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A normalized solitary wave solution of the Maxwell-Dirac equations ☆

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Abstract

We prove the existence of a L^2-normalized solitary wave solution for the Maxwell-Dirac equations in (3+1)-Minkowski space. In addition, for the Coulomb-Dirac model, describing fermions with attractive Coulomb interactions in the mean-field limit, we prove the existence of the (positive) energy minimizer.

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1. Introduction and main results

The Lagrangian for a charged, spin-$\frac{1}{2}$ relativistic particle (here $\hbar = c = 1$) interacting with its own electromagnetic field is given by

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi - \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu},$$

where we use the four-vector notations, $\mu, \nu \in \{0, 1, 2, 3\}$ and repeated index summation convention, with metric tensor $g^{\mu\nu} = \text{diag}\{1, -1 - 1, -1\}$ used to lower or raise the Lorentz indices. γ^μ are the 4×4 Dirac matrices given by $\gamma^0 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}$ and $\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$, $k = 1, 2, 3$, and σ_k are the 2×2-Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}.$$

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Ψ is the Dirac spinor taking values in $\mathbb{C}^4$ and $\bar{\Psi} = \Psi^\dagger \gamma^0$ is the Dirac adjoint, with $\Psi^\dagger$ the hermitian conjugate of Ψ; $m > 0$ is the particle's mass, $D_\mu = \partial_\mu + i e A_\mu$ is the gauge covariant derivative, with e the particle's charge ($e < 0$ for the electron) and A^μ is the electromagnetic 4-vector potential. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor field.

The Euler-Lagrange equations in the Lorenz gauge ($\partial_\mu A^\mu = 0$) are given by the Maxwell-Dirac equations

$$\begin{cases} (i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu)\Psi - m\Psi = 0 \\ \partial_\nu \partial^\nu A^\mu = 4\pi\, j^\mu \end{cases} \tag{MD}$$

where $j^\mu = e\,\bar{\Psi}\gamma^\mu \Psi$ is the conserved Dirac current ($\partial_\mu j^\mu = 0$). We look for solutions of (MD) stationary in time, localized and L^2-normalized in space, called *solitary waves*, and which can be seen as representations of the *extended particles*. Numerical evidence of the existence of solitary wave solutions of (MD) was obtained in [13]. The first proof of the existence of solitary waves is given by using variational methods (a linking argument) by M. Esteban V. Georgiev and E. Séré in [8]. They proved the existence of stationary solutions $\Psi(t, x) = \mathrm{e}^{i\omega t}\psi(x)$, for any $\omega \in (0, m)$, with ψ smooth, and exponentially decreasing at infinity together with all its derivatives. This result was later generalized to any $\omega \in (-m, m)$ in [1], using an axial symmetry ansatz on the class of solutions. Recently in [3] the authors prove the existence of solitary waves using a perturbative approach. In fact they prove the existence of a small amplitude stationary solution which bifurcates (via Implicit Function theorem) from the ground state of the Choquard's equation (see [11]). Let us remark that both the variational approach used in [8] (and also in [1]) and the perturbative approach used in [3] do not provide solutions with prescribed L^2-norm. Aim of this paper is to find one such L^2-normalized solution. We use a different variational characterization for critical points of the energy functional, inspired by the one used to characterize the first eigenvalue of the Dirac operators with Coulomb-type potentials (see e.g. [7], also [6] for an application in the nonlinear case) and we use of concentration-compactness-type arguments (see [12]). Note that in [6] the presence of an attractive external Coulomb potential (the dominant focusing term) allows one to recover compactness.

Let us also quote the article [2] where the authors study normalized solutions for a different problem which also has a strongly indefinite structure. In that paper the authors use a penalization method in the spirit of [9].

Our main result is the following.

Theorem. *There exists $\omega \in (0, m)$ and $\psi \in H^{1/2}(\mathbb{R}^3; \mathbb{C}^4)$, with $\|\psi\|^2_{L^2} = 1$, such that*

$$\begin{cases} \Psi(x^0, x) = \mathrm{e}^{i\omega x^0}\psi(x) \\ A^\mu(x^0, x) = A^\mu(x) = e(\gamma^0\psi, \gamma^\mu\psi)_{\mathbb{C}^4} * \frac{1}{|x|} \end{cases} \tag{1.1}$$

is a solution of (MD).

As already mentioned we prove this result by using a variational characterization of critical level of the energy functional introduced for the first eigenvalue of Dirac operators. Indeed let $A = (A^0, \boldsymbol{A})$, with $\boldsymbol{A} = (A^1, A^2, A^3)$, be the four-vector potential A^μ, clearly $A^0 = A_0$ and $A^k = -A_k$, $(k = 1, 2, 3)$, and let denote $\beta = \gamma^0$ and $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$, with $\alpha^k = \gamma^0\gamma^k$ $(k = 1, 2, 3)$. Then (Ψ, A) is a L^2-normalized stationary solution of (MD) of the form (1.1) if (ψ, ω) is a solution of the following (nonlinear) eigenvalue problem

$$\begin{cases} (i\boldsymbol{\alpha}\cdot\nabla - m\beta)\psi - eA_0\psi + e\boldsymbol{\alpha}\cdot\boldsymbol{A}\,\psi = \omega\,\psi \\ A_0(x) = e\,|\psi|^2 * \frac{1}{|x|}; \qquad \boldsymbol{A}(x) = e\,(\psi, \boldsymbol{\alpha}\psi)_{\mathbb{C}^4} * \frac{1}{|x|} \\ \|\psi\|^2_{L^2} = 1. \end{cases} \tag{E_ω}$$

We look for solutions of (E_ω) as (constrained) critical points of the functional

$$\mathcal{I}_{MD}(\psi) = \int_{\mathbb{R}^3} (\psi, D\psi)_{\mathbb{C}^4} - \frac{e^2}{2} \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_\psi(x)\rho_\psi(y) - J_\psi(x)\cdot J_\psi(y)}{|x-y|} dx dy$$

where $D = i\boldsymbol{\alpha}\cdot\nabla - m\beta$ and $\rho_\psi = |\psi|^2$ and $J_\psi = (\psi, \boldsymbol{\alpha}\psi)$, on the manifold

$$\Sigma = \{\psi \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4) : \|\psi\|^2_{L^2} = 1\}.$$

Note also that in the units we choose ($\hbar = c = 1$) the coupling constant e^2 is, as a matter of fact, the dimensionless fine structure constant $\frac{e^2}{\hbar c} \approx \frac{1}{137}$. The functional $\mathcal{I}_{MD}$ is strongly indefinite and presents a lack of compactness. Indeed, the operator $D = i\boldsymbol{\alpha} \cdot \nabla - m\beta$ is a first order, self-adjoint operator on $H^1(\mathbb{R}^3, \mathbb{C}^4)$ with purely absolutely continuous spectrum given by

$$\sigma(D) = (-\infty, -m] \cup [m, +\infty).$$

Let $\Lambda_\pm(D)$ be the two infinite rank orthogonal projectors on the positive/negative energies subspaces, then

$$D\Lambda_\pm(D) = \Lambda_\pm(D)D = \pm\sqrt{-\Delta + m}\,\Lambda_\pm(D) = \pm\Lambda_\pm(D)\sqrt{-\Delta + m},$$

hence for $\psi \in H^{1/2}(\mathbb{R}^3; \mathbb{C}^4)$ the operator form is given by

$$\int_{\mathbb{R}^3} (\psi, D\psi)_{\mathbb{C}^4} = \|(-\Delta + m)^{1/4}\Lambda_+(D)\psi\|^2_{L^2} - \|(-\Delta + m)^{1/4}\Lambda_-(D)\psi\|^2_{L^2}$$

and we denote $X_\pm(D) = \Lambda_\pm(D)H^{1/2}(\mathbb{R}^3; \mathbb{C}^4)$.

In fact we prove the existence of the L^2-normalized solitary wave solution of (MD) by means of the following variational characterization.

Theorem 1.2. *Let define*

$$E = \inf_{\substack{W \subset X_+(D) \\ \dim W = 1}} \sup_{\substack{\phi \in W \oplus X_-(D) \\ \|\phi\|_{L^2} = 1}} \mathcal{I}_{MD}(\phi)$$

then $E \in (0, m)$ and it is attained, namely there exists $\psi \in \Sigma$ such that $\mathcal{I}_{MD}(\psi) = E$. Moreover, there exists $\omega \in (0, m)$ (Lagrange multiplier) such that

$$d\mathcal{I}_{MD}(\psi)[h] = 2\omega \operatorname{Re}\langle\psi|h\rangle_{L^2}, \qquad \forall h \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$$

that is $(\psi, w) \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4) \times (0, m)$ is a solution of (E_ω) and

$$\begin{cases} \Psi(x^0, x) = \mathrm{e}^{i\omega x^0}\psi(x) \\ A^\mu(x^0, x) = A^\mu(x) = e(\gamma^0\psi, \gamma^\mu\psi)_{\mathbb{C}^4} * \frac{1}{|x|} \end{cases}$$

is a L^2-normalized solitary wave solution of (MD).

In addition, E is the lowest positive critical value of the functional $\mathcal{I}_{MD}$ on Σ.

As a byproduct of the proof of Theorem 1.2, we obtain also an interesting result for the Coulomb-Dirac model, describing fermions with attractive Coulomb interactions and that can be viewed as a semiclassical approximation of the (relativistically invariant) *polaron* model. We refer to [4] for a detailed discussion of this model and its solitary waves and to [1] for a multiplicity results of (not normalized) stationary solutions.

Denoting $H = -i\boldsymbol{\alpha} \cdot \nabla + m\beta = -D$, note that this is the operator usually called the (free) Dirac operator, clearly $\Lambda_\pm(H) = \Lambda_\mp(D)$ and $X_\pm(H) = X_\mp(D)$, then we have the following result.

Theorem 1.3. *There exists $\omega \in (0, m)$ and $\psi \in H^{1/2}(\mathbb{R}^3; \mathbb{C}^4)$ solution of*

$$\begin{cases} (-i\boldsymbol{\alpha} \cdot \nabla + m\beta)\psi + eA_0\psi = \omega\psi \\ A_0(x) = -e\,|\psi|^2 * \frac{1}{|x|} \\ \|\psi\|^2_{L^2} = 1. \end{cases} \tag{1.4}$$

Moreover,

$$\mathcal{I}_{CD}(\psi) = \inf_{\substack{W \subset X_+(H) \\ \dim W = 1}} \sup_{\substack{\phi \in W \oplus X_-(H) \\ \|\phi\|_{L^2} = 1}} \mathcal{I}_{CD}(\phi) = E \in (0, m),$$

where

$$\mathcal{I}_{CD}(\phi) = \int_{\mathbb{R}^3} (\phi, H\phi)_{\mathbb{C}^4} - e^2 \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\phi|^2(x)|\phi|^2(y)}{|x-y|} dxdy.$$

In addition, E is the lowest positive critical value of the energy functional $\mathcal{I}_{CD}$.

Let us mention a related result obtained in [10] where the authors prove the existence and orbital stability for the L^2-normalized, solitary wave solution, minimizer of the energy functional for the pseudo-relativistic model describing bosons with attractive Coulomb interactions.

2. Notation and preliminary results

From now on we take $m = 1$. We denote by $\hat{u}$ or $\mathcal{F}(u)$ the Fourier transform of u, defined by extending the formula

$$\hat{u}(p) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ip\cdot x} u(x)\, dx, \qquad \text{for } u \in \mathcal{S}(\mathbb{R}^3).$$

We denote

$$\langle f|g\rangle_{H^{1/2}} = \int_{\mathbb{R}^3} \sqrt{|p|^2+1}(\hat{f}(p), \hat{g}(p))\, dp$$

the scalar product in $H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ with $(\cdot\,,\cdot)$ the hermitian scalar product in $\mathbb{C}^4$.

Let $H = -i\boldsymbol{\alpha}\cdot\nabla + \beta$ be the (free) Dirac operator, in the (momentum) Fourier space we have the multiplication operator $\hat{H}(p) = \mathcal{F}H\mathcal{F}^{-1} = \boldsymbol{\alpha}\cdot p + \beta$ which, for each $p \in \mathbb{R}^3$, is an Hermitian 4×4-matrix with eigenvalues

$$\lambda_1(p) = \lambda_2(p) = -\lambda_3(p) = -\lambda_4(p) = \sqrt{|p|^2+1} \equiv \lambda(p).$$

The unitary transformation $U(p)$ which diagonalize $\hat{H}(p)$ is given explicitly by

$$U(p) = u_+(p)\mathbb{I}_4 + u_-(p)\beta\frac{\boldsymbol{\alpha}\cdot p}{|p|}$$
$$U^{-1}(p) = u_+(p)\mathbb{I}_4 - u_-(p)\beta\frac{\boldsymbol{\alpha}\cdot p}{|p|} = U^\dagger(p)$$

with $u_\pm(p) = \sqrt{\frac{1}{2}(1 \pm \frac{1}{\lambda(p)})}$. We have

$$U(p)\hat{H}(p)U^{-1}(p) = \lambda(p)\beta = \sqrt{|p|^2+1}\,\beta.$$

Hence the two orthogonal projectors $\Lambda_\pm(H)$ on $L^2(\mathbb{R}^3, \mathbb{C}^4)$ are given by

$$\Lambda_\pm(H) = \frac{1}{2}\mathcal{F}^{-1}U(p)^{-1}(\mathbb{I}_4 \pm \beta)U(p)\mathcal{F}. \tag{2.1}$$

We denote $X_\pm(H) = \Lambda_\pm(H)H^{1/2}(\mathbb{R}^3; \mathbb{C}^4)$. Clearly we have $\Lambda_\pm(H) = \Lambda_\mp(D)$ and $X_\pm(H) = X_\mp(D)$.

It may be useful consider the Foldy-Wouthuysen (FW) transformation (see e.g. [14]), namely the unitary transformation $U_{\text{FW}} = \mathcal{F}^{-1}U(p)\mathcal{F}$. Note that under the FW transformation the projectors $\Lambda_\pm(H)$ become simply

$$\Lambda(H)_\pm^{(FW)} = U_{\text{FW}}\Lambda_\pm(H)U_{\text{FW}}^{-1} = \frac{1}{2}(\mathbb{I}_4 \pm \beta). \tag{2.2}$$

Note that $\Lambda(D)_\pm^{(FW)} = \frac{1}{2}(\mathbb{I}_4 \mp \beta)$.

We consider the smooth functional $\mathcal{I}\colon H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)\to\mathbb{R}$ given by

$$\mathcal{I}(\psi)=\|\psi_+\|^2_{H^{1/2}}-\|\psi_-\|^2_{H^{1/2}}-\frac{\mathrm{e}^2}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\rho_\psi(x)\rho_\psi(y)-J_\psi(x)\cdot J_\psi(y)}{|x-y|}dxdy$$

where $\psi_\pm=\Lambda_\pm(D)\psi$, $\rho_\psi=|\psi|^2$ and $J_\psi=(\psi,\boldsymbol{\alpha}\psi)$.

The Frechét derivative $d\mathcal{I}(\phi):H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)\to\mathbb{R}$ is given by

$$\begin{aligned}d\mathcal{I}(\psi)[h]=&2\,\mathrm{Re}\langle\psi_+|h_+\rangle_{H^{1/2}}-2\,\mathrm{Re}\langle\psi_-|h_-\rangle_{H^{1/2}}\\&-2\mathrm{e}^2\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\rho_\psi(x)\,\mathrm{Re}(\psi,h)(y)-J_\psi(x)\cdot\mathrm{Re}(\psi,\boldsymbol{\alpha}h)(y)}{|x-y|}dxdy\end{aligned}$$

for any $h=h_++h_-\in H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$, with $h_\pm\in X_\pm(D)$.

Clearly $(\psi,\omega)\in H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)\times\mathbb{R}$ is a weak solution of (E_ω) if and only if

$$d\mathcal{I}(\psi)[h]=\omega\,2\,\mathrm{Re}\,\langle\psi|h\rangle_{L^2}\,,\qquad\forall h\in H^{1/2}(\mathbb{R}^3,\mathbb{C}^4).$$

Hence we look for (constrained) critical points of $\mathcal{I}$ on the manifold

$$\Sigma=\{\psi\in H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)\,:\,\|\psi\|^2_{L^2}=1\}.$$

Remark 2.3. Let us recall the following Hardy-type inequalities:

Hardy: $\||x|^{-1}\psi\|^2_{L^2}\le 4\|\nabla\psi\|^2_{L^2}$ for all $\psi\in H^1(\mathbb{R}^3)$;

Kato: $\||x|^{-\frac12}\psi\|^2_{L^2}\le\gamma_K\|(-\Delta)^{1/4}\psi\|^2_{L^2}$ for all $\psi\in H^{1/2}(\mathbb{R}^3)$, with $\gamma_K=\frac{\pi}{2}$.

Let us remark that $e^2\gamma_K<\frac{1}{87}$.

In view of Kato's inequality for any $\rho\in L^1(\mathbb{R}^3)$ and $\psi\in H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$ we have

$$\int_{\mathbb{R}^3}(\rho*\frac{1}{|x|})|\psi|^2(y)dy\le\gamma_K\|\rho\|_{L^1}\|(-\Delta)^{1/4}\psi\|^2_{L^2}. \tag{2.4}$$

Remark 2.5. Since $\mathcal{F}[\frac{1}{|x|}]=\sqrt{\frac{2}{\pi}}\frac{1}{|p|^2}$, for any $f\in L^1\cap L^{3/2}$ we have that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{f(x)\bar f(y)}{|x-y|}dxdy=4\pi\int_{\mathbb{R}^3}\frac{|\hat f|^2(p)}{|p|^2}\,dp\ge 0. \tag{2.6}$$

Hence in particular

$$\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{J_\psi(x)\cdot J_\psi(y)}{|x-y|}dxdy\ge 0. \tag{2.7}$$

Moreover since $|J_\psi|\le\rho_\psi$ for any $\psi\in H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$, we have that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\rho_\psi(x)\rho_\psi(y)-J_\psi(x)\cdot J_\psi(y)}{|x-y|}dxdy\ge 0. \tag{2.8}$$

Moreover we have the following useful result (see the Appendix for the proof).

Lemma 2.9. *For any $\psi = \psi_+ + \psi_- \in \Sigma$, let define $w = \frac{\psi_+}{\|\psi_+\|_{L^2}}$ we have*

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_\psi(x)\rho_\psi(y) - J_\psi(x)\cdot J_\psi(y)}{|x-y|}dxdy \geq \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_w(x)\rho_w(y) - J_w(x)\cdot J_w(y)}{|x-y|}dxdy$$
$$- 8\gamma_K(\|w\|^2_{H^{1/2}} - \|w\|^2_{L^2}) - 10\gamma_K(\|\psi_-\|^2_{L^2}\|w\|^2_{H^{1/2}} + \|\psi_-\|^2_{H^{1/2}}).$$

Moreover, if $v \in H^1(\mathbb{R}^3, \mathbb{C}^2)$, with $\|v\|^2_{L^2} = 1$, and $\frac{\psi_+}{\|\psi_+\|_{L^2}} = U^{-1}_{FW}\begin{pmatrix}0\\ v\end{pmatrix}$ we have

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_\psi(x)\rho_\psi(y) - J_\psi(x)\cdot J_\psi(y)}{|x-y|}dxdy \geq \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_v(x)\rho_v(y)}{|x-y|}dxdy$$
$$- 8\gamma_K\|\nabla v\|^2_{L^2} - 10\gamma_K(\|\psi_-\|^2_{L^2}\|v\|^2_{H^{1/2}} + \|\psi_-\|^2_{H^{1/2}}).$$

Finally we recall the following convergence result. Let $v \in H^{1/2}$, f_n, g_n, h_n bounded sequences in $H^{1/2}$ such that one of them converges weakly to zero in $H^{1/2}$, then we have (see for example [5, Lemma 4.1])

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|f_n|(x)|g_n|(x)|v|(y)|h_n|(y)}{|x-y|}dxdy \to 0, \qquad \text{as}\, n \to +\infty. \tag{2.10}$$

3. Maximization problem

We introduce the family of functionals $\mathcal{I}^{(m)}\colon H^{1/2}(\mathbb{R}^3, \mathbb{C}^4) \to \mathbb{R}$, with $m \in (0, 1]$,

$$\mathcal{I}^{(m)}(\psi) = \|\psi_+\|^2_{H^{1/2}} - \|\psi_-\|^2_{H^{1/2}} - m\frac{e^2}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_\psi(x)\rho_\psi(y) - J_\psi(x)\cdot J_\psi(y)}{|x-y|}dxdy,$$

where $\psi_\pm = \Lambda_\pm(D)\psi$, $\rho_\psi = |\psi|^2$ and $J_\psi = (\psi, \boldsymbol{\alpha}\psi)$. Clearly $\mathcal{I} = \mathcal{I}^{(1)}$.

Our first step will be to maximize the family of functionals $\mathcal{I}^{(m)}$ on the space

$$\mathcal{X}_W = \left\{ \psi \in \Sigma \;\middle|\; \psi_+ \in W \right\},$$

where $W \subset X_+(D)$ is a 1-dimensional vector space.

The tangent space of $\mathcal{X}_W$ at some point $\psi \in \mathcal{X}_W$ is the set

$$T_\psi\mathcal{X}_W = \left\{ h \in W \oplus X_-(D) \;\middle|\; \mathrm{Re}\langle\psi|h\rangle_{L^2} = 0 \right\}$$

and $\nabla_{\mathcal{X}_W}\mathcal{I}^{(m)}(\psi)$, the projection of the gradient $\nabla\mathcal{I}^{(m)}(\psi)$ on $T_\psi\mathcal{X}_W$, is given by

$$\mathrm{Re}\langle\nabla_{\mathcal{X}_W}\mathcal{I}^{(m)}(\psi)|h\rangle_{H^{1/2}} = d\mathcal{I}^{(m)}(\psi)[h] - 2\omega(\psi)\,\mathrm{Re}\langle\psi|h\rangle_{L^2}$$

for all $h \in W \oplus X_-(D)$ and $\omega(\psi) \in \mathbb{R}$ is such that $\nabla_{\mathcal{X}_W}\mathcal{I}^{(m)}(\psi) \in T_\psi\mathcal{X}_W$.

Let us introduce

$$\Sigma_+ = \left\{ w \in X_+(D) \;\middle|\; \|w\|^2_{L^2} = 1 \right\},$$

then, from now on, we characterize the 1-dimensional vector space $W \subset X_+(D)$ as $W = \mathrm{span}\{w\}$, with $w \in \Sigma_+$.

We begin giving a result on Palais-Smale sequences of $\mathcal{I}^{(m)}$ on $\mathcal{X}_W$, in particular we prove that the Palais-Smale condition holds on $\mathcal{X}_W$ for $\mathcal{I}^{(m)}$ at the positive levels.

Proposition 3.1. *For any $w \in \Sigma_+$ and for any $m \in (0, 1]$, let $\{\psi_n\} \subset \mathcal{X}_W$ be a Palais-Smale sequence of $\mathcal{I}^{(m)}$ on $\mathcal{X}_W$, that is*

$$\mathcal{I}^{(m)}(\psi_n) \to c \text{ and } \|\nabla_{\mathcal{X}_W}\mathcal{I}^{(m)}(\psi_n)\|_{H^{1/2}} \to 0, \qquad \text{as} \quad n \to +\infty.$$

Then,

(i) $\{\psi_n\} \subset \mathcal{X}_W$ *is bounded in* $H^{1/2}$*;*
(ii) $\omega(\psi_n)$ *is bounded and* $\liminf_{n\to+\infty} \omega(\psi_n)(\|(\psi_n)_+\|^2_{L^2} - \|(\psi_n)_-\|^2_{L^2}) > 0$*;*
(iii) *If* $c > 0$ *than* $\liminf_{n\to+\infty} \omega(\psi_n) > 0$ *and* $\{\psi_n\}$ *is pre-compact in* $H^{1/2}$.

Proof. (i) Since $\{(\psi_n)_+\} \subset W$ and $\|(\psi_n)_+\|^2_{L^2} \le 1$, we have $\|(\psi_n)_+\|_{H^{1/2}} \le \|w\|_{H^{1/2}}$. In view of (2.8) we have

$$\mathcal{I}^{(m)}(\psi_n) \le \|(\psi_n)_+\|^2_{H^{1/2}} - \|(\psi_n)_-\|^2_{H^{1/2}}$$

hence we get $\|(\psi_n)_-\|^2_{H^{1/2}} \le \|w\|^2_{H^{1/2}} - \mathcal{I}^{(m)}(\psi_n) \le \|w\|^2_{H^{1/2}} - c + o(1)$.

(ii) Since

$$\begin{aligned}\omega(\psi_n) + o(1) =& \frac{1}{2} d\mathcal{I}^{(m)}(\psi_n)[\psi_n] \\ =& \mathcal{I}^{(m)}(\psi_n) - m\frac{\mathrm{e}^2}{2} \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_{\psi_n}(x)\rho_{\psi_n}(y) - J_{\psi_n}(x)\cdot J_{\psi_n}(y)}{|x-y|} dxdy\end{aligned}$$

by (2.4) and (2.8) we have

$$\mathcal{I}^{(m)}(\psi_n) - m\frac{\mathrm{e}^2}{2}\gamma_K \|\psi_n\|^2_{H^{1/2}} \le \omega(\psi_n) \le \mathcal{I}^{(m)}(\psi_n).$$

Then since $\{\psi_n\}$ is a bounded sequence in $H^{1/2}$ we conclude that $\omega(\psi_n)$ is a bounded sequence. Moreover, we have

$$o(1) = d\mathcal{I}^{(m)}(\psi_n)[(\psi_n)_+ - (\psi_n)_-] - 2\omega(\psi_n)(\|(\psi_n)_+\|^2_{L^2} - \|(\psi_n)_-\|^2_{L^2})$$

and since $\mathrm{Re}(\psi_+ + \psi_-, \psi_+ - \psi_-) = |\psi_+|^2 - |\psi_-|^2$ again by (2.4) and (2.8) we get

$$\begin{aligned}&\omega(\psi_n)(\|(\psi_n)_+\|^2_{L^2} - \|(\psi_n)_-\|^2_{L^2}) + o(1) = \|(\psi_n)_+\|^2_{H^{1/2}} + \|(\psi_n)_-\|^2_{H^{1/2}} \\ &\quad - m\mathrm{e}^2 \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_{\psi_n}(x)(|(\psi_n)_+|^2 - |(\psi_n)_-|^2)(y)}{|x-y|} dxdy \\ &\quad + m\mathrm{e}^2 \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{J_{\psi_n}(x)\cdot(\mathrm{Re}((\psi_n)_+, \boldsymbol{\alpha}(\psi_n)_+) - \mathrm{Re}((\psi_n)_-, \boldsymbol{\alpha}(\psi_n)_-)(y)}{|x-y|} dxdy \\ &\ge \|(\psi_n)_+\|^2_{H^{1/2}} + \|(\psi_n)_-\|^2_{H^{1/2}} - 2m\mathrm{e}^2 \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_{\psi_n}(x)|(\psi_n)_+|^2(y)}{|x-y|} dxdy \\ &\ge (1 - 2\mathrm{e}^2\gamma_K)\|(\psi_n)_+\|^2_{H^{1/2}} + \|(\psi_n)_-\|^2_{H^{1/2}} > 1 - 2\mathrm{e}^2\gamma_K.\end{aligned}$$

(iii) If $c > 0$ clearly $\|(\psi_n)_+\|^2_{H^{1/2}} \ge \|(\psi_n)_-\|^2_{H^{1/2}}$ for n sufficiently large, and by (2.4) we have

$$\begin{aligned}\omega(\psi_n)\|(\psi_n)_+\|^2_{L^2} + o(1) =& \frac{1}{2} d\mathcal{I}^{(m)}(\psi_n)[(\psi_n)_+] \\ =& \|(\psi_n)_+\|^2_{H^{1/2}} - m\mathrm{e}^2 \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_{\psi_n}(x)\,\mathrm{Re}(\psi_n, (\psi_n)_+)(y)}{|x-y|} dxdy \\ &+ m\mathrm{e}^2 \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{J_{\psi_n}(x)\cdot \mathrm{Re}(\psi_n, \boldsymbol{\alpha}(\psi_n)_+)(y)}{|x-y|} dxdy \\ \ge& \|(\psi_n)_+\|^2_{H^{1/2}} - 2\mathrm{e}^2\gamma_K \|\psi_n\|_{H^{1/2}} \|(\psi_n)_+\|_{H^{1/2}} \\ \ge& \|(\psi_n)_+\|^2_{H^{1/2}} - 4\mathrm{e}^2\gamma_K \|(\psi_n)_+\|^2_{H^{1/2}} \ge (1 - 4\mathrm{e}^2\gamma_K)\|(\psi_n)_+\|^2_{H^{1/2}}.\end{aligned}$$

Hence we get $\omega(\psi_n) \ge (1 - 4\mathrm{e}^2\gamma_K) + o(1)$.

Since $\{(\psi_n)_+\} \subset W$ clearly $(\psi_n)_+ \to \psi_+$ in $H^{1/2}$ (up to subsequence), moreover $(\psi_n)_- \rightharpoonup \psi_-$ weakly in $H^{1/2}$ (up to subsequence) and since $\liminf_{n\to+\infty} \omega(\psi_n) > 0$, we have

$$\begin{aligned} o(1) &= -\frac{1}{2} d\mathcal{I}^{(m)}(\psi_n)[(\psi_n)_- - \psi_-] + \omega(\psi_n)\|(\psi_n)_- - \psi_-\|^2_{L^2} \\ &\geq \|(\psi_n)_- - \psi_-\|^2_{H^{1/2}} + me^2 \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(\rho_{\psi_n} - |J_{\psi_n}|)(x)|(\psi_n)_- - \psi_-|^2(y)}{|x-y|} dxdy + o(1) \\ &\geq \|(\psi_n)_- - \psi_-\|^2_{H^{1/2}} + o(1), \end{aligned}$$

and we may conclude that also $(\psi_n)_- \to \psi_-$ strongly in $H^{1/2}$. □

It turns out that all the critical points of $\mathcal{I}^{(m)}$ on $\mathcal{X}_W$ at positive levels are strict local maxima. More precisely we have the following result.

Proposition 3.2. *For any $m \in (0, 1]$ let $\psi^{(m)} \in H^{1/2}(\mathbb{R}^3; \mathbb{C}^4)$ be a critical point of $\mathcal{I}^{(m)}$ on $\mathcal{X}_W$ at a positive level, that is $d\mathcal{I}^{(m)}(\psi^{(m)})[h] - 2\omega(\psi^{(m)}) \operatorname{Re}\langle\psi^{(m)}|h\rangle_{L^2} = 0$ for any $h \in W \oplus X_-(D)$ and $\mathcal{I}^{(m)}(\psi^{(m)}) > 0$. Then there exists $\delta > 0$ such that*

$$d^2\mathcal{I}^{(m)}(\psi^{(m)})[h;h] - 2\omega(\psi^{(m)})\|h\|^2_{L^2} \leq -\delta\|h\|^2_{H^{1/2}}, \qquad \forall h \in T_{\psi^{(m)}}\mathcal{X}_W.$$

Hence in particular $\psi^{(m)}$ is a strict local maximum of $\mathcal{I}^{(m)}$ on $\mathcal{X}_W$.

Proof. Since $\mathcal{I}^{(m)}(\psi^{(m)}) > 0$ then by Proposition 3.1-(iii) we have that $\omega(\psi^{(m)}) > 0$. By the $U(1)$-invariance, a critical point $\psi \in \mathcal{X}_W$ (up to a phase factor) has the following form $\psi = aw + \eta$ with $a = a(\eta) = \sqrt{1 - \|\eta\|^2_{L^2}}$ and $\eta = \psi_-$. Now, any $h \in T_\psi\mathcal{X}_W$ takes the following form: $h = da(\eta)[\xi]w + \xi$ with $\xi \in X_-(D)$, and $da(\eta)[\xi] = -a^{-1} \operatorname{Re}\langle\eta|\xi\rangle_{L^2}$. We have

$$\begin{aligned} d^2\mathcal{I}^{(m)}(\psi)[h;h] &= a^{-1}da(\eta)[\xi]d^2\mathcal{I}^{(m)}(\psi)[\psi; (da(\eta)[\xi]w - a^{-1}da(\eta)[\xi]\eta)] \\ &\quad + 2d^2\mathcal{I}^{(m)}(\psi)[da(\eta)[\xi]w; \xi] + a^{-2}|da(\eta)[\xi]|^2 d^2\mathcal{I}^{(m)}(\psi)[\eta;\eta] + d^2\mathcal{I}^{(m)}(\psi)[\xi;\xi]. \end{aligned}$$

Since $d^2\mathcal{I}^{(m)}(\psi) : H^{1/2}(\mathbb{R}^3, \mathbb{C}^4) \times H^{1/2}(\mathbb{R}^3, \mathbb{C}^4) \to \mathbb{R}$ is given by

$$\begin{aligned} d^2\mathcal{I}^{(m)}(\psi)[h;k] &= 2\operatorname{Re}\langle k_+|h_+\rangle_{H^{1/2}} - 2\operatorname{Re}\langle k_-|h_-\rangle_{H^{1/2}} \\ &\quad - 2me^2 \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_\psi(x)\operatorname{Re}(h,k)(y) - J_\psi(x)\cdot\operatorname{Re}(h,\boldsymbol{\alpha}k)(y)}{|x-y|} dxdy \\ &\quad - 4me^2 \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\operatorname{Re}(\psi,h)(x)\operatorname{Re}(\psi,k)(y) - \operatorname{Re}(\psi,\boldsymbol{\alpha}h)(x)\cdot\operatorname{Re}(\psi,\boldsymbol{\alpha}k)(y)}{|x-y|} dxdy, \end{aligned}$$

we have in particular

$$\begin{aligned} d^2\mathcal{I}^{(m)}(\psi)[\psi;h] &= 2d\mathcal{I}^{(m)}(\psi)[h] - 2\operatorname{Re}\langle\psi_+|h_+\rangle_{H^{1/2}} + 2\operatorname{Re}\langle\psi_-|h_-\rangle_{H^{1/2}} \\ &\quad - 2me^2 \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_\psi(x)\operatorname{Re}(\psi,h)(y) - J_\psi(x)\cdot\operatorname{Re}(\psi,\boldsymbol{\alpha}h)(y)}{|x-y|} dxdy. \end{aligned}$$

Then, since $|da(\eta)[\xi]|^2 = -a^{-1}da(\eta)[\xi]\operatorname{Re}\langle\eta|\xi\rangle_{L^2}$ and $h = da(\eta)[\xi]w + \xi$, we have

$$\begin{aligned} &a^{-1}da(\eta)[\xi]d^2\mathcal{I}^{(m)}(\psi)[\psi; (da(\eta)[\xi]w - a^{-1}da(\eta)[\xi]\eta)] \\ &\quad = 4\omega(\psi)\|da(\eta)[\xi]w\|^2_{L^2} - 4\omega(\psi)\|a^{-1}da(\eta)[\xi]\eta\|^2_{L^2} \\ &\qquad - 2\|da(\eta)[\xi]w\|^2_{H^{1/2}} - 2\|a^{-1}da(\eta)[\xi]\eta\|^2_{H^{1/2}} \end{aligned}$$

$$
\begin{aligned}
&- 2m\mathrm{e}^2|da(\eta)[\xi]|^2 \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_\psi(x)\rho_w(y) - J_\psi(x)\cdot J_w(y)}{|x-y|}dxdy \\
&+ 2m\mathrm{e}^2 a^{-2}|da(\eta)[\xi]|^2 \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_\psi(x)\rho_\eta(y) - J_\psi(x)\cdot J_\eta(y)}{|x-y|}dxdy \\
\le & 2\omega(\psi)\|h\|^2_{L^2} - 2\|da(\eta)[\xi]w\|^2_{H^{1/2}} - 2\|a^{-1}da(\eta)[\xi]\eta\|^2_{H^{1/2}} \\
&- 2m\mathrm{e}^2|da(\eta)[\xi]|^2 \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_\psi(x)\rho_w(y) - J_\psi(x)\cdot J_w(y)}{|x-y|}dxdy \\
&+ 2m\mathrm{e}^2 a^{-2}|da(\eta)[\xi]|^2 \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_\psi(x)\rho_\eta(y) - J_\psi(x)\cdot J_\eta(y)}{|x-y|}dxdy.
\end{aligned}
$$

Finally we get

$$
\begin{aligned}
&d^2\mathcal{I}^{(m)}(\psi)[h;h] - 2\omega(\psi)\|h\|^2_{L^2} \le -2\|da(\eta)[\xi]w\|^2_{H^{1/2}} - 2\|a^{-1}da(\eta)[\xi]\eta\|^2_{H^{1/2}} \\
&\quad - 2m\mathrm{e}^2 \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_\psi(x)\rho_h(y) - J_\psi(x)\cdot J_h(y)}{|x-y|}dxdy \\
&\quad + 8\mathrm{e}^2\gamma_K(\|da(\eta)[\xi]w\|^2_{H^{1/2}} + \|\xi\|^2_{H^{1/2}}) - 2(1-2\mathrm{e}^2\gamma_K)\|\xi\|^2_{H^{1/2}} \\
&\quad - 2(1-2\mathrm{e}^2\gamma_K)\|a^{-1}da(\eta)[\xi]\eta\|^2_{H^{1/2}} \\
&\le -2(1-4\mathrm{e}^2\gamma_K)\|da(\eta)[\xi]w\|^2_{H^{1/2}} - 2(1-6\mathrm{e}^2\gamma_K)\|\xi\|^2_{H^{1/2}} \\
&\quad - 4(1-\mathrm{e}^2\gamma_K)\|a^{-1}da(\eta)[\xi]\eta\|^2_{H^{1/2}} \le -2(1-6\mathrm{e}^2\gamma_K)\|h\|^2_{H^{1/2}}. \quad \square
\end{aligned}
$$

For any $w \in \Sigma_+$ and $m \in (0,1]$ we consider the following maximization problem

$$\lambda_W(m) = \sup_{\psi\in\mathcal{X}_W} \mathcal{I}^{(m)}(\psi). \tag{3.3}$$

We have the following estimates on $\lambda_W(m)$.

Lemma 3.4. *For any $w \in \Sigma_+$ and $m \in (0,1]$, we have*

$$(1 - \frac{\mathrm{e}^2}{2}\gamma_K) \le (1 - m\frac{\mathrm{e}^2}{2}\gamma_K)\|w\|^2_{H^{1/2}} \le \lambda_W(m) \le \|w\|^2_{H^{1/2}}. \tag{3.5}$$

Proof. Clearly $\lambda_W(m) = \sup_{\psi\in\mathcal{X}_W} \mathcal{I}^{(m)}(\psi) \ge \mathcal{I}^{(m)}(w)$ and by (2.7), (2.4) we have

$$
\begin{aligned}
\mathcal{I}^{(m)}(w) &\ge \|w\|^2_{H^{1/2}} - m\frac{\mathrm{e}^2}{2} \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_w(x)\rho_w(y)}{|x-y|}dxdy \\
&\ge (1 - m\frac{\mathrm{e}^2}{2}\gamma_K)\|w\|^2_{H^{1/2}} \ge (1 - \frac{\mathrm{e}^2}{2}\gamma_K) > 0.
\end{aligned}
$$

Moreover, by (2.8) we have $\mathcal{I}^{(m)}(\psi) \le \|\psi_+\|^2_{H^{1/2}} \le \|w\|^2_{H^{1/2}}$ for any $\psi \in \mathcal{X}_W$, that is $\lambda_W(m) \le \|w\|^2_{H^{1/2}}$. $\square$

In view of all the above results we completely solve the maximization problem (3.3), more precisely we have the following.

Proposition 3.6. *For any $w \in \Sigma_+$ and $m \in (0,1]$ there exists, unique (up to a phase factor), $\psi^{(m)}(w) \in \mathcal{X}_W$, the strict global maximum of $\mathcal{I}^{(m)}$ on $\mathcal{X}_W$, namely*

$$\mathcal{I}^{(m)}(\psi^{(m)}(w)) = \sup_{\psi\in\mathcal{X}_W} \mathcal{I}^{(m)}(\psi) = \lambda_W(m).$$

Moreover, we have

$$d\mathcal{I}^{(m)}(\psi^{(m)}(w))[h] - 2\omega(\psi^{(m)}(w))\,\mathrm{Re}\langle\psi^{(m)}(w)|h\rangle_{L^2} = 0 \qquad \forall h \in W \oplus X_-(D)$$

and

(i) $0 < \omega(\psi^{(m)}(w)) \le \lambda_W(m)$ *and* $\|\psi_+^{(m)}(w)\|_{L^2}^2 > \|\psi_-^{(m)}(w)\|_{L^2}^2$;
(ii) $\|\psi_+^{(m)}(w)\|_{H^{1/2}}^2 - \|\psi_-^{(m)}(w)\|_{H^{1/2}}^2 \ge 1$;
(iii) $\|\psi_-^{(m)}(w)\|_{H^{1/2}}^2 \le m\frac{e^2}{2}\gamma_K\|w\|_{H^{1/2}}^2$;
(iv) *the map* $v \in X_+(D)\setminus\{0\} \to \psi^{(m)}(P(v))$*, with* $P(v) = \|v\|_{L^2}^{-1}v \in \Sigma_+$*, is smooth.*

Proof. *Existence:* Since, by Lemma 3.4, $\lambda_W(m) > 0$, by Ekeland's variational principle, there exists a Palais-Smale, maximizing sequence $\{\psi_n^{(m)}\}$ of $\mathcal{I}^{(m)}$ on $\mathcal{X}_W$, at a positive level. Then, by Proposition 3.1, $\psi_n^{(m)} \to \psi^{(m)}$ in $H^{1/2}$ (up to subsequence), $\omega(\psi_n^{(m)}) \to \omega(\psi^{(m)}) > 0$ and $\|\psi_+^{(m)}\|_{L^2}^2 > \|\psi_-^{(m)}\|_{L^2}^2$. Therefore we conclude that

$$\mathcal{I}^{(m)}(\psi^{(m)}) = \sup_{\psi\in\mathcal{X}_W} \mathcal{I}^{(m)}(\psi) = \lambda_W(m)$$

and

$$d\mathcal{I}^{(m)}(\psi^{(m)})[h] - 2\omega(\psi^{(m)})\,\mathrm{Re}\langle\psi^{(m)}|h\rangle_{L^2} = 0 \qquad \forall h \in W \oplus X_-(D).$$

(i) Note that

$$0 < \omega(\psi^{(m)}) = \frac{1}{2}d\mathcal{I}^{(m)}(\psi^{(m)})[\psi^{(m)}] \le \mathcal{I}^{(m)}(\psi^{(m)}) = \lambda_W(m).$$

(ii) Since by the $U(1)$-invariance, we can assume that

$$\psi^{(m)} = \psi_+^{(m)} + \psi_-^{(m)} = (1 - \|\psi_-^{(m)}\|_{L^2}^2)^{\frac{1}{2}} w + \psi_-^{(m)}$$

(up to a phase factor), with $w = \frac{\psi_+}{\|\psi_+\|_{L^2}} \in \Sigma_+$ then, since we have

$$\mathcal{I}^{(m)}(\psi^{(m)}) \ge \|w\|_{H^{1/2}}^2 - m\frac{e^2}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_w(x)\rho_w(y) - J_w(x)\cdot J_w(y)}{|x-y|}dxdy,$$

by Lemma 2.9 we get

$$\begin{aligned}
\|\psi_+^{(m)}\|_{H^{1/2}}^2 - \|\psi_-^{(m)}\|_{H^{1/2}}^2 - 1 \ge{}& \|w\|_{H^{1/2}}^2 - \|w\|_{L^2}^2 \\
&+ m\frac{e^2}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_{\psi^{(m)}}(x)\rho_{\psi^{(m)}}(y) - J_{\psi^{(m)}}(x)\cdot J_{\psi^{(m)}}(y)}{|x-y|}dxdy \\
&- m\frac{e^2}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_w(x)\rho_w(y) - J_w(x)\cdot J_w(y)}{|x-y|}dxdy \\
\ge{}& \|w\|_{H^{1/2}}^2 - \|w\|_{L^2}^2 - 4me^2\gamma_K(\|w\|_{H^{1/2}}^2 - \|w\|_{L^2}^2) \\
&- 5me^2\gamma_K(\|\psi_-^{(m)}\|_{L^2}^2\|w\|_{H^{1/2}}^2 + \|\psi_-^{(m)}\|_{H^{1/2}}^2). \\
\ge{}& (1 - 9me^2\gamma_K)(\|w\|_{H^{1/2}}^2 - \|w\|_{L^2}^2) \\
&+ 5me^2\gamma_K(\|\psi_+^{(m)}\|_{H^{1/2}}^2 - \|\psi_-^{(m)}\|_{H^{1/2}}^2 - 1)
\end{aligned}$$

since $(1 - 5me^2\gamma_K) > (1 - 9me^2\gamma_K) > 0$ we may conclude that

$$\|\psi_+^{(m)}\|_{H^{1/2}}^2 - \|\psi_-^{(m)}\|_{H^{1/2}}^2 - 1 \ge 0.$$

(iii) Since

$$(1 - m\frac{e^2}{2}\gamma_K)\|w\|^2_{H^{1/2}} \leq \mathcal{I}^{(m)}(\psi^{(m)}) \leq \|\psi_+^{(m)}\|^2_{H^{1/2}} - \|\psi_-^{(m)}\|^2_{H^{1/2}}$$
$$\leq \|w\|^2_{H^{1/2}} - \|\psi_-^{(m)}\|^2_{H^{1/2}}$$

we get also $\|\psi_-^{(m)}\|^2_{H^{1/2}} \leq m\frac{e^2}{2}\gamma_K\|w\|^2_{H^{1/2}}$.

Uniqueness: Suppose we have two different maximizers $\psi_1^{(m)}, \psi_2^{(m)} \in \mathcal{X}_W$. We use the Mountain Pass Theorem to reach a contradiction. Indeed, we consider the set

$$\Gamma^{(m)} = \left\{ \gamma : [0,1] \to \mathcal{X}_W \;\middle|\; \gamma(0) = \psi_1^{(m)},\ \gamma(1) = \psi_2^{(m)} \right\}$$

and the min-max level

$$c^{(m)} = \sup_{\gamma \in \Gamma^{(m)}} \min_{t\in[0,1]} \mathcal{I}^{(m)}(\gamma(t))$$

We have $c^{(m)} > 0$, indeed, by the $U(1)$-invariance, let $\psi_1^{(m)} = a(\eta_1)w + \eta_1$ and $\psi_2^{(m)} = a(\eta_2)w + \eta_2$, with $\eta_1, \eta_2 \in X_-(D)$ and $a(\eta_i) = (1 - \|\eta_i\|^2_{L^2})^{\frac{1}{2}}$ $(i = 1, 2)$, define $\eta(t) = t\eta_2 + (1-t)\eta_1 \in X_-(D)$, then $g(t) = a(\eta(t))w + \eta(t) \in \Gamma^{(m)}$.

Since $a(\eta_i)^2 > \frac{1}{2}$ and by (iii) we have $\|\eta_i\|^2_{H^{1/2}} \leq m\frac{e^2}{2}\gamma_K\|w\|^2_{H^{1/2}}$ $(i = 1, 2)$, then for any $t \in [0, 1]$ we have

$$\mathcal{I}^{(m)}(g(t)) \geq a(\eta(t))^2\|w\|^2_{H^{1/2}} - \|\eta(t)\|^2_{H^{1/2}} - m\frac{e^2}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_{g(t)}(x)\rho_{g(t)}(y)}{|x-y|}dxdy$$
$$\geq (1 - m\frac{e^2}{2}\gamma_K)a(\eta(t))^2\|w\|^2_{H^{1/2}} - (1 + m\frac{e^2}{2}\gamma_K)\|\eta(t)\|^2_{H^{1/2}}$$
$$\geq (1 - m\frac{e^2}{2}\gamma_K)ta(\eta_2)^2\|w\|^2_{H^{1/2}} - (1 + m\frac{e^2}{2}\gamma_K)t\|\eta_2\|^2_{H^{1/2}}$$
$$+ (1 - m\frac{e^2}{2}\gamma_K)(1-t)a(\eta_1)^2\|w\|^2_{H^{1/2}} - (1 + m\frac{e^2}{2}\gamma_K)(1-t)\|\eta_1\|^2_{H^{1/2}}$$
$$\geq \frac{1}{2}(1 - m\frac{e^2}{2}\gamma_K)\|w\|^2_{H^{1/2}} - (1 + m\frac{e^2}{2}\gamma_K)m\frac{e^2}{2}\gamma_K\|w\|^2_{H^{1/2}}$$
$$\geq \frac{1}{2}(1 - 2e^2\gamma_K)\|w\|^2_{H^{1/2}} > 0,$$

hence in particular we get $c^{(m)} \geq \min_{t\in[0,1]} \mathcal{I}^{(m)}(g(t)) > 0$.

By Propositions 3.2 and 3.1-(iii), we may conclude that $c^{(m)}$ is a Mountain pass critical level, and that there is $\phi^{(m)} \in \mathcal{X}_W$, a Mountain pass critical point for $\mathcal{I}^{(m)}$ on $\mathcal{X}_W$, with $\mathcal{I}^{(m)}(\phi^{(m)}) = c^{(m)} > 0$, namely a contradiction with Proposition 3.2, since a Mountain pass critical point cannot be a strict local maximum.

(iv) To prove that the map $v \to \psi^{(m)}(P(v))$ is smooth we use the Implicit Function Theorem. Fix $w_0 \in \Sigma_+$ and let $\psi^{(m)}(w_0) = a(\eta_0)w_0 + \eta_0$ be the unique maximizer (up to a phase factor) of $\mathcal{I}^{(m)}$ on $\mathcal{X}_W$. Let $V \subset X_+(D) \setminus \{0\}$ and $U \subset X_-(D)$ be, respectively, the small neighborhoods of w_0 and η_0, such that for any $(v, \eta) \in V \times U$ and, setting $\psi = a(\eta)P(v) + \eta$, with $a(\eta) = (1 - \|\eta\|^2_{L^2})^{1/2}$, we have $\|\eta\|^2_{L^2} < \frac{1}{2}$, $\mathcal{I}^{(m)}(\psi) > 0$ and $\|\eta\|^2_{H^{1/2}} \leq me^2\gamma_K\|P(v)\|^2_{H^{1/2}}$.

We consider the smooth maps $F^{(m)} : V \times U \to L(X_-(D))$ given by

$$F^{(m)}(v, \eta)[\xi] = d\mathcal{I}^{(m)}(a(\eta)P(v) + \eta)[da(\eta)[\xi]P(v) + \xi]$$

for any $\xi \in X_-(D)$. Clearly, we have $P(w_0) = w_0$ and $F^{(m)}(w_0, \eta_0) = 0$.

The operator $d_\eta F^{(m)}(w_0, \eta_0) : X_-(D) \to L(X_-(D))$ is given by

$$(d_\eta F^{(m)}(w_0, \eta_0)[\xi])[k] = d^2\mathcal{I}^{(m)}(\psi^{(m)}(w_0))[da(\eta_0)[\xi]w_0 + \xi; da(\eta_0)[k]w_0 + k]$$
$$+ d\mathcal{I}^{(m)}(\psi^{(m)}(w_0))[d^2a(\eta_0)[\xi; k]w_0] \qquad \forall \xi, k \in X_-(D)$$

To prove that $d_\eta F^{(m)}(w_0, \eta_0)$ is invertible we apply the Lax-Milgram theorem to the quadratic form $Q^{(m)}: X_-(D) \times X_-(D) \to \mathbb{R}$ defined by

$$Q^{(m)}[\xi; k] = -(d_\eta F^{(m)}(w_0, \eta_0)[\xi])[k].$$

Note that, since

$$\begin{aligned} d\mathcal{I}^{(m)}(\psi^{(m)}(w_0))[d^2 a(\eta_0)[\xi;\xi]w_0] &= 2\omega(\psi^{(m)}(w_0))a(\eta_0)d^2a(\eta_0)[\xi;\xi] \\ &= -2\omega(\psi^{(m)}(w_0))(|da(\eta_0)[\xi]|^2 + \|\xi\|^2_{L^2}), \end{aligned}$$

setting $h = da(\eta_0)[\xi]w_0 + \xi \in T_{\psi^{(m)}(w_0)}\mathcal{X}_W$, we have

$$\begin{aligned} Q^{(m)}[\xi;\xi] &= -(d_\eta F^{(m)}(w_0, \eta_0)[\xi])[\xi] \\ &= -(d^2\mathcal{I}^{(m)}(\psi^{(m)}(w_0))[h;h] - 2\omega(\psi^{(m)}(w_0))\|h\|^2_{L^2}). \end{aligned}$$

In view of Proposition 3.2, there exists $\delta > 0$ such that $Q^{(m)}[\xi;\xi] \geq \delta\|\xi\|^2_{H^{1/2}}$ for any $\xi \in X_-(D)$. Hence, by the Lax-Milgram theorem we may conclude that for any $f \in L(X_-(D))$ there exists unique $k \in X_-(D)$ such that $Q^{(m)}[k;\xi] = f[\xi]$ for any $\xi \in X_-(D)$, namely such that $d_\eta F^{(m)}(w_0, \eta_0)[k] = -f$. Finally, by the Implicit Function theorem, there exist $V_0 \subseteq V$ and $U_0 \subseteq U$, neighborhoods, respectively, of w_0 and η_0 and a smooth map $\eta^{(m)}: V_0 \to U_0$ such that $F^{(m)}(v, \eta^{(m)}(v)) = 0$ for all $v \in V_0$, that is, $\psi^{(m)}(P(v)) = a(\eta^{(m)}(v))P(v) + \eta^{(m)}(v)$ is a critical point of $\mathcal{I}^{(m)}$ on $\mathcal{X}_W$, with $W = \text{span}\{P(v)\}$, at a positive level. Then, by Proposition 3.2, $\psi^{(m)}(P(v))$ is a strict local maximum of $\mathcal{I}^{(m)}$ on $\mathcal{X}_W$.

Again by a contradiction argument, applying the Mountain Pass theorem as above, we may conclude that for any $v \in V_0$, $\psi^{(m)}(P(v))$ is the unique maximizer (up to a phase factor) of $\mathcal{I}^{(m)}$ on $\mathcal{X}_W$, with $W = \text{span}\{P(v)\}$.

Moreover, we have that for $w \in \Sigma_+$, $d\psi^{(m)}(w): X_+(D) \to X_+(D)$ is given by

$$\begin{aligned} d\psi^{(m)}(w)[h] =& a(\psi_-(w))dP(w)[h] \\ &+ da(\psi_-(w))[d\psi_-(w)[dP(w)[h]]]w + d\psi_-(w)[dP(w)[h]], \end{aligned}$$

where $d\psi_-(P(v))[dP(v)[h]] = d\eta^{(m)}(v)[h]$ and

$$d\eta^{(m)}(v)[h] = -(d_\eta F^{(m)}(v, \eta^{(m)}(v)))^{-1}[d_v F^{(m)}(v, \eta^{(m)}(v))[h]], \qquad \forall h \in X_+(D). \quad \square$$

4. Proof of Theorem 1.2

In view of the results of Proposition 3.6 we consider the smooth functionals $\mathcal{E}^{(m)}: X_+(D) \setminus \{0\} \to \mathbb{R}$, for any $m \in (0, 1]$, given by

$$\mathcal{E}^{(m)}(v) = \mathcal{I}^{(m)}(\psi^{(m)}(P(v))) = \sup_{\psi \in \mathcal{X}_W} \mathcal{I}^{(m)}(\psi),$$

where $W = \text{span}\{w\}$, with $w = P(v) \in \Sigma_+$. We set $\mathcal{E} = \mathcal{E}^{(1)}$ and $\psi(w) = \psi^{(1)}(w)$.

For any $w \in \Sigma_+$ we have

$$d\mathcal{E}(w)[h] = d\mathcal{I}(\psi(w))[d\psi(w)[dP(w)[h]]],$$

and, setting $k = dP(w)[h]$,

$$d\psi(w)[k] = a(\psi_-(w))k + da(\psi_-(w))[d\psi_-(w)[k]]w + d\psi_-(w)[k].$$

Since $da(\psi_-(w))[d\psi_-(w)[k]]w + d\psi_-(w)[k] \in T_{\psi(w)}\mathcal{X}_W$, in view of Proposition 3.6 we get $d\mathcal{E}(w)[h] = d\mathcal{I}(\psi(w))[a(\psi_-(w))dP(w)[h]]$.

Therefore, since $dP(w)[h] = h - w\,\text{Re}\langle w|h\rangle_{L^2}$, we get

$$\begin{aligned} d\mathcal{E}(w)[h] =& d\mathcal{I}(\psi(w))[a(\psi_-(w))h] - d\mathcal{I}(\psi(w))[a(\psi_-(w))w]\,\text{Re}\langle w|h\rangle_{L^2} \\ =& d\mathcal{I}(\psi(w))[a(\psi_-(w))h] - 2\omega(\psi(w))a(\psi_-(w))^2\,\text{Re}\langle w|h\rangle_{L^2} \\ =& a(\psi_-(w))(d\mathcal{I}(\psi(w))[h] - 2\omega(\psi(w))\,\text{Re}\langle\psi_+(w)|h\rangle_{L^2}) \end{aligned}$$

for all $h \in X_+(D)$. Since the tangent space of Σ_+ at $w \in \Sigma_+$ is the space

$$T_w\Sigma_+ = \left\{ h \in X_+(D) \,\middle|\, \mathrm{Re}\langle w|h\rangle_{L^2} = 0 \right\},$$

and $d\mathcal{E}(w)[w] = 0$, clearly Σ_+ is a natural constraint for $\mathcal{E}$. Therefore we may conclude that if $w \in \Sigma_+$ is a critical point for $\mathcal{E}$ then $\psi(w) = a(\psi_-(w))w + \psi_-(w)$ (as given in Proposition 3.6) is a critical point for $\mathcal{I}$ on Σ, namely

$$d\mathcal{I}(\psi(w))[h] - 2\omega(\psi(w))\,\mathrm{Re}\langle\psi(w)|h\rangle_{L^2} = 0, \qquad \forall h \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4).$$

For any $m \in (0, 1]$, we define the minimization problem

$$e(m) = \inf_{\substack{W\subset X_+ \\ \dim W=1}} \sup_{\psi\in\mathcal{X}_W} \mathcal{I}^{(m)}(\psi) = \inf_{w\in\Sigma_+} \mathcal{E}^{(m)}(w), \tag{4.1}$$

and $E(m) = m\,e(m)$, clearly $E = E(1) = e(1)$.

We have the following estimates on $e(m)$.

Lemma 4.2. *For any $m \in (0, 1]$ we have $0 < e(m) < 1$.*

Proof. By Lemma 3.4 we have $\lambda_W(m) = \sup_{\psi\in\mathcal{X}_W} \mathcal{I}^{(m)}(\psi) \geq (1 - \frac{\mathrm{e}^2}{2}\gamma_K)$, hence in particular we get $e(m) \geq (1 - \frac{\mathrm{e}^2}{2}\gamma_K) > 0$, for any $m \in (0, 1]$.

Now, since $\Lambda_+(D) = \frac{1}{2}U_{\mathrm{FW}}^{-1}(\mathbb{I}_4 - \beta)U_{\mathrm{FW}}$, we consider $w = U_{\mathrm{FW}}^{-1}\begin{pmatrix} 0 \\ v \end{pmatrix} \in \Sigma_+$, with $v \in H^1(\mathbb{R}^3, \mathbb{C}^2)$ and $\|v\|_{L^2}^2 = 1$. In view of Lemma 2.9, since $\|w\|_{H^{1/2}}^2 = \|v\|_{H^{1/2}}^2$ and $0 \leq \|w\|_{H^{1/2}}^2 - \|w\|_{L^2}^2 \leq \frac{1}{2}\|\nabla v\|_{L^2}^2$, for any $\psi \in \mathcal{X}_W$ we have

$$\begin{aligned}
\mathcal{I}^{(m)}(\psi) =& \|\psi_+\|_{H^{1/2}}^2 - \|\psi_-\|_{H^{1/2}}^2 - m\frac{\mathrm{e}^2}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_\psi(x)\rho_\psi(y) - J_\psi(x)\cdot J_\psi(y)}{|x-y|}dxdy \\
\leq& \|w\|_{H^{1/2}}^2 - \|w\|_{L^2}^2 - \|\psi_-\|_{L^2}^2\|w\|_{H^{1/2}}^2 - \|\psi_-\|_{H^{1/2}}^2 + 1 \\
&- m\frac{\mathrm{e}^2}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_v(x)\rho_v(y)}{|x-y|}dxdy \\
&+ 4m\mathrm{e}^2\gamma_K\|\nabla v\|_{L^2}^2 + 5m\mathrm{e}^2\gamma_K(\|\psi_-\|_{L^2}^2\|w\|_{H^{1/2}}^2 + \|\psi_-\|_{H^{1/2}}^2) \\
\leq& 1 + \frac{1}{2}(1 + 8m\mathrm{e}^2\gamma_K)\|\nabla v\|_{L^2}^2 - m\frac{\mathrm{e}^2}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_v(x)\rho_v(y)}{|x-y|}dxdy \\
&- (1 - 5m\mathrm{e}^2\gamma_K)(\|\psi_-\|_{L^2}^2\|v\|_{H^{1/2}}^2 + \|\psi_-\|_{H^{1/2}}^2).
\end{aligned}$$

Now, for any $\epsilon > 0$ we consider $v_\epsilon(x) = \epsilon^{3/2}v(\epsilon|x|)$ and $w_\epsilon = U_{\mathrm{FW}}^{-1}\begin{pmatrix} 0 \\ v_\epsilon \end{pmatrix} \in \Sigma_+$, setting $W_\epsilon = \mathrm{span}\, w_\epsilon$, then

$$e(m) - 1 \leq \sup_{\psi\in\mathcal{X}_{W_\epsilon}} \mathcal{I}^{(m)}(\psi) - 1 \leq \epsilon^2\|\nabla v\|_{L^2}^2 - \epsilon\left(m\frac{\mathrm{e}^2}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_v(x)\rho_v(y)}{|x-y|}dxdy \right),$$

hence, taking $\epsilon > 0$ sufficiently small, we may conclude that $e(m) - 1 < 0$. □

In view of the above Lemma 4.2 and thanks to the estimate in (ii)-Proposition 3.6 we have the following result, essential to the discussion of the minimization problem (4.1) when using a concentration-compactness argument.

Proposition 4.3. *$E(m)$ satisfies the strict subadditivity condition*

$$E(m) < E(m_1) + E(m_2),$$

for any $m \in (0, 1]$ and $m_1, m_2 \in (0, 1)$ such that $m_1 + m_2 = m$.

Proof. For any $\theta > 1$ and $m \in (0, 1)$ such that $\theta m \in (0, 1]$, by Proposition 3.6-(ii), for any $w \in \Sigma_+$ we have

$$\theta(\mathcal{I}^{(\theta m)}(\psi^{(\theta m)}(w)) - 1) \leq \theta^2(\mathcal{I}^{(m)}(\psi^{(\theta m)}(w)) - 1) \leq \theta^2(\mathcal{I}^{(m)}(\psi^{(m)}(w)) - 1)$$

hence we get $\theta(e(\theta m) - 1) \leq \theta^2(e(m) - 1)$.

Since by Lemma 4.2 we have $e(m) - 1 < 0$, we get $\theta^2(e(m) - 1) < \theta(e(m) - 1)$, namely $\theta e(\theta m) < \theta e(m)$. Therefore we may conclude that $E(\theta m) < \theta E(m)$.

Then, for any $m_1, m_2 \in (0, 1)$ such that $m_1 + m_2 = m \in (0, 1]$, setting $\theta_i = \frac{m}{m_i}$, then $\theta_i > 1$ and $\theta_i m_i = m \in (0, 1]$, since $\frac{1}{\theta_i} E(\theta_i m_i) < E(m_i)$, for $i = 1, 2$, we may conclude that

$$E(m) = \frac{m_1}{m} E(m) + \frac{m_2}{m} E(m) < E(m_1) + E(m_2). \quad \square$$

Now, by Ekeland's variational principle, there exists a Palais-Smale, minimizing sequence $\{w_n\} \subset \Sigma_+$, namely $\mathcal{E}(w_n) = \mathcal{I}(\psi(w_n)) \to E$ and $\|d\mathcal{E}(w_n)\| \to 0$, then by Proposition 3.6, the sequence $\psi_n = \psi(w_n)$ satisfies

$$\sup_{\|h\|_{H^{1/2}}=1} |d\mathcal{I}(\psi_n)[h] - 2\omega(\psi_n)\,\mathrm{Re}\langle\psi_n|h\rangle_{L^2}| \to 0.$$

Since $(1 - 4\mathrm{e}^2\gamma_K) + o(1) \leq \omega(\psi_n) \leq E + o(1)$ we have that $\omega(\psi_n) \to \omega \in (0, 1)$ (up to subsequence) and since $\|\psi_-(w_n)\|^2_{H^{1/2}} \leq \frac{\mathrm{e}^2}{2}\gamma_K \|w_n\|^2_{H^{1/2}}$ by Lemma 3.4 we have

$$1 = \|\psi_n\|^2_{L^2} \leq \|\psi_n\|^2_{H^{1/2}} \leq (1 + \frac{\mathrm{e}^2}{2}\gamma_K)\|w_n\|^2_{H^{1/2}} \leq \frac{1 + \frac{\mathrm{e}^2}{2}\gamma_K}{1 - \frac{\mathrm{e}^2}{2}\gamma_K}(E + o(1)).$$

Therefore $\{\psi_n\}$ is a Palais Smale sequence for the functional

$$\mathcal{I}_\omega(\psi) = \mathcal{I}(\psi) - \omega\|\psi\|^2_{L^2},$$

satisfying

$$0 < \inf_n \|\psi_n\|^2_{H^{1/2}} \leq \sup_n \|\psi_n\|^2_{H^{1/2}} < +\infty.$$

By the classical concentration-compactness principle (see [12]) we have a precise characterization of the lack of compactness of bounded Palais Smale sequences of $\mathcal{I}_w$, as given in Proposition 3.6 in [8], namely there exists a finite integer $p \geq 1$, and $\phi_1, \ldots, \phi_p \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ non trivial critical points of $\mathcal{I}_\omega$, with $\|\phi_i\|^2_{L^2} = m_i$, and p-sequences $\{x_n^i\} \subset \mathbb{R}^3$ $(i = 1, \ldots, p)$ such that $|x_n^i - x_n^j| \to +\infty$ (for $i \neq j$), and, up to subsequence,

$$\|\psi_n - \sum_{i=1}^{p} \phi_i(\cdot - x_n^i)\|_{H^{1/2}} \to 0 \qquad \text{as} \quad n \to +\infty, \tag{4.4}$$

hence, in particular, $1 = \|\psi_n\|^2_{L^2} = \sum_{i=1}^p m_i$.

Moreover, by (4.4) we have

$$\langle\psi_n - \sum_{i=1}^{p} \phi_i(\cdot - x_n^i)|(\psi_n)_+ - (\psi_n)_-\rangle_{H^{1/2}} = o(1)$$

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_{\psi_n}(x)(\psi_n(y), \psi_n(y) - \sum_{i=1}^p \phi_i(y - x_n^i))}{|x - y|} dxdy = o(1)$$

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{J_{\psi_n}(x) \cdot (\boldsymbol{\alpha}\psi_n(y), \psi_n(y) - \sum_{i=1}^p \phi_i(y - x_n^i))}{|x - y|} dxdy = o(1),$$

then, since $(\psi_n)_\pm(\cdot + x_n^i) \rightharpoonup (\phi_i)_\pm$ weakly in $H^{1/2}$, we get

$$\|(\psi_n)_+\|^2_{H^{1/2}} - \|(\psi_n)_-\|^2_{H^{1/2}} = \langle \psi_n - \sum_{i=1}^{p} \phi_i(\cdot - x_n^i) | (\psi_n)_+ - (\psi_n)_- \rangle_{H^{1/2}}$$
$$+ \sum_{i=1}^{p} \langle \phi_i | (\psi_n)_+(\cdot + x_n^i) \rangle_{H^{1/2}} - \sum_{i=1}^{p} \langle \phi_i | (\psi_n)_-(\cdot + x_n^i) \rangle_{H^{1/2}}$$
$$= \sum_{i=1}^{p} \left(\|(\phi_i)_+\|^2_{H^{1/2}} - \|(\phi_i)_-\|^2_{H^{1/2}} \right) + o(1),$$

and, by (2.10),

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_{\psi_n}(x)\rho_{\psi_n}(y)}{|x-y|} dxdy = \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_{\psi_n}(x)(\psi_n(y), \psi_n(y) - \sum_{i=1}^{p} \phi_i(y - x_n^i))}{|x-y|} dxdy$$
$$+ \sum_{i=1}^{p} \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_{\psi_n}(x + x_n^i)(\psi_n(y + x_n^i), \phi_i(y))}{|x-y|} dxdy$$
$$= \sum_{i=1}^{p} \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_{\phi_i}(x)\rho_{\phi_i}(y)}{|x-y|} dxdy + o(1),$$

analogously,

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{J_{\psi_n}(x)\cdot J_{\psi_n}(y)}{|x-y|} dxdy$$
$$= \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{J_{\psi_n}(x)\cdot(\boldsymbol{\alpha}\psi_n(y), \psi_n(y) - \sum_{i=1}^{p} \phi_i(y - x_n^i))}{|x-y|} dxdy$$
$$+ \sum_{i=1}^{p} \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{J_{\psi_n}(x + x_n^i)\cdot(\psi_n(y + x_n^i), \boldsymbol{\alpha}\phi_i(y))}{|x-y|} dxdy$$
$$= \sum_{i=1}^{p} \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{J_{\phi_i}(x)\cdot J_{\phi_i}(y)}{|x-y|} dxdy + o(1),$$

which implies $\mathcal{I}(\psi_n) = \sum_{i=1}^{p} \mathcal{I}(\phi_i) + o(1)$ and hence $E = \sum_{i=1}^{p} \mathcal{I}(\phi_i)$.

For any $i = 1, \dots, p$, we define $\psi_i = \frac{\phi_i}{\|\phi_i\|_{L^2}} = \frac{\phi_i}{\sqrt{m_i}} \in \Sigma$ then we have

$$\mathcal{I}(\phi_i) = \mathcal{I}(\sqrt{m_i}\psi_i) = m_i \mathcal{I}^{(m_i)}(\psi_i)$$

and

$$0 = d\mathcal{I}_\omega(\phi_i)[h] = \sqrt{m_i}(d\mathcal{I}^{(m_i)}(\psi_i)[h] - 2\omega \operatorname{Re}\langle \psi_i | h \rangle_{L^2}), \qquad \forall h \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4),$$

namely, for any $i = 1, \dots, p$, $\psi_i \in \Sigma$ is a critical point of $\mathcal{I}^{(m_i)}$ on Σ, with the Lagrange multiplier $\omega \in (0, 1)$.

Now, we have the following result, interesting in itself.

Lemma 4.5. *Let $\psi \in \Sigma$ be such that*

$$d\mathcal{I}^{(m)}(\psi)[h] - 2\omega \operatorname{Re}\langle \psi | h \rangle_{L^2} = 0, \qquad \forall h \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4),$$

for $m \in (0, 1]$ and $\omega \in (0, 1)$. Setting $w = \frac{\psi_+}{\|\psi_+\|_{L^2}} \in \Sigma_+$, then $\psi = \psi^{(m)}(w)$ is the unique (up to a phase factor) maximizer of $\mathcal{I}^{(m)}$ on $\mathcal{X}_W$, with $W = \operatorname{span}\{w\}$, namely

$$\mathcal{I}^{(m)}(\psi) = \mathcal{I}^{(m)}(\psi^{(m)}(w)) = \sup_{\phi \in \mathcal{X}_W} \mathcal{I}^{(m)}(\phi) = \mathcal{E}^{(m)}(w).$$

Proof. Clearly $\psi \in \mathcal{X}_W$ and it is a critical point for $\mathcal{I}^{(m)}$ on $\mathcal{X}_W$, moreover

$$\mathcal{I}^{(m)}(\psi) \geq \frac{1}{2} d\mathcal{I}^{(m)}(\psi)[\psi] = \omega > 0.$$

Therefore by Proposition 3.2 we have that ψ is a strict local maximum for $\mathcal{I}^{(m)}$ on $\mathcal{X}_W$. Moreover, by (2.4) and (2.8) we have

$$\begin{aligned}
\|\psi_+\|^2_{H^{1/2}} \geq & \omega(\|\psi_+\|^2_{L^2} - \|\psi_-\|^2_{L^2}) = \frac{1}{2} d\mathcal{I}^{(m)}(\psi)[\psi_+ - \psi_-] = \|\psi_+\|^2_{H^{1/2}} + \|\psi_-\|^2_{H^{1/2}} \\
& - me^2 \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_\psi(x)(|\psi_+|^2 - |\psi_-|^2)(y)}{|x-y|} dxdy \\
& + me^2 \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{J_\psi(x)\cdot(\mathrm{Re}(\psi_+, \boldsymbol{\alpha}\psi_+) - \mathrm{Re}(\psi_-, \boldsymbol{\alpha}\psi_-)(y)}{|x-y|} dxdy \\
\geq & \|\psi_+\|^2_{H^{1/2}} + \|\psi_-\|^2_{H^{1/2}} - 2me^2 \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_\psi(x)|\psi_+|^2(y)}{|x-y|} dxdy \\
\geq & (1 - 2me^2\gamma_K)\|\psi_+\|^2_{H^{1/2}} + \|\psi_-\|^2_{H^{1/2}},
\end{aligned}$$

that is $\|\psi_-\|^2_{H^{1/2}} \leq 2me^2\gamma_K\|\psi_+\|^2_{H^{1/2}}$.

Now, suppose that ψ is not the (unique up to a phase factor) maximizer of $\mathcal{I}^{(m)}$ on $\mathcal{X}_W$, then, again by a contradiction argument, applying the Mountain Pass theorem, as in the proof of Proposition 3.6, we find a contradiction, namely $\psi = \psi^{(m)}(w)$ and we may conclude that

$$\mathcal{I}^{(m)}(\psi) = \mathcal{I}(\psi^{(m)}(w)) = \sup_{\psi\in\mathcal{X}_W} \mathcal{I}^{(m)}(\psi). \quad \square$$

Now in view of Lemma 4.5, setting $w_i = \frac{(\psi_i)_+}{\|(\psi_i)_+\|_{L^2}}$ and $W_i = \mathrm{span}\{w_i\}$, for any $i = 1, \ldots, p$, we have that $\psi_i = \psi^{(m_i)}(w_i)$ (as in Proposition 3.6) and

$$\mathcal{I}^{(m_i)}(\psi_i) = \sup_{\psi\in\mathcal{X}_{W_i}} \mathcal{I}^{(m_i)}(\psi) = \mathcal{E}^{(m)}(w_i) \geq \inf_{w\in\Sigma_+} \mathcal{E}^{(m)}(w) = e(m_i).$$

Therefore we may conclude that

$$E = \sum_{i=1}^{p} \mathcal{I}(\phi_i) = \sum_{i=1}^{p} m_i \mathcal{I}^{(m_i)}(\psi_i) \geq \sum_{i=1}^{p} m_i e(m_i) = \sum_{i=1}^{p} E(m_i)$$

a contradiction with the strict subadditivity condition in Proposition 4.3, unless we have $p = 1$, that is $\psi_n \to \psi_1$ strongly in $H^{1/2}$, hence $\|\psi_1\|^2_{L^2} = m_1 = 1$ and

$$\mathcal{I}(\psi_1) = E = \inf_{\substack{W\subset X_+(D)\\ \dim W=1}} \sup_{\substack{\phi\in W\oplus X_-(D)\\ \|\phi\|_{L^2}=1}} \mathcal{I}(\phi).$$

Moreover

$$d\mathcal{I}(\psi_1)[h] - 2\omega\,\mathrm{Re}\langle\psi_1|h\rangle_{L^2} = 0 \qquad \forall h \in H^{1/2}(\mathbb{R}^3;\mathbb{C}^4),$$

namely $(\psi_1, \omega) \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4) \times (0, 1)$ is a weak solution of (E_ω).

Finally, to prove that E is the lowest positive critical value of the functional $\mathcal{I}$ on Σ, suppose by contrary that there exists $0 < \lambda < E$ and $\phi \in \Sigma$ such that $\mathcal{I}(\phi) = \lambda$ and

$$d\mathcal{I}(\phi)[h] - 2\mu\,\mathrm{Re}\langle\phi|h\rangle_{L^2} = 0, \qquad \forall h \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$$

with $\mu \in \mathbb{R}$ the Lagrange multiplier. Since $\lambda > 0$ we have that $\mu > 0$ (see (iii)-Proposition 3.1) and clearly $\mu \leq \lambda < 1$. Then $\mu \in (0, 1)$ and, setting $w = \frac{\phi_+}{\|\phi_+\|_{L^2}} \in \Sigma_+$, we apply Lemma 4.5 to conclude that ϕ is the unique (up to a phase factor) maximizer of $\mathcal{I}$ on $\mathcal{X}_W$, with $W = \text{span}\{w\}$, that is $\lambda = \mathcal{I}(\phi) = \sup_{\psi \in \mathcal{X}_W} \mathcal{I}(\psi) \geq E$, a contradiction. □

As a byproduct of all the previous results, with some minor changes, we obtain Theorem 1.3. Let us briefly list the differences.

Sketch of the proof of Theorem 1.3. In the Coulomb-Dirac model the self-interaction is attractive, so that we formulate the (nonlinear) eigenvalue problem using the operator $H = -D$. Clearly

$$\Lambda_\pm(H) = \Lambda_\mp(D) \qquad \text{and} \qquad X_\pm(H) = X_\mp(D),$$

then one follows the proof of Theorem 1.2 simply by exchanging the role of $\psi_\pm$ with $\psi_\mp$, indeed all the variational arguments and all the lemmata proved can be carried out, with no other changes, to deal with the functional $\mathcal{I}_{CD}$. Note that the term

$$\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{J_\psi(x)\cdot J_\psi(y)}{|x-y|} dxdy$$

is not present in the functional $\mathcal{I}_{CD}$, this in particular implies that some of the estimates provided are in fact simplified. □

Declaration of competing interest

There is no competing interest.

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Appendix A. Proof of Lemma 2.9

Lemma 2.9. *For any $\psi = \psi_+ + \psi_- \in \Sigma$, let define $w = \frac{\psi_+}{\|\psi_+\|_{L^2}}$ we have*

$$\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_\psi(x)\rho_\psi(y) - J_\psi(x)\cdot J_\psi(y)}{|x-y|} dxdy \geq \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_w(x)\rho_w(y) - J_w(x)\cdot J_w(y)}{|x-z|} dxdy$$
$$- 8\gamma_K(\|w\|^2_{H^{1/2}} - \|w\|^2_{L^2}) - 10\gamma_K(\|\psi_-\|^2_{L^2}\|w\|^2_{H^{1/2}} + \|\psi_-\|^2_{H^{1/2}}).$$

Moreover, if $v \in H^1(\mathbb{R}^3, \mathbb{C}^2)$, with $\|v\|^2_{L^2} = 1$, and $\frac{\psi_+}{\|\psi_+\|_{L^2}} = U_{FW}^{-1}\begin{pmatrix} 0 \\ v \end{pmatrix}$ we have

$$\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_\psi(x)\rho_\psi(y) - J_\psi(x)\cdot J_\psi(y)}{|x-y|} dxdy \geq \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_v(x)\rho_v(y)}{|x-y|} dxdy$$
$$- 8\gamma_K\|\nabla v\|^2_{L^2} - 10\gamma_K(\|\psi_-\|^2_{L^2}\|v\|^2_{H^{1/2}} + \|\psi_-\|^2_{H^{1/2}}).$$

Proof. For any $\psi = \psi_+ + \psi_- \in \Sigma$ with $\psi_+ = (1 - \|\psi_-\|^2_{L^2})^{1/2} w$, we have

$$\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_\psi(x)\rho_\psi(y) - J_\psi(x)\cdot J_\psi(y)}{|x-y|} dxdy$$
$$= (1 - \|\psi_-\|^2_{L^2})^2 \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_w(x)\rho_w(y) - J_w(x)\cdot J_w(y)}{|x-y|} dxdy$$

$$
\begin{aligned}
&+4 \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_{\psi_+}(x)\operatorname{Re}(\psi_+,\psi_-)(y) - J_{\psi_+}(x)\cdot\operatorname{Re}(\psi_+,\boldsymbol{\alpha}\psi_-)(y)}{|x-y|}dxdy \\
&+2 \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_{\psi}(x)\rho_{\psi_-}(y) - J_{\psi}(x)\cdot J_{\psi_-}(y)}{|x-y|}dxdy \\
&- \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_{\psi_-}(x)\rho_{\psi_-}(y) - J_{\psi_-}(x)\cdot J_{\psi_-}(y)}{|x-y|}dxdy \\
&+4 \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\operatorname{Re}(\psi_+,\psi_-)(x)\operatorname{Re}(\psi_+,\psi_-)(y)}{|x-y|}dxdy \\
&-4 \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\operatorname{Re}(\psi_+,\boldsymbol{\alpha}\psi_-)(x)\operatorname{Re}(\psi_+,\boldsymbol{\alpha}\psi_-)(y)}{|x-y|}dxdy,
\end{aligned}
$$

hence by (2.4), (2.6), (2.7) and (2.8) we get

$$
\begin{aligned}
&\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_{\psi}(x)\rho_{\psi}(y) - J_{\psi}(x)\cdot J_{\psi}(y)}{|x-y|} \ge \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_{w}(x)\rho_{w}(y) - J_{w}(x)\cdot J_{w}(y)}{|x-y|} \\
&- 2\gamma_K\|\psi_-\|^2_{L^2}\|w\|^2_{H^{1/2}} - \gamma_K\|\psi_-\|^2_{L^2}\|\psi_-\|^2_{H^{1/2}} - 4\gamma_K\|\psi_-\|_{H^{1/2}}\|\psi_-\|_{L^2}\|w\|_{H^{1/2}} \\
&- 4\Big|\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_w(x)\operatorname{Re}(w,\psi_-)(y)}{|x-y|}\Big| - 4\Big|\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{J_w(x)\cdot\operatorname{Re}(w,\boldsymbol{\alpha}\psi_-)(y)}{|x-y|}\Big|.
\end{aligned}
$$

Since $\Lambda_\pm(D) = \frac{1}{2}U_{\mathrm{FW}}^{-1}(\mathbb{I}_4 \mp \beta)U_{\mathrm{FW}}$, we set $w = U_{\mathrm{FW}}^{-1}\begin{pmatrix}0\\ v\end{pmatrix}$ and $\psi_- = U_{\mathrm{FW}}^{-1}\begin{pmatrix}\eta\\ 0\end{pmatrix}$ with $v,\eta \in H^{1/2}(\mathbb{R}^3,\mathbb{C}^2)$ and $\|v\|^2_{L^2} = 1$, in view of Remark 2.5 we have

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_w(x)\operatorname{Re}(w,\psi_-)(y)}{|x-y|}dxdy\Big| &= (2\pi)^{\frac{3}{2}}\sqrt{\frac{2}{\pi}}\left|\int_{\mathbb{R}^3} \frac{\hat\rho_w(p)\mathcal{F}[\operatorname{Re}(w,\psi_-)](p)}{|p|^2}dp\right| \\
&\le \sqrt{\frac{2}{\pi}}\|\rho_w\|_{L^1}\int_{\mathbb{R}^3}\frac{|\mathcal{F}[\operatorname{Re}(w,\psi_-)](p)|}{|p|^2}dp \\
&\le \sqrt{\frac{2}{\pi}}\int_{\mathbb{R}^3}\frac{1}{|p|^2}\left(\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}|(\hat w(p-q),\hat\psi_-(q))|\,dq\right)dp.
\end{aligned}
$$

Since $U^{-1}(p) = u_+(p)\mathbb{I}_4 - u_-(p)\beta\frac{\boldsymbol{\alpha}\cdot p}{|p|}$ with $u_\pm(p) = \sqrt{\frac{1}{2}(1\pm\frac{1}{\lambda(p)})}$ we have

$$
\begin{aligned}
(\hat w(p-q),\hat\psi_-(q)) &= \left(U^{-1}(p-q)\begin{pmatrix}0\\ \hat v(p-q)\end{pmatrix}, U^{-1}(q)\begin{pmatrix}\hat\eta(q)\\ 0\end{pmatrix}\right) \\
&= -u_+(q)u_-(p-q)\frac{p-q}{|p-q|}\cdot(\boldsymbol{\sigma}\hat v(p-q),\hat\eta(q)) \\
&\quad + u_+(p-q)u_-(q)\frac{q}{|q|}\cdot(\hat v(p-q),\boldsymbol{\sigma}\hat\eta(q)) \\
&= u_-(p-q)(u_+(p-q)\frac{q}{|q|} - u_+(q)\frac{p-q}{|p-q|})\cdot(\boldsymbol{\sigma}\hat v(p-q),\hat\eta(q)) \\
&\quad + u_+(p-q)(u_-(q)-u_-(p-q))\frac{q}{|q|}\cdot(\hat v(p-q),\boldsymbol{\sigma}\hat\eta(q)),
\end{aligned}
$$

and, since $u_-(q) \geq \frac{|q|}{2\lambda(q)}$, we have

$$|u_-(q) - u_-(p-q)| = \frac{\frac{1}{2}|\frac{1}{\lambda(q)} - \frac{1}{\lambda(p-q)}|}{u_-(q) + u_-(p-q)} \leq \frac{|\lambda(p-q) - \lambda(q)|}{|q|\lambda(p-q) + |p-q|\lambda(q)}$$
$$= \frac{||q|^2 - |p-q|^2|}{(|q|\lambda(p) + |p-q|\lambda(q))(\lambda(q) + \lambda(p-q))}$$
$$\leq \frac{|p|}{(\lambda(p-q)+1)^{1/2}(\lambda(q)+1)^{1/2}}.$$

Then we get

$$|(\hat{w}(p-q), \hat{\psi}_-(q))| \leq 2u_-(p-q)|\hat{v}(p-q)||\hat{\eta}(q))| + |p|\frac{|\hat{v}(p-q)|}{(\lambda(p-q)+1)^{\frac{1}{2}}}\frac{|\hat{\eta}(q))|}{(\lambda(q)+1)^{\frac{1}{2}}},$$

and we obtain

$$|\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_w(x)\,\mathrm{Re}(w,\psi_-)(y)}{|x-y|}dxdy| \leq \sqrt{\frac{2}{\pi}}\int_{\mathbb{R}^3} \frac{2}{|p|^2}\mathcal{F}\left[\mathcal{F}^{-1}[u_-(p)|\hat{v}|]\mathcal{F}^{-1}[|\hat{\eta}|]\right]dp$$
$$+\sqrt{\frac{2}{\pi}}\int_{\mathbb{R}^3}\frac{1}{|p|}\mathcal{F}\left[\mathcal{F}^{-1}\left[\frac{|\hat{v}|}{(\lambda(p)+1)^{\frac{1}{2}}}\right]\mathcal{F}^{-1}\left[\frac{|\hat{\eta}|}{(\lambda(p)+1)^{\frac{1}{2}}}\right]\right]dp.$$

Now by Kato's inequality we have

$$\sqrt{\frac{2}{\pi}}\int_{\mathbb{R}^3} \frac{2}{|p|^2}\mathcal{F}\left[\mathcal{F}^{-1}[u_-(p)|\hat{v}|]\mathcal{F}^{-1}[|\hat{\eta}|]\right]dp = \int_{\mathbb{R}^3}\frac{2}{|x|}\mathcal{F}^{-1}[u_-(p)|\hat{v}|]\mathcal{F}^{-1}[|\hat{\eta}|]\,dx$$
$$\leq 2\left\|\frac{\mathcal{F}^{-1}[u_-(p)|\hat{v}|]}{|x|^{\frac{1}{2}}}\right\|_{L^2}\left\|\frac{\mathcal{F}^{-1}[|\hat{\eta}|]}{|x|^{\frac{1}{2}}}\right\|_{L^2} \leq \sqrt{2}\gamma_K\|(\lambda(p)-1)^{\frac{1}{2}}\hat{v}\|_{L^2}\|\eta\|_{H^{1/2}}$$
$$\leq \frac{1}{2}\gamma_K(\|w\|^2_{H^{1/2}} - \|w\|^2_{L^2}) + \gamma_K\|\psi_-\|^2_{H^{1/2}},$$

since $\|(\lambda(p)-1)^{\frac{1}{2}}\hat{v}\|^2_{L^2} = \|v\|^2_{H^{1/2}} - \|v\|^2_{L^2} = \|w\|^2_{H^{1/2}} - \|w\|^2_{L^2}$.

Moreover, by Hardy's inequality we have

$$\sqrt{\frac{2}{\pi}}\int_{\mathbb{R}^3}\frac{1}{|p|}\mathcal{F}\left[\mathcal{F}^{-1}\left[\frac{|\hat{v}|}{(\lambda(p)+1)^{\frac{1}{2}}}\right]\mathcal{F}^{-1}\left[\frac{|\hat{\eta}|}{(\lambda(p)+1)^{\frac{1}{2}}}\right]\right]dp$$
$$=\frac{2}{\pi}\int_{\mathbb{R}^3}\frac{1}{|x|^2}\mathcal{F}^{-1}\left[\frac{|\hat{v}|}{(\lambda(p)+1)^{\frac{1}{2}}}\right]\mathcal{F}^{-1}\left[\frac{|\hat{\eta}|}{(\lambda(p)+1)^{\frac{1}{2}}}\right]dx$$
$$\leq\frac{2}{\pi}\left\|\frac{1}{|x|}\mathcal{F}^{-1}\left[\frac{|\hat{v}|}{(\lambda(p)+1)^{\frac{1}{2}}}\right]\right\|_{L^2}\left\|\frac{1}{|x|}\mathcal{F}^{-1}\left[\frac{|\hat{\eta}|}{(\lambda(p)+1)^{\frac{1}{2}}}\right]\right\|_{L^2}$$
$$\leq\frac{8}{\pi}\left\|\frac{|p||\hat{v}|}{(\lambda(p)+1)^{\frac{1}{2}}}\right\|_{L^2}\left\|\frac{|p||\hat{\eta}|}{(\lambda(p)+1)^{\frac{1}{2}}}\right\|_{L^2} \leq \gamma_K(\|w\|^2_{H^{1/2}} - \|w\|^2_{L^2}) + \gamma_K\|\psi_-\|^2_{H^{1/2}}$$

since $\left\|\frac{|p||\hat{v}|}{(\lambda(p)+1)^{\frac{1}{2}}}\right\|^2_{L^2} = \|(\lambda(p)-1)^{\frac{1}{2}}\hat{v}\|^2_{L^2}$ and $\gamma_K = \frac{\pi}{2}$.

Analogously we have

$$
\begin{aligned}
&\Big| \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\mathrm{Re}(w,\boldsymbol{\alpha}\psi_-)(x)\cdot J_w(y)}{|x-y|} dxdy\Big| \\
&\qquad = (2\pi)^{\frac{3}{2}}\sqrt{\frac{2}{\pi}}\left|\int\limits_{\mathbb{R}^3} \frac{\mathcal{F}[\mathrm{Re}(w,\boldsymbol{\alpha}\psi_-)](p)\cdot \mathcal{F}[\mathrm{Re}(w,\boldsymbol{\alpha}w)](p)}{|p|^2}\, dp\right| \\
&\qquad \le \sqrt{\frac{2}{\pi}}\|(w,\boldsymbol{\alpha}\psi_-)\|_{L^1}\int\limits_{\mathbb{R}^3} \frac{|\mathcal{F}[\mathrm{Re}(w,\boldsymbol{\alpha}w)](p)|}{|p|^2}\, dp \\
&\qquad \le \sqrt{\frac{2}{\pi}}\|w\|_{L^2}\|\psi_-\|_{L^2}\int\limits_{\mathbb{R}^3}\frac{1}{|p|^2}\left(\frac{1}{(2\pi)^{\frac{3}{2}}}\int\limits_{\mathbb{R}^3}|(\hat{w}(p-q),\boldsymbol{\alpha}\hat{w}(q))|\,dq\right) dp,
\end{aligned}
$$

and since

$$
\begin{aligned}
|(\hat{w}(p-q),\boldsymbol{\alpha}\hat{w}(q))| &\le (u_+(p-q)u_-(q)+u_+(q)u_-(p-q))|\hat{v}(q)||\hat{v}(p-q)| \\
&\le (u_-(q)+u_-(p-q))|\hat{v}(q)||\hat{v}(p-q)|,
\end{aligned}
$$

then by Kato's inequality we have

$$
\begin{aligned}
&\Big| \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\mathrm{Re}(w,\boldsymbol{\alpha}\psi_-)(x)\cdot J_w(y)}{|x-y|} dxdy\Big| \\
&\qquad \le 2\sqrt{\frac{2}{\pi}}\|\psi_-\|_{L^2}\int\limits_{\mathbb{R}^3}\frac{1}{|p|^2}\mathcal{F}\left[\mathcal{F}^{-1}[u_-(p)|\hat{v}|]\mathcal{F}^{-1}[|\hat{v}|]\right] dp \\
&\qquad \le 2\|\psi_-\|_{L^2}\left\|\frac{\mathcal{F}^{-1}[u_-(p)|\hat{v}|]}{|x|^{\frac{1}{2}}}\right\|_{L^2}\left\|\frac{\mathcal{F}^{-1}[|\hat{v}|]}{|x|^{\frac{1}{2}}}\right\|_{L^2} \\
&\qquad \le \sqrt{2}\gamma_K\|(\lambda(p)-1)^{\frac{1}{2}}\hat{v}\|_{L^2}\|\psi_-\|_{L^2}\|v\|_{H^{1/2}} \\
&\qquad \le \frac{1}{2}\gamma_K(\|w\|^2_{H^{1/2}}-\|w\|^2_{L^2})+\gamma_K\|\psi_-\|^2_{L^2}\|w\|^2_{H^{1/2}}.
\end{aligned}
$$

Therefore we may conclude that

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\rho_\psi(x)\rho_\psi(y)-J_\psi(x)\cdot J_\psi(y)}{|x-y|}dxdy - \int\limits_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\rho_w(x)\rho_w(y)-J_w(x)\cdot J_w(y)}{|x-y|}dxdy \\
&\quad \ge -8\gamma_K(\|w\|^2_{H^{1/2}}-|w|^2_{L^2})-10\gamma_K\|\psi_-\|^2_{H^{1/2}}-10\gamma_K|\psi_-|^2_{L^2}\|w\|^2_{H^{1/2}}.
\end{aligned}
$$

Moreover, by [8, Lemma 2.1] we have that

$$
\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\rho_w(x)\rho_w(y)-J_w(x)\cdot J_w(y)}{|x-y|}dxdy \ge \int\limits_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(w,\beta w)(x)(w,\beta w)(y)}{|x-y|}dxdy.
$$

Now for $w=U_{\mathrm{FW}}^{-1}\begin{pmatrix}0\\ v\end{pmatrix}$ with $v\in H^1(\mathbb{R}^3,\mathbb{C}^2)$ and $\|v\|^2_{L^2}=1$, we have

$$
\begin{aligned}
(w,\beta w)(x) &= \left(\begin{pmatrix}\mathcal{F}^{-1}[u_-(p)\frac{p}{|p|}\cdot\boldsymbol{\sigma}\hat{v}]\\ \mathcal{F}^{-1}[u_+(p)\hat{v}]\end{pmatrix},\begin{pmatrix}\mathcal{F}^{-1}[u_-(p)\frac{p}{|p|}\cdot\boldsymbol{\sigma}\hat{v}]\\ -\mathcal{F}^{-1}[u_+(p)\hat{v}]\end{pmatrix}\right)(x) \\
&= |\mathcal{F}^{-1}[u_-(p)\frac{p}{|p|}\cdot\boldsymbol{\sigma}\hat{v}]|^2(x)-|\mathcal{F}^{-1}[u_+(p)\hat{v}]|^2(x)=|\xi|^2(x)-|f|^2(x)
\end{aligned}
$$

where $f=\mathcal{F}^{-1}[u_+(p)\hat{v}]$ and $\xi=\mathcal{F}^{-1}[u_-(p)\frac{p}{|p|}\cdot\boldsymbol{\sigma}\hat{v}]$, then we have

$$\begin{aligned}\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(w,\beta w)(x)(w,\beta w)(y)}{|x-y|}dxdy&=\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|f|^2(x)|f|^2(y)}{|x-y|}dxdy\\&+\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|\xi|^2(x)|\xi|^2(y)}{|x-y|}dxdy-2\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|f|^2(x)|\xi|^2(y)}{|x-y|}dxdy\\&\geq\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|f|^2(x)|f|^2(y)}{|x-y|}dxdy-2\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|f|^2(x)|\xi|^2(y)}{|x-y|}dxdy\\&\geq\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|f|^2(x)|f|^2(y)}{|x-y|}dxdy-2\gamma_K\|f\|^2_{L^2}\|\xi\|^2_{H^{1/2}}.\end{aligned}$$

Moreover, setting $\chi=\mathcal{F}^{-1}[(1-u_+(p))\hat{v}]$, since $f=v-\chi$, by (2.6) we have

$$\begin{aligned}\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|f|^2(x)|f|^2(y)}{|x-y|}dxdy&=\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|v|^2(x)|v|^2(y)}{|x-y|}dxdy\\&-4\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|v|^2(x)\,\mathrm{Re}(v,\chi)(y)}{|x-y|}dxdy+2\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|f|^2(x)|\chi|^2(y)}{|x-y|}dxdy\\&-\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|\chi|^2(x)|\chi|^2(y)}{|x-y|}dxdy+4\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\mathrm{Re}(v,\chi)(x)\,\mathrm{Re}(v,\chi)(y)}{|x-y|}dxdy\\&\geq\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|v|^2(x)|v|^2(y)}{|x-y|}dxdy-\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|\chi|^2(x)|\chi|^2(y)}{|x-y|}dxdy\\&-4\left|\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|v|^2(x)\,\mathrm{Re}(v,\chi)(y)}{|x-y|}dxdy\right|.\end{aligned}$$

Since by Hardy's inequality we have

$$\Big|\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|v|^2(x)\,\mathrm{Re}(v,\chi)(y)}{|x-y|}dxdy\Big|\leq 2\|v\|^2_{L^2}\|\chi\|_{L^2}\|\nabla v\|_{L^2},$$

then, since $\gamma_K=\frac{\pi}{2}$, by (2.4) we get

$$\begin{aligned}\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|f|^2(x)|f|^2(y)}{|x-y|}dxdy\geq&\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|v|^2(x)|v|^2(y)}{|x-y|}dxdy\\&-\gamma_K\|\chi\|^2_{L^2}\|\chi\|^2_{H^{1/2}}-\frac{16}{\pi}\gamma_K\|\chi\|_{L^2}\|\nabla v\|_{L^2}.\end{aligned}$$

Moreover we have

$$\begin{aligned}\|\xi\|^2_{H^{1/2}}=&\|\lambda(p)^{1/2}u_-(p)|\frac{p}{|p|}\cdot\boldsymbol{\sigma}\hat{v}|\|^2_{L^2}=\|\lambda(p)^{1/2}u_-(p)\hat{v}\|^2_{L^2}\\=&\frac{1}{2}(\|v\|^2_{H^{1/2}}-\|v\|^2_{L^2})\leq\frac{1}{4}\|\nabla v\|^2_{L^2}\end{aligned}$$

and $\|f\|^2_{L^2}\leq\|v\|^2_{L^2}=1$. Since

$$|u_+(p)-1| = \frac{1-u_+(p)^2}{1+u_+(p)} = \frac{u_-(p)^2}{1+u_+(p)} \le u_-(p)^2 \le \frac{1}{2}\frac{|p|^2}{\lambda(p)+1} \le \frac{1}{2}|p|$$

we have $\|\chi\|_{L^2}^2 \le \frac{1}{4}\|v\|_{L^2}^2 = \frac{1}{4}$, but also

$$\|\chi\|_{L^2}^2 = \|(u_+(p)-1)\hat{v}\|_{L^2}^2 \le \frac{1}{4}\|\nabla v\|_{L^2}^2$$

and

$$\begin{aligned}\|\chi\|_{H^{1/2}}^2 =& \|\lambda(p)^{1/2}|(u_+(p)-1)\hat{v}|\|_{L^2}^2 \le \|\lambda(p)^{1/2}u_-(p)\hat{v}\|_{L^2}^2 \\ =& \frac{1}{2}(\|v\|_{H^{1/2}}^2 - \|v\|_{L^2}^2) \le \frac{1}{4}\|\nabla v\|_{L^2}^2.\end{aligned}$$

Therefore, we may conclude that

$$\begin{aligned}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(w,\beta w)(x)(w,\beta w)(y)}{|x-y|}dxdy \ge& \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|f|^2(x)|f|^2(y)}{|x-y|}dxdy - 2\gamma_K\|\xi\|_{H^{1/2}}^2 \\ \ge& \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|v|^2(x)|v|^2(y)}{|x-y|}dxdy - \gamma_K\|\chi\|_{L^2}^2\|\chi\|_{H^{1/2}}^2 \\ &- \frac{16}{\pi}\gamma_K\|\chi\|_{L^2}\|\nabla v\|_{L^2} - 2\gamma_K\|\xi\|_{H^{1/2}}^2 \\ \ge& \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|v|^2(x)|v|^2(y)}{|x-y|}dxdy - 4\gamma_K\|\nabla v\|_{L^2}^2. \quad \square\end{aligned}$$

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