Research Article

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Butterfly support for off diagonal coefficients and boundedness of solutions to quasilinear elliptic systems

https://doi.org/10.1515/anona-2021-0205
Received April 21, 2021; accepted August 5, 2021.

Abstract: We consider quasilinear elliptic systems in divergence form. In general, we cannot expect that weak solutions are locally bounded because of De Giorgi's counterexample. Here we assume that off-diagonal coefficients have a "butterfly support": this allows us to prove local boundedness of weak solutions.

Keywords: Quasilinear, elliptic, system, weak, solution, regularity

MSC: Primary: 35J47; Secondary: 35B65, 49N60

1 Introduction

This paper deals with quasilinear elliptic systems in divergence form

$$-div(a(x, u(x))Du(x)) = 0, \quad x \in \Omega, \tag{1.1}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^{N^2 n^2}$ is matrix valued with components $a_{i,j}^{\alpha,\beta}(x, y)$ where $i, j \in \{1, \dots, n\}$ and $\alpha, \beta \in \{1, \dots, N\}$.

On the coefficients $a_{i,j}^{\alpha,\beta}(x, y)$ we set the usual conditions, that is they are measurable with respect to x, continuous with respect to y, bounded and elliptic. When $N = 1$, that is in the case of one single equation, the celebrated De Giorgi-Nash-Moser theorem ensures that weak solutions $u \in W^{1,2}(\Omega)$ are locally bounded and even Hölder continuous, see section 2.1 in [27].

But in the vectorial case $N \geq 2$, the aforementioned result is no longer true due to the De Giorgi's counterexample, see [6], section 3 in [27] and the recent paper [29]; see also [32] and [20].

So it arises the question of finding additional structural restrictions on the coefficients $a_{i,j}^{\alpha,\beta}$ that keep away De Giorgi's counterexample and allow for local boundedness of weak solutions u, see Section 3.9 in [28].

In the present work we assume a condition on the support of off-diagonal coefficients: there exists $L_0 \in [0, +\infty)$ such that $\forall\, L \geq L_0$, when $\alpha \neq \beta$,

$$(a_{i,j}^{\alpha,\beta}(x, y) \neq 0 \text{ and } |y^\alpha| > L) \Rightarrow \left|y^\beta\right| > L, \tag{1.2}$$

(see Figure 1 and note that the support has the shape of a butterfly in the plane $y^\beta - y^\alpha$).

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Under such a restriction we are able to prove local boundedness of weak solutions. All the necessary assumptions and the result will be listed in section 2 while proofs will be performed in section 3.

It is worth to stress out that systems with special structure have been studied in [33], [26] and off-diagonal coefficients with a particular support have been successfully used when proving maximum principles in [21], L^∞-regularity in [22], when obtaining existence for measure data problems in [23], [24] and, for the degenerate case, in [7].

Higher integrability has been studied as well in [10] when off-diagonal coefficients are small and have staircase support and in [11] when off-diagonal coefficients are proportional to diagonal ones.

Let us mention as well that when the ratio between the largest and the smallest eigenvalues of $a_{i,j}^{\alpha,\beta}$ is close to 1, then regularity of u is studied at page 183 of [12]; see also [31], [18], [17], [19].

Let us also say that proving boundedness for weak solutions could be an important tool for getting fractional differentiability, see the estimate after (4.15) in [8]. In the present paper we deal with local boundedness of solutions. If the reader is interested in regularity up to a rough boundary it could be worth looking at [25].

2 Assumptions and Result

Assume Ω is an open bounded subset of $\mathbb{R}^n$, with $n \geq 3$. Consider the system of $N \geq 2$ equations

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{\alpha,\beta=1}^{N}\sum_{j=1}^{n} a_{i,j}^{\alpha,\beta}(x,u)\frac{\partial}{\partial x_j}u^\beta\right) = 0 \text{ in } \Omega, \text{ for } \alpha = 1,\dots,N. \tag{2.1}$$

Note that u^β is the β component of $u = (u^1, u^2, \dots, u^N)$. We list our structural conditions.

($\mathcal{A}$) *For all $i, j \in \{1,\dots,n\}$ and all $\alpha, \beta \in \{1,\dots,N\}$, we require that $a_{i,j}^{\alpha,\beta} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following conditions:*

($\mathcal{A}_0$)*$x \mapsto a_{i,j}^{\alpha,\beta}(x,y)$ is measurable and $y \mapsto a_{i,j}^{\alpha,\beta}(x,y)$ is continuous;*

($\mathcal{A}_1$)*(boundedness of all the coefficients) for some constant $c > 0$, we have*

$$|a_{i,j}^{\alpha,\beta}(x,y)| \leq c$$

for almost all $x \in \Omega$ and for all $y \in \mathbb{R}^N$;

($\mathcal{A}_2$)*(ellipticity of all the coefficients) for some constant $\nu > 0$, we have*

$$\sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x,y)\,\xi_i^\alpha \xi_j^\beta \geq \nu|\xi|^2$$

for almost all $x \in \Omega$, for all $y \in \mathbb{R}^N$ and for all $\xi \in \mathbb{R}^{N\times n}$;

($\mathcal{A}_3$)*("butterfly" support of off-diagonal coefficients) there exists $L_0 \in [0,+\infty)$ such that $\forall\, L \geq L_0$, when $\alpha \neq \beta$,*

$$(a_{i,j}^{\alpha,\beta}(x,y) \neq 0 \text{ and } |y^\alpha| > L) \Rightarrow |y^\beta| > L\,,$$

(see Figure 1).

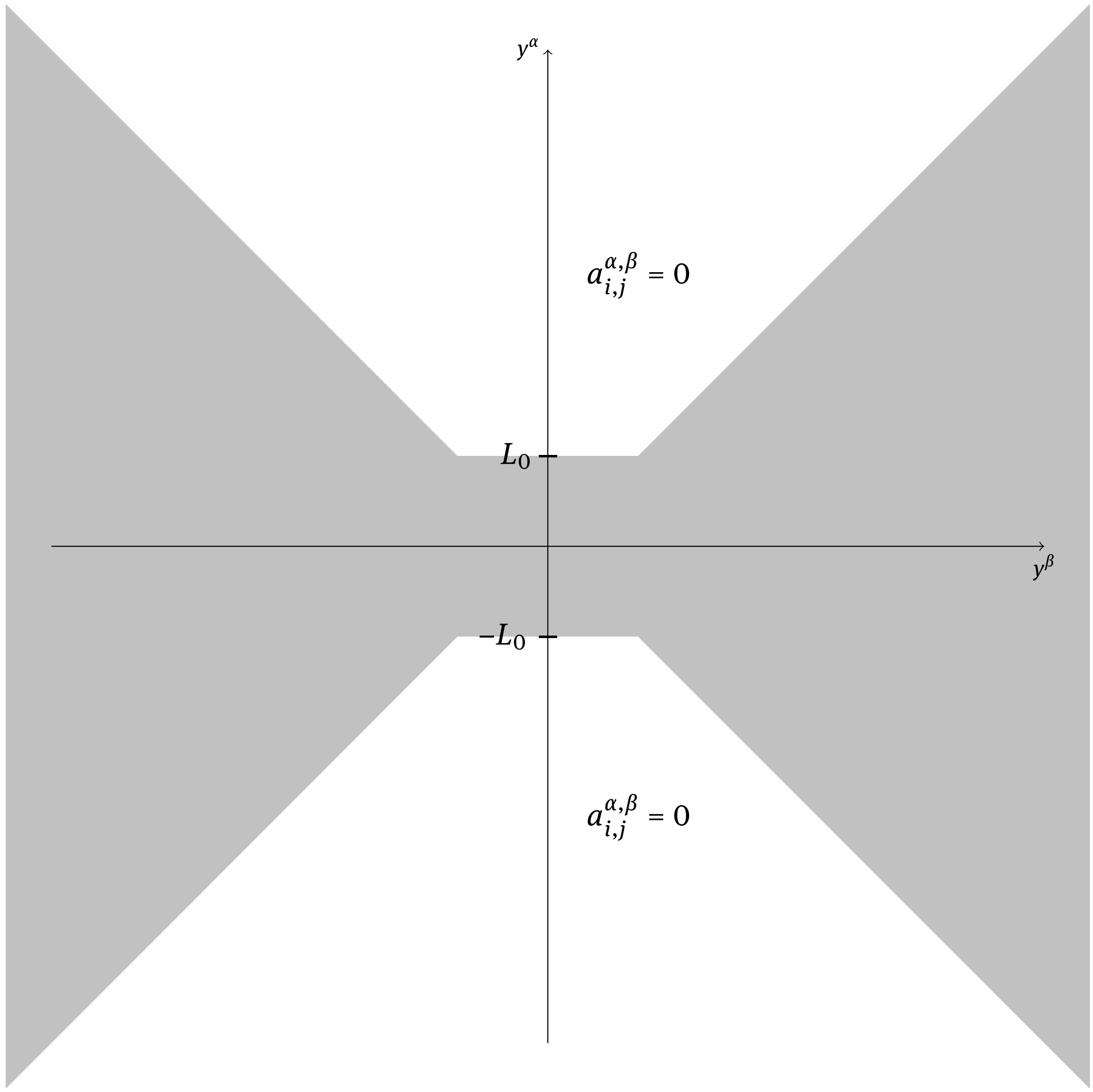

Fig. 1: Assumption $(\mathcal{A}_3)$: off-diagonal coefficients $a_{i,j}^{\alpha,\beta}$ vanish on the white part of the picture; they might be non zero only on the grey part.

Remark 2.1. *Assumption $(\mathcal{A}_3)$ guarantees equality (3.2): such an equality is a basic tool for proving boundedness of solutions.*

Example 2.2. *An example of coefficients which readily satisfy the aforementioned assumptions are defined as follows:*

$$a_{i,j}^{\alpha,\beta}(x,y) = a_{i,j}^{\alpha,\beta}(y) = \begin{cases} \delta_{ij} \dfrac{\max(|y^\beta| - |y^\alpha|, 0)}{1 + 2|y|} & \text{if } \alpha \neq \beta \\ \delta_{ij} & \text{if } \alpha = \beta \end{cases}$$

where $\alpha, \beta = 1, 2$ and $i, j = 1, \dots, n$ with $n \geq 3$ and $N = 2$. In this case we have $c = 1$, $\nu = 1/2$ and we can pick for instance $L_0 = 0$.

We say that a function $u : \Omega \to \mathbb{R}^N$ is a weak solution of the system (2.1), if $u \in W^{1,2}\left(\Omega, \mathbb{R}^N\right)$ and

$$\int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x, u(x)) D_j u^\beta(x) D_i \varphi^\alpha(x) dx = 0, \tag{2.2}$$

for all $\varphi \in W_0^{1,2}\left(\Omega, \mathbb{R}^N\right)$.

Theorem 2.3. *Let $u \in W^{1,2}\left(\Omega, \mathbb{R}^N\right)$ be a weak solution of system (2.1) under the set $(\mathcal{A})$ of assumptions. Then $u \in L^\infty_{loc}\left(\Omega, \mathbb{R}^N\right)$ and we have the following estimate*

$$\sup_{B(x_0,r)} |u^\alpha| \le 2 \max \left\{ L_0; \left(\frac{\left[\frac{2(n-1)}{(n-2)}\right]^n \left[4 + \frac{16c^2 n^4 N^4}{\nu^2}\right]^{n/2} 2^{4n+2+nn/2}}{(R-r)^n} \sum_{\beta=1}^{N} \int_{B(x_0,R)} |u^\beta|^2 \right)^{1/2} \right\} \tag{2.3}$$

for every $\alpha = 1, \dots, N$ and for every r, R with $0 < r < R$ and $B(x_0, R) \subset \Omega$, where c is the constant involved in assumption $(\mathcal{A}_1)$, ν is given in $(\mathcal{A}_2)$ and L_0 appears in $(\mathcal{A}_3)$.

Remark 2.4. *The present local L^∞–regularity result improves on [22] since assumption $(\mathcal{A}_3)$ allows off diagonal coefficients to have a larger support than in [22].*

Remark 2.5. *"Butterfly" support $(\mathcal{A}_3)$ has been used in [7] when proving the existence of* **at least one** *globally bounded solution to a (possibly) degenerate problem with zero boundary value problem. In the present work we prove local boundedness of* **every** *solution to a non degenerate system regardless of boundary values.*

3 Proof of the result

The proof of Theorem 2.3 will be performed in several steps

STEP 1. Caccioppoli inequality

Lemma 3.1. *(Caccioppoli inequality on superlevel sets) Let $u \in W^{1,2}\left(\Omega, \mathbb{R}^N\right)$ be a weak solution of system (2.1) under assumptions $(\mathcal{A}_0)$, $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_3)$. For $0 < s < t$, let $B(x_0, s)$ and $B(x_0, t)$ be concentric open balls centered at x_0 with radii s and t respectively. Assume that $B(x_0, t) \subset \Omega$ and $L \ge L_0$. Then*

$$\sum_{\alpha=1}^{N} \int_{\{|u^\alpha|>L\}\cap B(x_0,s)} \left|D\left|u^\alpha\right|\right|^2 dx \le \frac{16c^2 n^4 N^4}{\nu^2} \sum_{\alpha=1}^{N} \int_{\{|u^\alpha|>L\}\cap B(x_0,t)} \left(\frac{|u^\alpha| - L}{t-s}\right)^2 dx, \tag{3.1}$$

where c is the constant involved in assumption $(\mathcal{A}_1)$, ν is given in $(\mathcal{A}_2)$ and L_0 appears in $(\mathcal{A}_3)$.

Proof of Lemma 3.1 Let $u \in W^{1,2}\left(\Omega, \mathbb{R}^N\right)$ be a weak solution of system (2.1). Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be the standard cut-off function such that $0 \le \eta \le 1$, $\eta \in C_0^1(B(x_0, t))$, with $B(x_0, t) \subset \Omega$ and $\eta = 1$ in $B(x_0, s)$. Moreover, $|D\eta| \le 2/(t-s)$ in $\mathbb{R}^n$. For every level $L \ge L_0$, consider

$$T_L(s) = \begin{cases} -L & \text{if} \quad s < -L \\ s & \text{if} \quad -L \le s \le L \\ L & \text{if} \quad s > L \end{cases}$$

and

$$G_L(s) = s - T_L(s).$$

We define $\varphi : \mathbb{R}^n \to \mathbb{R}^N$ with $\varphi = (\varphi^1, \dots, \varphi^N)$, where

$$\varphi^\alpha := \eta^2 G_L\left(u^\alpha\right), \qquad \text{for all } \alpha \in \{1, \dots, N\}.$$

Then

$$D_i \varphi^\alpha = \eta^2 1_{\{|u^\alpha|>L\}} D_i u^\alpha + 2\eta(D_i\eta) 1_{\{|u^\alpha|>L\}} G_L\left(u^\alpha\right) \qquad \text{for all } i \in \{1, \dots, n\} \text{ and } \alpha \in \{1, \dots, N\}.$$

Using this test function in the weak formulation (2.2) of system (2.1), we have

$$\begin{aligned} 0 = \int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta} D_j u^\beta D_i \varphi^\alpha \, dx = \\ \int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta} D_j u^\beta \eta^2 1_{\{|u^\alpha|>L\}} D_i u^\alpha \, dx + \int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta} D_j u^\beta 2\eta(D_i\eta) 1_{\{|u^\alpha|>L\}} G_L\left(u^\alpha\right) dx. \end{aligned}$$

Now, assumption $(\mathcal{A}_3)$ guarantees that

$$a_{i,j}^{\alpha,\beta}(x, u(x))\, 1_{\{|u^\alpha|>L\}}(x) = a_{i,j}^{\alpha,\beta}(x, u(x))\, 1_{\{|u^\beta|>L\}}(x)\, 1_{\{|u^\alpha|>L\}}(x) \tag{3.2}$$

when $\beta \neq \alpha$ and $L \geq L_0$. It is worthwhile to note that (3.2) holds true when $\alpha = \beta$ as well; then

$$\begin{aligned} &\int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta} 1_{\{|u^\beta|>L\}} D_j u^\beta \eta^2 1_{\{|u^\alpha|>L\}} D_i u^\alpha \, dx \\ &\qquad = -\int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta} 1_{\{|u^\beta|>L\}} D_j u^\beta 2\eta(D_i\eta) 1_{\{|u^\alpha|>L\}} G_L\left(u^\alpha\right) \, dx. \end{aligned} \tag{3.3}$$

Now we can use ellipticity assumption $(\mathcal{A}_2)$ with $\xi_i^\alpha = 1_{\{|u^\alpha|>L\}} D_i u^\alpha$ and we get

$$\nu \int_\Omega \eta^2 \sum_{\alpha=1}^N 1_{\{|u^\alpha|>L\}} \left|Du^\alpha\right|^2 \, dx \leq \int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta} 1_{\{|u^\beta|>L\}} D_j u^\beta \eta^2 1_{\{|u^\alpha|>L\}} D_i u^\alpha \, dx. \tag{3.4}$$

Moreover

$$\left|G_L\left(u^\alpha\right)\right| = \left|u^\alpha\right| - L \text{ where } \left|u^\alpha\right| > L \tag{3.5}$$

and

$$\begin{aligned} -\int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta} 1_{\{|u^\beta|>L\}} D_j u^\beta 2\eta(D_i\eta) 1_{\{|u^\alpha|>L\}} G_L\left(u^\alpha\right) \, dx \leq \\ \int_\Omega c \sum_{\beta=1}^N \sum_{j=1}^n 1_{\{|u^\beta|>L\}} |D_j u^\beta| \sum_{\alpha=1}^N \sum_{i=1}^n 2\eta |D_i\eta| 1_{\{|u^\alpha|>L\}} \left|G_L\left(u^\alpha\right)\right| \, dx \leq \\ \int_\Omega c \sum_{\beta=1}^N n 1_{\{|u^\beta|>L\}} |Du^\beta| \sum_{\alpha=1}^N n 2\eta |D\eta| 1_{\{|u^\alpha|>L\}} \left|G_L\left(u^\alpha\right)\right| \, dx \leq \\ \int_\Omega cn^2 \epsilon \eta^2 \left(\sum_{\beta=1}^N 1_{\{|u^\beta|>L\}} |Du^\beta|\right)^2 + \int_\Omega \frac{cn^2}{\epsilon} |D\eta|^2 \left(\sum_{\alpha=1}^N 1_{\{|u^\alpha|>L\}} \left|G_L\left(u^\alpha\right)\right|\right)^2 \, dx \leq \\ \int_\Omega cn^2 N^2 \epsilon \eta^2 \sum_{\beta=1}^N 1_{\{|u^\beta|>L\}} |Du^\beta|^2 + \int_\Omega \frac{cn^2 N^2}{\epsilon} |D\eta|^2 \sum_{\alpha=1}^N 1_{\{|u^\alpha|>L\}} \left|G_L\left(u^\alpha\right)\right|^2 \, dx, \end{aligned} \tag{3.6}$$

where we used the inequality $2ab \le \epsilon a^2 + b^2/\epsilon$, provided $\epsilon > 0$. Merging (3.5), (3.4) and (3.6) into (3.3) we get

$$\nu \int_\Omega \eta^2 \sum_{\alpha=1}^N 1_{\{|u^\alpha|>L\}} |D u^\alpha|^2 \, dx \le$$
$$\int_\Omega c n^2 N^2 \epsilon \eta^2 \sum_{\beta=1}^N 1_{\{|u^\beta|>L\}} |Du^\beta|^2 + \int_\Omega \frac{cn^2N^2}{\epsilon} |D\eta|^2 \sum_{\alpha=1}^N 1_{\{|u^\alpha|>L\}} (|u^\alpha| - L)^2 \, dx.$$

We choose $\epsilon = \nu/(2cn^2N^2)$ and we have

$$\frac{\nu}{2} \int_\Omega \eta^2 \sum_{\alpha=1}^N 1_{\{|u^\alpha|>L\}} |D u^\alpha|^2 \, dx \le \int_\Omega \frac{2c^2n^4N^4}{\nu} |D\eta|^2 \sum_{\alpha=1}^N 1_{\{|u^\alpha|>L\}} (|u^\alpha| - L)^2 \, dx.$$

Using the properties of the cut off function η we deduce

$$\sum_{\alpha=1}^N \int_{\{|u^\alpha|>L\} \cap B(x_0,s)} |D u^\alpha|^2 \, dx \le \frac{16c^2n^4N^4}{\nu^2} \sum_{\alpha=1}^N \int_{\{|u^\alpha|>L\} \cap B(x_0,t)} \left(\frac{|u^\alpha| - L}{t - s} \right)^2 dx. \tag{3.7}$$

Note that

$$|D_i |u^\alpha|| = |D_i u^\alpha|;$$

this ends the proof of Lemma 3.1. □

STEP 2. Sup estimate for general vectorial functions

In the next Lemma we state and prove a general result that holds true for some general vectorial function $v \in W^{1,p}(\Omega, \mathbb{R}^N)$. Eventually, we will use such a result with $v = (|u^1|, \dots, |u^N|)$ and $p = 2$.

Lemma 3.2. *Assume that Ω is a bounded open subset of $\mathbb{R}^n$ and $v = (v^1, \dots, v^N) \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $1 < p < n$. We require the existence of constants $c_1 > 0$ and $L_0 \ge 0$ such that*

$$\sum_{\alpha=1}^N \int_{\{v^\alpha>L\} \cap B(x_0,s)} |Dv^\alpha|^p \, dx \le c_1 \sum_{\alpha=1}^N \int_{\{v^\alpha>L\} \cap B(x_0,t)} \left(\frac{v^\alpha - L}{t - s} \right)^p dx, \tag{3.8}$$

for every s, t, L, where $0 < s < t$, $B(x_0, t) \subset \Omega$ and $L \ge L_0$. Then,

$$\sup_{B(x_0,r)} v^\alpha \le 2 \max \left\{ L_0; \left(\frac{\left[\frac{(n-1)p}{(n-p)}\right]^n [2^p + c_1]^{n/p} \, 2^{4n+p+nn/p}}{(R - r)^n} \sum_{\beta=1}^N \int_{B(x_0,R)} (\max\{v^\beta; 0\})^p \right)^{1/p} \right\} \tag{3.9}$$

for every $\alpha = 1, \dots, N$ and for every r, R with $0 < r < R$ and $B(x_0, R) \subset \Omega$.

Proof of Lemma 3.2 Let us consider balls $B(x_0, r_1)$ and $B(x_0, r_2)$ with $0 < r_1 < r_2$ and $B(x_0, r_2) \subset \Omega$. Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be the standard cut-off function such that $0 \le \eta \le 1$, $\eta \in C_0^1(B(x_0, (r_1 + r_2)/2))$, with $\eta = 1$ in $B(x_0, r_1)$. Moreover, $|D\eta| \le 4/(r_2 - r_1)$ in $\mathbb{R}^n$. Let us set

$$A^\alpha_{L,r} =: \{x \in B(x_0, r) : v^\alpha > L\}.$$

Then, using Hölder inequality, Sobolev embedding and the properties of the cut-off function,

$$\int_{A^\alpha_{L,r_1}} (v^\alpha - L)^p \le \left(\int_{A^\alpha_{L,r_1}} (v^\alpha - L)^{p^*} \right)^{p/p^*} |A^\alpha_{L,r_1}|^{1-(p/p^*)} =$$

$$\left(\int_{A^\alpha_{L,r_1}} [\eta(v^\alpha - L)]^{p^*} \right)^{p/p^*} |A^\alpha_{L,r_1}|^{1-(p/p^*)} = \left(\int_{B(x_0,r_1)} [\eta(\max\{v^\alpha - L; 0\})]^{p^*} \right)^{p/p^*} |A^\alpha_{L,r_1}|^{1-(p/p^*)} \le$$

$$\left(\int_{B(x_0,(r_1+r_2)/2)} [\eta(\max\{v^\alpha - L; 0\})]^{p^*} \right)^{p/p^*} |A^\alpha_{L,r_1}|^{1-(p/p^*)} \le$$

$$c_2 \int_{B(x_0,(r_1+r_2)/2)} |D[\eta(\max\{v^\alpha - L; 0\})]|^p |A^\alpha_{L,r_1}|^{1-(p/p^*)} =$$

$$c_2 \int_{B(x_0,(r_1+r_2)/2)} |(D\eta)(\max\{v^\alpha - L; 0\}) + \eta D(\max\{v^\alpha - L; 0\})|^p |A^\alpha_{L,r_1}|^{1-(p/p^*)} =$$

$$c_2 \int_{A^\alpha_{L,(r_1+r_2)/2}} |(D\eta)(v^\alpha - L) + \eta D v^\alpha|^p |A^\alpha_{L,r_1}|^{1-(p/p^*)} \le$$

$$c_2\, 2^p \left(\int_{A^\alpha_{L,(r_1+r_2)/2}} |(D\eta)(v^\alpha - L)|^p + \int_{A^\alpha_{L,(r_1+r_2)/2}} |\eta D v^\alpha|^p \right) |A^\alpha_{L,r_1}|^{1-(p/p^*)} \le$$

$$c_2\, 2^p \left(4^p \int_{A^\alpha_{L,(r_1+r_2)/2}} \left(\frac{v^\alpha - L}{r_2 - r_1} \right)^p + \int_{A^\alpha_{L,(r_1+r_2)/2}} |Dv^\alpha|^p \right) |A^\alpha_{L,r_1}|^{1-(p/p^*)} \tag{3.10}$$

where $c_2 = [(n-1)p/(n-p)]^p$. Now we sum upon α from 1 to N obtaining

$$\sum_{\alpha=1}^N \int_{A^\alpha_{L,r_1}} (v^\alpha - L)^p \le$$

$$c_2\, 2^p \sum_{\alpha=1}^N \left(4^p \int_{A^\alpha_{L,(r_1+r_2)/2}} \left(\frac{v^\alpha - L}{r_2 - r_1} \right)^p + \int_{A^\alpha_{L,(r_1+r_2)/2}} |Dv^\alpha|^p \right) |A^\alpha_{L,r_1}|^{1-(p/p^*)} \le$$

$$c_2\, 2^p \sum_{\alpha=1}^N \left(4^p \int_{A^\alpha_{L,(r_1+r_2)/2}} \left(\frac{v^\alpha - L}{r_2 - r_1} \right)^p + \int_{A^\alpha_{L,(r_1+r_2)/2}} |Dv^\alpha|^p \right) \left(\sum_{\beta=1}^N |A^\beta_{L,r_1}| \right)^{1-(p/p^*)} . \tag{3.11}$$

In order to control $\sum \int |Dv^\alpha|^p$ we use our assumption (3.8) with $s = (r_1 + r_2)/2$ and $t = r_2$: we get

$$\sum_{\alpha=1}^N \int_{A^\alpha_{L,r_1}} (v^\alpha - L)^p \le$$

$$c_2\, 2^p \left(4^p \sum_{\alpha=1}^N \int_{A^\alpha_{L,(r_1+r_2)/2}} \left(\frac{v^\alpha - L}{r_2 - r_1} \right)^p + c_1 2^p \sum_{\alpha=1}^N \int_{A^\alpha_{L,r_2}} \left(\frac{v^\alpha - L}{r_2 - r_1} \right)^p \right) \left(\sum_{\beta=1}^N |A^\beta_{L,r_1}| \right)^{1-(p/p^*)} \le$$

$$c_2\, 2^p [4^p + c_1 2^p] \left(\sum_{\alpha=1}^{N} \int_{A^\alpha_{L,r_2}} \left(\frac{v^\alpha - L}{r_2 - r_1} \right)^p \right) \left(\sum_{\beta=1}^{N} |A^\beta_{L,r_1}| \right)^{1-(p/p^*)} . \tag{3.12}$$

We want to estimate $|A^\beta_{L,r_1}|$ by means of $\int (v^\beta - L)^p$. We are able to do that for a lower level $\tilde{L}$. Indeed, for $L > \tilde{L} \geq L_0$, we have

$$|A^\beta_{L,r_1}| = \frac{1}{(L-\tilde{L})^p} (L-\tilde{L})^p |A^\beta_{L,r_1}| = \frac{1}{(L-\tilde{L})^p} \int_{A^\beta_{L,r_1}} (L-\tilde{L})^p \leq$$

$$\frac{1}{(L-\tilde{L})^p} \int_{A^\beta_{L,r_1}} \left(v^\beta - \tilde{L} \right)^p \leq \frac{1}{(L-\tilde{L})^p} \int_{A^\beta_{\tilde{L},r_1}} \left(v^\beta - \tilde{L} \right)^p \leq \frac{1}{(L-\tilde{L})^p} \int_{A^\beta_{\tilde{L},r_2}} \left(v^\beta - \tilde{L} \right)^p . \tag{3.13}$$

Note that

$$1 - (p/p^*) = p/n. \tag{3.14}$$

Inserting (3.14) and (3.13) into (3.12) we deduce

$$\sum_{\alpha=1}^{N} \int_{A^\alpha_{L,r_1}} (v^\alpha - L)^p \leq$$

$$\frac{c_2\, 2^p [4^p + c_1 2^p]}{(r_2 - r_1)^p \, (L-\tilde{L})^{pp/n}} \left(\sum_{\alpha=1}^{N} \int_{A^\alpha_{L,r_2}} (v^\alpha - L)^p \right) \left(\sum_{\beta=1}^{N} \int_{A^\beta_{\tilde{L},r_2}} \left(v^\beta - \tilde{L} \right)^p \right)^{p/n} . \tag{3.15}$$

We want to estimate $\int (v^\alpha - L)^p$ with $\int (v^\alpha - \tilde{L})^p$. Since $L > \tilde{L}$, we have

$$\int_{A^\alpha_{L,r_2}} (v^\alpha - L)^p \leq \int_{A^\alpha_{L,r_2}} (v^\alpha - \tilde{L})^p \leq \int_{A^\alpha_{\tilde{L},r_2}} (v^\alpha - \tilde{L})^p . \tag{3.16}$$

Inserting (3.16) into (3.15) we get

$$\sum_{\alpha=1}^{N} \int_{A^\alpha_{L,r_1}} (v^\alpha - L)^p \leq \frac{c_2\, 2^p [4^p + c_1 2^p]}{(r_2 - r_1)^p \, (L-\tilde{L})^{pp/n}} \left(\sum_{\beta=1}^{N} \int_{A^\beta_{\tilde{L},r_2}} \left(v^\beta - \tilde{L} \right)^p \right)^{1+(p/n)} . \tag{3.17}$$

Now we fix $0 < r < R$, with $B(x_0, R) \subset \Omega$, and we take the following sequence of radii

$$\rho_i = r + \frac{R - r}{2^i} \tag{3.18}$$

for $i = 0, 1, 2, \ldots$; then $\rho_0 = R$ and $\rho_i - \rho_{i+1} = (R-r)/2^{i+1} > 0$, so ρ_i strictly decreases and $r < \rho_i \leq R$.

Let us fix a level $d \geq L_0$ and we take the following sequence of levels

$$k_i = 2d \left(1 - \frac{1}{2^{i+1}} \right) \tag{3.19}$$

for $i = 0, 1, 2, \ldots$; then $k_0 = d$ and $k_{i+1} - k_i = d/2^{i+1} > 0$, so k_i strictly increases and $L_0 \leq d \leq k_i < 2d$. We can use (3.17) with levels $L = k_{i+1} > k_i = \tilde{L}$ and radii $r_1 = \rho_{i+1} < \rho_i = r_2$:

$$\sum_{\alpha=1}^{N} \int_{A^\alpha_{k_{i+1},\rho_{i+1}}} (v^\alpha - k_{i+1})^p \leq \frac{c_2\, 2^p [4^p + c_1 2^p]}{((R-r)/2^{i+1})^p \, (d/2^{i+1})^{pp/n}} \left(\sum_{\beta=1}^{N} \int_{A^\beta_{k_i,\rho_i}} \left(v^\beta - k_i \right)^p \right)^{1+(p/n)} =$$

$$\frac{c_2\,4^p[2^p + c_1]2^{(i+1)p}\,2^{(i+1)pp/n}}{(R-r)^p\,d^{pp/n}}\left(\sum_{\beta=1}^{N}\int_{A^{\beta}_{k_i,\rho_i}}\left(v^{\beta}-k_i\right)^p\right)^{1+(p/n)}. \tag{3.20}$$

Let us set

$$J_i =: \sum_{\alpha=1}^{N}\int_{A^{\alpha}_{k_i,\rho_i}}(v^{\alpha}-k_i)^p; \tag{3.21}$$

then (3.20) can be written as follows

$$J_{i+1} \le \frac{c_2\,4^p[2^p + c_1]2^{(1+(p/n))p}}{(R-r)^p\,d^{pp/n}}\left(2^{(1+(p/n))p}\right)^i (J_i)^{1+(p/n)}. \tag{3.22}$$

We would like to get

$$\lim_{i\to\infty} J_i = 0; \tag{3.23}$$

this is true provided

$$J_0 \le \left(\frac{c_2\,4^p[2^p + c_1]2^{(1+(p/n))p}}{(R-r)^p\,d^{pp/n}}\right)^{-n/p}\left(2^{(1+(p/n))p}\right)^{-nn/(pp)}, \tag{3.24}$$

as Lemma 7.1 says at page 220 in [13]. Let us try to check (3.24): we first rewrite it as follows

$$\sum_{\alpha=1}^{N}\int_{A^{\alpha}_{k_0,\rho_0}}(v^{\alpha}-k_0)^p \le \left(\frac{c_2\,4^p[2^p + c_1]2^{(1+(p/n))p}}{(R-r)^p\,d^{pp/n}}\right)^{-n/p}\left(2^{(1+(p/n))p}\right)^{-nn/(pp)}; \tag{3.25}$$

we keep in mind that $k_0 = d$ and $\rho_0 = R$; so, (3.25) can be written in the following way

$$\left(\frac{c_2\,4^p[2^p + c_1]2^{(1+(p/n))p}}{(R-r)^p}\right)^{n/p}\left(2^{(1+(p/n))p}\right)^{nn/(pp)}\sum_{\alpha=1}^{N}\int_{A^{\alpha}_{d,R}}(v^{\alpha}-d)^p \le d^p. \tag{3.26}$$

Note that $d \ge L_0 \ge 0$ so, when $v^{\alpha} > d$, we have $v^{\alpha} - d \le v^{\alpha} = \max\{v^{\alpha}; 0\}$; then

$$\int_{A^{\alpha}_{d,R}}(v^{\alpha}-d)^p \le \int_{A^{\alpha}_{d,R}}(\max\{v^{\alpha};0\})^p \le \int_{B(x_0,R)}(\max\{v^{\alpha};0\})^p. \tag{3.27}$$

Using (3.27), we get the following sufficient condition when checking (3.26):

$$\frac{\left(c_2\,4^p[2^p + c_1]2^{(1+(p/n))p}\right)^{n/p}\;2^{(1+(p/n))nn/p}}{(R-r)^n}\sum_{\alpha=1}^{N}\int_{B(x_0,R)}(\max\{v^{\alpha};0\})^p \le d^p. \tag{3.28}$$

Then, we fix d verifying (3.28) and $L_0 \le d$; then (3.24) is satisfied and (3.23) holds true. We keep in mind that $r < \rho_i$ and $k_i < 2d$, so we can use (3.16) with $r_2 = r < \rho_i$, $L = 2d$ and $\tilde{L} = k_i$:

$$\int_{\{v^{\alpha}>2d\}\cap B(x_0,r)}(v^{\alpha}-2d)^p \le \int_{\{v^{\alpha}>k_i\}\cap B(x_0,r)}(v^{\alpha}-k_i)^p \le \int_{\{v^{\alpha}>k_i\}\cap B(x_0,\rho_i)}(v^{\alpha}-k_i)^p, \tag{3.29}$$

so that

$$0 \le \sum_{\alpha=1}^{N}\int_{\{v^{\alpha}>2d\}\cap B(x_0,r)}(v^{\alpha}-2d)^p \le \sum_{\alpha=1}^{N}\int_{\{v^{\alpha}>k_i\}\cap B(x_0,\rho_i)}(v^{\alpha}-k_i)^p = J_i; \tag{3.30}$$

since (3.23) holds true, we have $\lim_i J_i = 0$, so

$$\sum_{\alpha=1}^{N}\int_{\{v^{\alpha}>2d\}\cap B(x_0,r)}(v^{\alpha}-2d)^p = 0; \tag{3.31}$$

this means that $|\{v^\alpha > 2d\} \cap B(x_0, r)| = 0$, so that

$$v^\alpha \le 2d \qquad \text{almost everywhere in } B(x_0, r). \tag{3.32}$$

Level d can be selected as follows

$$d = \max \left\{ L_0; \left(\frac{\left(c_2\, 4^p [2^p + c_1] 2^{(1+(p/n))p}\right)^{n/p} 2^{(1+(p/n))nn/p}}{(R-r)^n} \sum_{\beta=1}^{N} \int_{B(x_0,R)} (\max\{v^\beta; 0\})^p \right)^{1/p} \right\}$$

and claim (3.9) is proved after noting that $(4^p\, 2^{(1+(p/n))p})^{n/p}\, 2^{(1+(p/n))nn/p} = 2^{4n+p+nn/p}$ and $c_2 = [(n-1)p/(n-p)]^p$. This ends the proof of Lemma 3.2. □

STEP 3. Proof of Theorem 2.3

Caccioppoli inequality proved in Lemma 3.1 allows us to use Lemma 3.2 with $v^\alpha = |u^\alpha|$, $p = 2$ and $c_1 = \frac{16c^2 n^4 N^4}{\nu^2}$: this gives estimate (2.3) and the proof of Theorem 2.3 ends here. □

Remark 3.3. *In the present work we used a test function φ that modifies every component of u; this gives the summation on the index α in Caccioppoli's inequality (3.1). In [4], [1] and [3] only one component of u is modified and a Caccioppoli's inequality without the summation on α is proved.*

Moreover, the Caccioppoli's inequality proved in [4] and [1] has an exponent p^* on the right–hand side in contrast with the same p that we have on both sides of (3.8), see also [30], [9], [2], [5], [14], [15], [16].

Remark 3.4. *In [22] it is used $\max\{u^\alpha - L; 0\}$ in the test function φ, see Figure 2 (left), while in the present paper we use $G_L(u^\alpha)$ instead, see Figure 2 (right). Such a function $G_L(u^\alpha)$ allows us to deal with support larger than in [22] for off diagonal coefficients.*

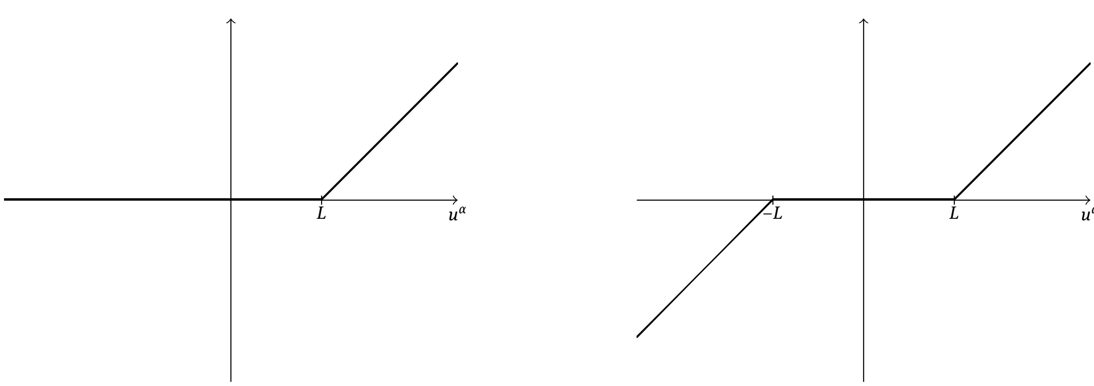

Fig. 2: (left) graph of $u^\alpha \to \max\{u^\alpha - L; 0\}$; (right) graph of $u^\alpha \to G_L(u^\alpha)$.

Acknowledgement: Leonardi has been supported by Piano della Ricerca di Ateneo 2020-2022–PIACERI: Project MO.S.A.I.C. "Monitoraggio satellitare, modellazioni matematiche e soluzioni architettoniche e urbane per lo studio, la previsione e la mitigazione delle isole di calore urbano".

Leonetti acknowledges the support from RIA-UNIVAQ.

Leonardi and Leonetti have been partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Rocha and Staicu acknowledge the partial support by the Portuguese Foundation for Science and Technology (FCT), through CIDMA - Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2019(CIDMA).

Conflict of interest statement. Authors state no conflict of interest.

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