

Spectral gap for Glauber type dynamics for a special class of potentials.

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Abstract

We consider an equilibrium birth and death type process for a particle system in infinite volume, the latter is described by the space of all locally finite point configurations on $\mathbb{R}^d$. These Glauber type dynamics are Markov processes constructed for pre-given reversible measures. A representation for the “carré du champ” and “second carré du champ” for the associate infinitesimal generators L are calculated in infinite volume and a corresponding coercivity identity is derived. The latter is used to give explicit sufficient conditions for the appearance and bounds for the size of the spectral gap of L . These techniques are applied to Glauber dynamics associated to Gibbs measure and conditions are derived extending all previous known results. In the high temperature regime now potentials also with a non-trivial negative part can be treated. Furthermore, a special class of potentials is defined for which the size of the spectral gap is as least as large as for the free system and, surprisingly, is independent of the activity. This type of potentials should not show any phase transition for a given temperature at any activity.

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1 Introduction

The process studied in this paper is an analogue for continuous systems of the well-known Glauber dynamics for lattice systems. The main focus of the paper is on the spectral properties of the associated infinitesimal generator L . Such kind of dynamics were introduced for the first time by C. Preston in [19, 8] for systems in finite volume, such that for each finite time interval at most a finite number of particles appear in the system. By construction, equilibrium states of classical statistical mechanics, Gibbs measures, are formally reversible measures for such processes. Gibbs measures are perturbations of Poisson point processes, though they are in general inequivalent to all Poisson point processes, highly correlated and do not have necessarily nice decay of correlation properties. Gibbs measures are constructed using a pair potential ϕ and an activity z . In [14], Yu. Kondratiev and E. Lytvynov constructed the Glauber dynamics in infinite volume using Dirichlet-form techniques. In any finite time interval, an infinite number of birth and death events happen, therefore this process cannot be considered as a birth and death process in the classical sense. In infinite volume, the processes exist only in an L^2 -sense with respect to a chosen invariant measure μ . For a more general construction in special cases, see [6, 16, 12, 13].

The infinitesimal generator L associated to these dynamics have a spectral gap for small positive potentials and small activity (high temperature regime). In [4], L. Bertini, N. Cancrini and F. Cesi derived a Poincare inequality in finite volume and a bound on the spectral gap uniform in the volume. They pointed out that typically a log-Sobolev type inequality will not hold, cf. [17] for Poisson processes. In [14] the technique of coercivity identity was used to improve the result and to give a clear estimate for the spectral gap. In [5], A.-S. Boudou, P. Caputo, P. dai Pra and G. Posta derived a general framework for this technique for general jump-type processes and rederived the result for the Glauber dynamics in finite volume. In [15], Yu. Kondratiev, R. Minlos and E. Zhizhina derive the one particle space invariant subspace and estimated the next gap in the spectrum.

In [2], D. Bakry and M. Emery calculated the “second carré du champ” generalizing the Bochner-Lichérowicz-Weitzenböck formula and in this way related the spectral gap of the Laplacian on a manifold with the underlying curvature. Therefore, it seems quite natural to apply these techniques also in the case of Glauber dynamics in the continuum.

In Section 3, we consider, slightly more general, all measure which have an integration by parts formula with respect to the considered difference operator, in other words measures which have a Papangelou kernel. We calculate the “second carré du champ” in infinite volume under very mild assumptions on the invariant measure μ and the associated Papangelou kernel exploiting fundamentally the pointwise nature of the “second carré du champ”. We recover in an equivalent form the coercivity identity given in [14] and exactly the one given in [5], however in infinite volume. This tech-

nique has the advantage to provide a motivation which particular form of the coercivity identity to use, although a geometrical justification could not be given. However, the results presented in this paper may motivate an adequate geometrical structure on configuration spaces. Sufficient criteria for the presence of a spectral gap are derived from the coercivity identity.

In Section 4, we study the case of operators L associated to Gibbs measures in more details. Sufficient conditions for the presence of a spectral gap are derived and bounds on the size of the gap in terms of the potential and the activity are given. We introduce a class of non-trivial potentials for which the spectral gap has at least the size as in the free case and, even more surprisingly, the derived bound on the size of the spectral gap is independent of the activity. The definition of this class is based upon Fourier transform and hence the continuous space structure of the system is essential. Even more surprisingly, there are potentials with non-trivial negative part in this class. Furthermore, we show that an increase in the temperature will not alter these estimates as well. For positive potentials from this class, this result improves essentially the bound given in [14].

Finally, we derive a bound for potentials which are the sum of a potential from the aforementioned special class and a usual regular and stable potential in a generalized high temperature regime. The size of the spectral gap is estimated in terms of the density and not of the activity, which is more satisfying from the viewpoint of physics. This result gives, in particular, an improved estimate on the size of the spectral gap even if one just considers a generic stable and regular potentials alone. Till now only non-negative potentials could be treated and even for general positive potentials the previous results are improved.

Precisely speaking we do not derive a spectral gap but a coercivity inequality on cylinder functions. If L is essentially self-adjoint on this domain, as proven for positive potentials in [14], then the coercivity identity is equivalent to spectral gap. Essential self-adjointness for non-positive potentials will be subject of future investigations.

Assuming essential self-adjointness, we found a class of potentials with a very interesting thermodynamical property. These potentials have a non-trivial attractive part, nevertheless there will be no phase transition of any kind for all values of the activity z .

2 States and dynamics

2.1 Configuration space

The configuration space $\Gamma := \Gamma_{\mathbb{R}^d}$ over $\mathbb{R}^d$ is defined as the set of all Radon measures with values in $\mathbb{N} \cup \{0, \infty\}$, i.e. for any $\gamma \in \Gamma$ there exists a sequence $(x_i)_{i \in I}$ of vectors from $\mathbb{R}^d$ and an index set $I \subset \mathbb{N}$ such that $\gamma = \sum_{i \in I} \varepsilon_{x_i}$. Conversely, any sequence without accumulation points can be associated to a configuration by the above formula.

Modulo renumeration there is only one sequence representing γ . The space Γ is Polish in the relative topology as a subset of the space of all Radon measures $\mathcal{M}(\mathbb{R}^d)$ endowed with the vague topology, i.e. the topology generated by the mappings

$$\gamma \mapsto \langle f, \gamma \rangle := \int_{\mathbb{R}^d} f(x) \gamma(dx) \quad \mathcal{C}_0(\mathbb{R}^d),$$

where $\mathcal{C}_0(\mathbb{R}^d)$ denotes the set of all continuous functions on $\mathbb{R}^d$ with compact support. The corresponding Borel σ -algebra on Γ is denoted by $\mathcal{B}(\Gamma)$. A probability measure on $(\Gamma, \mathcal{B}(\Gamma))$ is called a point process (random field). A measurable function $r : \mathbb{R}^d \times \Gamma \rightarrow [0, \infty]$ is the Papangelou intensity of a point process μ if

$$\int_{\Gamma} \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx) F(x, \gamma) = \int_{\Gamma} \mu(d\gamma) \int_{\mathbb{R}^d} dx \cdot r(x, \gamma) F(x, \gamma + \delta_x) \quad (2.1)$$

for any measurable function $F : \mathbb{R}^d \times \Gamma \rightarrow [0, +\infty]$. Let us fix a point process μ which has Papangelou intensity r and for which the first correlation function exists. The first n correlation functions exist exactly iff μ has all local moments up to degree n , that is, for all bounded measurable subsets $\Lambda \subset \mathbb{R}^d$ the following integral $\int_{\Gamma} \gamma(\Lambda)^n \mu(d\gamma)$ is finite.

2.2 Glauber dynamics

In this subsection we introduce the Glauber dynamics, a birth and death type dynamics in the continuum via Dirichlet form techniques, for details cf. [14]. For this purpose we first introduce the set $\mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$ of all functions of the form

$$\Gamma \ni \gamma \mapsto F(\gamma) = g_F(\langle \varphi_1, \gamma \rangle, \dots, \langle \varphi_N, \gamma \rangle),$$

where $N \in \mathbb{N}$, $\varphi_1, \dots, \varphi_N \in \mathcal{C}_0(\mathbb{R}^d)$ and $g_F \in \mathcal{C}_b(\mathbb{R}^N)$. Here $\mathcal{C}_b(\mathbb{R}^N)$ denotes the set of all continuous bounded functions on $\mathbb{R}^N$. The dynamics is constructed using two types of difference operators which are in some sense adjoint to each other: for $F : \Gamma \rightarrow \mathbb{R}$, $\gamma \in \Gamma$, and $x, y \in \mathbb{R}^d$

$$(D_x^- F)(\gamma) := F(\gamma - \delta_x) - F(\gamma), \quad (D_x^+ F)(\gamma) := F(\gamma) - F(\gamma + \delta_x). \quad (2.2)$$

As we want to consider the dynamics only in an L^2 -framework, we use the following bilinear form, cf. [14]

$$\mathcal{E}(F, G) := \int_{\Gamma} \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx) (D_x^- F)(\gamma) (D_x^- G)(\gamma), \quad F, G \in \mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d), \Gamma), \quad (2.3)$$

The following properties of the $\mathcal{E}$, which are useful for our considerations, were proved in [14].

Using the associated integration by parts formula for a measure μ with a Pangelou intensity r and first local moments in [14] it was proven that the bilinear form $(\mathcal{E}, \mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d), \Gamma))$ is closable on $L^2(\Gamma, \mu)$ and its closure is a Dirichlet form also denoted by $(\mathcal{E}, D(\mathcal{E}))$. The generator $(L, D(L))$ associated to $(\mathcal{E}, D(\mathcal{E}))$, i.e. $\mathcal{E}(F, G) = (-LF, G)_{L^2(\Gamma, \mu)}$ is for functions $F \in \mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d), \Gamma) \subset D(L)$ given by

$$(LF)(\gamma) = \int_{\mathbb{R}^d} \gamma(dx) (D_x^- F)(\gamma) - \int_{\mathbb{R}^d} r(x, \gamma) (D_x^+ F)(\gamma) dx \quad \mu\text{-a.e.} \quad (2.4)$$

Following the usual techniques for Dirichlet forms, in [14], for the case, that μ is a Gibbs measure, the associated conservative Hunt process was constructed, that is,

$$\mathbf{M} = (\boldsymbol{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t \geq 0}, (\boldsymbol{\Theta}_t)_{t \geq 0}, (\mathbf{X}(t))_{t \geq 0}, (\mathbf{P}_\gamma)_{\gamma \in \Gamma})$$

on Γ (see e.g. [18, p. 92]) which is properly associated with $(\mathcal{E}, D(\mathcal{E}))$, i.e., for all $(\mu\text{-versions of}) F \in L^2(\Gamma, \mu)$ and all $t > 0$ the function

$$\Gamma \ni \gamma \mapsto p_t F(\gamma) := \int_{\boldsymbol{\Omega}} F(\mathbf{X}(t)) d\mathbf{P}_\gamma$$

is an $\mathcal{E}$ -quasi-continuous version of $\exp(tL)F$. $\boldsymbol{\Omega}$ is the set of all *cadlag* functions $[0, \infty[\rightarrow \Gamma$. The processes $\mathbf{M}$ is up to μ -equivalence unique (cf. [18, Chap. IV, Sect. 6]). In particular, $\mathbf{M}$ is μ -symmetric (i.e., $\int G p_t F d\mu = \int F p_t G d\mu$ for all $F, G : \Gamma \rightarrow \mathbb{R}_+$, $\mathcal{B}(\Gamma)$ -measurable), and thus has μ as an invariant measure.

3 Coercivity identity for Glauber dynamics

3.1 Carré du champ

In this subsection we compute two quadratic forms associated to L , the generator of Glauber dynamics given by (2.4), the so-called “carré du champ”, the “second carré du champ” and hence an analogue of the Bochner-Lichnérowicz-Weitzenböck formula in this context, cf. e.g. [1]. As this is essentially an algebraic calculation most details are omitted and we give just the main steps of these computation, which should allow the interested reader to easily reconstruct the missing details.

In this subsection we essentially need only the following assumption on $r : \mathbb{R}^d \times \Gamma \rightarrow [0, \infty]$: There exists a subset $\Gamma_{\text{temp}} \subset \Gamma$ such that

1. $r(x, \gamma) < \infty$ for all $(x, \gamma) \in \mathbb{R}^d \times \Gamma_{\text{temp}}$
2. for all $\gamma \in \Gamma_{\text{temp}}$, the function $x \mapsto r(x, \gamma)$ is locally integrable
3. for all $\gamma \in \Gamma_{\text{temp}}$ and all $x \in \gamma$ and $y \in \mathbb{R}^d$ also $\gamma - \delta_x$ and $\gamma + \delta_y$ are in Γ_{temp} .

For $F, G \in \mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$ we define the “carré du champ” corresponding to L as

$$\square(F, G) := \frac{1}{2}(L(FG) - FLG - GLF). \quad (3.1)$$

Let us split the generator L into its death and birth part

$$L^- F(\gamma) := \sum_{x \in \gamma} D_x^- F(\gamma), \quad L^+ F(\gamma) := \int_{\mathbb{R}^d} r(x, \gamma) D_x^+ F(\gamma) dx, \quad (3.2)$$

such that $L = L^- - L^+$. Due to linearity one obtains that $\square(F, G) = \square^-(F, G) + \square^+(F, G)$, where $\square^-$ and $-\square^+$ are the “carré du champ” corresponding to the death and birth parts

$$\square^-(F, G) := \frac{1}{2} \int_{\mathbb{R}^d} \gamma(dx) D_x^- F(\gamma) D_x^- G(\gamma), \quad \square^+(F, G) := \frac{1}{2} \int_{\mathbb{R}^d} r(x, \gamma) D_x^+ F(\gamma) D_x^+ G(\gamma) dx.$$

Iterating in some sense the definition of “carré du champ” one may introduce the so-called $\square_2$, cf. [1], as follows

$$2\square_2(F, F) := L\square(F, F) - 2\square(F, LF). \quad (3.3)$$

The splitting in birth and death part allows us to split $\square_2$ correspondingly in the following way:

$$\begin{aligned} 2\square_2(F, F) &= (L^-\square^-(F, F) - 2\square^-(F, L^-F)) \\ &\quad - (L^+\square^+(F, F) - 2\square^+(F, L^+F)) \\ &\quad + (L^-\square^+(F, F) - 2\square^+(F, L^-F)) \\ &\quad - (L^+\square^-(F, F) - 2\square^-(F, L^+F)) \end{aligned} \quad (3.4)$$

All brackets will be calculated separately using the following product rules type formulas

Lemma 3.1 *If $H : \mathbb{R}^d \times \Gamma_{\text{temp}} \rightarrow \mathbb{R}$ is locally bounded and for fixed $\gamma \in \Gamma_{\text{temp}}$ the function $x \mapsto H_x(\gamma)$ has compact support, then*

$$D_x^+ \sum_{y \in \gamma} H_y(\gamma) = \sum_{y \in \gamma} D_x^+ H_y(\gamma) - H_x(\gamma + \delta_x) \quad (3.5)$$

$$D_x^- \sum_{y \in \gamma} H_y(\gamma) = \sum_{y \in \gamma - \delta_x} D_x^- H_y(\gamma) - H_x(\gamma) \quad (3.6)$$

$$D_x^+ \left(\int_{\mathbb{R}^d} r(y, \gamma) H_y(\gamma) dy \right) = \int_{\mathbb{R}^d} r(y, \gamma) D_x^+ H_y(\gamma) dy + \int_{\mathbb{R}^d} D_x^+ r(y, \gamma) H_y(\gamma + \delta_x) dy, \quad (3.7)$$

$$D_x^- \left(\int_{\mathbb{R}^d} r(y, \gamma) H_y(\gamma) dy \right) = \int_{\mathbb{R}^d} r(y, \gamma) D_x^- H_y(\gamma) dy + \int_{\mathbb{R}^d} D_x^- r(y, \gamma) H_y(\gamma - \delta_x) dy. \quad (3.8)$$

Computing the first summand of (3.4) we obtain

$$L^-\square^-(F, F)(\gamma) - 2\square^-(L^-F, F)(\gamma) = \frac{1}{2} \sum_{x \in \gamma} \sum_{y \in \gamma - \delta_x} (D_x^- D_y^- F)^2(\gamma) + \square^-(F, F)(\gamma),$$

whereas for the second summand we may derive the following expression

$$\begin{aligned} L^+\square^+(F)(\gamma) - 2\square^+(F, L^+F)(\gamma) &= -\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} r(x, \gamma) r(y, \gamma) (D_x^+ D_y^+ F)^2(\gamma) dx dy \\ &+ \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} r(x, \gamma) D_x^+ r(y, \cdot)(\gamma) (D_y^+ F)^2(\gamma + \delta_x) dx dy \\ &- \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} r(x, \gamma) D_x^+ F(\gamma) D_x^+ r(y, \cdot)(\gamma) D_y^+ F(\gamma + \delta_x) dx dy \end{aligned}$$

Finally, calculating the mixed terms in (3.4), we obtain

$$\begin{aligned} (L^-\square^+(F) - 2\square^+(F, L^-F)) &= \sum_{x \in \gamma} \int_{\mathbb{R}^d} r(y, \gamma) (D_x^- D_y^+ F)^2(\gamma) dy \\ &+ \frac{1}{2} \sum_{x \in \gamma} \int_{\mathbb{R}^d} D_x^- r(y, \cdot)(\gamma) (D_y^+ F)^2(\gamma - \delta_x) dy \\ -L^+\square^-(F) + 2\square^-(F, L^+F)(\gamma) &= \sum_{x \in \gamma} \int_{\mathbb{R}^d} r(y, \gamma) (D_x^- D_y^+ F)^2(\gamma) dy \\ &+ \frac{1}{2} \int_{\mathbb{R}^d} r(y, \gamma) (D_y^+ F)^2(\gamma) dy \\ &+ \sum_{y \in \gamma} D_y^- F(\gamma) \int_{\mathbb{R}^d} D_y^- r(x, \cdot)(\gamma) D_x^+ F(\gamma - \delta_y) dx \end{aligned}$$

Summarizing, adding all four parts we gain the following expression for $\square_2$

$$\square_2(F, F)(\gamma) \tag{3.9}$$

$$= \frac{1}{2} \square(F, F)(\gamma) + \square(F, F)^+(\gamma) \tag{3.10}$$

$$\begin{aligned} &+ \frac{1}{4} \sum_{x \in \gamma} \sum_{y \in \gamma - \delta_x} (D_x^- D_y^- F)^2(\gamma) + \frac{1}{2} \sum_{y \in \gamma} \int_{\mathbb{R}^d} r(x, \gamma) (D_x^+ D_y^- F)^2(\gamma) dx \\ &+ \frac{1}{4} \int_{\mathbb{R}^d} \sum_{x \in \gamma} D_x^- r(y, \cdot)(\gamma) [(D_y^+ F)^2(\gamma - \delta_x) + 2D_y^+ F(\gamma - \delta_x) D_x^- F(\gamma)] dy \\ &+ \frac{1}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} r(x, \gamma) r(y, \gamma) (D_x^+ D_y^+ F)^2(\gamma) dx dy \\ &+ \frac{1}{4} \int_{\mathbb{R}^d} r(x, \gamma) \int_{\mathbb{R}^d} D_x^+ r(y, \cdot)(\gamma) [-(D_y^+ F)^2(\gamma + \delta_x) + 2D_y^+ F(\gamma + \delta_x) D_x^+ F(\gamma)] dy dx \end{aligned}$$

This representation is still not in a convenient form. For Gaussian type measures there is a Bochner-Lichnérowicz-Weitzenböck kind formula and an associated Bakry-Emery criterium for $\square_2$ in terms of geometrical quantities like the underlying curvature and the Hessian. Unfortunately, in our case we lack this understanding of the associated geometrical structure. However, we observe that we have three terms of fourth order in the differential operator. One may expect that in a natural representation they all would have all the same integral w.r.t. the reversible measure μ , which, as we will see, is not the case for the second but last summand, cf. (3.12). Therefore, we rearrange the last and second but last summand in (3.10) and obtain the following Bochner-Lichnérowicz-Weitzenböck formula

Theorem 3.1 *For all $F, G \in \mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$ it holds that*

$$\begin{aligned} \square_2(F, F)(\gamma) &= \frac{1}{2} \square(F, F)(\gamma) + \square^+(F, F)(\gamma) \\ &+ \frac{1}{4} \sum_{x \in \gamma} \sum_{y \in \gamma - \delta_x} (D_x^- D_y^- F)^2(\gamma) + \frac{1}{2} \sum_{y \in \gamma} \int_{\mathbb{R}^d} r(x, \gamma) (D_x^+ D_y^- F)^2(\gamma) dx \\ &+ \frac{1}{4} \int_{\mathbb{R}^d} \sum_{x \in \gamma} D_x^- r(y, \cdot)(\gamma) [(D_y^+ F)^2(\gamma - \delta_x) + 2D_y^+ F(\gamma - \delta_x) D_x^- F(\gamma)] dy \\ &+ \frac{1}{4} \int_{\mathbb{R}^d} r(x, \gamma) \int_{\mathbb{R}^d} r(y, \gamma + \delta_x) (D_x^+ D_y^+ F)^2(\gamma) dy dx \\ &+ \frac{1}{4} \int_{\mathbb{R}^d} r(x, \gamma) \int_{\mathbb{R}^d} D_x^+ r(y, \cdot)(\gamma) [-(D_y^+ F)^2(\gamma) + 2D_y^+ F(\gamma) D_x^+ F(\gamma)] dy dx \end{aligned}$$

Proof: Using just the definition of D_x^+ the last two summand in (3.10) can be rewritten as follows

$$\begin{aligned} &\frac{1}{4} \int_{\mathbb{R}^d} r(x, \gamma) \int_{\mathbb{R}^d} r(y, \gamma + \delta_x) (D_x^+ D_y^+ F)^2(\gamma) dy dx \\ &+ \frac{1}{4} \int_{\mathbb{R}^d} r(x, \gamma) \int_{\mathbb{R}^d} D_x^+ r(y, \cdot)(\gamma) [(D_x^+ D_y^+ F)^2(\gamma) - (D_y^+ F)^2(\gamma + \delta_x) + 2D_y^+ F(\gamma + \delta_x) D_x^+ F(\gamma)] dy dx \end{aligned} \quad (3.11)$$

It remains to simplify the last bracket. Expanding the first summand of the bracket and using

$$\begin{aligned} -D_x^+ (D_y^+ F)^2(\gamma) &= (D_y^+ F)^2(\gamma + \delta_x) - (D_y^+ F)^2(\gamma) \\ 2D_x^+ (D_y^+ F)(\gamma) D_y^+ F(\gamma) &= 2(D_y^+ F(\gamma) - D_y^+ F(\gamma + \delta_x)) D_y^+ F(\gamma) \end{aligned}$$

we get

$$\begin{aligned} &(D_x^+ D_y^+ F)^2(\gamma) - (D_y^+ F)^2(\gamma + \delta_x) + 2D_y^+ F(\gamma + \delta_x) D_x^+ F(\gamma) \\ &= -D_x^+ (D_y^+ F)^2(\gamma) + 2D_x^+ D_y^+ F(\gamma) D_y^+ F(\gamma) - (D_y^+ F)^2(\gamma + \delta_x) + 2D_y^+ F(\gamma + \delta_x) D_x^+ F(\gamma) \\ &= -(D_y^+ F)^2(\gamma) + 2D_x^+ F(\gamma) D_y^+ F(\gamma) - 2D_x^+ F(\gamma + \delta_x) D_y^+ F(\gamma) + 2D_y^+ F(\gamma + \delta_x) D_x^+ F(\gamma) \end{aligned}$$

According to Lemma 3.2 below the integral expression w.r.t. which one has to integrate the afore calculated summand is symmetric under the interchange of x and y . Hence the last two terms in the previous calculation cancel each other and the second summand in (3.11) can be simplified to

$$\int_{\mathbb{R}^d} r(x, \gamma) \int_{\mathbb{R}^d} D_x^+ r(y, \cdot)(\gamma) [-(D_y^+ F)^2(\gamma) + 2D_x^+ F(\gamma) D_y^+ F(\gamma)] dy dx,$$

which yields the result. ■

Lemma 3.2 *For $\mu \otimes dx$ -a.a. (γ, x) holds that*

$$r(x, \gamma) D_x^+ r(y, \cdot)(\gamma) dx dy = r(y, \gamma) D_y^+ r(x, \cdot)(\gamma) dy dx$$

Proof: As the above equality has to be interpreted a.s. it is sufficient to show that the following expression is invariant under the interchange of x and y for any cylinder function H . This is obvious after the following rewriting

$$\begin{aligned} & \int_{\Gamma} \int_{\mathbb{R}^d} r(x, \gamma) \int_{\mathbb{R}^d} D_x^+ r(y, \cdot)(\gamma) H(\gamma + \delta_x + \delta_y, x, y) dy dx \mu(d\gamma) \\ &= \int_{\Gamma} \int_{\mathbb{R}^d} r(x, \gamma) \int_{\mathbb{R}^d} r(y, \gamma) H(\gamma + \delta_x + \delta_y, x, y) dy dx \mu(d\gamma) \\ & \quad - \int_{\Gamma} \sum_{\substack{x, y \in \gamma \\ x \neq y}} H(\gamma, x, y) \mu(d\gamma) \end{aligned}$$
■

3.2 Coercivity identity

In order to study spectral properties of L we consider integrals of $\square$ and $\square_2$ with respect to an associated probability μ , that is a probability measure with a Papangelou intensities r , cf. (2.1). The representation given in Theorem 3.1 yields a particular representation useful for this purpose.

In this subsection we need to assume that μ has not only local moments up to first but up to second order. In particular, then for all compact $\Lambda \subset \mathbb{R}^d$ holds that $\gamma \mapsto \int_{\Lambda} \int_{\Lambda} r(y, \gamma) r(y, \gamma + \delta_x)$ is integrable w.r.t μ . In order that $\mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d), \Gamma) \subset D(L^2)$, we have additionally to assume that $\gamma \mapsto \int_{\Lambda} r(x, \gamma) dx$ is in $L^2(\Gamma, \mu)$. Then r has a version which fulfills all assumptions used in Subsection 3.1 hold for a set Γ_{temp} of full measure.

Recall that L is symmetric with respect to μ and L applied to constant functions is zero. Using that we get the following relations for $\square$ and $\square_2$: for all $F \in \mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$ holds

$$\mathcal{E}(F, F) = \int_{\Gamma} F(\gamma) L F(\gamma) \mu(d\gamma) = \int_{\Gamma} \square(F, F)(\gamma) \mu(d\gamma).$$

$$\int_{\Gamma} (LF)^2(\gamma) \mu(d\gamma) = \int_{\Gamma} \square_2(F, F)(\gamma) \mu(d\gamma)$$

The following identities are derived using repeatedly the identity $D_x^+ F(\gamma - \delta_x) = D_x^- F(\gamma)$. and the definition of the Papangelou intensities, cf. (2.1). For all $F \in \mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$ holds

$$\frac{1}{2} \int_{\Gamma} \square(F, F)(\gamma) \mu(d\gamma) = \int_{\Gamma} \square^{\pm}(F, F)(\gamma) \mu(d\gamma) = \int_{\Gamma} \int_{\mathbb{R}^d} r(x, \gamma) (D_x^+ F)^2(\gamma) dx \mu(d\gamma)$$

and, in particular, one gets the representation (2.3) for the Dirichlet form $\mathcal{E}$.

Furthermore, in the representation given in Theorem 3.1 the expectations of all fourth order terms coincides, that is, for all $F \in \mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$ holds

$$\begin{aligned} & \int_{\Gamma} \sum_{y \in \gamma} \int_{\mathbb{R}^d} r(x, \gamma) (D_x^+ D_y^- F)^2(\gamma) dx \mu(d\gamma) \\ &= \int_{\Gamma} \sum_{x \in \gamma} \sum_{y \in \gamma - \delta_x} (D_x^+ D_y^- F)^2(\gamma - \delta_x) \mu(d\gamma) \\ &= \int_{\Gamma} \sum_{x \in \gamma} \sum_{y \in \gamma - \delta_x} (D_x^- D_y^- F)^2(\gamma) \mu(d\gamma). \end{aligned}$$

and indeed in Subsection 3.1 the last fourth order term in (3.10) was rearranged in such a form in Theorem 3.1 that now holds

$$\int_{\Gamma} \int_{\mathbb{R}^d} r(x, \gamma) \int_{\mathbb{R}^d} r(y, \gamma + \delta_x) (D_x^+ D_y^+ F)^2(\gamma) dy dx \mu(d\gamma) = \int_{\Gamma} \sum_{y \in \gamma} \sum_{x \in \gamma - \delta_y} (D_x^- D_y^- F)^2(\gamma) \mu(d\gamma) \quad (3.12)$$

For the remaining second order terms in Theorem 3.1 one can find some cancelations. For all $F \in \mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$ holds

$$\begin{aligned} & \int_{\Gamma} \int_{\mathbb{R}^d} \sum_{x \in \gamma} D_x^- r(y, \cdot)(\gamma) [(D_y^+ F)^2(\gamma - \delta_x) + 2D_y^+ F(\gamma - \delta_x) D_x^- F(\gamma)] dy \mu(d\gamma) \\ &= \int_{\Gamma} \int_{\mathbb{R}^d} r(x, \gamma) \int_{\mathbb{R}^d} D_x^- r(y, \cdot)(\gamma + \delta_x) [(D_y^+ F)^2(\gamma) + 2D_y^+ F(\gamma) D_x^- F(\gamma + \delta_x)] dy dx \mu(d\gamma) \\ &= \int_{\Gamma} \int_{\mathbb{R}^d} r(x, \gamma) \int_{\mathbb{R}^d} D_x^+ r(y, \cdot)(\gamma) [(D_y^+ F)^2(\gamma) + 2D_y^+ F(\gamma) D_x^+ F(\gamma)] dy dx \mu(d\gamma). \end{aligned}$$

Note that the first summand in the last term has the opposite sign as the first summand in the last term of the representation given in Theorem 3.1. Summarizing one obtains the coercivity identity

Theorem 3.2 *For all $F \in \mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$ holds that*

$$\begin{aligned} \int_{\Gamma} (LF)^2(\gamma) \mu(d\gamma) &= \int_{\Gamma} \square_2(F, F)(\gamma) \mu(d\gamma) \\ &= \int_{\Gamma} \square(F, F)(\gamma) \mu(d\gamma) + \int_{\Gamma} \sum_{x \in \gamma} \sum_{y \in \gamma - \delta_x} (D_x^- D_y^- F)^2(\gamma) \mu(d\gamma) \\ &\quad + \int_{\Gamma} \int_{\mathbb{R}^d} r(x, \gamma) \int_{\mathbb{R}^d} D_x^+ r(y, \cdot)(\gamma) D_y^+ F(\gamma) D_x^+ F(\gamma) dy dx \mu(d\gamma). \end{aligned}$$

3.3 Sufficient condition for spectral gap

Instead of proving spectral gap directly using the Poincare inequality, we consider the following approach, see [10] and [3, Chapter. 6, Section 4].

Let L be a nonnegative self-adjoint operator which maps the constant functions to zero. Let $D(L)$ be a core of L and $c > 0$. Then L has a spectral gap of at least c if and only if the following so-called coercivity inequality holds

$$\int_{\Gamma} (LF)^2(\gamma) \mu(d\gamma) \geq c \mathcal{E}(F, F), \quad \forall F \in D(L). \quad (3.13)$$

The latter inequality can be expressed in terms of the “carré du champ” $\square$ and $\square_2$

$$\int_{\Gamma} \square_2(F, F)(\gamma) \mu(d\gamma) \geq c \int_{\Gamma} \square(F, F)(\gamma) \mu(d\gamma). \quad (3.14)$$

For diffusions D. Bakry and M. Emery could derive directly an inequality for $\square$ and $\square_2$, cf. [2], which we are not able to do.

Inserting in (3.14) the representations of the previous sections one easily derives the following inequality

$$\begin{aligned} (1 - c) \int_{\Gamma} \int_{\mathbb{R}^d} r(x, \gamma) (D_x^+ F)^2(\gamma) dx \mu(d\gamma) \\ + \int_{\Gamma} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} r(x, \gamma) D_x^+ r(y, \cdot)(\gamma) D_y^+ F(\gamma) D_x^+ F(\gamma) dy dx \mu(d\gamma) \geq 0 \end{aligned} \quad (3.15)$$

using that

$$\sum_{x \in \gamma} \sum_{y \in \gamma - \delta_x} (D_x^- D_y^- F)^2(\gamma) \geq 0$$

Considering the integrand (3.15) for fixed γ and denoting by

$$K_{\gamma}(x, y) = r(x, \gamma)(r(y, \gamma) - r(y, \gamma + \delta_x)), \quad \psi_{\gamma}(x) = D_x^+ F(\gamma).$$

we can give a sufficient condition for the inequality (3.15) to hold for all $F \in \mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$, namely for all $\psi \in \mathcal{C}_0(\mathbb{R}^d)$ holds

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (K_\gamma(x, y) + (1 - c)\sqrt{r(x, \gamma)}\sqrt{r(y, \gamma)}\delta(x - y))\psi(y)\psi(x)dx dy \geq 0. \quad (3.16)$$

This can be formulate more elegantly using the following definition

Definition 3.1 *A Radon measure K on $\mathbb{R}^d \times \mathbb{R}^d$ is called a positive definite kernel if for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ holds*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(x)\psi(y)K(dx, dy) \geq 0 \quad (3.17)$$

Theorem 3.3 *If there is a $c > 0$ such that for μ -a.a. γ the kernel*

$$r(x, \gamma)(r(y, \gamma) - r(y, \gamma + \delta_x)) + (1 - c)\sqrt{r(x, \gamma)}\sqrt{r(y, \gamma)}\delta(x - y) \quad (3.18)$$

is positive definite then the coercivity inequality (3.13) for L with constant c holds for all $F \in \mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$.

4 Coercivity identity for Gibbs measures

In this section we demonstrate that the sufficient condition for the coercivity inequality developed in Theorem 3.3 gives surprising results for the Glauber dynamics associated to Gibbs measures.

4.1 Gibbs measures

Gibbs measures are just the measures with Papangelou intensities of the form $r(x, \gamma) = z \exp[-E(x, \gamma)]$, where $z > 0$ and

$$E(x, \gamma) := \begin{cases} \sum_{y \in \gamma} \phi(x - y), & \text{if } \sum_{y \in \gamma} |\phi(x - y)| < \infty, \\ +\infty, & \text{otherwise,} \end{cases}$$

for a measurable symmetric function $\phi : \mathbb{R}^d \rightarrow (-\infty, \infty]$. One calls such a measure a Gibbs measure to the activity z and pair potential ϕ . Sometimes it is useful to introduce an extra parameter, the inverse temperature β , and consider Gibbs measures for $\beta\phi$.

To guarantee existence of a measure with such Papangelou intensities, we need to require further conditions on the pair potential ϕ . For every $r \in \mathbb{Z}^d$, define a cube $\Delta_r = \{x \in \mathbb{R}^d : r_i - \frac{1}{2} \leq x_i < r_i + \frac{1}{2}\}$. These cubes form a partition of $\mathbb{R}^d$. Denote by

$N_r(\gamma) = \gamma(\Delta_r)$. One says that ϕ is superstable (SS) if there exist $A > 0$, $B \geq 0$ such that, for all $\gamma \in \Gamma$ such that $\gamma(\mathbb{R}^d) < \infty$ holds

$$\sum_{\{x,y\} \subset \gamma} \phi(x-y) \geq \sum_{r \in \mathbb{Z}^d} AN_r^2(\gamma) - BN_r(\gamma).$$

ϕ is called stable (S) if the above condition holds just for $A = 0$. One says that ϕ is regular (R) if ϕ is bounded below and there exists an $R > 0$ and a positive decreasing function φ on $[0, +\infty)$ such that $|\phi(x)| \leq \varphi(|x|)$ for all $x \in \mathbb{R}^d$ with $|x| \geq R$ and

$$\int_R^\infty t^{d-1} \varphi(t) dt < \infty. \quad (4.1)$$

For the notion of tempered Gibbs measure and the following theorem, see [21].

Theorem 4.1 *Let ϕ be (SS) and (R), then the set $\mathcal{G}_{\text{temp}}(z, E)$ of all tempered Gibbs measures is non-empty and for each measure from $\mathcal{G}_{\text{temp}}(z, E)$ all correlation functions exist and satisfy the so-called Ruelle bound.*

A Gibbs measure that fulfills the Ruelle bound has all (local) moments and one can see quite easily that also $\gamma \mapsto \int_{\mathbb{R}^d} r(x, \gamma) dx$ is in $L^2(\Gamma, \mu)$, cf. e.g. [14]. Hence all assumptions of Subsection 3.1 are fulfilled. Hence, in the sequel, we will restrict ourself to Gibbs measures which fulfill a Ruelle bound.

4.2 Coercivity inequality

For Gibbs measures condition (3.18) takes the following form

Theorem 4.2 *Let μ be a Gibbs measure for a pair potential ϕ and activity z which fulfills a Ruelle bound. If for a.a. γ the kernel*

$$e^{-E(x, \gamma)} e^{-E(y, \gamma)} z (1 - e^{-\phi(x-y)}) + (1 - c) e^{-\frac{1}{2}E(x, \gamma)} e^{-\frac{1}{2}E(y, \gamma)} \delta(x - y) \quad (4.2)$$

is positive definite then the coercivity inequality (3.13) for L with constant c holds for all $F \in \mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$.

The following easy reformulation will become very fruitful later on. Using in (3.16) the function $e^{-\frac{1}{2}E(x, \gamma)} \psi(x)$ instead of ψ gives

Corollary 4.3 *Let μ be a Gibbs measure for a pair potential ϕ and activity z which fulfills a Ruelle bound. If for a.a. γ the kernel*

$$e^{-\frac{1}{2}E(x, \gamma)} e^{-\frac{1}{2}E(y, \gamma)} z (1 - e^{-\phi(x-y)}) + (1 - c) \delta(x - y) \quad (4.3)$$

is positive definite then the coercivity inequality (3.13) for L with constant c holds for all $F \in \mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$.

4.3 Potentials increasing the spectral gap

For the Poisson point process, i.e. the Gibbs measure for the potential $\phi = 0$, one has the spectral gap $c = 1$, which follows also directly from condition (4.2). In order to prove condition (4.3) for $c = 1$ it is obviously sufficient to prove non-negativity (for a.a. γ) of the expression for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\frac{1}{2}E(x,\gamma)} e^{-\frac{1}{2}E(y,\gamma)} (1 - e^{-\phi(x-y)}) \psi(y) \psi(x) dx dy. \quad (4.4)$$

and considering this a bilinear form in $e^{-\frac{1}{2}E(x,\gamma)} \psi(x)$ and recalling that due to Ruelle bound and regularity the latter function is integrable, one is lead to the following sufficient condition

$$\int_{\mathbb{R}^d} (1 - e^{-\phi(x)}) \psi * \psi(x) dx, \quad (4.5)$$

where $\psi * \psi$ denotes the convolution of ψ with ψ . Recalling the following definition

Definition 4.1 *A locally bounded measurable function $u : \mathbb{R}^d \mapsto \mathbb{C}$ is called positive definite if for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ holds*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(x) \psi * \psi(x) dx \geq 0$$

and $u(0) \leq 1$.

As $1 - e^{-\phi}$ is bounded, condition (4.5) means that $f : x \mapsto 1 - e^{-\phi}$ is a positive definite function.

Remark 4.1 *Note that the condition (4.5) does not depend on z .*

To apply this condition we now investigate if there exists any potential ϕ such that f is positive definite and ϕ fulfills the conditions guaranteeing the existence of a Gibbs measure, namely (SS) and (R).

Theorem 4.4 *Let f be a continuous positive definite function which is (R). Define*

$$\phi := -\ln(1 - f). \quad (4.6)$$

*Then ϕ fulfills (4.5) and is (SS) and (R). For every Gibbs measure μ for the potential ϕ and **for any** activity z which fulfills a Ruelle bound the associated generator L of the Glauber dynamics fulfills a coercivity inequality for $c = 1$ and all $F \in \mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$.*

Proof. Due to positive definiteness $|f(x)| \leq f(0) \leq 1$. Defining for $x \in [-1, 1]$ the function $h(x) = -\ln(1-x)$ one can write $\phi = h \circ f$. First, we show that ϕ is regular. As f is regular there exists an $\tilde{R} > 0$ and a positive decreasing function φ on $[0, +\infty)$ which fulfills (4.1) and such that $|f(x)| \leq \varphi(|x|)$ for all $x \in \mathbb{R}^d$ with $|x| \geq \tilde{R}$. Note that for $x \in [-1, 1/2]$ it holds that $|h(x)| \leq 2x$. Choose an $R \geq \tilde{R}$ such that $\varphi(R) \leq 1/2$. Then for all $x \in \mathbb{R}^d$ with $|x| \geq R$ it holds $|f(x)| \leq 1/2$ and hence

$$|\phi(x)| \leq 2f(x) \leq 2\varphi(|x|),$$

which implies that ϕ is regular.

Second, we show that ϕ is superstable. One easily sees that $h(x) \geq x + \mathbb{1}_{[2/3, 1]}(x)(-\ln(1-x) - x)$. Shorthanding $g(x) = -\ln(1-x) - x$ one obtains $\phi(x) \geq f(x) + \mathbb{1}_{[2/3, 1]}(f(x))g(f(x))$. Hence, ϕ is bigger then the sum of a positive definite function and a continuous function, which is positive in 0, therefore due to Lemma 1.2 in [21] the potential ϕ is a superstable. $\square$

We now try to understand the structure of potentials fulfilling condition (4.5). For that let us recall the following definition

Definition 4.2 A generalized function (distribution) $u \in \mathcal{D}(\mathbb{R}^d)$ is called positive definite if for all $\varphi \in \text{mathcal{C}}_0^\infty(\mathbb{R}^d)$

$$\langle u, \tilde{\varphi} \star \varphi \rangle \geq 0 \tag{4.7}$$

holds, where $\tilde{\varphi}(x) := \overline{\varphi(-x)}$.

Proposition 4.1 Let ϕ be a potential fulfilling condition (4.5) which is (S), (R), and lower semi-continuous at zero. Then it is of the form (4.6) and hence also (SS). Furthermore, ϕ is integrable, itself positive definite in the sense of generalized functions, and

$$\limsup_{x \downarrow 0} (\phi(x) + 2 \ln(x)) < \infty \tag{4.8}$$

Proof. Let us define $f := 1 - e^{-\phi}$ and show that the function f fulfils the conditions of Theorem 4.4. As ϕ is stable it is non-negative in 0 and hence $|f(0)| \leq 1$. Furthermore, f is lower semi-continuous at zero. Due to the positive definiteness of f one has that f is continuous and $|f(x)| \leq f(0) \leq 1$. One obtains the representation (4.6) by inverting the definition of f . As in the proof of Theorem 4.4 one can check that f also fulfills (R). Then Theorem 4.4 implies that ϕ is also (SS).

Using that $1 - \cos(x) \geq \frac{x^2}{2}$ for small enough x , f is non-negative, the positive definiteness and $f(0) \leq 1$, we obtain that there exists a constant $c > 0$ such that $1 - f(x) \geq c|x|^2$ for small enough x . Hence $\phi(x) \leq -2 \ln(|x|) - \ln(c)$. As ϕ is bounded below and regular, it is integrable.

Writing again $\phi = h \circ f$, we note that $h(x) = -\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ with radius of convergence 1. Approximate ϕ by the functions $\phi_\delta(x) := h \circ ((1-\delta)f(x))$ for $0 < \delta < 1$. Since $|(1-\delta)f(x)| < 1$ and h has a Taylor series with non-negative coefficients, for all $0 < \delta < 1$ the function ϕ_δ is positive definite, cf. e.g. [9, Proposition 3.5.17]. As h is monotone increasing $|\phi_\delta| \leq |\phi|$ and the latter function is integrable. Hence ϕ_δ is also positive definite in the sense of generalized functions. Since ϕ_δ converge pointwise to ϕ for $\delta \rightarrow 0$ uniformly bounded by ϕ , by Lebesgue's dominated convergence ϕ is also positive definite in the sense of generalized functions. $\square$

4.4 Parameter dependence

A typical question in statistical mechanics is to study the behavior of the system under change of a parameters. In the previous subsection we identify potentials which fulfill (4.5) for all z and hence will show no phase transition even for large z . To investigate the temperature dependence we reintroduce the inverse temperature $\beta > 0$ in to our consideration, that is we consider instead of ϕ the potential $\beta\phi$. We consider ϕ as fix and vary β and z . The corresponding Papangelou intensity is $r(x, \gamma) = ze^{-\beta E(x)}$ and hence condition 4.5 takes the form

$$\int_{\mathbb{R}^d} (1 - e^{-\beta\phi(x)}) \psi * \psi(x) dx. \quad (4.9)$$

If (4.9) is positive for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ then we say that ϕ fulfills condition (4.9) for β . Note, that the condition is independent of the activity z .

Proposition 4.2 *Let ϕ be a potential which fulfills condition (4.5) for a $\bar{\beta} > 0$ and is (S), (R), and lower semi-continuous at zero. Then ϕ fulfills condition (4.5) for all $0 < \beta \leq \bar{\beta}$.*

Proof. Denote by $f := 1 - e^{-\bar{\beta}\phi}$ the function considered in condition (4.5), which is positive definite by assumption. One the one hand, it is easy to see that $f_\beta(x) := 1 - e^{-\beta\phi(x)}$ are also continuous and (R). One the other hand, $f_\beta(x) = 1 - (1 - f(x))^{\beta/\bar{\beta}}$ has a power series expansion $f_\beta(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \beta/\bar{\beta} (\beta/\bar{\beta} - 1) \dots (\beta/\bar{\beta} - n + 1) (f(x))^n$ with radius of convergence 1. All the coefficients of the series are nonnegative, if $\beta/\bar{\beta} \leq 1$. Proceeding as in Proposition 4.1, one proves that f_β is the pointwise limit of positive definite functions. As f_β is itself bounded and a limit of positive definite functions, it is a positive definite in the sense of functions. $\square$

4.5 Examples

For concreteness we give a small collection of potentials which fulfills the condition of Theorem 4.4 to get a better feeling how such potentials may look like. Especially

interesting is that among them are potentials, which have a non-trivial negative part.

$\phi(x)$	$f(x)$	Parameters
$-\ln(1 - e^{-tx^2} \cos(ax)),$	$e^{-tx^2} \cos(ax),$	$t > 0, a \in \mathbb{R}$
$-\ln(1 - e^{-t x } \cos(ax)),$	$e^{-t x } \cos(ax),$	$t > 0, a \in \mathbb{R}$
$-\ln\left(1 - \frac{\cos(ax)}{1 + \sigma^2 x^2}\right),$	$\frac{1}{1 + \sigma^2 x^2} \cos(ax),$	$\sigma > 0, a \in \mathbb{R}$
$-\ln(1 - (1 - \frac{ x }{a}) \mathbb{1}_{[-a,a]}(x) \cos(bx)),$	$(1 - \frac{ x }{a}) \mathbb{1}_{[-a,a]}(x) \cos(bx),$	$a > 0, b \in \mathbb{R},$

In all examples above one can exchange $\cos(ax)$ by $\frac{\sin(ax)}{ax}$.

In the d -dimensional case we can give following examples:

$\phi(x)$	$f(x)$	Parameters
$-\ln(1 - e^{-t x ^2} \cos(a \cdot x))$	$e^{-t x ^2} \cos(a \cdot x)$	$x \in \mathbb{R}^d, t > 0, a \in \mathbb{R}^d$
$-\ln\left(1 - e^{-t x ^2} \prod_{j=1}^d \frac{\sin(a_j x_j)}{a_j x_j}\right)$	$e^{-t x ^2} \prod_{j=1}^d \frac{\sin(a_j x_j)}{a_j x_j}$	$x \in \mathbb{R}^d, t > 0$ where
$-\ln\left(1 - \left(\frac{r}{ x }\right)^{n/2} J_{n/2}(r x)\right)$	$\left(\frac{r}{ x }\right)^{n/2} J_{n/2}(r x)$	$r \geq 0, n > 2d - 1$
$-\ln\left(1 - \frac{2^{n/2} \Gamma(\frac{n+1}{2})}{\sqrt{\pi}} \cdot \frac{t}{(x ^2 + t^2)^{\frac{n+1}{2}}}\right)$	$\frac{2^{n/2} \Gamma(\frac{n+1}{2})}{\sqrt{\pi}} \cdot \frac{t}{(x ^2 + t^2)^{\frac{n+1}{2}}}$	$t > 0, n > d - 1$

$J_{n/2}$ is the Bessel function of the first kind of order $n/2$. One can multiply f in any of the examples with factors of the form $\cos(a \cdot x)$ and $\prod_{j=1}^d \frac{\sin(a_j x_j)}{a_j x_j}$.

All these examples are constructed by choosing a positive definite function f and express $\phi(x) = -\ln(1 - f(x))$.

4.6 High temperature and low densities

In the previous subsection we considered potentials which increase the spectral gap. Such potentials admit at most a logarithmic singularity at zero. In this subsection we will show that one may add a non-negative potential to these kind of potentials. However, the constant c in the coercivity inequality will decrease and the spectral gap will depend on the activity of z .

Theorem 4.5 *Let ϕ_1 be bounded below, (R) and (S) and ϕ_2 a potential fulfilling the conditions of Theorem 4.4. Then for every Gibbs measure μ for the potential $\phi_1 + \phi_2$ and*

the activity z , the associated generator L of the Glauber dynamics fulfills a coercivity inequality for the constant

$$c = 1 - \left(\sup_{y \in \mathbb{R}^d} \right) \rho_\mu^{(1)}(y) \int_{\mathbb{R}^d} dx e^{-\phi_2(x)} |1 - e^{-\phi_1(x)}|,$$

where $\rho_\mu^{(1)}(y) := \int_{\Gamma} e^{-E(y|\gamma)} \mu(d\gamma)$ is the first correlation function.

Proof. The main idea is to apply condition 4.3 directly. In order to prove positive definiteness of the kernel (4.3) one has to prove non-negativity of the following expression for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \psi(x) \psi(y) \left[e^{-\frac{1}{2}E(x,\gamma)} e^{-\frac{1}{2}E(y,\gamma)} z (1 - e^{-\phi(x-y)}) + (1 - c) \delta(x - y) \right] \quad (4.10)$$

Rewriting

$$1 - e^{-\phi} = 1 - e^{-\phi_2} + e^{-\phi_2}(1 - e^{-\phi_1}).$$

the first part of (4.3) takes the form

$$e^{-\frac{1}{2}E(x,\gamma)} e^{-\frac{1}{2}E(y,\gamma)} z (1 - e^{-\phi_2(x-y)}) + e^{-\frac{1}{2}E(x,\gamma)} e^{-\frac{1}{2}E(y,\gamma)} z e^{-\phi_2(x-y)} (1 - e^{-\phi_1(x-y)})$$

As in the beginning of Subsection 4.3 the first summand is a positive definite due to the assumptions on ϕ_2 . The second summand can be bounded as follows

$$\begin{aligned} & \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \psi(x) e^{-\frac{1}{2}E(x,\gamma)} \psi(y) e^{-\frac{1}{2}E(y,\gamma)} z e^{-\phi_2(x-y)} (1 - e^{-\phi_1(x-y)}) \\ & \geq -z \int_{\mathbb{R}^d} dx e^{-\phi_2(x)} |1 - e^{-\phi_1(x)}| \int_{\mathbb{R}^d} dy |\psi|(x+y) e^{-\frac{1}{2}E(x+y,\gamma)} |\psi(y)| e^{-\frac{1}{2}E(y,\gamma)} \end{aligned}$$

Applying Cauchy-Schwarz inequality to the last factor one obtains

$$\begin{aligned} & z \int_{\mathbb{R}^d} dy |\psi|(x+y) e^{-\frac{1}{2}E(x+y,\gamma)} |\psi(y)| e^{-\frac{1}{2}E(y,\gamma)} \\ & \leq \int_{\mathbb{R}^d} dy \rho^{(1)}(y) \psi^2(y). \end{aligned}$$

Summarizing (4.10) can be bounded below by

$$\int_{\mathbb{R}^d} dy \left[-\rho^{(1)}(y) \int_{\mathbb{R}^d} dx e^{-\phi_2(x)} |1 - e^{-\phi_1(x)}| + (1 - c) \right] \psi^2(y) \quad (4.11)$$

which is non-negative if and only if the bracket is non-negative. $\square$

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