

Research Article

Unrefinable partitions into distinct parts in a normalizer chain

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Abstract

Unrefinable partitions into distinct parts are those in which no part x can be replaced with integers whose sum is x obtaining a new partition into distinct parts. Such a relationship between the parts does not seem to be much investigated and consequently very little is known on general properties of unrefinable partitions. However, they play a role in the combinatorial nature of a certain chain of subgroups. More precisely, in a recent paper on a study of the Sylow 2-subgroups of the symmetric group with 2^n elements it has been shown that the growth of the first $(n - 2)$ consecutive indices of a certain normalizer chain is linked to the sequence of partitions of integers into distinct parts. We prove here that the $(n - 1)$ -th index of the previously mentioned chain is related to the number of unrefinable partitions into distinct parts satisfying a condition on the minimal excludant.

Keywords: symmetric group on 2^n elements; partitions into distinct parts; unrefinable partitions; minimal excludant; Sylow 2-subgroups.

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1. Introduction

The sequence (b_j) of the number of partitions of integers into distinct parts has been extensively studied in past and recent years and is a well-understood integer sequence [1, 11] appearing in different areas of mathematics. Recently, triggered by a problem in algebraic cryptography related to translation subgroups in the symmetric group with 2^n elements [4, 8–10], it has been shown that such a sequence is related to the growth of the indices of consecutive terms in a chain of normalizers [5]. More precisely, let Σ_n be a Sylow 2-subgroup of the symmetric group $\text{Sym}(2^n)$ and T be an elementary abelian regular subgroup of Σ_n . Defining $N_n^0 = N_{\Sigma_n}(T)$ and recursively $N_n^i = N_{\Sigma_n}(N_n^{i-1})$, the authors proved that the number

$$\log_2 |N_n^i : N_n^{i-1}|$$

is independent of n for $1 \leq i \leq n - 2$, and is equal to the $(i + 2)$ -th term of the sequence (a_j) of the partial sums of (b_j) (cfr. Table 1).

The result is obtained by proving that the terms in the chain are *saturated* subgroups, i.e. generated by *rigid commutators*, a family of left-normed commutators involving a special set of generators of Σ_n . We invite the interested reader to refer to Aragona et al. [5], where rigid commutators and saturated groups are introduced and described in detail.

When $i > n - 2$, the behavior of the chain does not seem to show any recognizable pattern and the study of its combinatorial nature is still open. Nonetheless, further investigations on experimental evidences led us to notice that the first exception to the rule, i.e. $\log_2 |N_n^{n-1} : N_n^{n-2}|$, is linked to the number of partitions of n into distinct parts which do not admit a *refinement*, i.e. those partitions $n = p_1 + p_2 + \dots + p_t$ where no integer p_i can be replaced by $l \geq 2$ distinct integers q_1, q_2, \dots, q_l , whose sum is p_i and such that the resulting partition

$$n = p_1 + p_2 + \dots + p_{i-1} + (q_1 + q_2 + \dots + q_l) + p_{i+1} + \dots + p_t$$

is still a partition into distinct parts. In our knowledge, *unrefinable* partitions have not been investigated so far.

We prove here (cfr. Theorem 3.1) that a transversal of N_n^{n-2} in N_n^{n-1} is made of rigid commutators which are in one-to-one correspondence with unrefinable partitions into distinct parts with a condition on their minimal excludants, where the minimal excludant of a partition is the least positive integer which does not appear in the partition.

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j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
b_j	0	0	0	1	1	2	3	4	5	7	9	11	14	17	21
a_j	0	0	0	1	2	4	7	11	16	23	32	43	57	74	95

Table 1: First values of the sequences (b_j) and (a_j) .

A brief summary on rigid commutators and saturated subgroups is presented in Section 2, which also contains a representation of the Sylow 2-subgroup of the symmetric group and to the precise definition of the normalizer chain under investigation. The proof of Theorem 3.1 and considerations on unrefinable partitions and rigid commutators can be found in Section 3.

2. Preliminaries

Let n be a non-negative integer. Let us define a set of permutations $\{s_i \mid 1 \leq i \leq n\}$ which generates a Sylow 2-subgroup of the symmetric group on 2^n letters.

The Sylow 2-subgroup

Let us consider the set

$$\mathcal{T}_n = \{w_1 \dots w_n \mid w_i \in \{0, 1\}\}$$

of binary words of length n , where \mathcal{T}_0 contains only the empty word. The infinite rooted binary tree \mathcal{T} is defined as the graph whose vertices are $\bigcup_{j \geq 0} \mathcal{T}_j$ and where two vertices, say $w_1 \dots w_n$ and $v_1 \dots v_m$, are connected by an edge if $|m - n| = 1$ and $w_i = v_i$ for $1 \leq i \leq \min(m, n)$. The empty word is the root of the tree and it is connected with both the two words of length 1.

We can define a sequence $\{s_i\}_{i \geq 1}$ of automorphisms of this tree. Each s_i necessarily fixes the root, which is the only vertex of degree 2. The automorphism s_1 changes the value w_1 of the first letter of every non-empty word into $\bar{w}_1 \stackrel{\text{def}}{=} (w_1 + 1) \bmod 2$ and leaves the other letters unchanged. If $i \geq 2$, we define

$$(w_1 \dots w_k)_{s_i} \stackrel{\text{def}}{=} \begin{cases} \text{empty word} & \text{if } n = 0 \\ w_1 \dots \bar{w}_i \dots w_k & \text{if } k \geq i \text{ and } w_1 = \dots = w_{i-1} = 0 \\ w_1 \dots w_k & \text{otherwise.} \end{cases}$$

In general, s_i leaves a word unchanged unless the word has length at least i and the letters preceding the i -th one are all zero, in which case the i -th letter is increased by 1 modulo 2. If $i \leq n$ and the word $w_1 \dots w_n \in \mathcal{T}_n$ is identified with the integer $1 + \sum_{i=1}^n 2^{n-i} w_i \in \{1, \dots, 2^n\}$, then s_i acts on \mathcal{T}_n as the permutation whose cyclic decomposition is

$$\prod_{j=1}^{2^{n-i}} (j, j + 2^{n-i})$$

which has order 2. In particular, the group $\langle s_1, \dots, s_n \rangle$ acts faithfully on the set \mathcal{T}_n , whose cardinality is 2^n , as a Sylow 2-subgroup Σ_n of the symmetric group $\text{Sym}(2^n)$.

Rigid commutators

The *commutator* of two elements h and k in a group G is defined as $[h, k] \stackrel{\text{def}}{=} h^{-1}k^{-1}hk = h^{-1}h^k$. The *left-normed commutator* of the m elements $g_1, \dots, g_m \in G$ is the usual commutator if $m = 2$ and is recursively defined by

$$[g_1, \dots, g_{n-1}, g_n] \stackrel{\text{def}}{=} [[g_1, \dots, g_{n-1}], g_n]$$

if $m \geq 3$. In this paper we will only focus on left-normed commutators in s_1, \dots, s_n , therefore, for the sake of simplicity, we write $[i_1, \dots, i_k]$ to denote the left-normed commutator $[s_{i_1}, \dots, s_{i_k}]$, when $k \geq 2$. We also write $[i]$ to denote the element s_i and we set $[\]$ to be the identity permutation.

Definition 2.1. A left-normed commutator $r = [i_1, \dots, i_k] \in \Sigma_n$ is called *rigid* when we have $i_1 > i_2 > \dots > i_k$ or $r = [\]$. The set of all the rigid commutators of Σ_n is denoted by \mathcal{R} and we let $\mathcal{R}^* \stackrel{\text{def}}{=} \mathcal{R} \setminus \{[\]\}$.

Definition 2.2. A subgroup of Σ_n is said to be saturated if it is generated by rigid commutators.

Let us define a special set $\{t_1, t_2, \dots, t_n\}$ of rigid commutators where

$$t_i \stackrel{\text{def}}{=} [i, i - 1, \dots, 2, 1]. \tag{1}$$

Remark 2.1. The saturated subgroup $T \stackrel{\text{def}}{=} \langle t_1, t_2, \dots, t_n \rangle$ is an elementary abelian regular subgroup of Σ_n .

Theorem 2.1 ([5]). The normalizer N in Σ_n of a saturated subgroup of Σ_n is also saturated, provided that N contains T .

Let $i_1 > i_2 > \dots > i_k$ and let $X = \{1, \dots, i_1\} \setminus \{i_1, \dots, i_k\}$. For the purposes of this work it is more convenient to use the notation

$$\vee[i_1; X]$$

to denote the rigid commutator $[i_1, \dots, i_k]$. The commutator of two rigid commutators is again a rigid commutator which, in the above notation, can be computed in the following way:

Proposition 2.1 ([5]). Let $1 \leq a, b \leq n$ and let I and J be subsets of $\{1, 2, \dots, a - 1\}$ and $\{1, 2, \dots, b - 1\}$ respectively. Then

$$[\vee[a; I], \vee[b; J]] = \begin{cases} \vee[\max(a, b); (I \cup J) \setminus \{\min(a, b)\}] & \text{if } \min(a, b) \in I \cup J \\ 1 & \text{otherwise.} \end{cases}$$

The normalizer chain

The normalizer chain starting at T is defined as

$$N_n^i \stackrel{\text{def}}{=} \begin{cases} N_{\Sigma_n}(T) & \text{if } i = 0, \\ N_{\Sigma_n}(N_n^{i-1}) & \text{if } i \geq 1. \end{cases} \tag{2}$$

Notice that N_n^0 is the Sylow 2-subgroup U_n of the affine group $\text{AGL}(2, n)$. It is worth noticing here that

$$N_{\Sigma_n}(N_n^i) = N_{\text{Sym}(2^n)}(N_n^i),$$

for all $i \geq 0$ [6].

In order to show where partitions of integers come into play, let us briefly describe the generators of the first $n - 2$ normalizers of the chain. First, let us determine the permutations in Σ_n normalizing T : for $1 \leq j < i \leq n$ let us define

$$X_{ij} \stackrel{\text{def}}{=} \{1, \dots, i\} \setminus \{j\} \text{ and } u_{ij} \stackrel{\text{def}}{=} [X_{ij}] = \vee[i; \{j\}] \in \mathcal{R}^*, \tag{3}$$

and let us set

$$\mathcal{U}_n \stackrel{\text{def}}{=} \{t_1, \dots, t_n, u_{ij} \mid 1 \leq j < i \leq n\} \subseteq \mathcal{R}^*.$$

Next, let us define

$$\mathcal{W}_{ij} \stackrel{\text{def}}{=} \left\{ \vee[i; I] \in \mathcal{R}^* \mid I \subseteq \{1, 2, \dots, i - 1\}, |I| \geq 2, \sum_{x \in I} x = j \right\} \tag{4}$$

for each $1 \leq i \leq n$ and j , and

$$\mathcal{N}_n^i \stackrel{\text{def}}{=} \begin{cases} \mathcal{U}_n & \text{if } i = 0 \\ \mathcal{N}_n^{i-1} \cup \left(\bigcup_{j=1}^i \mathcal{W}_{n+j-i, j+2} \right) & \text{for } i > 0. \end{cases} \tag{5}$$

Note that, if $j \leq i - 2$, then $|\mathcal{W}_{i,j}| = b_j$, i.e. it corresponds to the number of partitions of j into at least two distinct parts.

The previous elements generate the subgroups in the normalizer chain:

Theorem 2.2 ([5]). For $i \leq n - 2$, the group $\langle \mathcal{N}_n^i \rangle$ is the i -th term N_n^i of the normalizer chain. In particular, the subgroup N_n^i of Σ_n is generated by U_n and the rigid commutators $\vee[a; X]$ such that

- $|X| \geq 2$,
- $\sum_{x \in X} x \leq i + 2 - (n - a)$.

The following straightforward consequence is derived:

Corollary 2.1 ([5]). For $1 \leq i \leq n - 2$, the number $\log_2 |N_n^i : N_n^{i-1}|$ is independent of n . It equals the $(i + 2)$ -th term of the sequence (a_j) of the partial sums of the sequence (b_j) counting the number of partitions of j into at least two distinct parts.

3. Unrefinable partitions

Every non-empty finite subset $X \subseteq \mathbb{N} \setminus \{0\}$ represents a partition of the integer

$$\sum X \stackrel{\text{def}}{=} \sum_{x \in X} x$$

into *distinct* parts. Some partitions, e.g. $7 = 1 + 2 + 4$, are not refinable, some others are. For example, in $10 = 1 + 4 + 5$, the part 5 can be replaced by $2 + 3$ obtaining a partition of $10 = 1 + 2 + 3 + 4$ into distinct parts. More precisely:

Definition 3.1. A finite non-empty subset $X \subseteq \mathbb{N} \setminus \{0\}$ is refinable if there exists $x \in X$ and a subset $Y \subseteq \mathbb{N} \setminus (X \cup \{0\})$ such that $x = \sum Y$. If so, X induces a refinable partition into distinct parts of $\sum X$. Equivalently, $(X \setminus \{x\}) \dot{\cup} Y$ is again a partition into distinct parts of $\sum X$, called a refinement of X . We say that X is an unrefinable partition of $\sum X$ into distinct parts if X is not refinable.

Remark 3.1. Notice that if X is a refinable partition, then there exists Y as in Definition 3.1, $|Y| = 2$, such that $x = \sum Y$.

Let us now recall the concept of minimal excludant, better known from its use in combinatorial game theory [12–14] and recently considered in the theory of partitions of integers [2, 3, 7].

Definition 3.2. The minimal excludant $\text{mex}(X)$ of a set X of positive integers is the least positive integer that is not an element of X .

Rigid commutators and unrefinable partitions are linked by the following consideration depending on Proposition 2.1. Suppose that $a > 1$ and that $X \subseteq \{1, \dots, a - 1\}$ is refinable. If $x \in X$ and Y are as in Definition 3.1, then

$$[\vee[a; X], \vee[x; Y]] = \vee[a; Z]$$

where $Z \stackrel{\text{def}}{=} (X \setminus \{x\}) \dot{\cup} Y$ is the refinement of X obtained by replacing x with Y . Notice that $\sum Z = \sum X$. Since $\sum Y = x$, by Theorem 2.2 the rigid commutator $\vee[x; Y]$ belongs to N_n^{n-2} for $n \geq x$. Conversely, if

$$[\vee[a; X], \vee[x; Y']] = \vee[a; Z]$$

for some Y' and if $\sum Z = \sum X$, then $x \in X$ and $Z = (X \setminus \{x\}) \dot{\cup} Y$ is a refinement of X , where $Y \stackrel{\text{def}}{=} Y' \setminus X$ is such that $\sum Y = x$.

Bearing this in mind, let us prove the following:

Theorem 3.1. Let $a > 1$, $X \subseteq \{1, 2, \dots, a - 1\}$ and let $k = \text{mex}(X)$ be the minimal excludant of X . The rigid commutator $\vee[a; X]$ belong to $N_n^{n-1} \setminus N_n^{n-2}$ if and only if

1. $\sum X = a + 1$,
2. X is an unrefinable partition,
3. $a \leq n < a + k$.

Proof. Suppose that $c \stackrel{\text{def}}{=} \vee[a; X]$ belongs to $N_n^{n-1} \setminus N_n^{n-2}$ and let $b = \sum X$ and $h = \min(X)$. By Theorem 2.2, since $c \notin N_n^{n-2}$, we have $b \geq a + 1$. Let us define

$$d \stackrel{\text{def}}{=} \begin{cases} u_{h,h-1} & \text{if } h > 1 \\ t_1 & \text{if } h = 1, \end{cases}$$

where $u_{h,h-1}$ and t_1 are respectively defined in Eq. (3) and Eq. (1). From Proposition 2.1, the commutator $[c, d]$ is rigid of the form $\vee[a; Y]$, where $\sum Y = \sum X - 1 = b - 1$. Since $d \in N_n^{n-2}$, we have that $[c, d] \in N_n^{n-2}$, and in particular $\sum Y = b - 1 \leq a$. We already know that $a \leq b - 1$, and so we have the equality $\sum X = b = a + 1$, as claimed in (1).

Suppose now that X is refinable. We have already observed that there exist $x \in X$ and a rigid commutator $\vee[x; Y] \in N_n^{n-2}$, i.e. $\sum Y = x$, such that

$$[c, \vee[x; Y]] = \vee[a; Z],$$

with $\sum Z = \sum X = a + 1$. This implies that $\vee[a; Z] \notin N_n^{n-2}$, and so c does not normalize N_n^{n-2} , in contradiction to the assumption. This proves (2).

Finally, we obviously have $k \leq a$. Let us prove that the equality cannot hold. Suppose indeed that $k = a$, then X is the set $\{1, \dots, a - 1\}$. By (1), we have the quadratic equation

$$a + 1 = \sum X = (a^2 - a)/2$$

which admits no integer solutions. Now, let $l \stackrel{\text{def}}{=} a + k$ and let us prove that $n < l$. Suppose that $n \geq l$. By definition of l and by Theorem 2.2 we have that $d \stackrel{\text{def}}{=} \vee[l; \{a, k\}] \in N_n^{n-2} \setminus N_n^{n-3}$. Since $c \in N_n^{n-1}$, then $N_n^{n-2} \ni [c, d] = \vee[l; X \cup \{k\}]$. This is possible only if $l \geq \sum X + k = a + 1 + k = l + 1$, a contradiction. The claim (3) is then proved.

Conversely, suppose that (1), (2) and (3) hold and let us consider a generic rigid commutator in N_n^{n-2}

$$d \stackrel{\text{def}}{=} \vee[m; V] \in N_n^{n-2}.$$

We have $\sum V \leq m$ and $m \leq n < a + k$. Let us prove first that $[c, d] \in N_n^{n-2}$. To do so we may assume $[c, d] \neq 1$, otherwise there is nothing to prove. Three cases need to be distinguished.

Assume first that $m < a$. Since $[c, d] \neq 1$, then $[c, d] = \vee[a; Z]$, where $Z = (X \setminus \{m\}) \cup V$. Notice that $\sum Z < \sum X = a + 1$, where the equality is excluded since Z would otherwise represent a refinement of X , which is unrefinable by hypothesis. Hence $\sum Z \leq a$, and so $[c, d] \in N_n^{n-2}$, as claimed. The case $m = a$ has already been discussed since it corresponds to $[c, d] = 1$. To conclude, suppose that $a < m \leq n < a + k$. Since $[c, d] \neq 1$, by Proposition 2.1 we have $a \in V$, and consequently

$$\sum(V \setminus \{a\}) \leq m - a < k. \tag{6}$$

By Eq. (6), we have that $V \setminus \{a\}$ contains only elements smaller than k and, since $k = \text{mex}(X)$ is minimum positive integer in the complementary set of X in \mathbb{N} , the set $V \setminus \{a\}$ has to be a subset of X . Thus we have $[c, d] = \vee[m; Z]$, where $Z = X \cup (V \setminus \{a\}) = X$. Hence $\sum Z = \sum X = a + 1 \leq m$ and so $[c, d] \in N_n^{n-2}$.

We proved that $[c, d] \in N_n^{n-2}$ for all $d \in N_n^{n-2} \cap \mathcal{R}^*$. Now, since N_n^{n-2} is a saturated subgroup of Σ_n we have, by Theorem 2.1, that $c \in N_{\Sigma_n}(N_n^{n-2}) = N_n^{n-1}$. Moreover, $c \notin N_n^{n-2}$ since $\sum X = a + 1 > a$. \square

Using the GAP implementation of the algorithmic version of Theorem 2.2, available at GITHUB (<https://github.com/ngunivag/normalizer-chain>), it is easy to compute the first normalizers in the chain of Eq.(2). The first values of $\log_2 |N_n^i : N_n^{i-1}|$ are shown in Table 2. Notice that for $i \leq n - 2$ the blue numbers correspond to those of the sequence (a_j) , whereas black bold numbers represent $\log_2 |N_n^{n-1} : N_n^{n-2}|$ and correspond to the number of unrefinable partitions as in Theorem 3.1. Moreover, notice that the diagonals in the blue area of Table 2 contain the same sequence of numbers. This is not the case

n	$\log_2 N_n^i : N_n^{i-1} $ for $1 \leq i \leq 14$													
3	1	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	2	1	1	0	0	0	0	0	0	0	0	0	0
5	1	2	4	1	2	2	1	1	1	1	0	0	0	0
6	1	2	4	7	2	4	4	1	1	2	2	2	2	1
7	1	2	4	7	11	4	7	3	4	2	2	4	4	4
8	1	2	4	7	11	16	7	5	6	2	6	6	3	3
9	1	2	4	7	11	16	23	4	9	4	11	4	12	9
10	1	2	4	7	11	16	23	32	4	14	5	20	7	19
11	1	2	4	7	11	16	23	32	43	5	22	7	32	4
12	1	2	4	7	11	16	23	32	43	57	7	32	12	43
13	1	2	4	7	11	16	23	32	43	57	74	12	42	18
14	1	2	4	7	11	16	23	32	43	57	74	95	8	24
15	1	2	4	7	11	16	23	32	43	57	74	95	121	8

Table 2: Values of $\log_2 |N_n^i : N_n^{i-1}|$ for small i and n .

when looking at the bold diagonal related to the $(n - 1)$ -th normalizer. The reason for this is explained below.

Remark 3.2. Recall that, if $\sum X \leq a$, then the rigid commutator $c = \vee[a; X]$ belongs to N_n^{n-2} . In particular $c \in N_m^{m-2}$ for all $m \geq n$. A similar property does not hold for rigid commutators in N_n^{n-1} . In this case, we can show that if $c \in N_n^{n-1} \setminus N_n^{n-2}$, then $c \in N_m^{m-1}$ for finitely many m . Indeed, if $a \leq m$, by Theorem 3.1 we have that $c \in N_m^{m-1}$ only if m is in the interval $a \leq m < a + \text{mex}(X)$.

It is natural to ask when $k = \text{mex}(X)$ is the largest possible with respect to a . Since $\sum X = a + 1$, it is clear that this happens exactly when X is a triangular partition, i.e. $X = \{1, \dots, k - 1\}$. We then have that

$$a = \sum X - 1 = k(k - 1)/2 - 1 = (k + 1)(k - 2)/2$$

is the integer preceding the $(k - 1)$ -th triangular number $k(k - 1)/2$. Since $a = (k + 1)(k - 2)/2 > (k - 2)^2/2$ we have that $\sqrt{2a} + 2$ is an upper bound for the largest possible value of k . In this case, if $a \geq 3$, then $c \in N_n^{n-1} \setminus N_n^{n-2}$ implies

$$n < a + k \leq a + \sqrt{2a} + 2 \leq (\sqrt{a} + 1)^2,$$

i.e. $a > n - 1 - 2\sqrt{n}$.

Example 3.1. When $n = 8$ we have computed that $\log_2 |N_8^7 : N_8^6| = 7$. Indeed, applying Theorem 3.1 it can be shown that the normalizer N_8^7 can be generated by the generators of N_8^6 and by the following 7 rigid commutators of the type $\vee[a; X]$

$$\begin{aligned} \vee[5; \{3, 2, 1\}] &= [5, 4], \\ \vee[6; \{4, 2, 1\}] &= [6, 5, 3], \\ \vee[7; \{5, 2, 1\}] &= [7, 6, 4, 3], \\ \vee[7; \{4, 3, 1\}] &= [7, 6, 5, 2], \\ \vee[8; \{6, 2, 1\}] &= [8, 7, 5, 4, 3], \\ \vee[8; \{5, 3, 1\}] &= [8, 7, 6, 4, 2], \\ \vee[8; \{4, 3, 2\}] &= [8, 7, 6, 5, 1]. \end{aligned}$$

Open problem

It is natural to wonder whether a general closed formula for $\log_2 |N_n^{n-1} : N_n^{n-2}|$ may be found, even though at the time of writing this does not seem an easy task. To our knowledge, the problem of determining a closed formula or a generating function for the sequence c_j of the number of unrefinable partitions into distinct parts is open. The first values of c_j are shown here in Table 3, while a list of the first 1000 integers can be found in the On-Line Encyclopedia of Integer Sequences [15, <https://oeis.org/A179009>]. The related problem of determining a closed formula for the number of unrefinable partitions with a given minimal excludant, related to the number of partitions of Theorem 3.1, seems at the moment out of reach.

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
c_j	1	1	1	1	1	2	1	1	2	3	1	2	2	3	5

Table 3: First values of the sequences (c_j) .

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