of continuous-time systems can be obtained, the asymptotic behavior of the two types of systems do not often coincide. It is well known that in the theory of functional differential equations, it is useful to employ differential inequalities involving the max functional, e.g., [CG94], [FMK21], to investigate the asymptotic stability. The well known Halanay's inequality condition plays an important role in the analysis of dynamics with switching or delays, since it provides an alternative to the oftentimes difficult task of constructing Lyapunov functions. In the following, we present to the reader a chronological overview of the development the discrete-time Halanay-type inequality.

2.1.1 Linear discrete-time Halanay inequalities

A discrete version of the Halanay inequality continuous counterpart is proved in [LF02]. For the reader's convenience, we report here the proof provided in [LF02].

Theorem 2.1.1. *([LF02]) Let us consider $a, b \in \mathbb{R}^+$ and a positive sequence of real numbers*

$y : \{-\Delta, -\Delta+1, \dots, 0, 1, \dots\} \to \mathbb{R}^+$ that satisfies the following inequality, for $k \geq 0$:

$$y(k+1) - y(k) \leq -ay(k) + b \max_{j=0,1,\dots,\Delta}\{y(k-j)\}. \tag{2.1.1}$$

If $0 < b < a \leq 1$, then there exists a constant $\lambda_0 \in (0,1)$ such that, for $k \geq 0$,

$$y(k) \leq \lambda_0^k \|y_0\|_\infty. \tag{2.1.2}$$

Moreover, λ_0 can be chosen as the smallest root in the interval $(0,1)$ of the equation

$$\lambda^{\Delta+1} + (a-1)\lambda^\Delta - b = 0. \tag{2.1.3}$$

Proof. Let us consider a positive sequence of real numbers $z : \{-\Delta, -\Delta+1, \dots, 0, 1, \dots\} \to \mathbb{R}^+$, such that the following equality holds, for $k \geq 0$

$$z(k+1) - z(k) = -az(k) + b \max_{j=0,1,\dots,\Delta}\{z(k-j)\}. \tag{2.1.4}$$

Since $1 - a \geq 0$ and $b > 0$, by induction on k, it easy to prove that if the sequence $y(k)$ satisfies (2.1.1) and $y(k) \leq z(k)$ for $k = -\Delta, \dots, 0$ then $y(k) \leq z(k)$ for all $k \geq 0$. Let define $z(k) := K\lambda^k$, with $K > 0$ and $\lambda^k \in (0,1)$, we now prove that the sequence $z(k)$ is a solution of (2.1.4) if and only if λ is a solution of (2.1.3). Let us consider a continuous function $F : (0,1] \to \mathbb{R}$, defined as

$$F(\lambda) = \lambda^{\Delta+1} + (a-1)\lambda^\Delta - b. \tag{2.1.5}$$

The behavior of F to the limits is:

$$\lim_{\lambda \to 0^+} F(\lambda) = -b < 0 \quad F(1) = a - b < 0.$$

From the continuity of F, it follows that there exist $\lambda_0 \in (0,1)$ such that $F(\lambda_0) = 0$. Furthermore, we can choose the smallest value of λ satisfying this equation since $F(\lambda)$ is a polynomial and it has at most $\Delta + 1$ real roots. Thus, for every $K > 0$, the sequence $z(k) = K\lambda_0^k$ is a solution of (2.1.4). Finally, letting $K = \max_{j=0,1,\dots,\Delta}\{\|y(-j)\|_\infty\}$, we obtain that $z(k) \geq y(k)$ for all $k = -\Delta, -\Delta+1, \dots 0$. Hence, using the first part of the proof, $y(k) \leq z(k) = K\lambda_0^k$ for all $k \geq 0$. □

The Halanay-type approach is based on monotonicity arguments for positively homogeneous operators (like the max functional). The proof of Theorem 2.1.1 is based on a comparison result, which is demonstrated using two basic facts:

a) the difference equation

$$y(k) \leq \lambda_0^k \|y_0\|_\infty$$

is monotone with respect to the usual pointwise ordering in $\mathbb{R}^{k+1}$ if $a, b \geq 0$;

b) equation (2.1.5) has a unique real root $\lambda_0 \in (0,1)$ since $b > 0$ and $a + b < 1$.

Theorem 2.1.1 is considered as the starting point of further research steps conducted in Chapter 3 and Chapter 4. Indeed, we derive a nonlinear Halanay-type inequality with and without forcing terms using comparison principle.

Another version of the linear discrete-time Halanay-type inequality is given in [UN09]. In this case, the sequence is considerate on sub levels and instead of having max approximation the authors obtain a summation approximation. For the reader's convenience, we report here the proof provided in [UN09].

Theorem 2.1.2. *([UN09]) Let $b_j \in \mathbb{R}^+$, $h_j \in \mathbb{N}$ for $j = 1, \ldots, \Delta$, a and $b_\Delta \in \mathbb{R}^+$ such that the following inequalities hold:*

$$0 = h_0 < h_1 < \cdots < h_\Delta \tag{2.1.6}$$

$$\sum_{i=0}^{\Delta} b_i < a \leq 1. \tag{2.1.7}$$

Let $y : \{-h_\Delta, \ldots, 0\} \to \mathbb{R}^+$ be a sequence of positive reals such that the following inequality holds, for $k \geq 0$:

$$y(k+1) - y(k) \leq -ay(k) + \sum_{j=0}^{\Delta} b_j y(k - h_j). \tag{2.1.8}$$

Then, there exists $\lambda_0 \in (0,1)$ such that:

$$y(k) \leq \lambda_0^k \max_{j=0,1,\ldots,\Delta} \{y(k - h_\Delta)\}. \tag{2.1.9}$$

Moreover, λ_0 can be chosen as the smallest root in the interval $(0,1)$ of the equation

$$F(\lambda) = \lambda^{h_\Delta+1} - (1 - a + b_0)\lambda^{h_\Delta} - b_1\lambda^{h_\Delta - h_{\Delta+1}} - \cdots - b_{\Delta-1}\lambda^{h_\Delta - h_{\Delta-1}} - b_\Delta = 0. \tag{2.1.10}$$

Proof. Let the positive sequence $z : \{-\Delta, -\Delta+1, \ldots, 0\} \to \mathbb{R}^+$ be the solution of the following equation, for $k \geq 0$:

$$z(k+1) - z(k) = -az(k) + \sum_{j=0}^{\Delta} b_j z(k-h_j). \tag{2.1.11}$$

Since $b_i \in \mathbb{R}^+$ and $0 < p < 1$, if the sequence y satisfies (2.1.8) and $y(k) \leq z(k)$ for $k = 0, 1, \ldots, h_\Delta$, then $y(k) \leq z(k)$ for all $k \leq 0$. For a given $K > 0$ and $\lambda \in (0,1)$, the sequence $z(k) = K\lambda^k$ is a solution of (2.1.10) if and only if λ is a root of the polynomial (2.1.10). Indeed, since

$$\lim_{\lambda \to 0^+} F(\lambda) = -b_\Delta < 0,$$
$$F(1) = a - \sum_{j=0}^{r} b_j > 0,$$

from the continuity of F, there exists a smallest real number $\lambda_0 \in (0,1)$ such that (2.1.10) is satisfied. Thus, for any $K > 0$, the sequence $z(k) = K\lambda_0^k$ is a solution of (2.1.5). Let $K_0 = \|y_0\|_\infty$. Then $z(k) = K\lambda_0^k$ is a solution of (2.1.5), and it follows that $y(k) \leq z(k)$ for $k = 0, 1, \ldots, h_\Delta$. Therefore, by using the first part of the proof, we conclude that $y(k) \leq z(k) = K_0\lambda_0^k$, for all $k \geq 0$. □

Remark 2.1.1. In the case of positive sequences, the linear discrete-time Halanay inequality in (2.1.8) is less conservative than the one reported in (2.1.1). In fact, if a positive sequence $y : \{-\Delta, -\Delta+1, \ldots, 0, 1, \ldots\} \to \mathbb{R}^+$ satisfies (2.1.1) then it also satisfies (2.1.8). On the other hand, defining the sequence $y(k) = \frac{1}{2^k}$, $k \geq -1$, for $\Delta = 1$, $a = \frac{5}{6}$, $b = b_0 = b_1 = \frac{1}{7}$, the inequality (2.1.8) is satisfied while the inequality (2.1.1) not.

2.1.2 Nonlinear discrete-time Halanay inequality

The first work that prove a nonlinear Halanay inequality is in reported [RAS09], where a system of inequalities is provided. For the reader's convenience we report here the proof given in [RAS09].

Theorem 2.1.3. *([RAS09]) Let $a_i, b_i, c_i, d_i, e_i, f_i \in \mathbb{R}^+$, $\sum_{j=0}^{\Delta} d_i e > 0$, $h_i \in \mathbb{Z}$ for $i = 0, 1, \ldots, \Delta$ such that the following inequalities hold*

$$0 = h_0 < h_1 < \cdots < h_r$$
$$b + (c+d)(e+f) < a \leq 1;$$

where $a = \sum_{i=0}^{\Delta} a_i$, $b = \sum_{i=0}^{\Delta} b_i$, $c = \sum_{i=0}^{\Delta} c_i$, $d = \sum_{i=0}^{\Delta} d_i$, $e = \sum_{i=0}^{\Delta} e_i$ and $f = \sum_{i=0}^{\Delta} f_i$. Furthermore, let us consider the positive sequences $y, z : \{-h_\Delta, -h_\Delta + 1, \ldots,\} \to \mathbb{R}^+$ that satisfy the following inequalities, *for every* $k \geq 0$:

$$y(k+1) - y(k) \leq \sum_{j=0}^{\Delta} \Big(- a_j y(k) + b_j (y(k-h_j)^p) + c_j z(k) + d_j z(k-h_j) \Big)$$

$$z(k) \leq \sum_{j=0}^{\Delta} \Big(e_j y(k) + f_j (y(k-h_j))^p \Big)$$

where $p \in \mathbb{R}$. *Then there exists constants* $K_i \geq 0$, $i = 1, 2$, *and* $\lambda_0 \in (0,1)$ *such that, for* $k \geq 0$*:*

$$y(k) \leq \max_{j=0,1,\ldots,\Delta} \{y(k-h_j), g^{-1} z(k-h_j)\} \lambda_0^k \,, \quad z(k) \leq g \max_{j=0,1,\ldots,\Delta} \{y(k-h_j)\}, g^{-1} z(k-h_j)\} \lambda_0^k,$$

where $g = e + \sum_{j=0}^{\Delta} f_j \lambda_0^{-k+(k-\Delta)p}$. *Moreover,* λ_0 *can be chosen as the smallest root in the interval* $(0,1)$ *of the following equation, for* $k \geq 0$*:*

$$\lambda - (1 - a - ce) - \sum_{j=0}^{\Delta} (b_j + c f_j) \lambda^{(k-h_j)p-k} - \sum_{j=0}^{\Delta} d_i e \lambda^{-h_j} - \sum_{j=0}^{\Delta} \Big(d_j \sum_{i=0}^{\Delta} f_i \lambda^{(k-h_i-h_j)p-k} \Big) = 0. \qquad (2.1.12)$$

Proof. Let us consider the following positive sequences $u, v : \{-h_\Delta, -h_\Delta + 1, \ldots\} \to \mathbb{R}^+$, such that the following inequalities hold, $k \geq 0$:

$$u(k) = (1-a)^k u(0) + \sum_{i=0}^{k-1} (1-a)^{k-i-1} \times \sum_{j=0}^{\Delta} \Big(- a_j u_i + b_j (x(i-h_j)))^p + c_j v_i + d_j v(i-h_j)) \Big) \qquad (2.1.13)$$

$$v(k) = \sum_{j=0}^{\Delta} \Big(e_i u(k) + f_j (u(k-h_j))^p \Big). \qquad (2.1.14)$$

Since $(1-a) \geq 0$, by induction it follows that, if $y(k) \leq u(k)$ and $z(k) \leq v(k)$ for $k = -\Delta, \ldots, 0$, then $y(k) \leq u(k)$ and $z(k) \leq v(k)$ for all $k \geq 0$. We observe that, the system (2.1.13), for $k \geq 0$, is equivalent to:

$$u(k+1) - u(k) = \sum_{j=0}^{\Delta} \Big(- a_i u(k) + b_i (u(k-h_j))^p + c_i v(k) + d_i v(k-h_j) \Big) \qquad (2.1.15)$$

$$v(k) = \sum_{j=0}^{\Delta} \Big(e_i u(k) + f_j (u(k-h_j))^p \Big). \qquad (2.1.16)$$

Now, we prove that, under the assumption of the theorem, there exists a solution $(u(k), z(k))$ to the system (2.1.15) in the form of $u(k) = \lambda_0^k$, $v(k) = g\lambda_0^k$, with $g > 0$, $\lambda_0 \in (0,1)$. The pair $(u(k), z(k))$

is a solution of the systems (2.1.15) if and only if the following equalities hold, for $k \geq 0$:

$$\lambda_0^{k+1} = (1-a)\lambda_0^k + \sum_{j=0}^{\Delta}\left(b_i\lambda_0^{(k-h_j)p} + c_i g\lambda_0^k + d_j g\lambda_0^{k-h_j}\right) \tag{2.1.17}$$

$$g\lambda_0^k = \sum_{j=0}^{\Delta}\left(e_j\lambda_0^k + f_j\lambda_0^{(k-h_j)p}\right). \tag{2.1.18}$$

This is equivalent to the existence of a solution $\lambda_0 \in (0,1)$, of equation (2.1.12). Let us consider a continuous function $F : (0,1] \to \mathbb{R}$, defined as

$$F(\lambda) = \lambda - (1-a-ce) - \sum_{j=0}^{\Delta}(b_j + cf_j)\lambda^{(k-h_j)p-k}$$
$$-\sum_{j=0}^{\Delta} d_i e\lambda^{-h_j} - \sum_{j=0}^{\Delta}\left(d_j\sum_{i=0}^{\Delta} f_i\lambda^{(k-h_i-h_j)p-k}\right).$$

Notice that

$$F(0) = \lim_{\lambda\to 0^+} F(\lambda) = -\infty$$
$$F(1) = a - b - (c+d)(e+f) > 0,$$

then, by continuity, there exists $\lambda_0 \in (0,1)$ such that $F(\lambda_0) = 0$. Hence, the pair (λ_0, g) is a solution of (2.1.17) with $g = e + \sum_{j=0}^{\Delta} f_j\lambda_0^{-k+(k-\Delta)p} > 0$. For this value of λ_0, the pair $(K\lambda_0^k, Kg\lambda_0^k)$ is a solution of (2.1.15) for every $K \geq 0$. Thus, choosing $K = \max_{j=0,1,\ldots,\Delta}\{y(-h_j), g^{-1}z(-h_j)\}$ we have that $y(k) \leq K\lambda_0^k = u(k)$ and $z(k) \leq Kg\lambda_0^k = v(k)$ for all $k = -\Delta, -\Delta+1, \ldots, 0$. Hence using the first part of the proof, we conclude that $y(k) \leq u(k)$ and $z(k) \leq v(k)$ for all $k \geq 0$. □

The next result [Bak10] concerns a nonlinear version of the discrete-time Halanay inequality in (2.1.1). In this case, the author presents a sequence with increment bounds. For the reader's convenience we report here the proof given in [Bak10].

Theorem 2.1.4. *([Bak10]) Let $\eta^\natural$ be a positive real number, $\omega(k,s,t)$ a function that is continuous for all (s,t) monotonic decreasing in s, and monotonic non-decreasing in t, with $\omega(k,0,0) = 0$ for every $k \geq 0$ such that, if $0 \leq u \leq \eta^\natural$,the following inequality holds, for $k \geq 0$:*

$$\omega(k,s,t) \leq 0. \tag{2.1.19}$$

Let us consider a positive sequence y satisfying the following inequality, for $k \geq 0$:

$$y(k+1) - y(k) \leq \omega(k, y_k, \max_{k-N(k)\leq j\leq k}\{y(j)\}),$$

then, if $\max_{k_0-N(k_0)\le j\le k_0}\{y(j)\} \le \eta^\natural$*, it follows that, for* $k \ge k_0$

$$y(k) \le \max_{k_0-N(k_0)\le j\le k_0}\{y(j)\}.$$

Proof. Let us assume that $\max_{k_0-N(k_0)\le j\le k_0}\{y(j)\} \le \eta^\natural$. Using (2.1.19), it follows that

$$y(k_0) \le \max_{k_0-N(k_0)\le j\le k_0}\{y(j)\}$$

and

$$\omega(k_0, \max_{k_0-N(k_0)\le j\le k_0}\{y(j)\}, \max_{k_0-N(k_0)\le j\le k_0}\{y(j)\}) \le 0.$$

Moreover, we have that

$$\begin{aligned} y(k_0+1) - y(k_0) &\le \omega(k_0, y(k_0), \max_{k_0-N(k_0)\le j\le k_0}\{y(j)\}) \\ &\le y_{k_0} + \omega(k_0, \max_{k_0-N(k_0)\le j\le k_0}\{y(j)\}, \max_{k_0-N(k_0)\le j\le k_0}\{y(j)\}) \\ &+ \{\omega(k_0, y(k_0), \max_{k_0-N(k_0)\le j\le k_0}\{y(j)\}) - \omega(k_0, \max_{k_0-N(k_0)\le j\le k_0}\{y(j)\}, \max_{k_0-N(k_0)\le j\le k_0}\{y(j)\})\}. \end{aligned} \tag{2.1.20}$$

The last term in (2.1.20) is non-positive and it follows that $y(k_0+1) \le y(k(0)) \le \eta^\natural$. We have that $y(k_0+1) \le \max_{k_0-N(k_0+1)\le j\le k_0+1}\{y(j)\} \le \eta^\natural$. By induction the theorem is proved. □

In Theorem 2.1.4, most attention is devoted to provide a bound for the involved non-negative functions. Theorem 2.1.4 will be object of further investigation in Chapter 3, where we provide also results concerning the asymptotic behavior of involved functions, as well as results concerning the uniform convergence to the origin.

2.2 Stability analysis via Halanay inequality

In this section, we present some stability results concerning the applicability of the Halanay-type inequalities in the contest of Lyapunov stability theory for the asymptotic estimation of the solution of difference systems. In what follow, we consider a specific class of discrete time-delay systems. This class of discrete-time delay systems is often encountered in practical applications such as networked control systems and dynamical networks with time delays [BLW05], [ZM03], [GCL04], [LG04]. We consider a specific case of the discrete-time delay systems in (1.3.1), describe by the following equation:

$$x(k+1) = f(x(k), x(k-h_1), \dots, x(k-h_{m_0}), u(k)),$$
$$x(\theta) = \xi_0(\theta),\ \theta \in \{-\Delta, -\Delta+1, \dots, 0\}, \tag{2.2.1}$$

2.2.1 Global asymptotic stability

The next result gives and asymptotic estimate of the the difference equation described by

$$x(k+1) = (1-a)x(k) + f(k, x(k), x(k-1), \dots, x(k-\Delta)). \tag{2.2.2}$$

this equation is a particular case of the system plat (2.2.1). Here, there is a simplification of the system in order to prove the asymptotic stability of the solution having a heavy bound on the function describing the dynamic of the system. By simple application of linear Halanay-type inequality in (2.1.1), the following theorem provides the 0-GAS of (2.2.2). For the reader's convenience we report here the proof given in [LF02].

Theorem 2.2.1. *([LF02]) Let us consider the difference equation described by (2.2.2). Let $a, b \in \mathbb{R}^+$ such that*

$$0 < a \leq 1 \quad b < a.$$

If, for all $x(j) \in \mathbb{R}^{\Delta+1}$, the following inequality holds:

$$|f(k, x(k), \dots, x(k-\Delta))| \leq b\|x(k-j)\|_\infty, \tag{2.2.3}$$

then there exists $\lambda_0 \in (0,1)$ such that for every solution of (2.2.2), the following inequality holds, for $k \geq 0$:

$$|x(k)| \leq \lambda_0^k \max_{j=0,1,\dots,\Delta} \|x\|_\infty.$$

Then, the systems described by (2.2.2) is 0-GAS.

Proof. The proof consists in a discretization of (2.2.2), using results in [AW97]. Let the positive sequence of real numbers $y : \{-\Delta, -\Delta+1, \dots, 0, 1, \dots\} \to \mathbb{R}^+$ be a solution of (2.2.2). From [AW97], we know that, for $k \geq 0$:

$$x(k) = x_0(1-a)^k + \sum_{j=0}^{k-1}(1-a)^{k-j-1} f(j, x(j), \dots, x(j-\Delta)). \tag{2.2.4}$$

Using the upper bound in (2.2.3), from (2.2.4), we have

$$|x(k)| \leq |x_0|(1-a)^k + \sum_{j=0}^{k-1}(1-a)^{k-j-1} b \max_{j=0,1,\ldots,\Delta} |x(k-j)|. \tag{2.2.5}$$

Choosing $V(k) = |x(k)|$ for $k = 0, 1, \ldots, \Delta$, using (2.2.5), we have, for $k > 0$:

$$V(k) = |x_0|(1-a)^k + \sum_{j=0}^{k-1}(1-a)^{k-j-1} b \max_{j=0,1,\ldots,\Delta} |x(k-j)|.$$

Hence, it follows that $|x(k)| \leq V(k)$, and:

$$V(k+1) - V(k) = -aV(k) + b \max_{j=0,1,\ldots,\Delta}\{\|x(k-j)\|_\infty\} \leq -av(k) + b \max_{j=0,1,\ldots,\Delta}\{V(k-j)\}.$$

Then, by using Theorem 2.1.1, we have, for $K \geq 0$:

$$|x(k)| \leq V(k) \leq \lambda_0^k \max_{j=0,1,\ldots,\Delta}\{V(k-j)\} = \lambda_0^k \max_{j=0,1,\ldots,\Delta}\{|x(k-j)|\},$$

where λ_0 is chosen as indicated in Theorem 2.1.1. □

Next, we report another results regarding the global asymptotic stability of (2.2.2) that make use of the Halanay-type inequality in Theorem 2.1.3.

Theorem 2.2.2. *([RAS09]) Let us consider the difference equation described by (2.2.2). Assume that f satisfies the following inequalities*

$$|f(k, x(k), \ldots, x(k-\Delta))| \leq \sum_{j=0}^{\Delta} l_j |x(k-\Delta)|^p,$$

$$|f(k, x(k), \ldots, x(k-\Delta)) - x(k)| \leq \sum_{j=0}^{\Delta} q_j |x(k-\Delta) - x(k-1-\Delta)|^p,$$

where $l_j, q_j, p \in \mathbb{R}^+$, $\sum_{i=0}^{\Delta} q_i|a| > 0$, $h_j \in \mathbb{Z}$, $j = 0, 1, \ldots, \Delta - 1$ and $h_\Delta \in \mathbb{N}$. If one of the following conditions hold:

(a) $0 \leq a \leq 1 - b$, $0 < bq < 1$, *and* $0 < l \leq 1$,

(b) $a < 0$ *and* $0 < bq < (a+b)(-a+bl)^{-1}$,

then there exists a constant $\lambda_0 \in (0,1)$ *such that*

$$|x(k)| \leq \lambda_0^k \max_{j=0,1,\ldots,\Delta} \{|x_j|, \bar{a}|x(k-j) - x(k-1-j)|\},$$

where $\bar{a} = |a| + b\sum_{i=0}^{\Delta} l_i \lambda_0^{-k+(k-h_i)p}$, $l = \sum_{i=0}^{\Delta} l_i$, $q = \sum_{i=0}^{\Delta} q_i$ *and* λ_0 *can be chosen sa the smallest root in* $(0,1)$ *of the following equation:*

$$F(\lambda) = \lambda - (1-a-b) - \sum_{i=0}^{\Delta} b|a|q_i\lambda^{-h_i} - \sum_{i=0}^{\Delta}\Big(\sum_{j=0}^{\Delta} bl_j\lambda^{(k-h_j-h_i)p-k}\Big).$$

Then, the systems described by (2.2.2) is 0*-GAS.*

2.2.2 Global uniform exponential stability

In this section, we report some results regarding Razumikhin-type theorems that guarantee uniformly exponential stability for the system described by (2.2.1).

Definition 2.2.1. The system described by (2.2.1) is said to be uniformly exponentially stable if, for any initial data $k_0 \in \mathbb{N}$, $x_{k_0} = \xi_0$, there exist two positive numbers $\alpha > 0$, $M > 0$, where both α and M are independent of k_0, such that the following inequality holds, for all $k \geq k_0$, $k \in \mathbb{N}$

$$||x(k, k_0, \xi_0)|| \leq M||\xi_0||e^{-\alpha(k-k_0)}.$$

For the reader's convenience we report here the proof given in [LM07].

Theorem 2.2.3. *([LM07]) Assume that there exist a positive defined function* $V : C \times \mathbb{R}^n \to \mathbb{R}^+$, *positive reals* c_1, c_2, *a constant* $\lambda \in (0,1)$, *constants* $\lambda_i \in (0,1)$ *for* $i = 1,2,\ldots,r_0$ *such that the following conditions holds:*

i) $c_1||x||^r \leq V(k,x) \leq c_2||x||^r$

ii) $V(k+1, x(k+1)) \leq \lambda V(k, x(k)) + \sum_{i=1}^{r_0} \lambda_i V(k - h_i(k), x(k - h_i(k)))$.

If $\lambda + \sum_{i=1}^{r_0} \lambda_i < 1$, *then the system described by (2.2.1) with* $u(k) \equiv 0$ *is UES and its Lyapunov exponent is less than or equal to* $-\ln p/r(m+1)$, *where* $p > 1$ *is the unique root of the equation*

$$1 - \lambda - \sum_{i=1}^{m_0} \lambda_i = \frac{\ln p}{m+1} \tag{2.2.6}$$

Proof. The equation (2.2.6) has a unique solution $1 < p < 1 - \dfrac{\lambda}{\sum_{i=1}^{m_0} \lambda_i}$.

Then, for $h_i(k) \in \{-m, \ldots, 0\}$, we have

$$V(k - h_i(k), x(k - h_i(k))) \leq pV(k, x(k)).$$

Using (*ii*), we obtain

$$V(k+1, x(k+1)) \leq \Big(\lambda + p \sum_{i=1}^{m_0} \lambda_i\Big) V(k, x(k)).$$

Observing that $\lambda + p \sum_{i=1}^{m_0} \lambda_i < 1$, then we have that for any ϕ and $s \in \{-m, -m+1, \ldots, 0\}$ if

$$V(k+s, \phi(s)) \leq pV(k, \phi(0))$$

then

$$V(k+1, f(k, \phi)) \leq \lambda V(k, \phi(0)).$$

Let us denote $\bar{\lambda} = \lambda + p \sum_{i=0}^{m_0} \lambda_i$. From (2.2.6), we have that $\dfrac{\ln p}{(m+1)} = 1 - \bar{\lambda}$. Using continuity argumentation, we obtain that, for any $0 < \bar{\lambda} < 1$,

$$1 - \bar{\lambda} < -\ln \bar{\lambda} \Longrightarrow \frac{\ln p}{(m+1)} \leq -\ln \bar{\lambda}.$$

Hence $v = \min\{\ln(1/\bar{\lambda}, \ln(p/(m+1)))\} = \ln(p)/(m+1)$. Using (2.1.3), we have

$$\sum_{i=0}^{m_0} \lambda_i = \frac{(1-\lambda) - \ln(p)/(m+1)}{p}$$

From the previous equation and considering the fact that $e^{-v} = 1/p^v < 1/p$, we have

$$\lambda e^{-v} + \sum_{i=0}^{m_0} \lambda_i < \frac{1}{p}.$$

Thus, if some $-h_i$ satisfies

$$V(k - h_i(k), x(k - h_i(k))) > e^v V(k, x(k))$$

then using (*ii*) and the previous equation, it follows that

$$\begin{aligned} V(k+1, x(k+1)) &\leq (\lambda e^{-v} + \sum \lambda_i) \max\{V(k - h_j, x(k - h_j))\} \\ &\leq \frac{1}{p} \max_{\theta}\{V(k+\theta, x(k+\theta))\}. \end{aligned}$$

Using the previous inequality and (i), we obtain

$$\|x\| \leq \left(\frac{c_1}{c_2}\right)e^{-(v/r)k}\|\phi\|_m$$

Then the system described by (2.2.1) is UES. □

Remark 2.2.1. Theorem 2.2.3 can be seen as the counterpart of the Halanay-type inequality on continuous delay systems [Nic01].

In Theorem 2.2.3, the discrete-time linear Halanay-type inequality (2.1.1) in Theorem 2.1.1 is employed, and using time-varying Lyapunov functions the uniform exponential stability of nonlinear discrete-time delay systems with time-varying delays is derived. In Chapter 3, we investigate the application of the linear Halanay-type inequality (2.1.1) for deriving Lyapunov based conditions in a class of systems where the delays are subjects to follow a dynamic driven by a digraph. Nevertheless, we recognize that the linear Halanay's based methodology in Theorem 2.2.3 would allow the deduction of conditions for generally constrained time delays.

2.2.3 Input-to-state stability

In section, we report some results from [LH09] that investigate Razumikhin-type ISS criteria for the class of discrete time-delay systems with disturbance inputs or control inputs in (2.2.1).

Theorem 2.2.4. *([LH09]) Assume that there exist a positive defined function $V : C \times \mathbb{R}^n \to \mathbb{R}^+$, positive reals c_1, c_2, a constant $\lambda \in (0,1)$, a constant $p > 1$, a function γ of class $\mathcal{K}$ such that of any $\phi \in C$ the following inequalities hold:*

- *(i)* $c_1\|x\|^r \leq V(k,x) \leq c_2\|x\|^r$
- *(ii)* $V(k,x(k)) \leq \lambda \max_\theta\{e^{(v\theta)}V(k+\theta, x(k+\theta))\} + \gamma(\|u(k)\|)$
  whenever $V(k+s, x(k+s)) \leq pV(k,x(k))$

where $v = \min\{\ln(\frac{1}{\lambda}), \dfrac{\ln p}{(m+1)}\}$. *Then the system described by (2.2.1) is globally exponential ISS.*

For the reader's convenience, we report here the proof given in [LH09].

Theorem 2.2.5. *([LH09]) Assume that there exist a positive defined function $V : C \times \mathbb{R}^n \to \mathbb{R}^+$, positive reals c_1, c_2, a constant $\lambda \in (0,1)$, constants $\lambda_i \in (0,1)$ for $i = 1, 2, \ldots, r_0$, a function γ of class $\mathcal{K}$ such that the following conditions holds:*

- *i)* $c_1\|x\|^r \leq V(k,x) \leq c_2\|x\|^r$

ii) $V(k+1, x(k+1)) \leq \lambda V(k, x(k)) + \sum_{i=1}^{m_0} \lambda_i V(k - h_i(k), x(k - h_i(k))) + \gamma(\|u(k)\|)$.

If $\lambda + \sum_{i=1}^{r_0} \lambda_i < 1$*, then the system described by (2.2.1) is globally exponentially ISS with* $v = lnp/r(m+1)$*, where* $p > 1$ *is the unique root of the equation*

$$1 - \lambda - \sum_{i=1}^{m_0} \lambda_i = \frac{\ln p}{m+1} \tag{2.2.7}$$

Proof. The equation (2.2.7) has a unique solution $1 < p < 1 - \dfrac{\lambda}{\sum_{i=1}^{m_0} \lambda_i}$.
Then, for $h_i(k) \in \{-m, \ldots, 0\}$, we have

$$V(k - h_i(k), x(k - h_i(k))) \leq pV(k, x(k)).$$

Using (*ii*), we obtain

$$V(k+1, x(k+1)) \leq \Big(\frac{\lambda}{1 - p\sum_{i=1}^{m_0} \lambda_i}\Big) V(k, x(k)) + \Big(\frac{1}{1 - p\sum_{i=1}^{m_0} \lambda_i}\Big) \gamma(\|u(k)\|).$$

Observing that $\lambda + p \sum \lambda_i < 1$, and that $V(k, x(k)) \leq \max_\theta \{e^{(v\theta)} V(k+\theta, x(k+\theta))\}$, using (*ii*) it follows that

$$V(k, x(k)) \leq \lambda \max_\theta \{e^{(v\theta)} V(k-1+\theta, x(k-1+\theta))\} + \gamma(\|u(k-1)\|).$$

Then the system described by (2.2.1) is globally exponential ISS. □

In Theorem 2.2.5, we find another application of the linear Halanay-type inequality (2.1.1) in Theorem 2.1.1 to prove the global exponential ISS property of nonlinear discrete-time delay systems with time-varying delays. In Chapter 4, we discuss a deep comparison of Theorem 2.2.5 and our results that make use of a nonlinear version of the Halanay-type inequality (2.1.1). The main advantage lies on the computational aspect of finding a suitable Lyapunov function ensuring the ISS property.

Chapter 3

Stability Analysis of Nonlinear Discrete-time Time-Delay Systems via Nonlinear Halanay Inequality

Note 3.0.1. This chapter entails the results from the research paper titled *Stability Analysis of Nonlinear Discrete-time Time-Delay Systems via Nonlinear Halanay Inequality* by Maria Teresa Grifa (the PhD candidate), Pierdomenico Pepe (the thesis advisor), available on IEEEXplore, which got 2 citations. To read the full reference disclaimer, please refer to [0.1].

Chapter Description

This chapter deals with the stability analysis of fully nonlinear discrete-time delay systems with constrained time-varying delays. The plant here considered can be obtained, for instance, by suitable discretization of linear continuous-time systems affected by network induced time delays, when sensor/actuator synchronization and uniform sampling are employed [Pep19]. A delays digraph is used to model the topology of the delay signals. We provide a novel nonlinear Halanay-type inequality based on comparison functions. Using this novel result, we prove some sufficient conditions for the global asymptotic stability, uniform global asymptotic stability and global exponential stability are established.

In Section 3.1, the plant of the system under study is illustrated. In Section 3.2, the stability definitions aimed to be proved are stated. In Section 3.3, some preliminary results useful in the description of the chapter are reported. In Section 3.4, the main result of chapter is provided. In Section 3.5, the stability analysis for the system under study is reported.
In Section 3.6, a matrix inequality approach on the stability analysis of linear discrete-time delay systems is shown. In Section 3.9, some meaningful examples are illustrated.

3.1 Problem formulation

Let us consider a nonlinear discrete-time delay system described by the following equations (see [PP17] and reference therein):

$$\begin{aligned} x(k+1) &= f(x(k), x(k-d_1(k)), \ldots, x(k-d_r(k))), \\ x(\theta) &= \xi_0(\theta), \ \theta \in \{-\Delta, -\Delta+1, \ldots, 0\}, \end{aligned} \tag{3.1.1}$$

where: $k \in \mathbb{N}$; Δ is a known positive integer, the maximum involved time delay; $x(j) \in \mathbb{R}^n$, $j \geq -\Delta$; for $1 \leq i \leq r$, $d_i(k) \in \{0, 1, \ldots, \Delta\}$ is a time-varying time delay, r is a known positive integer; the function $f : \mathbb{R}^{n(r+1)} \to \mathbb{R}^n$ satisfies the equality $f(0, 0, \ldots, 0) = 0$; $\xi_0 \in \mathcal{C}$.
Let $d(k) = [d_1(k)\ d_2(k)\ \ldots\ d_r(k)]^T$, $k \in \mathbb{N}$, denote the vector collecting all time delays at time k. Let $D \subseteq \{0, 1, \ldots, \Delta\}^r$ be the non-empty subset of allowed values for the time-delays vector $d(k)$. That is, for any $k \in \mathbb{N}$, $d(k) \in D$.

3.1.1 Delay digraph

The non-empty set D takes into account a prior information regarding the delays.
Let $E(D)$ be the finite set of all pairs $(\delta_1, \delta_2) \in D \times D$ such that, for any $k \in \mathbb{N}$, if $d(k) = \delta_1$, it is

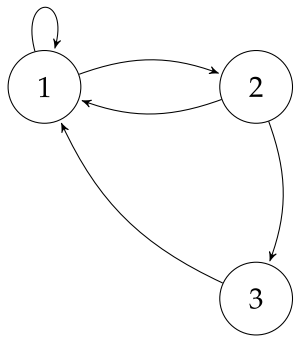


Figure 1: Delay digraph. $D = \{1, 2, 3\}$ and $E(D) = \{(1, 1), (1, 2), (2, 1), (2, 3), (3, 1)\}$

allowed $d(k + 1) = \delta_2$.
We associate a digraph $G(D, E(D))$ to the system (3.1.1) with the following identifications ([Pep19]):

1. the vertex set of the digraph is the set of delays D;
2. the directed edges set of the digraph is the set $E(D)$.

We recall a standard assumption (see [KC16]) which ensures that the solution of the systems describe by (3.1.1) is in $\mathbb{N}$.

Assumption 3.1.1. ([KC16]) Let $G(D, E(D))$ the delay digraph associated to the system (3.1.1). For any vertex $\delta \in D$ at least one of the following two cases occurs:

1. there exist vertexes ρ_1 and ρ_2, such that (ρ_1, δ) and (δ, ρ_2) are directed edges, i.e., belong to $E(D)$;
2. $(\delta, \delta) \in E(D)$,i.e., $\delta, \delta)$ is a self-loop.

We denote the set of time-delay signals as the set:

$$M_D = \{d : \mathbb{N} \to D \mid (d(k), d(k + 1)) \in E(D), k \in \mathbb{N}\}.$$

For $\delta \in D$, we define the set $S(\delta)$ of all possible walks of length Δ with ending point δ:

$$S(\delta) = \Big\{(v_0, \dots, v_{\Delta-1}, v_\Delta) : v_i \in D,\ 0 \le i \le \Delta\ , (v_i, v_{i-1}) \in E(D),\ 1 \le i \le \Delta,\ v_0 = \delta\Big\}. \tag{3.1.2}$$

Example 3.1.1. We report an example of delay digraph and we build the set of all possible walks $S(\delta)$ with $\delta \in D$. Consider the delay digraph in Fig. 1, with vertex set $D = \{1, 2, 3\}$ and edges set

$E(D) = \{(1,1),(1,2),(2,1),(2,3),(3,1)\}$. The length of a walk is $\Delta = 3$. The sets with ending point $\delta \in D$, are listed below:

$$S(1) = \{(1,1,1),(1,2,1),(1,2,3)\}$$
$$S(2) = \{((2,1,1),(2,3,1),(2,1,2)\}$$
$$S(3) = \{(3,1,1),(3,1,2)\}$$

3.2 Stability definitions

We denote with $x(k, \xi_0, d)$, $k \in \mathbb{N}$, the unique solution of (3.1.1) corresponding to initial state ξ_0 and time-delay signal $d \in M_D$. We recall here some standard stability definitions adapted to the problem plant depicted in Section 3.1.

Definition 3.2.1. The system described by (3.1.1) is stable if $\forall\ \epsilon > 0$ there exists $\delta > 0$ such that, for any $\xi_0 \in C$ satisfying $\|\xi_0\|_\infty \leq \delta$, for any $d \in M_D$, the following inequality holds, for all $k \in \mathbb{N}$

$$\|x(k, \xi_0, d)\| \leq \epsilon.$$

Definition 3.2.2. The system described by (3.1.1) is globally attractive if for any $\xi_0 \in C$, $d \in M_D$, the following limit holds:

$$\lim_{k\to\infty} x(k, \xi_0, d) = 0.$$

Definition 3.2.3. The system described by (3.1.1) is 0-GAS if it is stable and globally attractive.

Definition 3.2.4. The system described by (3.1.1) is uniformly 0-GAS if there exists a function β of class $\mathcal{KL}$ such that, for any initial state ξ_0 and for any delay signal $d \in M_D$, the corresponding solution $x(k, \xi_0, d)$ satisfies the following inequality, $\forall k \in \mathbb{N}$:

$$\|x(k, \xi_0, d)\| \leq \beta(\|\xi_0\|_\infty, k).$$

Definition 3.2.5. The system described by (3.1.1) is 0-GES if there exist positive reals M, p, with $M \geq 1$ and $0 \leq p < 1$, such that, for any initial state ξ_0 and for any delay signal $d \in M_D$, the corresponding solution $x(k, \xi_0, d)$ satisfies the following inequality, $\forall k \in \mathbb{N}$:

$$\|x(k, \xi_0, d)\| \leq Mp^k \|\xi_0\|_\infty.$$

3.3 Preliminary results

The following result is instrumental for proving the forthcoming analysis.

Lemma 3.3.1. ([Pep14]) Let $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ be a function of class $\mathcal{K}$ such that $(I_d - \alpha)$ is a function of class $\mathcal{K}$. Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined, for $t,\ s \in \mathbb{R}^+$, as $\omega(s,t) = \alpha^{[\frac{t}{\Delta}]}(s)$.
Then, there exists a function β of class $\mathcal{KL}$ such that

$$\omega(s,t) \leq \beta(s,t) \quad \forall s,t \in \mathbb{R}^+. \tag{3.3.1}$$

Proof. For any positive integer k, α^k is a function of class $\mathcal{K}$ and $\alpha^{k+1}(s) < \alpha^k(s)$, for all $s > 0$, $k = 1, 2, \ldots$. Moreover, for any positive real s,

$$\lim_{k \to +\infty} \alpha^k(s) = 0.$$

Indeed, since the positive sequence $k \to \alpha^k(s)$ is strictly decreasing, it admits a finite limit as $k \to +\infty$. Such limit cannot be positive. Indeed, by contradiction, let us suppose that such limit is equal to $c > 0$. Then a positive real ϵ would exist such that $\alpha(c + \epsilon) < c$, and there would exist a positive integer $\bar{k}$ such that, $\forall\ k \geq \bar{k}$, the contradiction would follow

$$c \leq \alpha^{k+1}(s) = \alpha \circ \alpha^k \leq \alpha(c + \epsilon) < c.$$

Therefore, the function $s \to \omega(s,t)$ is a function of class $\mathcal{K}$ and, for any fixed $s \in \mathbb{R}^+$, the function $t \to \omega(s,t)$ is a piece-wise constant, right continuous function (discontinuous at $k\delta$, $k = 0, 1, \ldots$), monotonically decreasing to 0. Now, let $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be defined, for $s,\ t \in \mathbb{R}^+$, as

$$\beta(s,t) = \begin{cases} \omega(s,t) & t \in [0, \Delta) \\ \omega(s, t - \Delta) + (\frac{t}{\Delta} - [\frac{t}{\Delta}])(\omega(s,t) - \omega(s, t - \Delta)) & t \in [k\Delta, (k+1)\Delta],\ k \in \mathbb{N} - \{0\} \ldots \end{cases}.$$

The function β is of class $\mathcal{KL}$ and the inequality (3.3.1) holds. □

3.4 Main Results

3.4.1 Nonlinear discrete-time Halanay-type inequality

In this section, we introduce a nonlinear discrete-time Halanay-type inequality.

Theorem 3.4.1. *Let α, β be functions of class $\mathcal{K}$ such that*

$$(\alpha - \beta) \in \mathcal{P} \quad \text{and} \quad (I_d - \alpha) \in \mathcal{K}. \tag{3.4.1}$$

Let $y : \{-\Delta, -\Delta+1, \dots\} \to \mathbb{R}^+$ be a sequence satisfying the following inequality, for $k \geq 0$,

$$y(k+1) - y(k) \leq -\alpha(y(k)) + \beta(\|y_k\|_\infty). \tag{3.4.2}$$

Then:

$$i)\ \|y(k)\| \leq \|y_0\|_\infty,\ k \in \mathbb{N}; \quad ii)\ \lim_{k\to\infty} y(k) = 0. \tag{3.4.3}$$

Furthermore, if

$$(\alpha - \beta) \in \mathcal{K} \quad \textit{and} \quad (I_d - \alpha + \beta) \in \mathcal{K}, \tag{3.4.4}$$

then, there exists a function ω of class $\mathcal{KL}$ such that, for any sequence $y : \{-\Delta, -\Delta+1, \dots\} \to \mathbb{R}^+$ satisfying (3.4.2), the following inequality holds, for $k \geq 0$:

$$y(k) \leq \omega(\|y_0\|_\infty, k). \tag{3.4.5}$$

Proof. The part (i) in (3.4.3) follows by induction on k.
For $k = 0$, (i) is trivially true. Let $y(j) \leq \|y_0\|_\infty$ for $j = 0, 1, \dots, k$.
For $k+1$, we obtain:

$$\begin{aligned} y(k+1) \leq & y(k) - \alpha(y(k)) + \beta(\|y_k\|_\infty) \leq (I_d - \alpha)(y(k)) + \beta(\|y_k\|_\infty) \leq (I_d - \alpha)\|y_0\|_\infty + \beta(\|y_0\|_\infty) \\ & \leq (I_d - \alpha + \beta)\|y_0\|_\infty \leq \|y_0\|_\infty. \end{aligned}$$

Thus, the condition (i) is proved. The proof of (ii) in (3.4.3) follows by contradiction. Let assume that the following limit holds:

$$\limsup_{k\to\infty} y(k) = l \neq 0.$$

Using (3.4.2), we have:

$$\limsup_{k\to\infty} y(k+1) \leq \limsup_{k\to\infty} (I_d - \alpha)(y(k)) + \limsup_{k\to\infty} \beta(\|y_k\|_\infty). \tag{3.4.6}$$

From (3.4.6), we obtain that

$$l \leq (I_d - \alpha)(l) + \beta(l)$$

and therefore

$$-\alpha(l) + \beta(l) \geq 0. \tag{3.4.7}$$

By the inequality (3.4.7) it follows that $(\alpha - \beta) \notin \mathcal{P}$, which is absurd. Therefore, $l = 0$.
Now, let also conditions (3.4.4) be satisfied. Then, for any sequence $y : \{-\Delta, -\Delta+1, \dots\} \to \mathbb{R}^+$

satisfying (3.4.2), for $k \geq 0$, the following inequality holds

$$\|y_{k(\Delta+1)}\|_\infty \leq (I_d - \alpha + \beta)^{\{k\}}(\|y_0\|_\infty), \tag{3.4.8}$$

where $(I_d - \alpha + \beta)^{\{0\}} = I_d$, and $(I_d - \alpha + \beta)^{\{k+1\}} = (I_d - \alpha + \beta) \circ (I_d - \alpha + \beta)^{\{k\}}$, $k \geq 0$.
The proof of the inequality (3.4.8) follows by induction on k. For $k = 0$, the inequality (3.4.8) is verified. Let us assume that the inequality (3.4.8) holds for some $k \geq 0$. We prove that the inequality (3.4.8) holds for $k+1$. It is sufficient to show that, for $j = -\Delta, -\Delta+1, \ldots, 0$ the following inequality holds:

$$y((k+1)(\Delta+1)+j) \leq (I_d - \alpha + \beta)(\|y_{k(\Delta+1)}\|_\infty). \tag{3.4.9}$$

Indeed, from (3.4.9) and the induction hypothesis, the following inequalities/equality hold:

$$\begin{aligned} \|y_{(k+1)(\Delta+1)}\|_\infty &\leq (I_d - \alpha + \beta)(\|y_{k(\Delta+1)}\|_\infty) \leq I_d - \alpha + \beta) \circ (I_d - \alpha + \beta)^{\{k\}}(\|y_0\|_\infty) \\ &= (I_d - \alpha + \beta)^{\{k+1\}}(\|y_0\|_\infty). \end{aligned}$$

The inequality (3.4.9) is proved by induction. We show first that the inequality (3.4.9) holds for $j = -\Delta$. Taking into account of (3.4.2), conditions (3.4.4), we have:

$$\begin{aligned} y((k+1)(\Delta+1)-\Delta) &\leq (I_d - \alpha)(y((k+1)(\Delta+1)-\Delta-1)) + \beta(\|y_{k(\Delta+1)}\|_\infty)) \\ &\leq (I_d - \alpha)(\|y_{k(\Delta+1)}\|_\infty) + \beta(\|y_{k(\Delta+1)}\|_\infty) \\ &= (I_d - \alpha + \beta)(\|y_{k(\Delta+1)}\|_\infty). \end{aligned}$$

Thus, the inequality (3.4.9) holds for $j = -\Delta$. Let us now assume that the inequality (3.4.9) holds for $j = -\Delta, -\Delta+1, \ldots, i$, for some $i \in \{-\Delta, -\Delta+1, \ldots, -1\}$, and let us show that the inequality (3.4.9) holds for $j = i+1$. From (3.4.2), taking into account the induction assumption and conditions (3.4.1), (3.4.4), we have:

$$\begin{aligned} y((k+1)(\Delta+1)+(i+1)) &\leq (I_d - \alpha)(y((k+1)(\Delta+1)+i)) + \beta(\|y_{(k+1)(\Delta+1)+i}\|_\infty)) \\ &\leq (I_d - \alpha)(\|y_{(k+1)(\Delta+1)+i}\|_\infty) + \beta(\|y_{(k+1)(\Delta+1)+i}\|_\infty) \\ &= (I_d - \alpha + \beta)(\|y_{(k+1)(\Delta+1)+i}\|_\infty) \\ &= (I_d - \alpha + \beta)\left(\sup_{\theta \in \{-\Delta, -\Delta+1, \ldots, 0\}} y((k+1)(\Delta+1)+i+\theta) \right) \\ &\leq (I_d - \alpha + \beta)\left(\max\left\{ \|y_{k(\Delta+1)}\|_\infty, (I_d - \alpha + \beta)(\|y_{k(\Delta+1)}\|_\infty) \right\} \right) \\ &= (I_d - \alpha + \beta)(\|y_{k(\Delta+1)}\|_\infty). \end{aligned}$$

Therefore, the inequality (3.4.9) holds, and thus the inequality (3.4.8) is proved. From (3.4.8), we obtain the following inequality, for $k \geq 0$:

$$y(k) \leq (I_d - \alpha + \beta)^{\{[\frac{k}{\Delta+1}]\}}(\|y_0\|_\infty). \tag{3.4.10}$$

The inequality (3.4.5), with suitable function ω of class $\mathcal{KL}$ depending on α and β, follows from (3.4.10) and Lemma 3.3.1. The proof of the theorem is complete. □

Remark 3.4.1. We would like to underline that the result in Theorem 3.4.1 is can be deduced by Theorem 2.1.4. Here, we provide also results concerning the asymptotic behavior of involved functions, as well as results concerning the uniform convergence to the origin.

3.4.2 Global asymptotic stability

In this section, we employ the provided nonlinear Halanay inequality in Theorem 3.4.1, in order to establish Lyapunov stability conditions for the system described by (3.1.1).

Theorem 3.4.2. *Assume that there exist a function $V : D \times \mathbb{R}^n \to \mathbb{R}^+$, functions γ_1, γ_2 of class $\mathcal{K}_\infty$, functions α, β of class $\mathcal{K}$ with the properties*

$$(\alpha - \beta) \in \mathcal{P} \quad \text{and} \quad (Id - \alpha) \in \mathcal{K}, \tag{3.4.11}$$

such that, for all $x \in \mathbb{R}^n$, $\phi \in C$, $\delta = [\ \delta_1 \quad \delta_2 \quad \ldots \quad \delta_r\ ]^T \in D$, $(\delta, \rho) \in E(D)$, the following inequalities hold:

i) $\gamma_1(\|x\|) \leq V(\delta, x) \leq \gamma_2(\|x\|)$

ii) $V(\rho, f(\phi(0), \phi(-\delta_1), \ldots, \phi(-\delta_r))) - V(\delta, \phi(0)) \leq -\alpha(V(\delta, \phi(0))) + \beta\left(\inf_{p \in S(\delta)} \max_{j=0,1,\ldots,\Delta} V(p(j+1), \phi(-j))\right).$

Then, the system described by (3.1.1) is 0*-GAS.*
Furthermore, if

$$(\alpha - \beta) \in \mathcal{K} \quad \text{and} \quad (I_d - \alpha + \beta) \in \mathcal{K}, \tag{3.4.12}$$

then the system described by (3.1.1) is uniformly 0*-GAS.*

Proof. Let $d \in \mathcal{M}_D$. Let $p_0 \in S(d(0))$. Let $w : \{-\Delta, -\Delta + 1, \ldots, 0\} \to \mathbb{R}^+$ be the function defined as follows, for $j \in \{-\Delta, -\Delta + 1, \ldots, 0\}$:

$$w(j) = V(p_0(-j + 1), \xi_0(j)).$$

Let $y : \{-\Delta, -\Delta+1, \ldots\} \to \mathbb{R}^+$ be the function defined, for $k \in \{-\Delta, -\Delta+1, \ldots\}$, as

$$y(k) = \begin{cases} w(k), & k \in \{-\Delta, -\Delta+1, \ldots, 0\} \\ V(d(k), x(k, \xi_0, d)), & k \geq 0. \end{cases} \tag{3.4.13}$$

Notice that $w(0) = V(d(0), x(0, \xi_0, d))$, no conflict of definition arises in (3.4.13).
Let $L : \{0, 1, \ldots, \Delta\} \to \mathbb{R}^+$ be the function defined, for $j \in \{0, 1, \ldots, \Delta\}$, as

$$L(j) = \max_{l \in \{-\Delta, -\Delta+1, \ldots, -j\}} \{w(j+l)\}.$$

Then, using the underlying delay digraph and the inequality (*ii*), we have, for $k \geq 0$,

$$\begin{aligned} y(k+1) - y(k) &= V(d(k+1), x(k+1, \xi_0, d)) - V(d(k), x(k, \xi_0, d)) \\ &\leq -\alpha(V(d(k), x(k, \xi_0, d))) \\ &+ \begin{cases} \beta\Big(\max\Big\{L(k), \max\limits_{j=0,1,\ldots,k}\{V(d(k-j), x(k-j, \xi_0, d))\}\Big\}\Big), & k = 0, 1, \ldots, \Delta, \\ \beta\Big(\max\limits_{j=0,1,\ldots,\Delta}\Big\{V(d(k-j), x(k-j, \xi_0, d))\Big\}\Big), & k \geq \Delta. \end{cases} \end{aligned}$$

Therefore, we have, for $k \geq 0$,

$$y(k+1) - y(k) \leq -\alpha(y(k)) + \beta(\|y_k\|_\infty).$$

Using (*i*) and results in Theorem 3.4.1, for $k \geq 0$, we obtain:

$$y(k) \leq \|y_0\|_\infty \leq \gamma_2(\|\xi_0\|_\infty) \tag{3.4.14}$$

and

$$\lim_{k \to \infty} y(k) = 0. \tag{3.4.15}$$

From (*i*) and (3.4.14), it follows, for $k \geq 0$:

$$\|x(k, \xi_0, d)\| \leq \gamma_1^{-1}(y(k)) \leq \gamma_1^{-1} \circ \gamma_2(\|\xi_0\|_\infty). \tag{3.4.16}$$

Thus, the 0-GAS property of system (3.1.1) follows from (3.4.16), (3.4.15) and (*i*). Let us now assume that conditions (3.4.12) are also satisfied. We prove the uniform 0-GAS property of (3.1.1). From (3.4.5) in Theorem 3.4.1, for some function ω of class $\mathcal{KL}$, for $k \geq 0$, we have:

$$y(k) \leq \omega(\|y_0\|_\infty, k). \tag{3.4.17}$$

From (i), we obtain, for $k \geq 0$:

$$\|x(k)\| \leq \gamma_1^{-1}(y(k)) \leq \gamma_1^{-1} \circ \omega(\|y_0\|_\infty, k) \leq \gamma_1^{-1} \circ \omega(\gamma_2(\|\xi_0\|_\infty), k).$$

The function $(s,t) \to \gamma_1^{-1} \circ \omega(\gamma_2(s), t)$, $s, t \in \mathbb{R}^+$, is a function of class $\mathcal{KL}$. Thus, the system described by (3.1.1) is uniformly 0-GAS. The proof of the theorem is complete. □

3.4.3 Global exponential stability

The following Corollary concerns the 0-GES property. A similar result, for the case of time-delay signals adhering to a delays digraph, can be deduced by Theorem 2.2.5.

Corollary 3.4.1. Assume that there exist a function $V : D \times \mathbb{R}^n \to \mathbb{R}^+$; positive reals m, c_1, c_2, a, b, with $b < a < 1$, such that, for all $x \in \mathbb{R}^n$, $\phi \in C$, $\delta = [\ \delta_1 \quad \delta_2 \quad \ldots \quad \delta_r\ ]^T \in D$; $(\delta, \rho) \in E(D)$, the following inequalities hold:

i) $c_1\|x\|^m \leq V(\delta, x) \leq c_2\|x\|^m$

ii) $V(\rho, f(\phi(0), \phi(-\delta_1), \ldots, \phi(-\delta_r))) - V(\delta, \phi(0)) \leq -aV(\delta, \phi(0)) + b \inf_{p \in S(\delta)} \Big(\max_{j=0,1,\cdots,\Delta} \{V(p(j+1), \phi(-j))\} \Big).$

Then, the system described by (3.1.1) is 0-GES.

Proof. The 0-GES property of (3.1.1) follows from the same argumentation used for the proof of Theorem 3.4.2, taking into account (i) and that, in this case, the function $I_d - \alpha + \beta$ in (3.4.8) (see the proof of Theorem 3.4.1) is defined, for $s \in \mathbb{R}^+$, as $(1 - a + b)s$. □

Remark 3.4.2. An alternative proof of Theorem 3.4.1 can be given employing the linear Halanay inequality in Theorem 2.1.1.

3.5 A matrix inequality for GES

In this section, sufficient conditions as novel matrix inequalities, inherited by Halanay's linear inequality, for the global exponential stability of linear discrete-time delay systems, with delay signals constrained to follow a delay digraph, are investigated. In comparison with other techniques in [LHI08], [Pep19] which make use of Lyapunov-Krasovskii functionals approach, the almost-linear matrix inequalities here provided can be competitive for large time-delay bounds and rich digraph information, at least as far as the computational burden is concerned. Let us consider the linear discrete-time delay system described by the equations (see [Fri14] and reference therein)

$$x(k+1) = A_0x(k) + \sum_{i=1}^{r} A_i x(k - d_i(k))$$
$$x(\theta) = \xi_0(\theta),\ \theta \in \{-\Delta, -\Delta+1, \ldots, 0\} \tag{3.5.1}$$

where: Δ is a known positive integer; $x(j) \in \mathbb{R}^n$, $j \in \{-\Delta, -\Delta+1, \ldots\}$; for $1 \le i \le r$, $d_i(k) \in \{0, 1, \ldots, \Delta\}$ is a time-varying time delay, r is a known positive integer; $A_j \in \mathbb{R}^{n\times n}$, $0 \le j \le r$; $\xi_0 \in C$. Let $d(k) = [d_1(k)\ d_2(k)\ \ldots\ d_r(k)]^T \in D \subset \{0, 1, \ldots, \Delta\}^r$, $k \in \mathbb{N}$, be the delays vector at time k. Let $G(D, E(D))$ be the delays digraph associated to the system described by (3.5.1). For $\delta = [\delta_1\ \delta_2\ \ldots\ \delta_r]^T \in D$, let us consider $A_\delta \in \mathbb{R}^{n\times(\Delta+1)n}$ defined by means of the following equality, which must hold for all $\phi \in C$:

$$A_\delta\, \mathcal{I}^{-1}(\phi) = A_0\phi(0) + A_1\phi(-\delta_1) + A_2\phi(-\delta_2) + \cdots + A_r\phi(-\delta_r). \tag{3.5.2}$$

Theorem 3.5.1. *Assume that there exist card(D) positive symmetric matrices $Q_\delta \in \mathbb{R}^{n\times n}$, $\delta \in D$, positive reals $\lambda_j \ge 0$, $1 \le j \le \Delta+1$, with $\sum_{j=1}^{\Delta+1} \lambda_j \le 1$, positive reals a, b with $b < a$ and $a < 1$, such that, for every $(\delta, \rho) \in E(D)$ and for every $p \in S(\delta)$, the following inequality holds*

$$A_\delta^T Q_\rho A_\delta - G(\delta, p) \le 0, \tag{3.5.3}$$

where $G(\delta, p) \in \mathbb{R}^{n(\Delta+1)\times n(\Delta+1)}$ is defined as follows:

$$G(\delta, p) = \mathrm{diag}\Big((1-a)Q_\delta + b\lambda_1 Q_{p(1)},\ b\lambda_2 Q_{p(2)}, \ldots, b\lambda_{\Delta+1} Q_{p(\Delta+1)}\Big).$$

Then, the system described by (3.5.1) is 0-GES.

Proof. Let us consider the quadratic Lyapunov functions $V(\delta, x) = x^T Q_\delta x$, $\delta \in D$, $x \in \mathbb{R}^n$.
Let $\phi \in C$, $(\delta, \rho) \in E(D)$, $p \in S(\delta)$. From inequality (3.5.3), we obtain:

$$(\mathcal{I}^{-1}(\phi))^T A_\delta^T Q_\rho A_\delta \mathcal{I}^{-1}(\phi) \le (\mathcal{I}^{-1}(\phi))^T G(\delta, p) \mathcal{I}^{-1}(\phi). \tag{3.5.4}$$

From inequality (3.5.2), for every $p \in S(\delta)$, we have:

$$(A_\delta \mathcal{I}^{-1}(\phi))^T Q_\rho A_\delta \mathcal{I}^{-1}(\phi) - \phi(0)^T Q_\delta \phi(0)$$
$$\le -a\phi(0)^T Q_\delta \phi(0) + b \sum_{j=0}^{\Delta} \lambda_{j+1} \phi(-j)^T Q_{p(j+1)} \phi(-j). \tag{3.5.5}$$

Observe that, since $\sum_{j=1}^{\Delta+1} \lambda_j \le 1$, for any non-negative reals α_j, $j = 1, 2, \ldots, \Delta + 1$, the following inequality holds:

$$\sum_{j=1}^{\Delta+1} \lambda_j \alpha_j \le \max_{j=1,\ldots,\Delta+1} \alpha_j.$$

From (3.5.5), we have:

$$\begin{aligned} V\Big(\rho, A_0\phi(0) + \sum_{i=1}^{r} A_i\phi(-\delta_i)\Big) - V(\delta, \phi(0)) &\le -aV(\delta, \phi(0)) + b \inf_{p\in S(\delta)} \sum_{j=0}^{\Delta} \lambda_{j+1} V(p(j+1), \phi(-j)) \\ &\le -aV(\delta, \phi(0)) + b \inf_{p\in S(\delta)} \max_{j=0,1,\ldots,\Delta} V(p(j+1), \phi(-j)). \end{aligned} \tag{3.5.6}$$

Thus, from (3.5.6) we obtain the inequality (*ii*) in Corollary 3.4.1. The condition (*i*) in Corollary 3.4.1 is satisfied with $m = 2$. By Corollary 3.4.1, the system described by (3.5.1) is 0-GES. □

Remark 3.5.1. A drawback in the use of the Lyapunov-Krasvoskii based LMIs in [Pep19], [LHI08] as selected by the delays digraph, is the possible unsatisfactory computational performance, due to the dimensions and the number of decision matrices involved in the numerical solver. It is important to notice that the dimension of symmetric positive definite matrices Q_δ involved in the inequalities (3.5.3) does not increase with the maximum delay Δ, as it does in [Pep19] and in Theorem 1.4.1. For instance, in the case of time-varying single time-delay, the involved *card*(D) symmetric positive definite decision matrices in Theorem 1.4.1 belong to $\mathbb{R}^{n(\Delta+1)\times n(\Delta+1)}$, while the involved *card*(D) symmetric positive definite decision matrices in (3.5.3) belong to $\mathbb{R}^{n\times n}$. On the other hand, the number of matrix inequalities (3.5.3) increases more than the ones in Theorem 1.4.1 as the delay digraph provides less information (see the role of $S(\delta)$ in (3.5.3)). These facts, for sufficiently large time-delays and sufficient information from the delays digraph, can yield better computation performance of (3.5.3) with respect to LMIs in Theorem 1.4.1. Notice also that the inequality (3.5.3) is not linear. It becomes a LMI when the parameters a, b and λ_j, $j = 1, 2, \ldots, \Delta + 1$, are preliminary chosen.

3.6 Applications

In this section, we provide some examples for illustrate our methodology.

3.6.1 Scalar nonlinear discrete-time delay system

Consider the following nonlinear discrete-time delay system:

$$\begin{aligned} x(k+1) &= \frac{1}{2}\tanh(x(k)) - \frac{1}{2}\text{sat}(x(k-d(k))) \\ x(\tau) &= \xi_0(\tau),\ \tau \in \{-1, 0\}. \end{aligned} \tag{3.6.1}$$

Note 3.6.1. In the paper [GP20], it was erroneously stated that the linear system obtained from (3.6.1), by linearization on the origin, is not globally asymptotically stable for $d(k) = 1$ for all $k \in \mathbb{N}$. Indeed, the linearized system is actually globally asymptotically stable also in this case.

Let us consider the delays digraph $G(D, E(D))$ where:

1. the vertexes set is $D = \{0, 1\}$
2. the edges set is $E(D) = \{(0,0), (0,1), (1,0)\}$.
3. the sets of all possible walks $S(\delta)$, $\delta \in D$ in in (3.1.2) are:

$$S(0) = \{(0,0), (0,1)\} \quad S(1) = \{(1,0)\}.$$

By $G(D, E(D))$, whenever $d(k) = 1$, for $k \in \mathbb{N}$, it must be $d(k+1) = 0$. Let us consider the Lyapunov function $V(d, x) = r_d|x|$, $d \in D$, $x \in \mathbb{R}$, with positive reals r_d.
In what follows, we apply the Lyapunov conditions $(i) - (ii)$ in Corollary 3.4.1 based on the linear discrete-time Halanay-type inequality in Theorem 2.1.1.
For $(\delta, \rho) = (0, 0)$, the following inequalities hold:

$$\begin{aligned} V(0, f(\phi(0), \phi(0))) - V(0, \phi(0)) &\leq r_0 \left| \frac{1}{2}\tanh(\phi(0)) - \frac{1}{2}\text{sat}\phi(0)) \right| - r_0|\phi(0)| \\ &\leq \frac{1}{2} r_0 \tanh(1)|\phi(0)| - r_0|\phi(0)| \leq -ar_0|\phi(0)| + br_0|\phi(0)|, \end{aligned}$$

then, we obtain the conditions:

$$0 < a < 1, \quad a > b, \quad b \geq \frac{1}{2}\tanh(1). \tag{3.6.2}$$

For $(\delta, \rho) = (0, 1)$,:

$$\begin{aligned} V(1, f(\phi(0), \phi(0))) - V(0, \phi(0)) &\leq r_1 \left| \frac{1}{2}\tanh(\phi(0)) - \frac{1}{2}\text{sat}(\phi(0)) + u \right| - r_0|\phi(0)| \\ &\leq \frac{1}{2} r_1 \tanh(1)|\phi(0)| - r_0|\phi(0)| \leq -ar_0|\phi(0)| + br_0|\phi(0)|, \end{aligned}$$

then, the following conditions hold:

$$0 < a < 1, \quad a > b, \quad br_0 \geq \frac{1}{2} r_1 \tanh(1).$$

For $(\delta, \rho) = (1, 0)$, we have:

$$\begin{aligned} V(0, f(\phi(0), \phi(-1))) - V(1, \phi(0)) &\leq r_0 \left| \frac{1}{2} \tanh(\phi(0)) - \frac{1}{2} \text{sat}(\phi(-1)) + u \right| - r_1 |\phi(0)| \\ &\leq \frac{1}{2} r_0 |\phi(0)| + \frac{1}{2} r_0 |\phi(-1)| - r_1 |\phi(0)| \\ &\leq -a r_1 |\phi(0)| + \lambda_1 b r_1 |\phi(0)| + \lambda_2 b r_0 |\phi(-1)|, \end{aligned}$$

then, the following conditions hold:

$$0 < a < 1, \quad a > b, \quad br_1 \geq \frac{1}{2} r_0, \quad \lambda_2 b \geq \frac{1}{2},$$

and

$$\lambda_i \geq 0, i = 1, 2, \quad \lambda_1 + \lambda_2 \leq 1.$$

Choosing:

$$r_0 = 2.2,\ r_1 = 4.6,\ a = 0.9,\ b = 0.8,\ \lambda_1 = 0.3,\ \lambda_2 = 0.7, \tag{3.6.3}$$

the 0-GES property of the system described by (3.6.1) follows from Corollary 3.4.1

3.6.2 Scalar nonlinear discrete-time delay system with arbitrary time-varying delays

Consider the following nonlinear discrete-time delay system with arbitrary, as long as bounded, time-varying delays:

$$\begin{aligned} x(k+1) &= \tanh(x(k)) + \tanh \circ \tanh(x(k - d(k))) - \tanh(x(k - d(k))) \\ x(\tau) &= \xi_0(\tau),\ \tau \in \{-\Delta, -\Delta + 1, \ldots, 0\} \end{aligned} \tag{3.6.4}$$

Let $D = \{0, 1, \ldots, \Delta\}$ and $E(D) = \{(i, j) : i, j \in D\}$.
Let the Lyapunov function be chosen as $V(d, x) = |x|$, $d \in D$, $x \in \mathbb{R}$.
For every $(\delta, \rho) \in E(D)$, every $p \in S(\delta)$, every $\phi \in C$, we have:

$$\begin{aligned} V(\rho, f(d, \phi)) - V(\delta, \phi(0)) &\leq \tanh(|\phi(0)|) - |\phi(0)| + \tanh(|\phi(-\delta)|) - \tanh \circ \tanh(|\phi(-\delta)|) \\ &\leq -\alpha(|\phi(0)|) + \beta\Big(\inf_{p \in S(\delta)} \max_{j=0,1,\ldots,\Delta} V(p(j+1), \phi(-j)) \Big), \end{aligned}$$

where, for $s \in \mathbb{R}^+$, $\alpha(s) = s - \tanh(s)$, $\beta = \tanh(s) - \tanh \circ \tanh(s)$. Observing that $\alpha \in \mathcal{K}$, $\beta \in \mathcal{K}$, $(\alpha - \beta) \in \mathcal{K}$, $(I_d - \alpha) \in \mathcal{K}$, $(I_d - \alpha + \beta) \in \mathcal{K}$, the conditions in Theorem 3.4.2 are satisfied and thus the system described by (3.6.4) is uniformly 0-GAS. We would like to address that the nonlinearity of functions α and β, which does not allow the use of Corollary 3.4.1 (i.e., of linear Halanay's inequality). Furthermore, the application of Theorem 3.4.2 to this example is extremely easy. The choice of functions α and β comes very natural. No suitable functions as per the Razumikhin condition are here necessary to find (see for instance functions p and q in Theorem 2.2.5). Razumikhin conditions in Theorem 1.3.4 concern constant time delays and thus cannot be used for system 3.6.4.

3.6.3 Linear discrete-time delay system

In the following, we consider a linear discrete-time delay system studied in [Fri14], [LHI08], [Pep18].
A comparison between our methodology in Theorem 3.5.1 with the ones in [Fri14], [LHI08], [Pep18] is given.

Consider the linear system:

$$\begin{aligned} x(k+1) &= \eta\, A_0 x(k) + \eta\, A_1 x(k - d(k)) \\ x(\tau) &= \xi_0(\tau),\ \ \tau \in \{-\Delta, -\Delta + 1, \dots, 0\} \end{aligned} \tag{3.6.5}$$

where $\eta \in \mathbb{R}^+$, and

$$A_0 = \begin{bmatrix} 0.8 & 0; \\ 0 & 0.97 \end{bmatrix} \quad A_1 = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}$$

We first consider the case $\Delta = 20$, $D = \{1, 20\}$, $E(D) = \{(1, 20), (20, 1)\}$, and $\eta = 1$.
We fix $a = 0.5$, $b = 0.49$, $\lambda_j = \frac{1}{\Delta+1}$ for $j = 1, 2, \dots, 21$, in (3.5.3). The resulting linear matrix inequalities (3.5.3) are marginally feasible, thus the 0-GES property cannot be established. Instead, the LMIs in Theorem 1.4.1 (and in [Pep19]) are feasible, thus the 0-GES property is established.
Let us now consider system (3.6.5) with $\eta = 0.1$, and different values of Δ. With this value of η the LMIs in Proposition 1.4.1 are feasible by Matlab LMI toolbox, and therefore, by Proposition 1.4.1, for any given Δ, the system described by (3.6.5) is 0-GAS for arbitrary time-varying delay. As well, we observe that with $\Delta = 50$, the approach in Theorem 1.4.1 yields related LMIs feasibility in 1034.48 seconds. With our methodology, the feasibility problem of LMIs (3.5.3) (as per the same above choice of parameters a, b, λ_j, $j = 1, 2, \dots, \Delta + 1$) is solved in 1.36 seconds.
For $\Delta = 400$, the feasibility problem of LMIs (3.5.3) (again as per the same above choice of

parameters a, b, λ_j, $j = 1, 2, \dots, \Delta + 1$), is successfully solved by Matlab LMI toolbox, and thus the 0-GES property is proved. The elapsed time for the feasibility solution results to be less than 2 minutes. Instead, the LMIs in Theorem 1.4.1 remain still unsolved after 10 hours of computation. In this case, the computational improvement provided by the matrix inequalities (3.5.3) with respect to the ones in Theorem 1.4.1 is evident. With this example we want just to show that, though the Halanay's based methodology is more conservative in general, the matrix inequalities here provided can have better performance, by Matlab LMI toolbox, than the LMIs provided in Theorem 1.4.1, for sufficiently large delays and sufficiently rich information from the delays digraph. Notice that the 0-GES property is here proved to hold, which is not considered, for instance, in [LM08], [JSL18], [Pep19], [CdSC18], [LHI08], where the 0-GAS and the uniform 0-GAS properties are investigated.

Chapter 4

On a Novel Nonlinear Discrete-Time Halanay's Inequality with Forcing Term and Applications to the Input-to-State Stability of Delay Systems

Note 4.0.1. This chapter is collated from the research paper titled *"On a Novel Nonlinear Discrete-Time Halanay's Inequality with Forcing Term and Applications to the Input-to-State Stability of Delay Systems"* by Maria Teresa Grifa (the PhD candidate) and Pierdomenico Pepe (the thesis advisor), submitted to *International Journal of Control*. To read the full reference disclaimer, please refer to [0.1].

Chapter Description

In this chapter, we present a study on the input-to-state stability for nonlinear discrete-time delay systems. A nonlinear discrete-time Halanay-type inequality with forcing term is provided. Sufficient Lyapunov conditions for the input-to-state stability are given. The provided sufficient Lyapunov conditions cover all cases with possible constraints on time-delay signals, as expressed by a suitable delays digraph. The delay digraph can guarantee the stability even when instability occurs for one or more values of the delays. A matrix inequality is derived to prove the exponential input-to-state stability of linear discrete-time delay systems with delay signals obeying to a delays digraph. Finally, the link between global exponential stability and input-to-state stability is shown in the linear case with arbitrary delay signals.

In Section 4.1, the plant of the system under study is illustrated. In Section 4.2, the stability definitions aimed to be proved are stated. In Section 4.3, some preliminary results useful in the description of the chapter are reported. In Section 4.4, the main result of chapter is provided. In Section 4.5, the stability analysis for the system under study is reported. In Section 4.6, a matrix inequality approach on the stability analysis of linear discrete-time delay systems is shown. In Section 4.9, some meaningful examples are illustrated.

4.1 Problem formulation

The notion of input-to-state stability was proposed in [Son89] for continuous-time nonlinear systems to investigate how an external input affects the system stability. To our best knowledge, there are no results available in the literature concerning nonlinear discrete-time Halanay-type inequalities with arbitrarily large and bounded forcing term. Consequently, no results for the input-to-state stability of discrete-time delay systems, with or without constrained delays, based on nonlinear discrete-time Halanay-type techniques, exist in the literature. Our aim is to fill the gap in the literature regarding the nonlinear discrete-time Halanay-type inequality in the case with forcing term, and the related applications to the study of the input-to-state stability for discrete-time delay systems equipped with delays digraphs.

Let us consider the discrete-time delay system with time-varying-delays and input signals of the form (see [Pep19, Fri14, Zho18] and references therein)

$$\begin{aligned} x(k+1) &= f(x(k), x(k-d_1(k)), \dots, x(k-d_r(k)), u(k)) \\ x(\theta) &= \xi_0(\theta),\ \ \theta \in \{-\Delta, -\Delta+1, \dots, 0\} \end{aligned} \tag{4.1.1}$$

where: $k \in \mathbb{N}$; Δ is a known positive integer, the maximum involved time delay; $x(j) \in \mathbb{R}^n,\ j \geq -\Delta$;

$u(k) \in \mathbb{R}^m$ is the input; for $1 \le i \le r$, $d_i(k) \in \{0, 1, \ldots, \Delta\}$ is a time-varying time delay, r is a known positive integer; the function $f : \mathbb{R}^{n(r+1)} \times \mathbb{R}^m \to \mathbb{R}^n$ satisfies the equality $f(0, 0, \ldots, 0) = 0$; $\xi_0 \in \mathcal{C}$. Let $d(k) = [d_1(k)\ d_2(k)\ \cdots\ d_r(k)]^T$, $k \in \mathbb{N}$, denote the vector collecting all time delays at time k. Let $D \subseteq \{0, 1, \ldots, \Delta\}^r$ be the non-empty subset of allowed values for the time-delays vector $d(k)$. That is, for any $k \in \mathbb{N}$, $d(k) \in D$.

4.2 Stability definitions

In this section, we provide to the reader some stability definitions adapted to the contest where a digraph is used to model the delays in the system described by (4.1.1). We adopt the same delay-digraph formulation used in Subsection 3.1.1 in Chapter 3. Under the Assumption 3.1.1 in Chapter 3, we associate to the set D the set $E(D)$ for building the delay digraph $G(D, E(D))$. Furthermore, we consider the set of all possible walk of length Δ with ending point in $\delta \in D$ as defined in (3.1.2).
The set of time-delay signals, which are allowed by the delays digraph, is denoted as:

$$M_D = \{d : \mathbb{N} \to D \mid (d(k), d(k+1)) \in E(D), k \in \mathbb{N}\}.$$

The set of input signals is defined as:

$$M_U = \{u : \mathbb{N} \to \mathbb{R}^m\}$$

We denote with $x(k, \xi_0, d, u)$, $k \in \mathbb{N}$, the unique solution of (4.1.1) corresponding to initial state ξ_0, time-delay signal $d \in M_D$ and input signal $u \in M_U$.

Definition 4.2.1. The system described by (4.1.1) is said to be ISS if there exists a function ω of class $\mathcal{KL}$ and a function σ of class $\mathcal{K}$ such that, for any initial state ξ_0, for any input signal $u \in M_U$, and for any time-delay signal $d \in M_D$, the corresponding solution $x(k, \xi_0, d, u)$ of (4.1.1) satisfies the following inequality, $\forall k \in \mathbb{N}$,:

$$\|x(k, \xi_0, d, u)\| \le \omega\Big(\|\xi_0\|_\infty, k\Big) + \sigma\Big(\max_{j=0,1,\cdots,k-1} |u(j)|\Big) \tag{4.2.1}$$

where the second term of the sum in the right-hand side of (4.2.1) is taken equal to 0 for $k = 0$.

Definition 4.2.2. The system described by (4.1.1) is said to be exponentially ISS if there exist reals M, p, with $M \ge 1$ and $0 \le p < 1$, and a function μ of class $\mathcal{K}$ such that, for any initial state ξ_0, for any input signal $u \in M_U$, and for any time-delay signal $d \in M_D$, the corresponding solution $x(k, \xi_0, d, u)$ of (4.1.1) satisfies the following inequality, for $k \ge 0$:

$$\|x(k, \xi_0, d, u)\| \le Mp^k\|\xi_0\|_\infty + \mu\Big(\max_{j=0,1,\cdots,k-1} |u(j)|\Big), \tag{4.2.2}$$

where the second term of the sum in the right-hand side of (4.2.2) is taken equal to 0 for $k = 0$.

4.3 Main Results

4.3.1 Nonlinear discrete-time Halanay-type inequality with forcing term

The discrete-time Halanay-type inequality in nonlinear form, with forcing term, is a significant step forward with respect to the one without forcing term stated in Theorem 3.4.1, and requires a completely different proof. In this section, we prove a novel discrete-time nonlinear Halanay-type inequality with forcing term.

Theorem 4.3.1. *Let α, β be functions of class $\mathcal{K}$ such that*

$$(\alpha - \beta) \in \mathcal{K} \quad \text{and} \quad (I_d - \alpha) \in \mathcal{K}. \tag{4.3.1}$$

Then, there exists a function ω of class $\mathcal{KL}$ and a function σ of class $\mathcal{K}$ such that: for any $c \in \mathbb{R}^+$ and any function $y : \{-\Delta, -\Delta + 1, \dots\} \to \mathbb{R}^+$ satisfying the following inequality, for $k \geq 0$,

$$y(k+1) - y(k) \leq -\alpha(y(k)) + \beta(\|y_k\|_\infty) + c, \tag{4.3.2}$$

the following inequality holds, for $k \geq 0$,

$$y(k) \leq \omega(\|y_0\|_\infty, k) + \sigma(c). \tag{4.3.3}$$

Proof. The proof is based on a transformation of inequality (4.3.2), involving past values of the sequence y, into a memory-less inequality for which the comparison principle in Lemma 1.2.2 can be applied.

Let us consider any sequence $y : \{-\Delta, -\Delta + 1, \dots\} \to \mathbb{R}^+$ satisfying (4.3.2). Using (4.3.2) with $k = 0$, we have:

$$y(1) \leq (I_d - \alpha)y(0) + \beta(\|y_0\|_\infty) + c \leq (I_d - \alpha + \beta)(\|y_0\|_\infty) + c. \tag{4.3.4}$$

Using (4.3.2) with $k = 1$, by (4.3.4), taking (4.3.1) into account, we obtain:

$$\begin{aligned} y(2) &\leq (I_d - \alpha)\big(y(1)\big) + \beta(\|y_1\|_\infty) + c \leq (I_d - \alpha + \beta)(\|y_1\|_\infty) + c \\ &\leq (I_d - \alpha + \beta)\max\{\|y_0\|_\infty, (I_d - \alpha + \beta)(\|y_0\|_\infty) + c\} + c \\ &\leq \max\left\{\begin{matrix} (I_d - \alpha + \beta)(\|y_0\|_\infty) + c \\ (I_d - \alpha + \beta)(\|y_0\|_\infty) + 2c \end{matrix}\right\} \leq (I_d - \alpha + \beta)\big(\|y_0\|_\infty\big) + 2c. \end{aligned} \tag{4.3.5}$$

Using (4.3.2) with $k = 2$, by (4.3.5), taking (4.3.1) into account, the following inequalities hold:

$$\begin{aligned} y(3) &\le (I_d - \alpha)\big(y(2)\big) + \beta(\|y_2\|_\infty) + c \le (I_d - \alpha + \beta)(\|y_2\|_\infty) + c \\ &\le (I_d - \alpha + \beta)\max\{y(2), y(1), \|y_0\|_\infty\} + c \\ &\le (I_d - \alpha + \beta)\max\begin{Bmatrix} (I_d - \alpha + \beta)(\|y_0\|_\infty) + 2c \\ (I_d - \alpha + \beta)(\|y_0\|_\infty) + c \\ \|y_0\|_\infty \end{Bmatrix} + c \\ &\le (I_d - \alpha + \beta)(\|y_0\|_\infty) + 3c. \end{aligned} \tag{4.3.6}$$

Using (4.3.2) with $k = 3, 4, \ldots, \Delta$, by analogous reasoning as in (4.3.4), (4.3.5), (4.3.6), we obtain

$$y(j) \le (I_d - \alpha + \beta)(\|y_0\|_\infty) + jc, \ \ j = 4, 5, \ldots, \Delta + 1. \tag{4.3.7}$$

Since

$$\|y_{\Delta+1}\|_\infty = \max\{y(1), y(2), \ldots, y(\Delta + 1)\},$$

from (4.3.4), (4.3.5), (4.3.6), (4.3.7), we have

$$\|y_{\Delta+1}\|_\infty \le (I_d - \alpha + \beta)(\|y_0\|_\infty) + (\Delta + 1)c.$$

We repeat the reasoning using (4.3.2), for $k = \Delta + 1, \Delta + 2, \ldots, 2\Delta + 2$, and obtain the following inequality:

$$\|y_{2(\Delta+1)}\|_\infty \le (I_d - \alpha + \beta)(\|y_{\Delta+1}\|_\infty) + (\Delta + 1)c.$$

Then, we obtain recursively the following inequality, for $k \ge 0$,

$$\|y_{(k+1)(\Delta+1)}\|_\infty \le (I_d - \alpha + \beta)\|(y_{k(\Delta+1)}\|_\infty) + (\Delta + 1)c. \tag{4.3.8}$$

By (4.3.8), we have:

$$\begin{aligned} \|y_{(k+1)(\Delta+1)}\|_\infty - \|y_{k(\Delta+1)}\|_\infty &\le (I_d - \alpha + \beta)(\|y_{k(\Delta+1)}\|_\infty) + (\Delta + 1)c - \|y_{k(\Delta+1)}\|_\infty \\ &= -(\alpha - \beta)(\|y_{k(\Delta+1)}\|_\infty) + (\Delta + 1)c. \end{aligned}$$

Let us now consider the sequence $z : \mathbb{N} \to \mathbb{R}^+$ defined, for $k \in \mathbb{N}$, as $z(k) = \|y_{k(\Delta+1)}\|_\infty$. Then the following inequality holds, for $k \ge 0$:

$$z(k + 1) - z(k) \le -(\alpha - \beta)(z(k)) + (\Delta + 1)c. \tag{4.3.9}$$

By Lemma 1.2.2, there exist a function $\tilde{\omega}$ of class $\mathcal{KL}$ depending on α, β and a function $\tilde{\sigma}$ of class

$\mathcal{K}$ such that the following inequality holds, for $k \geq 0$:

$$z(k) \leq \tilde{\omega}(z(0), k) + \tilde{\sigma}((\Delta + 1)c). \tag{4.3.10}$$

From (4.3.10) it follows that, for $k \geq 0$:

$$y(k) \leq \tilde{\omega}\Big(\|y_0\|_\infty, \max\Big\{0, \frac{k}{\Delta + 1} - 1\Big\}\Big) + \tilde{\sigma}((\Delta + 1)c). \tag{4.3.11}$$

From (4.3.11), the inequality (4.3.3) follows with the functions ω of class $\mathcal{KL}$ and σ of class $\mathcal{K}$ defined as:

$$\begin{aligned} &\omega(s, t) = \tilde{\omega}\Big(s, \max\Big\{0, \frac{t}{\Delta + 1} - 1\Big\}\Big), \; s, t \in \mathbb{R}^+; \\ &\sigma(s) = \tilde{\sigma}((\Delta + 1)\, s), \; s \in \mathbb{R}^+. \end{aligned}$$

The proof of the Theorem is complete. □

4.3.2 Input-to-state stability analysis

In this section, by employing Theorem 4.3.1, Lyapunov conditions for the ISS of discrete-time delay systems with delay signals constrained to adhere to a delay digraph are established.
The results concerning the input-state-stability here established, by this novel discrete-time nonlinear Halanay-type inequality with forcing term, are a significant step forward with respect to the ones concerning the global asymptotic stability given in Chapter 3, and cannot be achieved neither by Theorems 3.4.2.

Theorem 4.3.2. *Assume that there exist a function $V : D \times \mathbb{R}^n \to \mathbb{R}^+$, functions η_1, η_2 of class $\mathcal{K}_\infty$, functions α, β, γ of class $\mathcal{K}$ with the properties*

$$(\alpha - \beta) \in \mathcal{K} \quad \text{and} \quad (I_d - \alpha) \in \mathcal{K}, \tag{4.3.12}$$

such that, for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $\phi \in \mathcal{C}$, $\delta = [\delta_1 \; \delta_2 \; \cdots \; \delta_r]^T \in D$, $(\delta, \rho) \in E(D)$, the following inequalities hold:

i) $\eta_1(\|x\|) \leq V(\delta, x) \leq \eta_2(\|x\|)$

ii) $V(\rho, f(\phi(0), \phi(-\delta_1), \dots, \phi(-\delta_r), u)) - V(\delta, \phi(0)) \leq -\alpha(V(\delta, \phi(0))) + \beta\left(\inf_{p \in S(\delta)} \max_{j=0,1,\dots,\Delta} V(p(j+1), \phi(-j))\right) + \gamma(\|u\|)$.

Then, the system described by (4.1.1) is ISS.

Proof. Let $d \in M_D$, $u \in M_U$ with $|u(k)| \le v$, $k \in \mathbb{N}$, $v \in \mathbb{R}^+$.
Let $p_0 \in S(d(0))$. Let $w : \{-\Delta, -\Delta+1, \dots, 0\} \to \mathbb{R}^+$ be the function defined, for $j \in \{-\Delta, -\Delta+1, \dots, 0\}$, as

$$w(j) = V(p_0(-j+1), \xi_0(j)).$$

Let $y : \{-\Delta, -\Delta+1, \dots\} \to \mathbb{R}^+$ be the function defined, for $k \in \{-\Delta, -\Delta+1, \dots\}$, as

$$y(k) = \begin{cases} w(k), & k \in \{-\Delta, -\Delta+1, \dots, 0\}, \\ V(d(k), x(k, \xi_0, d, u)), & k \ge 0. \end{cases} \tag{4.3.13}$$

Notice that $w(0) = V(d(0), x(0, \xi_0, d, u))$, so no conflict of definition arises in (4.3.13). Let $L : \{0, 1, \dots, \Delta\} \to \mathbb{R}^+$ be the function defined, for $j \in \{0, 1, \dots, \Delta\}$, as

$$L(j) = \max_{l \in \{-\Delta, -\Delta+1, \dots, -j\}} \{w(j+l)\}.$$

Using the underlying delay digraph and the inequality (*ii*), we have, for $k \ge 0$:

$$\begin{aligned} y(k+1) - y(k) &= V(d(k+1), x(k+1, \xi_0, d, u)) - V(d(k), x(k, \xi_0, d, u)) \\ &\le -\alpha(V(d(k), x(k, \xi_0, d, u))) + \\ &\begin{cases} \beta\Big(\max\{L(k), \max\limits_{j=0,1,\dots,k}\{V(d(k-j), x(k-j, \xi_0, d, u))\}\}\Big), & k = 0, 1, \dots, \Delta, \\ \beta\Big(\max\limits_{j=0,1,\dots,\Delta}\{V(d(k-j), x(k-j, \xi_0, d, u))\}\Big), & k \ge \Delta, \end{cases} \\ &+ \gamma(\|u(k)\|). \end{aligned}$$

Therefore, we have, for $k \ge 0$:

$$\begin{aligned} y(k+1) - y(k) &\le -\alpha(y(k)) + \beta(\|y_k\|_\infty) + \gamma(\|u(k)\|) \\ &\le -\alpha(y(k)) + \beta(\|y_k\|_\infty) + \gamma(v). \end{aligned}$$

From (4.3.3) in Theorem 4.3.1, for some function $\tilde{\omega} \in \mathcal{KL}$ and $\tilde{\sigma} \in \mathcal{K}$, for $k \ge 0$, we have

$$y(k) \le \tilde{\omega}(\|y_0\|_\infty, k) + \tilde{\sigma}(\gamma(v)). \tag{4.3.14}$$

Using (*i*) and (4.3.14) we obtain, for $k \ge 0$:

$$y(k) \le \tilde{\omega}(\eta_2(\|\xi_0\|), k) + \tilde{\sigma}(\gamma(v)). \tag{4.3.15}$$

From (i) and (4.3.15), it follows, for $k \geq 0$:

$$\|x(k, \xi_0, d, u)\| \leq \eta_1^{-1}(y(k)) \leq \eta_1^{-1}\Big(\tilde{\omega}(\eta_2(\|\xi_0\|), k) + \tilde{\sigma}(\gamma(v))\Big)$$
$$\leq \eta_1^{-1}\Big(2\tilde{\omega}(\eta_2(\|\xi_0\|), k)\Big) + \eta_1^{-1}\Big(2\tilde{\sigma}(\gamma(v))\Big) = \omega(\|\xi_0\|_\infty, k) + \sigma(v)$$

with

$$\omega(s,t) = \eta_1^{-1}\Big(2\tilde{\omega}(\eta_2(s), t)\Big), s, t \in \mathbb{R}^+$$
$$\sigma(s) = \eta_1^{-1}\Big(2\tilde{\sigma}(\gamma(s))\Big), s \in \mathbb{R}^+.$$

Since the function ω is of class $\mathcal{KL}$ and the function σ is of class $\mathcal{K}$, by causality arguments it follows that the system described by (4.1.1) is ISS. The proof of the Theorem is complete. □

4.3.3 Global exponential stability

For completeness, we report the following Corollary concerning the exponential ISS property.

Corollary 4.3.1. Let us consider the system described by (4.1.1). Assume that there exist a function $V : D \times \mathbb{R}^n \to \mathbb{R}^+$ continuous in the second argument, a function γ of class $\mathcal{K}$, positive reals m, c_1, c_2, a, b with $b < a < 1$, such that, for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $\phi \in \mathcal{C}$, $\delta = [\delta_1\ \delta_2\ \cdots\ \delta_r]^T \in D$, $(\delta, \rho) \in E(D)$, the following inequalities hold:

i) $c_1\|x\|^m \leq V(\delta, x) \leq c_2\|x\|^m$

ii) $V(\rho, f(\phi(0), \phi(-\delta_1), ..., \phi(-\delta_r), u)) - V(\delta, \phi(0)) \leq -aV(\delta, \phi(0)) + b \inf_{p \in S(\delta)} \Big(\max_{j=0,1,\cdots,\Delta} \{V(p(j+1), \phi(-j))\} \Big) + \gamma(\|u\|)$.

Then, the system described by (4.1.1) is exponentially *ISS*.

Proof. The same proof given for Theorem 4.3.2 can be repeated here setting $\alpha(s) = as$ and $\beta(s) = bs$, $s \in \mathbb{R}^+$. The exponential ISS follows by the fact that, in this case, the function $\tilde{\gamma}$ in (4.3.10), in the proof of Theorem 4.3.1, is defined for $s, t \in \mathbb{R}^+$, as $\tilde{\omega}(s,t) = (1 - a - b)^t s$. □

Remark 4.3.1. The exponential ISS proved for [LH09, Example 6.1] by Theorem 2.2.5 can be proved as well by the Corollary 4.3.1.

4.4 Matrix Inequality approach to ISS in the linear case

In this section, we show that the matrix inequality provided in Theorem 3.5.1, concerning the 0-GES property of linear discrete-time delay systems with delay signals constrained to follow a

delay digraph, is valid for proving the input-to-state stability.
Let us consider the linear discrete-time delay system described by the equations (see [Fri14] and reference therein)

$$x(k+1) = A_0 x(k) + \sum_{i=1}^{r} A_i x(k - d_i(k)) + Bu(k)$$
$$x(\theta) = \xi_0(\theta),\ \theta \in \{-\Delta, -\Delta+1, \dots, 0\} \tag{4.4.1}$$

where: Δ is a known positive integer; $x(j) \in \mathbb{R}^n$, $j \in \{-\Delta, -\Delta+1, \dots\}$; $u(k) \in \mathbb{R}^m$ is the input; for $1 \le i \le r$, $d_i(k) \in \{0, 1, \dots, \Delta\}$ is a time-varying time delay, r is a known positive integer; $A_j \in \mathbb{R}^{n\times n}$, $0 \le j \le r$; $B \in \mathbb{R}^{n\times m}$, $\xi_0 \in C$. Let $d(k) = [d_1(k)\ d_2(k)\ \dots\ d_r(k)]^T \in D \subset \{0, 1, \dots, \Delta\}^r$, $k \in \mathbb{N}$, be the delays vector at time k.
Let $G(D, E(D))$ be the delays digraph associated to the system described by (4.4.1). For $\delta = [\delta_1\ \delta_2\ \dots\ \delta_r]^T \in D$, let us consider $A_\delta \in \mathbb{R}^{n\times(\Delta+1)n}$ defined by means of the following equality, which must hold for all $\phi \in \Omega$:

$$A_\delta\ \mathcal{I}^{-1}(\phi) = A_0\phi(0) + A_1\phi(-\delta_1) + A_2\phi(-\delta_2) + \cdots + A_r\phi(-\delta_r). \tag{4.4.2}$$

Theorem 4.4.1. *Assume that there exist card(D) positive symmetric matrices $Q_\delta \in \mathbb{R}^{n\times n}$, $\delta = [\delta_1\ \delta_2\ \cdots\ \delta_r]^T \in D$, positive reals $\lambda_j \ge 0$, $1 \le j \le \Delta+1$, with $\sum_{j=1}^{\Delta+1} \lambda_j \le 1$, positive reals a, b, with $b < a < 1$, such that, for every $(\delta, \rho) \in E(D)$ and for every $p \in S(\delta)$, the following inequality holds*

$$A_\delta^T Q_\rho A_\delta - G(\delta, p) < 0, \tag{4.4.3}$$

where $G(\delta, p) \in \mathbb{R}^{n(\Delta+1)\times n(\Delta+1)}$ is defined as follows:

$$G(\delta, p) = \operatorname{diag}\Big((1-a)Q_\delta + b\lambda_1 Q_{p(1)},\ b\lambda_2 Q_{p(2)}, \dots, b\lambda_{\Delta+1} Q_{p(\Delta+1)}\Big).$$

Then, the system described by (4.4.1) is exponentially ISS.

Proof. For suitable small $\epsilon > 0$, the following matrix inequality holds:

$$A_\delta^T Q_\rho A_\delta + \epsilon I - G(\delta, p) < 0. \tag{4.4.4}$$

Let us consider the quadratic Lyapunov functions $V(\delta, x) = x^T Q_\delta x$, $x \in \mathbb{R}^n$, $\delta \in D$.
Let $\delta = [\delta_1\ \delta_2\ \cdots \delta_r]^T \in D$, $\phi \in C$, $(\delta, \rho) \in E(D)$, $p \in S(\delta)$.

Using Young's inequality, we obtain:

$$
\begin{aligned}
&V\Big(\rho, A_0\phi(0) + \sum_{i=1}^{r} A_i\phi(-\delta_i) + Bu\Big) - V(\delta, \phi(0)) = \\
&\mathcal{I}^{-1}(\phi)^T A_\delta^T Q_\rho A_\delta \mathcal{I}^{-1}(\phi) - \phi^T(0) Q_\delta \phi(0) + 2\mathcal{I}^{-1}(\phi)^T A_\delta^T Q_\omega Bu + u^T B^T Q_\rho Bu \leq \\
&\mathcal{I}^{-1}(\phi)^T A_\delta^T Q_\rho A_\delta \mathcal{I}^{-1}(\phi) - \phi^T(0) Q_\delta \phi(0) + \epsilon \|\mathcal{I}^{-1}(\phi)\|^2 + \frac{1}{\epsilon}\|A_\delta^T Q_\rho B\|^2 \|u\|^2 \\
&+ \|B^T Q_\rho B\| \|u\|^2 = \\
&\mathcal{I}^{-1}(\phi)^T \left[A_\delta^T Q_\rho A_\delta + \epsilon I\right] \mathcal{I}^{-1}(\phi) - \phi^T(0) Q_\delta \phi(0) + \frac{1}{\epsilon}\|A_\delta^T Q_\rho B\|^2 \|u\|^2 \\
&+ \|B^T Q_\rho B\| \|u\|^2
\end{aligned} \tag{4.4.5}
$$

From (4.4.4) and (4.4.5), it follows that

$$
\begin{aligned}
V\Big(\rho, A_0\phi(0) + \sum_{i=1}^{r} A_i\phi(-\delta_i) + Bu\Big) - V(\delta, \phi(0)) &\leq -aV(\delta, \phi(0)) + b\sum_{j=0}^{\Delta} \lambda_{j+1} V(p(j+1), \phi(-j)) + \gamma(\|u\|) \\
&\leq -aV(\delta, \phi(0)) + b \max_{j=0,1,\ldots,\Delta} V(p(j+1), \phi(-j)) + \gamma(\|u\|),
\end{aligned} \tag{4.4.6}
$$

where γ is the function of class $\mathcal{K}_\infty$ defined, for $s \in \mathbb{R}^+$, as

$$
\gamma(s) = \Big(\frac{1}{\epsilon}\|A_\delta^T Q_\rho B\|^2 + \|B^T Q_\rho B\|\Big) s^2.
$$

Condition (*i*) in Corollary 4.3.1 is satisfied with $m = 2$. Condition (*ii*) in Corollary 4.3.1 is satisfied by means of (4.4.6). Therefore, by Corollary 4.3.1 it follows that the system described by (4.4.1) is exponentially ISS. □

4.5 The linear case with arbitrary delays

In this section, we consider the case of linear system (4.4.1) with $D = \{0, 1, \ldots, \Delta\}^r$, Δ positive integer, and arbitrary delay signal $d : \mathbb{N} \to D$, i.e., $E(D) = \{(\delta, \rho),\ \delta \in D,\ \rho \in D\}$. In what follows we recall some results in [PP17] needed for proving our next result.

Claim 4.5.1. ([PP17]) Let M, p and q be the positive reals with $M \geq 1$ and $0 < p < 1$. There exists a functional $V : C \to \mathbb{R}^+$ such that, for any $\phi, \psi \in C$ and for any $d \in D$, the following inequalities hold:

i) $a_1\|\phi\|_\infty \leq V(\phi) \leq a_2\|\phi\|_\infty$

ii) $V(L(\phi, d, 0)) - V(\phi) \leq -a_3\|\phi\|_\infty$

iii) $|V(\phi) - V(\psi)| \leq a_4 |\phi - \psi|_\infty$

where

$$a_1 = 1,\ a_2 = \frac{M}{1-p},\ a_3 = 1 - Mp^{T+1}$$
$$a_4 = \sum_{k=0}^{T} (1+q)^k,\ T = \Big[-\frac{\log_n M}{\log_n p} \Big] + 1.$$

Proof. Let, for $\phi \in C$ and $d \in M_D$, $\xi_k(\phi, d, 0) \in C$, $k \in \mathbb{N}$ denote the solution of

$$\begin{aligned} \xi_{k+1} &= L(\xi_k, d(k), u(k)) \\ \xi_0 &= \psi. \end{aligned} \tag{4.5.1}$$

with $u \equiv 0$, initial state $\psi = \phi$ and delay signal d.
Let $V : C \to \mathbb{R}^+$ defined, for $\phi \in C$, as follows:

$$V(\phi) = \sum_{k=0}^{T} \sup_{d \in M_{S_r}} |\xi_k(\phi, d, 0).$$

Then, the following inequality holds:

$$\|\phi\|_\infty \leq V(\phi) \leq \Big(\sum_{k=0}^{T} Mp^k \Big) \|\phi\|_\infty \leq \frac{M}{1-p} \|\phi\|_\infty.$$

So the inequality (*i*) is proved.
For any $\phi \in C$ and $d \in D$, we have:

$$\begin{aligned} V(L(\phi, d, 0)) - V(\phi) &\leq \sum_{k=0}^{T} \sup_{\tilde{d} \in M_{S_r}} \|\xi_k(L(\phi, d, 0), \tilde{d}, 0)\|_\infty - \sum_{k=0}^{T} \sup_{\tilde{d} \in M_{S_r}} \|\xi_k(\phi, \tilde{d}, 0)\|_\infty \\ &\leq \sum_{k=0}^{T} \sup_{\tilde{d} \in M_{S_r}} \|\xi_{k+1}(\phi, \tilde{d}, 0)\|_\infty - \sum_{k=0}^{T} \sup_{\tilde{d} \in M_{S_r}} \|\xi_k(\phi, \tilde{d}, 0)\|_\infty \\ &= \sum_{k=0}^{T+1} \sup_{\tilde{d} \in M_{S_r}} \|\xi_k(\phi, \tilde{d}, 0)\|_\infty - \sum_{k=0}^{T} \sup_{\tilde{d} \in M_{S_r}} \|\xi_k(\phi, \tilde{d}, 0)\|_\infty \\ &= \sup_{\tilde{d} \in M_{S_r}} \|\xi_{T+1}(\phi, \tilde{d}, 0)\|_\infty - \|\phi\|_\infty \leq Mp^{T+1} \|\phi\|_\infty - \|\phi\|_\infty. \end{aligned}$$

Then, the inequality in (*ii*) is proved.
Next, we prove by induction on k that for any $\phi_i \in C$, $i = 1, 2$, and ant $d \in M_{S_r}$, the following

inequality holds, for $k \geq 0$:

$$\|\xi_k(\phi_1, d, 0) - \xi_k(\phi_2, d, 0)\|_\infty \leq (1+q)^k \|\phi_1 - \phi_2\|_\infty. \tag{4.5.2}$$

For $k = 0$, the inequality (4.5.2) is verified. For $k \geq 0$, we have:

$$\begin{aligned}
\|\xi_{k+1}(\phi_1, d, 0) - \xi_{k+1}(\phi_2, d, 0)\|_\infty &= \|L(\xi_{k+1}(\phi_1, d, 0), d(k), 0) - L(\xi_{k+1}(\phi_2, d, 0), d(k), 0)\|_\infty \\
&\leq \|\xi_k(\phi_1, d, 0) - \xi_k(\phi_2, d, 0)\|_\infty \\
&+ |l(\xi_k(\phi_1, d, 0), d(k), 0) - l(\xi_k(\phi_2, d, 0), d(k), 0)| \\
&\leq (1+q)\|\xi_k(\phi_1, d, 0) - \xi_k(\phi_2, d, 0)\|_\infty \leq (1+q)\|\phi_1 - \phi_2\|_\infty.
\end{aligned}$$

Thus, the inequality (4.5.2) holds for $k+1$. Using inequality (4.5.2), for $\phi, \psi \in C$, we have:

$$\begin{aligned}
|V(\phi) - V(\psi)| &= |\sum_{k=0}^{T} \sup_{\tilde{d} \in M_{S_r}} \|\xi_k(\phi, d, 0)\|_\infty - \sum_{k=0}^{T} \sup_{\tilde{d} \in M_{S_r}} \|\xi_k(\psi, d, 0)\|_\infty| \\
&\leq \sum_{k=0}^{T} | \sup_{\tilde{d} \in M_{S_r}} \|\xi_k(\phi, d, 0)\|_\infty - \sup_{\tilde{d} \in M_{S_r}} \|\xi_k(\psi, d, 0)\|_\infty| \\
&\leq \sum_{k=0}^{T} \sup_{\tilde{d} \in M_{S_r}} \|\xi_k(\phi, d, 0) - \xi_k(\psi, d, 0)\|_\infty \\
&\leq \sum_{k=0}^{T} (1+q)^k \|\phi - \psi\|_\infty.
\end{aligned}$$

Therefore, the inequality (*iii*) is proved. □

We obtain the following result, which links the 0-GES property to the ISS property for the system described by (4.4.1).

Theorem 4.5.1. *Let system (4.4.1), with arbitrary delay signal $d : \mathbb{N} \to D$, be 0-GES. Then, system (4.4.1) is ISS.*

Proof. Let us rewrite system (4.4.1) in the form (4.5.1):

$$\begin{aligned}
x_{k+1} &= L(x_k, d(k), u(k)), \ k \geq 0 \\
x_0 &= \xi_0
\end{aligned}$$

where $L : C \times D \times \mathbb{R}^m \to C$ is the map defined, for $\phi \in C$, $d = [d_1 \ d_2 \ \cdots \ d_r]^T \in D$, $u \in \mathbb{R}^m$, as follows

$$L(\phi, d, u)(\theta) = \begin{cases} A_0\phi(0) + \sum_{i=1}^{r} A_i\phi(-d_i) + Bu, \ \theta = 0 \\ \phi(\theta+1), \ \theta = -\Delta, -\Delta+1, \ldots, -1. \end{cases} \tag{4.5.3}$$

Then, there exist a functional $V : C \to \mathbb{R}^+$ and positive reals $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that, for any $\phi, \psi \in C$ and for any $d \in D$, the following inequalities hold , see Claim 4.5.1:

$$\begin{aligned} &i)\ \alpha_1\|\phi\|_\infty \leq V(\phi) \leq \alpha_2\|\phi\|_\infty; \\ &ii)\ V(L(\phi, d, 0)) - V(\phi) \leq -\alpha_3\|\phi\|_\infty; \\ &iii)\ |V(\phi) - V(\psi)| \leq \alpha_4\|\phi - \psi\|_\infty. \end{aligned} \tag{4.5.4}$$

Now, taking into account (4.5.4), for any $\phi \in C$, $d \in D$, $u \in \mathbb{R}^m$, we have:

$$\begin{aligned} V(L(\phi, d, u)) - V(\phi) &= V(L(\phi, d, u)) - V(L(\phi, d, 0)) + V(L(\phi, d, 0)) - V(\phi) \\ &\leq -\alpha_3\|\phi\|_\infty + |V(L(\phi, d, u)) - V(L(\phi, d, 0))| \\ &\leq -\alpha_3\|\phi\|_\infty + \alpha_4|L(\phi, d, u) - L(\phi, d, 0)| \\ &\leq -\alpha_3\|\phi\|_\infty + \alpha_4\|B\|\|u\|. \end{aligned} \tag{4.5.5}$$

By (i) in (4.5.4) and (4.5.5), the ISS property follows from [PP17, Theorem 3]. □

4.6 Applications

In this section, we prove the ISS property for some examples studied in Chapter 3.

4.6.1 Scalar nonlinear discrete-time delay system

Consider the following nonlinear discrete-time delay system:

$$\begin{aligned} x(k+1) &= \frac{1}{2}\tanh(x(k)) - \frac{1}{2}\text{sat}(x(k - d(k))) + u(k) \\ x(\tau) &= \xi_0(\tau), \ \tau \in \{-1, 0\}. \end{aligned} \tag{4.6.1}$$

Let $D = \{0, 1\}$. This nonlinear example is not trivial to be studied for both the global asymptotic stability and the ISS (provided that these properties hold), by Lyapunov methodologies available in the literature, when the delay signal can be arbitrary (see [LH09], [PP17]). Indeed it is very challenging to find a Lyapunov function satisfying the conditions in [LH09, Theorem 3.3, Theorem 3.6, Theorem 4.1] and [Pep19, Theorem 3], when the delay signal has no constraints.

We show here how Corollary 4.3.1 can be helpful in the case the delay signal has to obey to suitable constraints. Let us consider the delays digraph setting:

1. $G(D, E(D))$ with $D = \{0, 1\}$ and $E(D) = \{(0, 0), (0, 1), (1, 0)\}$

2. the sets $S(\delta)$, $\delta \in D$ are:

$$S(0) = \{(0, 0), (0, 1)\} \quad S(1) = \{(1, 0)\}$$

By $G(D, E(D))$, whenever $d(k) = 1$, for $k \in \mathbb{N}$, it must be $d(k + 1) = 0$.
For $(\delta, \rho) = (0, 0)$ we have, for suitable positive reals a, b as in Corollary 4.3.1,

$$\begin{aligned} V(0, f(\phi(0), \phi(0), u)) - V(0, \phi(0)) &\le r_0 \left| \frac{1}{2}\tanh(\phi(0)) - \frac{1}{2}\mathrm{sat}\phi(0)) + u \right| - r_0|\phi(0)| \\ &\le \frac{1}{2} r_0 \tanh(1)|\phi(0)| - r_0|\phi(0)| + r_0|u| \\ &\le -ar_0|\phi(0)| + br_0|\phi(0)| + r_0|u|, \end{aligned}$$

provided that the following conditions hold: $0 < a < 1$, $a > b$, $b \ge \frac{1}{2}\tanh(1)$.
For $(\delta, \rho) = (0, 1)$, we have

$$\begin{aligned} V(1, f(\phi(0), \phi(0), u)) - V(0, \phi(0)) &\le r_1 \left| \frac{1}{2}\tanh(\phi(0)) - \frac{1}{2}\mathrm{sat}(\phi(0)) + u \right| - r_0|\phi(0)| \\ &\le \frac{1}{2} r_1 \tanh(1)|\phi(0)| - r_0|\phi(0)| + r_1|u| \\ &\le -ar_0|\phi(0)| + br_0|\phi(0)| + r_1|u|, \end{aligned}$$

provided that, according to Corollary 4.3.1, the following conditions hold: $0 < a < 1$, $a > b$, $br_0 \ge \frac{1}{2} r_1 \tanh(1)$.
For $(\delta, \rho) = (1, 0)$, we have:

$$\begin{aligned} V(0, f(\phi(0), \phi(-1), u)) - V(1, \phi(0)) &\le r_0 \left| \frac{1}{2}\tanh(\phi(0)) - \frac{1}{2}\mathrm{sat}(\phi(-1)) + u \right| - r_1|\phi(0)| \\ &\le \frac{1}{2} r_0|\phi(0)| + \frac{1}{2} r_0|\phi(-1)| - r_1|\phi(0)| + r_0|u| \\ &\le -ar_1|\phi(0)| + \lambda_1 br_1|\phi(0)| + \lambda_2 br_0|\phi(-1)| + r_0|u| \end{aligned}$$

provided that, according to Corollary 4.3.1, the following conditions hold: $0 < a < 1$, $a > b$, $\lambda_i \ge 0$, $i = 1, 2$, $\lambda_1 + \lambda_2 \le 1$, $\lambda_1 br_1 \ge \frac{1}{2} r_0$, $\lambda_2 b \ge \frac{1}{2}$. The exponentially ISS property of system (4.6.1) follows from Corollary 4.3.1 with $r_0 = 2.2$, $r_1 = 4.6$, $a = 0.9$, $b = 0.8$, $\lambda_1 = 0.3$, $\lambda_2 = 0.7$ and $\sigma(s) = r_1 s$, $s \in \mathbb{R}^+$.

4.6.2 Scalar nonlinear discrete-time delay system with arbitrary time-varying delays

Consider the following nonlinear discrete-time delay system with arbitrary and bounded time-varying delays:

$$\begin{aligned} x(k+1) &= \tanh(x(k)) + \tanh\circ\tanh(x(k-d(k))) - \tanh(x(k-d(k))) + u(k) \\ x(\tau) &= \xi_0(\tau),\ \ \tau \in \{-\Delta, -\Delta+1, \dots, 0\}. \end{aligned} \tag{4.6.2}$$

Let $D = \{0, 1, \dots, \Delta\}$ and $E(D) = \{(i, j) : i, j \in D\}$, and $V(d, x) = |x|$, $d \in D$, $x \in \mathbb{R}$.
For every $(\delta, \rho) \in E(D)$, every $p \in S(\delta)$, every $\phi \in C$, we have:

$$\begin{aligned} V(\omega, f(d, \phi, u)) - V(\delta, \phi(0)) &\leq \tanh(|\phi(0)|) - |\phi(0)| + \tanh(|\phi(-\delta))| - \tanh\circ\tanh(|\phi(-\delta)|) + |u| \\ &\leq -\alpha(|\phi(0)|) + \beta\Big(\inf_{p\in S(\delta)} \max_{j=0,1,\dots,\Delta} V(p(j+1), \phi(-j))\Big) + \sigma(\|u\|), \end{aligned}$$

where, for $s \in \mathbb{R}^+$, $\alpha(s) = s - \tanh(s)$, $\beta = \tanh(s) - \tanh\circ\tanh(s)$ and $\sigma(s) = s$. The conditions in Theorem 4.3.2 are satisfied and thus the system described by (4.6.2) is ISS. The reader can appreciate the simplicity of the application of Theorem 4.3.2 , based on Theorem 4.3.1, in order to prove ISS of this example. No linear Halanay-type inequality available in the literature can be successful for proving the ISS result of this example with the chosen function V. If one would try to prove the ISS result of this example, by a linear Halanay-type inequality, then, for any involved function $V(d, x)$, $d \in D$, $x \in \mathbb{R}^n$, the inequalities (i) in Theorem 4.3.2 must be satisfied as well as the following limit must hold:

$$\lim_{x\to 0^+} \frac{V(d, \tanh(x))}{V(d, x)} < 1. \tag{4.6.3}$$

Finding a function V with these properties, provided it exists, is at least challenging. No function V of polynomial type of any order exists such that inequalities (i) in Theorem 4.3.2 and (4.6.3) are satisfied. So it is quite impossible to prove the ISS result of this example by means of Theorem 2.2.5. The same problem arises with Razumikhin methodologies for exponential ISS in [LH09, Theorems 3.3, 3.6]. This example validates the main advantage of the nonlinear Halanay-type inequality here provided, that is the simplicity of application, which is the main reason of the popularity for any Halanay-type inequality in the literature. The price to pay is in general more conservativeness with respect to other methodologies, which could require cumbersome computations (see for instance [LH09, Theorem 3.5] for the Razumikhin methodology, [PP17, Theorem 3], [Pep19, Theorem 2] for the Krasovskii one). In particular, Theorem 2 in [Pep19] involves a Lyapunov function defined in Ω, instead of the simple function defined in $\mathbb{R}$ here used. Though it is proved in Theorem 2 in [Pep19] that a successful function defined in $D \times C$ exists, finding it is not as easy as finding a function defined in $D \times \mathbb{R}$ and applying our methodology.

Results in [MSS18] do not apply to this example because of involved nonlinearities.

Chapter 5

Sufficient Lyapunov Conditions for Exponential Mean Square Stability of Discrete-Time Systems with Markovian Delays

Note 5.0.1. This chapter entails the results from the paper titled *Sufficient Lyapunov conditions for exponential mean square stability of discrete-time systems with markovian delays* by Anastasia Impicciatore (PhD student at University of L'Aquila), Maria Teresa Grifa (the PhD candidate), Alessandro D'Innocenzo (professor at University of L'Aquila), Pierdomenico Pepe (the thesis advisor), available on arXiv.org. To read the full reference disclaimer, please refer to [0.1].

Chapter Description

This chapter aims to investigate the stability analysis of nonlinear discrete-time delay systems affected by markovian delays. The mean square stability of discrete-time markovian switching systems has been extensively analyzed in the linear case [OC06],[YALB19], only few works presented in the literature investigate this stability notion in the nonlinear framework (see [PPB14],[ATG10]). Constraints provided by bounded delay variations are studied in [DBT18], [CdSC18], [JZL16]. In [DBT18], the regulation problem for discrete-time linear systems with bounded unknown random state delay is presented. In [JZL16], the problem of disturbance rejection control for markovian jump linear systems is investigated. Our purpose is to provide a Lyapunov characterization of exponential mean square stability and prove an extension of Lyapunov conditions presented for discrete-time systems with delay digraph [Pep19] to exponential mean square stability of discrete-time systems with markovian delays. We provide a transformation of the discrete-time system with markovian delays into a discrete-time Markov jump system, linking the methodologies available for Markov jump systems and discrete-time systems with constrained delays. The relationship between discrete-time systems with delays and switching delay-free systems is provided in [LHI08], [Pep18].

In Section 5.1, the system plant under study is defined. In Section 5.2, some preliminaries results mandatory for the proof contained in the core of the chapter are reported. In Section 5.3, the main result of chapter is provided. In Section 5.4, an example is illustrated.

5.1 Problem formulation

In this section, we illustrate the steps for building the system plant. We write the discrete-time delay system as a switching discrete-time system where the delays are constrained to adhere to a Markov chain. Then, we transform the switching system with markovian delays to a Markov jump system.

Let us consider the discrete-time delay system of the form (see [PP17]):

$$\begin{aligned} x(k+1) &= f(x(k), x(k-d_1(k)), \dots, x(k-d_r(k))), \\ x(\theta) &= \xi_0(\theta), \ \theta \in \{-\Delta, -\Delta+1, \dots, 0\}, \end{aligned} \tag{5.1.1}$$

where: $k \in \mathbb{N}$; Δ is a known positive integer, the maximum involved time delay; $x(j) \in \mathbb{R}^n$, $j \geq -\Delta$; for $1 \leq i \leq r$, $d_i(k) \in \{0, 1, \dots, \Delta\}$ is a time-varying time delay, r is a known positive integer; the function $f : \mathbb{R}^{n(r+1)} \to \mathbb{R}^n$ satisfies the equality $f(0, 0, \dots, 0) = 0$; $\xi_0 \in \mathcal{C}$.
Let $d(k) = [d_1(k) \ d_2(k) \ \dots \ d_r(k)]^T$, $k \in \mathbb{N}$, denote the vector collecting all time delays at time k.

Let $D \subset \{0, 1, \ldots, \Delta\}^r$ be the set of allowed values for the time-delays vector $d(k)$. That is, for any $k \in \mathbb{N}$, $d(k) \in D$. Using (1.3.3), the system (5.1.1) can be rewritten using the following equation:

$$\begin{aligned} x_{k+1} &= F(x_k, d(k)), \quad k \in \mathbb{N}, \\ x_0 &= \xi_0, \quad \xi_0 \in C, \end{aligned} \tag{5.1.2}$$

$x_k \in C$, $x_k(\theta) = x(k+\theta)$, $\theta \in \{-\Delta, -\Delta+1, \ldots, 0\}$, $k \in \mathbb{N}$.
The map $F : C \times D \to C$ is defined as follows, for $\phi \in C$, $d = [d_1\ d_2\ \ldots\ d_r]^T \in D$,

$$F(\phi, d)(\theta) = \begin{cases} f(\phi(0), \phi(-d_1), \ldots, \phi(-d_r)), & \theta = 0, \\ \phi(\theta+1), & \theta = -\Delta, -\Delta+1, \ldots, -1. \end{cases} \tag{5.1.3}$$

Let us define the Markov chain as the sequence $\eta : \mathbb{N} \to \mathcal{S}$, with $\mathcal{S} \triangleq \{1, 2, \ldots, s\}$, $s = card(D)$. The transition probability matrix of the MC is defined as, for all $i, j \in \mathcal{S}$:

$$P \triangleq \left[p_{ij}\right]_{i,j \in \mathcal{S}}, \quad p_{ij} \triangleq \mathbb{P}(\eta(k+1) = j | \eta(k) = i), \tag{5.1.4}$$

and

$$\sum_{j \in \mathcal{S}} p_{ij} = 1,\ \forall i \in \mathcal{S}, \quad 0 \leq p_{ij} \leq 1, \quad \forall i, j \in \mathcal{S} \tag{5.1.5}$$

Assumption 5.1.1. Assume that the delay $d(k+1)$, $k \in \mathbb{N}$, depends only on the delay at the previous step $d(k)$, $k \in \mathbb{N}$, and assume that our prior knowledge on the transition from $d(k)$ to $d(k+1)$ is given by a transition probability.

Let $H : D \to \mathcal{S}$ be a bijective function defined for all $\delta_i \in D$, and for all $i \in \mathcal{S}$ as

$$H(\delta_i) \triangleq i. \tag{5.1.6}$$

The inverse function of H is $H^{-1} : \mathcal{S} \to D$, defined for all $i \in \mathcal{S}$ and for all $\delta_i \in D$, as follows

$$H^{-1}(i) \triangleq \delta_i. \tag{5.1.7}$$

Consider p_{ij} defined in (5.1.4). By applying the definition of p_{ij} in (5.1.4) and the definition of the functions H and H^{-1} in (5.1.6) and (5.1.7) respectively, the following equalities hold:

$$\begin{aligned} p_{ij} &= \mathbb{P}\left(\eta(k+1) = j | \eta(k) = i\right) \\ &= \mathbb{P}\Big(H(d(k+1)) = H(\delta_j) | H(d(k)) = H(\delta_i)\Big) \\ &= \mathbb{P}\left(d(k+1) = \delta_j | d(k) = \delta_i\right), \end{aligned} \tag{5.1.8}$$

for all $\delta_i, \delta_j \in D$, for all $i, j \in \mathcal{S}$.
Consequently, the modes of the MC $\{\eta(k)\}_{k\in\mathbb{N}}$ with TPM defined by (5.1.4)-(5.1.5) are associated to the delays in the set D, through the function H^{-1}.
Let $\mathcal{E}(D)$ be the finite set of all pairs $(\delta_i, \delta_j) \in D \times D$, $i, j \in \mathcal{S}$ such that, for any $k \in \mathbb{N}$, if $d(k) = \delta_i$, it is allowed $d(k+1) = \delta_j$. We define the set $\mathcal{E}(D)$ as follows

$$\mathcal{E}(D) \triangleq \{(\delta_i, \delta_j) \in D \times D,\ \delta_i, \delta_j \in D,\ i, j \in \mathcal{S} \,|\, p_{ij} > 0\}. \tag{5.1.9}$$

We give the definition of Markov jump system adopted in [OC06], [YALB19].

Definition 5.1.1. Let Σ denote the Markov jump system defined on the stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_k, \mathbb{P})$ as follows

$$\Sigma \triangleq (\mathcal{D}, P, H)\,, \tag{5.1.10}$$

where $\mathcal{D}$ is the system described by (5.1.1) and rewritten in the form (5.1.2), P is a known TPM defined by (5.1.4)-(5.1.5) modeling the stochastic switching of the delays, and H is the bijective function defined by (5.1.6).

From (5.1.2), the Markov jump system Σ in (5.1.10) can be written as follows:

$$\begin{aligned} x_{k+1} &= F(x_k, H^{-1}(\eta(k))), \quad k \in \mathbb{N}, \\ x_0 &= \xi_0, \quad \xi_0 \in \mathcal{C}, \end{aligned} \tag{5.1.11}$$

where $x_k \in \mathcal{C}$, $x_k(\theta) = x(k+\theta)$, $\theta \in \{-\Delta, -\Delta+1, \ldots, 0\}$, $k \in \mathbb{N}$; $\eta(k) \in \mathcal{S}$, $k \in \mathbb{N}$ is a MC with TPM P defined by (5.1.4-5.1.5). The map F is defined by (5.1.3) and it can be rewritten as follows for $\phi \in \mathcal{C}$, and for $i \in \mathcal{S}$:

$$F(\phi, H^{-1}(i))(\theta) = \begin{cases} f(\phi(0), \phi(-H_1^{-1}(i)), \ldots, \phi(-H_r^{-1}(i))), & \theta = 0, \\ \phi(\theta+1), & \theta = -\Delta, -\Delta+1, \ldots, -1, \end{cases} \tag{5.1.12}$$

with

$$H^{-1}(i) = \left[H_1^{-1}(i) \ldots H_r^{-1}(i)\right]^T \in D,\ \forall i \in \mathcal{S}. \tag{5.1.13}$$

Let $x_k(\xi_0)$, $k \in \mathbb{N}$, denote the trajectory that evolves according to (5.1.11), corresponding to initial state $\xi_0 \in \mathcal{C}$. Recall that $x_k(\xi_0)(0) = x(k, \xi_0)$, $k \in \mathbb{N}$. The variable $x(k, \xi_0) \in \mathbb{R}^n$, $\xi_0 \in \mathcal{C}$, $k \in \mathbb{N}$, is a random variable on the stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_k, \mathbb{P})$, since the delay evolves according to a discrete-time MC, with given transition probabilities. Thus, we are interested in the behavior of the second moment of $x(k, \xi_0)$, $k \in \mathbb{N}$, $\xi_0 \in \mathcal{C}$.

5.2 Stability definitions

We focus our analysis on stability notions concerning the behavior of the second moment of the state of the system. We consider the exponential mean square stability: a stability notion involving the second moment of the state [OC06].

Definition 5.2.1. The Markov jump system Σ is EMSS if there exist $M, \zeta \in \mathbb{R}^+$ with $M \geq 1$ and $0 < \zeta < 1$, such that for any $\xi_0 \in C$, the following inequality holds for any $k \in \mathbb{N}$,

$$\mathbb{E}[\|x(k,\xi_0)\|^2] \leq M\zeta^k\Big(\|\xi_0\|_\infty\Big)^2. \tag{5.2.1}$$

5.3 Preliminary results

In the following, we introduce some technical results that are mandatory for proving the main result. We would like to underline the fact that the following results are heavily inspired by Lemma 1.3.1 concerning the case with delay-independent functions and with arbitrary time-delay signals in a given bounded set and Lemma 2 in [Pep19] concerning the case with delay-dependent functions and with a given delays digraph.

Lemma 5.3.1. Let there exist a function $V : C \times D \to \mathbb{R}^+$, real positive numbers α_i, $i = 1,2,3$, such that, for all $\phi \in C$, for all $i \in \mathcal{S}$, the following inequalities hold:

a_1) $\alpha_1\|\phi(0)\|^2 \leq V(\phi, H^{-1}(i)) \leq \alpha_2\|\phi\|_\infty^2$,

a_2) $\mathcal{L}V(\phi, H^{-1}(i)) \leq -\alpha_3\|\phi(0)\|^2$,

with the operator $\mathcal{L}V$ defined in (5.4.1). Then, there exist a function $W : C \times D \to \mathbb{R}^+$, real positive numbers β_i, $i = 1,2,3$, such that, for all $\phi \in C$, for all $i \in \mathcal{S}$, the following inequalities hold:

b_1) $\beta_1\|\phi(0)\|^2 \leq W(\phi, H^{-1}(i)) \leq \beta_2\|\phi\|_\infty^2$,

b_2) $\sum_{j\in\mathcal{S}} p_{ij} W\Big(F(\phi, H^{-1}(i)), H^{-1}(j)\Big) - W(\phi, H^{-1}(i)) \leq -\beta_3\|\phi\|_\infty^2$,

where H^{-1} is defined in (5.1.7).

Proof. Let us consider the function $W : C \times D \to \mathbb{R}^+$ defined, for $\phi \in C$, $i \in \mathcal{S}$, as

$$W(\phi, H^{-1}(i)) = V(\phi, H^{-1}(i)) + \max_{\theta=1,2,\ldots,\Delta} e^{-\theta}\alpha_3\|\phi(-\theta)\|^2. \tag{5.3.1}$$

Then, from (a_1), we have

$$\beta_1\|\phi(0)\|^2 \leq W(\phi, H^{-1}(i)) \leq \beta_2\|\phi\|_\infty^2, \tag{5.3.2}$$

with $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2 + \alpha_3$.
From (a_2), we obtain:

$$\begin{aligned}
&\sum_{j\in\mathcal{S}} p_{ij} W\Big(F(\phi, H^{-1}(i)), H^{-1}(j)\Big) - W(\phi, H^{-1}(i)) = \\
&\sum_{j\in\mathcal{S}} p_{ij} V\Big(F(\phi, H^{-1}(i)), H^{-1}(j)\Big) - V(\phi, H^{-1}(i)) + \max_{\theta=1,2,\ldots,\Delta} e^{-\theta}\alpha_3\|\phi(-\theta+1)\|^2 - \max_{\theta=1,2,\ldots,\Delta} e^{-\theta}\alpha_3\|\phi(-\theta)\|^2 \\
&\leq -\alpha_3\|\phi(0)\|^2 + e^{-1}\max_{\theta=1,2,\ldots,\Delta} e^{1-\theta}\alpha_3\|\phi(-\theta+1)\|^2 - \max_{\theta=1,2,\ldots,\Delta} e^{-\theta}\alpha_3\|\phi(-\theta)\|^2 \\
&\leq -\alpha_3\|\phi(0)\|^2 + e^{-1}\max_{\theta=0,1,\ldots,\Delta-1} e^{-\theta}\alpha_3\|\phi(-\theta)\|^2 - \max_{\theta=1,2,\ldots,\Delta} e^{-\theta}\alpha_3\|\phi(-\theta)\|^2 \\
&\leq -\alpha_3\|\phi(0)\|^2 + e^{-1}\alpha_3\|\phi(0)\|^2 + e^{-1}\max_{\theta=1,2,\ldots,\Delta} e^{-\theta}\alpha_3\|\phi(-\theta)\|^2 - \max_{\theta=1,2,\ldots,\Delta} e^{-\theta}\alpha_3\|\phi(-\theta)\|^2 \\
&\leq -(1-e^{-1})\alpha_3\|\phi(0)\|^2 - (1-e^{-1})\alpha_3 e^{-\Delta}\max_{\theta=1,2,\ldots,\Delta}\|\phi(-\theta)\|^2 \\
&\leq -(1-e^{-1})\alpha_3 e^{-\Delta}\|\phi\|_\infty^2.
\end{aligned}$$

Define $\beta_3 \triangleq (1-e^{-1})\alpha_3 e^{-\Delta}$. Then, the function W satisfies (b_1), (b_2). This completes the proof. □

Lemma 5.3.2. Assume that there exist a function $V : C \times D \to \mathbb{R}^+$, real positive numbers γ_i, $i = 1, 2, 3$ such that, for all $\phi \in C$, for all $i \in \mathcal{S}$, the following inequalities hold:

$c_1)$ $\gamma_1\|\phi(0)\|^2 \leq V(\phi, H^{-1}(i)) \leq \gamma_2\|\phi\|_\infty^2$,

$c_2)$ $\mathcal{L}V(\phi, H^{-1}(i)) \leq -\gamma_3\|\phi\|_\infty^2$,

where H^{-1} is defined in (5.1.7), and the operator $\mathcal{L}V$ is defined in (5.4.1).
Then, the system Σ is EMSS.

Proof. From (c_2), for all $x_k \in C$, for all $\eta(k) \in \mathcal{S}$, $k \in \mathbb{N}$, we have:

$$\begin{aligned}
\mathcal{L}V(x_k, H^{-1}(\eta(k))) &= \sum_{\eta(k+1)\in\mathcal{S}} p_{\eta(k)\eta(k+1)} V(x_{k+1}, H^{-1}(\eta(k+1))) - V(x_k, H^{-1}(\eta(k))) \\
&\leq -\gamma_3\|x_k\|_\infty^2.
\end{aligned} \tag{5.3.3}$$

By the Markov property, from (5.3.3), we obtain:

$$\mathbb{E}\Big[\Big(V(x_{k+1}, H^{-1}(\eta(k+1))) - V(x_k, H^{-1}(\eta(k)))\Big)|\mathcal{F}_k\Big] \leq -\gamma_3\|x_k\|_\infty^2. \tag{5.3.4}$$

From (5.3.4), applying the property of the expected value conditioned to a filtration, the following inequality holds:

$$\mathbb{E}\Big[V(x_{k+1}, H^{-1}(\eta(k+1))) - V(x_k, H^{-1}(\eta(k)))\Big] \leq -\gamma_3 \mathbb{E}[\|x_k\|_\infty^2]. \tag{5.3.5}$$

Using the linearity of the expected value, from (5.3.5) we obtain:

$$\mathbb{E}[V(x_{k+1}, H^{-1}(\eta(k+1)))] - \mathbb{E}[V(x_k, H^{-1}(\eta(k)))] \leq -\gamma_3 \mathbb{E}[\|x_k\|_\infty^2]. \tag{5.3.6}$$

From (c_1), it follows that

$$\mathbb{E}[\|x_k\|_\infty^2] \geq \frac{1}{\gamma_2} \mathbb{E}[V(x_k, H^{-1}(\eta(k))]. \tag{5.3.7}$$

Using (5.3.6) and (5.3.7), we have

$$\mathbb{E}[V(x_{k+1}, H^{-1}(\eta(k+1)))] - \mathbb{E}[V(x_k, H^{-1}(\eta(k)))] \leq -\frac{\gamma_3}{\gamma_2} \mathbb{E}[V(x_k, H^{-1}(\eta(k)))]. \tag{5.3.8}$$

Let $\gamma_4 \triangleq \dfrac{\gamma_3}{\gamma_2}$, notice that $\gamma_4 > 0$, since γ_3, $\gamma_2 > 0$. Without loss of generality, pick $\gamma_4 < 1$. From (5.3.8), it follows that

$$\mathbb{E}[V(x_{k+1}, H^{-1}(\eta(k+1)))] \leq (1-\gamma_4)\mathbb{E}[V(x_k, H^{-1}(\eta(k)))]. \tag{5.3.9}$$

Using recursive argument, from (5.3.9), we have

$$\mathbb{E}[V(x_k, H^{-1}(\eta(k)))] \leq (1-\gamma_4)^k \mathbb{E}[V(\xi_0, H^{-1}(\eta(0)))]. \tag{5.3.10}$$

From (c_1), for $k \in \mathbb{N}$,

$$\begin{aligned} &\gamma_1 \mathbb{E}[\|x(k)\|^2] \leq \mathbb{E}[V(x_k, H^{-1}(\eta(k)))], \\ &(1-\gamma_4)^k \mathbb{E}[V(\xi_0, H^{-1}(\eta(0)))] \leq (1-\gamma_4)^k \gamma_2 \mathbb{E}[\|\xi_0\|_\infty^2]. \end{aligned} \tag{5.3.11}$$

From (5.3.10) and (5.3.11), it follows that

$$\gamma_1 \mathbb{E}[\|x(k)\|^2] \leq (1-\gamma_4)^k \gamma_2 \mathbb{E}[\|\xi_0\|_\infty^2]. \tag{5.3.12}$$

From (5.3.12), the following inequality holds

$$\mathbb{E}[\|x(k)\|^2] \leq (1-\gamma_4)^k \frac{\gamma_2}{\gamma_1} \mathbb{E}[\|\xi_0\|_\infty^2] \ \leq (1-\gamma_4)^k \frac{\gamma_2}{\gamma_1} \|\xi_0\|_\infty^2. \tag{5.3.13}$$

By defining $M \triangleq \frac{\gamma_2}{\gamma_1} \geq 1$ and $\zeta \triangleq (1-\gamma_4)$, with $0 < \zeta < 1$, from (5.3.13),

$$\mathbb{E}[\|x(k)\|^2] \leq M\zeta^k(\|\xi_0\|_\infty)^2. \tag{5.3.14}$$

Thus, the system Σ is EMSS. □

5.4 Main Results

In this section, we define a methodology which makes use of multiple Lyapunov functions depending on the mode of the Markov chain, that governs the switching delay.
Then, we provide sufficient Lyapunov conditions guaranteeing the EMSS of system Σ.

Let us consider a scalar function $V : C \times D \to \mathbb{R}^+$. We associate to V the operator $\mathcal{L}V : C \times D \to \mathbb{R}$, defined for $\phi \in C$, $i \in \mathcal{S}$,

$$\mathcal{L}V(\phi, H^{-1}(i)) \triangleq \sum_{j\in\mathcal{S}} p_{ij} V\Big(F(\phi, H^{-1}(i)), H^{-1}(j)\Big) - V(\phi, H^{-1}(i)), \tag{5.4.1}$$

with H^{-1} defined in (5.1.7).

Theorem 5.4.1. *Assume there exist a function $V : C \times D \to \mathbb{R}^+$ and real positive numbers α_i, $i = 1, 2, 3$, such that, for all $\phi \in C$, for all $i \in \mathcal{S}$, the following inequalities hold:*

i) $\alpha_1\|\phi(0)\|^2 \leq V(\phi, H^{-1}(i)) \leq \alpha_2\|\phi\|^2_\infty$,

ii) $\mathcal{L}V(\phi, H^{-1}(i)) \leq -\alpha_3\|\phi(0)\|^2$,

where H^{-1} is defined in (5.1.7). Then, the system Σ is EMSS.

Proof. From (i)-(ii), by Lemma 5.3.1, it follows that there exist a function $V : C \times D \to \mathbb{R}^+$, $\gamma_i \in \mathbb{R}^+$, $i = 1, 2, 3$ such that, for all $\phi \in C$, for all $i \in \mathcal{S}$, the following inequalities hold:

c_1) $\gamma_1\|\phi(0)\|^2 \leq V(\phi, H^{-1}(i)) \leq \gamma_2\|\phi\|^2_\infty$,

c_2) $\mathcal{L}V(\phi, H^{-1}(i)) \leq -\gamma_3\|\phi\|^2_\infty$,

with $\mathcal{L}V$ defined in (5.4.1). From Lemma 5.3.2, we obtain that the system Σ is EMSS. □

Remark 5.4.1. Notice that the classical representation of system Σ using a Markov jump system allows to use the conditions in Theorem (1.5.1) and in [AIP20]. An important feature of the involved Lyapunov inequalities, shared with the cases of delay-dependent and delay-independent Lyapunov functions (see [PP17], [Pep19]), is that lower bound of Lyapunov functions, as well as of the related difference operators, are given in a weaker form with respect to the Lyapunov conditions for Markov jump systems (see for instance [PPB14], [AIP20]). Indeed, the lower bound of condition (*i*) in Theorem 5.4.1 and the inequality in condition (*ii*) in Theorem 5.4.1 do not involve $\phi \in \mathcal{C}$, but only $\phi(0) \in \mathbb{R}^n$. Indeed, condition (*ii*) of Theorem 5.4.1 do not involve $\phi \in \mathcal{C}$, but only $\phi(0) \in \mathbb{R}^n$.

5.5 Applications

In this section, we illustrate our methodology using the following scalar nonlinear system [Pep19]:

$$\begin{aligned} x(k+1) &= \mathrm{sat}(x(k)) - \gamma \mathrm{sat}(x(k-d(k))), \\ x(\tau) &= \xi_0(\tau),\ \tau \in \{-2,-1,0\}, \end{aligned} \tag{5.5.1}$$

with $\xi_0 \in \mathcal{C}$, $x(k) \in \mathbb{R}$, $\gamma \in [1, 1.2]$. We transform system (5.5.1) to a system defined in the space of initial conditions using equation (5.1.2):

$$F(\phi,d)(\theta) = \begin{cases} \mathrm{sat}(\phi(0)) - \gamma \mathrm{sat}(\phi(-d)), & \theta = 0, \\ \phi(\theta+1), & \theta = -2,-1. \end{cases} \tag{5.5.2}$$

Consider the MC $\{\eta_k\}_{k\in\mathbb{N}}$ in Figure 2, with set of states $\mathcal{S} = \{1,2\}$. We associate each delay in D to a mode of the Markov chain in $\mathcal{S}$. Let the bijective function $H : D \to \mathcal{S}$ be defined as

$$H(d) = \begin{cases} 1 & \text{if } d = 0, \\ 2 & \text{if } d = 2. \end{cases} \tag{5.5.3}$$

The function $H^{-1} : \mathcal{S} \to D$ is thus defined as

$$H^{-1}(i) = \begin{cases} 0 & \text{if } i = 1, \\ 2 & \text{if } i = 2. \end{cases} \tag{5.5.4}$$

The TPM associated with the MC $\{\eta_k\}_{k\in\mathbb{N}}$ is given by

$$P = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}, \quad p,q \in (0,1). \tag{5.5.5}$$

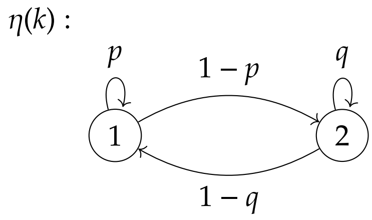


Figure 2: The Figure depicts the state diagram of the Markov chain $\eta(k)$ modeling the switching delay in the presented example: p stands for the probability of having a delay $d(k+1) = 0$ provided that the previous delay is $d(k) = 0$, while q stands for the probability of having a delay $d(k+1) = 2$, provided that the previous delay is $d(k) = 2$.

Hence, we obtain a Markov jump system Σ where $\mathcal{D}$ is (5.5.1), P is defined in (5.5.5) and H is defined in (5.5.3). We analyze the Markov jump system resulting by the application of our methodology. As mentioned before, we obtain a switching system with two modes: one is stable and the other one is unstable. Notice that, as consequence of the structure of P in (5.5.5), the set $\mathcal{E}(D)$ is given by

$$\mathcal{E}(D) = \{(0,0),(0,2),(2,0),(2,2)\}. \tag{5.5.6}$$

We aim to study the EMSS property of system (5.5.1) with switches delays governed by the MC $\{\eta(k)\}_{k\in\mathbb{N}}$, with TPM P defined in (5.5.5), and with the function H defined in (5.5.4).
In the following, we want to verify whether conditions (i) and (ii) of Theorem 5.4.1 are satisfied. We consider a candidate Lyapunov function $V : \mathcal{C} \times D \to \mathbb{R}^+$ defined, for $\phi \in \mathcal{C}$, $i \in \mathcal{S}$, as

$$V(\phi, H^{-1}(i)) = \lambda_i \sup_{j=0,1,2} 2^{j-1}\gamma^j c^{-j} \|\phi(-j)\|^2, \tag{5.5.7}$$

with $\lambda_i \in \mathbb{R}^+$, $i \in \mathcal{S}$, $\gamma \in [1, 1.2]$ and $1 < c \leq e$.
Pick

$$\alpha_1 = \min_{i\in\mathcal{S}} \lambda_i \quad \alpha_2 = 2\gamma^2 \max_{i\in\mathcal{S}} \lambda_i$$

Thus, condition (i) of Theorem 5.4.1 is satisfied.
In order to verify condition (ii), we consider the expression of $\mathcal{L}V(\phi, H^{-1}(i))$, for all $\phi \in \mathcal{C}$, $i \in \mathcal{S}$. When we consider the expression of $\mathcal{L}V(\phi, H^{-1}(1))$, from (5.4.1) we obtain the following equality:

$$\mathcal{L}V(\phi, H^{-1}(1)) = pV\left(F(\phi, H^{-1}(1)), H^{-1}(1)\right) + (1-p)V\left(F(\phi, H^{-1}(1)), H^{-1}(2)\right) - V(\phi, H^{-1}(1)). \tag{5.5.8}$$

From (5.5.7) and (5.5.8), we obtain the following equality:

$$\mathcal{L}V(\phi, H^{-1}(1)) = (p\lambda_1 + (1-p)\lambda_2) \sup_{j=0,1,2} 2^{j-1}\gamma^j c^{-j}\|F(\phi, H^{-1}(1))(-j)\|^2 - \lambda_1 \sup_{j=0,1,2} 2^{j-1}\gamma^j c^{-j}\|\phi(-j)\|^2. \tag{5.5.9}$$

By (5.5.4), we have:

$$\begin{aligned}\mathcal{L}V(\phi, H^{-1}(1)) \leq{}& (p\lambda_1 + (1-p)\lambda_2)2^{-1}\|(1-\gamma)\mathrm{sat}(\phi(0))\|^2 + (p\lambda_1 + (1-p)\lambda_2)\sup_{j=1,2} 2^{j-1}\gamma^j c^{-j}\|\phi(-j+1)\|^2 \\ & - \lambda_1 \sup_{j=0,1,2} 2^{j-1}\gamma^j c^{-j}\|\phi(-j)\|^2. \end{aligned} \tag{5.5.10}$$

By the properties of the supremum, the following inequality holds

$$\sup_{j=1,2} 2^{j-1}\gamma^j c^{-j}\|\phi(-j+1)\|^2 \leq \sup_{j=1,2,3} 2^{j-1}\gamma^j c^{-j}\|\phi(-j+1)\|^2. \tag{5.5.11}$$

By changing the index variable in the supremum, we can write

$$\begin{aligned}\sup_{j=1,2,3} 2^{j-1}\gamma^j c^{-j}\|\phi(-j+1)\|^2 &= 2\gamma c^{-1} \sup_{j=1,2,3} 2^{(j-1)-1}\gamma^{(j-1)} c^{-(j-1)}\|\phi(-j+1)\|^2 \\ &= 2\gamma c^{-1} \sup_{\theta=0,1,2} 2^{\theta-1}\gamma^{\theta} c^{-\theta}\|\phi(-\theta)\|^2. \end{aligned} \tag{5.5.12}$$

Thus, from (5.5.10), (5.5.11), (5.5.12), we obtain the following inequalities

$$\begin{aligned}\mathcal{L}V(\phi, H^{-1}(1)) \leq{}& (p\lambda_1 + (1-p)\lambda_2)\Big(2^{-1}(1-\gamma)^2\|\phi(0)\|^2 + +2\gamma c^{-1} \sup_{\theta=0,1,2} 2^{\theta-1}\gamma^{\theta} c^{-\theta}\|\phi(-\theta)\|^2\Big) + \\ & - \lambda_1 \sup_{\theta=0,1,2} 2^{\theta-1}\gamma^{\theta} c^{-\theta}\|\phi(-\theta)\|^2 \\ \leq{}& (p\lambda_1 + (1-p)\lambda_2)\left((1-\gamma)^2 + 2\gamma c^{-1}\right) \times \sup_{\theta=0,1,2} 2^{\theta-1}\gamma^{\theta} c^{-\theta}\|\phi(-\theta)\|^2 \\ & - \lambda_1 \sup_{\theta=0,1,2} 2^{\theta-1}\gamma^{\theta} c^{-\theta}\|\phi(-\theta)\|^2. \end{aligned} \tag{5.5.13}$$

By defining ω_1 as follows,

$$\omega_1 \triangleq \lambda_1 \left[1 - \left(p + (1-p)\frac{\lambda_2}{\lambda_1}\right)\left((1-\gamma)^2 + 2\gamma c^{-1}\right)\right], \tag{5.5.14}$$

we get

$$\mathcal{L}V(\phi, H^{-1}(1)) \leq -\omega_1 \sup_{\theta=0,1,2} 2^{\theta-1}\gamma^{\theta} c^{-\theta}\|\phi(-\theta)\|^2. \tag{5.5.15}$$

When we consider the expression of $\mathcal{L}V(\phi, H^{-1}(2))$, we obtain:

$$\mathcal{L}V(\phi, H^{-1}(2)) = (1-q)V(F(\phi, H^{-1}(2)), H^{-1}(1)) + qV(F(\phi, H^{-1}(2)), H^{-1}(2)) - V(\phi, H^{-1}(2)). \tag{5.5.16}$$

From (5.5.16), and (5.5.7), the following equality holds:

$$\mathcal{L}V(\phi, H^{-1}(2)) == ((1-q)\lambda_1 + q\lambda_2) \sup_{j=0,1,2} 2^{j-1}\gamma^j c^{-j} \|F(\phi, H^{-1}(2))(-j)\|^2 - \lambda_2 \sup_{j=0,1,2} 2^{j-1}\gamma^j c^{-j} \|\phi(-j)\|^2. \tag{5.5.17}$$

From (5.5.17), applying the properties of the supremum, it follows that

$$\begin{aligned} \mathcal{L}V(\phi, H^{-1}(2)) \leq ((1-q)\lambda_1 + q\lambda_2) \Big(2^{-1} \|F(\phi, H^{-1}(2))(0)\|^2 + \sup_{j=1,2} 2^{j-1}\gamma^j c^{-j} \|\phi(-j+1)\|^2 \Big) \\ - \lambda_2 \sup_{j=0,1,2} 2^{j-1}\gamma^j c^{-j} \|\phi(-j)\|^2. \end{aligned} \tag{5.5.18}$$

From (5.5.18), we have

$$\begin{aligned} \mathcal{L}V(\phi, H^{-1}(2)) \leq ((1-q)\lambda_1 + q\lambda_2) \Big(2^{-1} \|\text{sat}(\phi(0)) - \gamma \text{sat}(\phi(-2))\|^2 \\ + 2\gamma c^{-1} \sup_{j=1,2} 2^{(j-1)-1}\gamma^{j-1} c^{-j+1} \|\phi(-j+1)\|^2 \Big) \\ - \lambda_2 \sup_{\theta=0,1,2} 2^{\theta-1}\gamma^\theta c^{-\theta} \|\phi(-\theta)\|^2. \end{aligned} \tag{5.5.19}$$

From (5.5.19), by applying the properties of the Euclidean norm, Young's inequality and the properties of the function *sat* the following inequalities hold

$$\begin{aligned} \mathcal{L}V(\phi, H^{-1}(2)) &\leq ((1-q)\lambda_1 + q\lambda_2) \Big(\|\phi(0)\|^2 + \gamma^2 \|\phi(-2)\|^2 \\ &+ 2\gamma c^{-1} \sup_{\theta=0,1,2} 2^{\theta-1}\gamma^\theta c^{-\theta} \|\phi(-\theta)\|^2 \Big) \\ &- \lambda_2 \sup_{\theta=0,1,2} 2^{\theta-1}\gamma^\theta c^{-\theta} \|\phi(-\theta)\|^2 \\ &\leq ((1-q)\lambda_1 + q\lambda_2) \Big(\big(2 + 2^{-1}c^2 + 2\gamma c^{-1} \big) \\ &\times \sup_{\theta=0,1,2} 2^{\theta-1}\gamma^\theta c^{-\theta} \|\phi(-\theta)\|^2 \Big) \\ &- \lambda_2 \sup_{\theta=0,1,2} 2^{\theta-1}\gamma^\theta c^{-\theta} \|\phi(-\theta)\|^2. \end{aligned} \tag{5.5.20}$$

From (5.5.20), by defining ω_2 as follows,

$$\omega_2 \triangleq \lambda_2 \left[1 - \left(q + (1-q)\frac{\lambda_1}{\lambda_2} \right) \left(2 + 2^{-1}c^2 + 2\gamma c^{-1} \right) \right], \tag{5.5.21}$$

we obtain

$$\mathcal{L}V(\phi, H^{-1}(2)) \leq -\omega_2 \sup_{\theta=0,1,2} 2^{\theta-1}\gamma^{\theta}c^{-\theta}\|\phi(-\theta)\|^2. \tag{5.5.22}$$

Under the following constraints

$$L_B < \frac{\lambda_2}{\lambda_1} < U_B, \tag{5.5.23}$$

with

$$U_B = \frac{1-((1-\gamma)^2+2\gamma c^{-1})p}{((1-\gamma)^2+2\gamma c^{-1})(1-p)}, \tag{5.5.24}$$

$$L_B = \frac{\left(4+e^2+4\gamma c^{-1}\right)(1-q)}{2-\left(4+c^2+4\gamma c^{-1}\right)q}, \tag{5.5.25}$$

$$(p,q) \in (0,1)\times(0,1),\ q < \frac{2}{4+c^2+4\gamma c^{-1}}; \tag{5.5.26}$$

we obtain that ω_1, $\omega_2 \in \mathbb{R}^+$. Thus, from (5.5.15) and (5.5.22) the following inequality holds, for all $\phi \in \mathcal{C}$, for all $i \in \mathcal{S}$,

$$\mathcal{L}V\left(\phi, H^{-1}(i)\right) \leq -\alpha_3\|\phi(0)\|^2, \tag{5.5.27}$$

with $\alpha_3 \in \mathbb{R}^+$, defined as

$$\alpha_3 \triangleq \frac{1}{2}\min\{\omega_1, \omega_2\}. \tag{5.5.28}$$

Thus, condition (*ii*) of Theorem 5.4.1 is satisfied and the system (5.5.1) with switches delays governed by the MC $\{\eta(k)\}_{k\in\mathbb{N}}$, with TPM P defined in (5.5.5), and with the function H defined in (5.5.4) is EMSS.

Remark 5.5.1. Notice that, the conditions for the exponential mean square stability of Markov jump systems in Theorem 1.5.1 (see also [AIP20]) cannot be applied in the example (5.5.1) on the extended state in $\mathbb{R}^{n(\Delta+1)}$.

5.5.1 Statistical results

In Figure 3, we present Montecarlo simulations of the trajectories generated by the system (5.5.1), considering values of the pairs (p, q) such that conditions (*i*) and (*ii*) of Theorem 5.4.1 are satisfied.

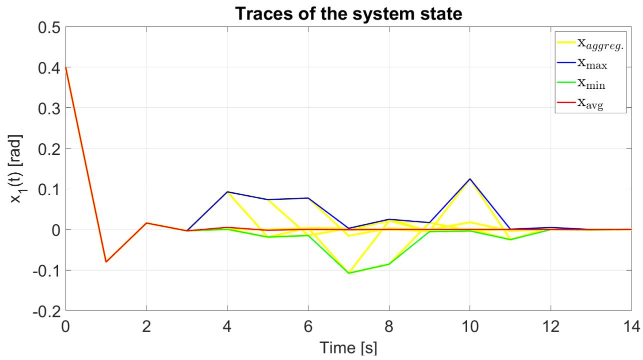
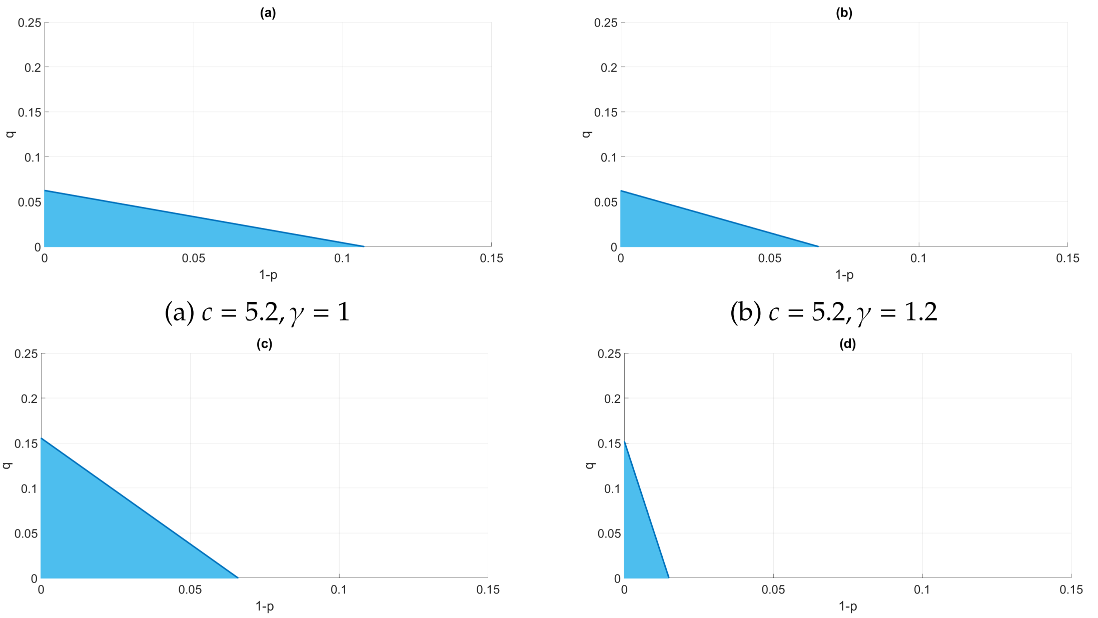


Figure 3: Traces of system state obtained with $\gamma = 1.2$, $p = 0.95$, and $q = 0.01$.

The yellow trajectories correspond to the state trajectories associated with different switching paths (that are admissible according to P), the maximum and the minimum trajectory are plotted in blue and green, respectively. Finally, the red line corresponds to the average evolution of the state trajectories. From Figure 3, we observe that trajectories decrease exponentially and converge to zero. This result reflects the analysis presented in this section. In Figures 4, for different values of c in the candidate Lyapunov function (5.5.7), we show the regions of pairs (p, q) such that conditions $(i) - (ii)$ of Theorem 5.4.1 are satisfied (light blue region) and the evolution of the maximum q with respect to $(1 - p)$ such that the conditions $(i) - (ii)$ of Theorem 5.4.1 are satisfied (dark blue line). From Figures 4, by comparing row-wise, we observe that when c spans from e to 5.2, the segment on $1 - p$ shrinks while segment on q expands. From Figures 4, by comparing column-wise, when γ goes from 1 to 1.2 the segment on $1 - p$ shrinks, the aforementioned light blue region becomes smaller and smaller when the parameter γ increases. Choosing $c > e$, the conditions (5.5.25) lead to a widen set of values for $1 - p$, while, by condition (5.5.26), the values of q are restricted.



(a) $c = 5.2, \gamma = 1$

(b) $c = 5.2, \gamma = 1.2$

(c) $c = e, \gamma = 1$

(d) $c = e, \gamma = 1.2$

Figure 4: The figure shows the regions of pairs $(p, q) \in (0, 1) \times (0, 1)$ such that conditions of Theorem 5.4.1 are satisfied .

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Some Other Academic Works

1. *Electric Energy Price Forecasting: Descriptive Analysis and Features Selection*

This is a work of the PhD candidate Maria Teresa Grifa, available online on International Journal of Pure and Applied Mathematics, 2017.

2. *MTA-KDD'19: A Dataset for Malware Traffic Detection*

This is a join work of the PhD candidate Maria Teresa Grifa with Prof. Giuseppe Della Penna (University of L'Aquila) and researcher Ivan Letteri (University of L'Aquila), available online on Research Gate, which got 1 citation. This work was presented at Italian Conference of Cybersecurity 2020, Ancona, Italy, February 2020.

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