BOUSSINESQ-TYPE APPROXIMATION FOR THE FREE OVERFALL IN OPEN CHANNELS[1]

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ABSTRACT

This brief note examines the free boundary problem concerning the steady 2D inviscid potential fluid flow. The case study used in this analysis pertains to the free overfall in open channels. The flow problem solution is obtained by using the Boussinesq-type approximation for the solitary wave [J.V. Boussinesq, Théorie de l'intumescence liquide appelée onde solitaire ou de translation se propageant dans un canal rectangulaire, C. R. Acad. Sci. Paris 72 (1871) 755-759]. In terms of the free boundary location and bed pressure profile, the computational results are in good agreement with test data available in literature.

KEYWORDS

free boundary problem, two-dimensional potential flow, fluid mechanics, open channel flow, free overfall, Boussinesq-type model

1. Introduction

This brief note examines the free boundary problem concerning the steady 2D inviscid potential fluid flow. The case study used in this analysis pertains to the fluid flow in the vicinity of the abrupt end of a long open channel. In literature, this problem is usually referred as the free overfall problem. The free overfall problem is closely related to the flow measuring and controlling systems. Starting from the experiments carried out by Rouse [1], a large number of investigations on the free overfall problem have been performed (see, e.g., [2]). The flow problem solution can be obtained by using numerical [3-5] or approximate methods [see, e.g., 6-11]. An approximate solution of this free boundary problem can be obtained by using the method proposed by Boussinesq to study the solitary wave [12-19]. By adopting Boussinesq's method, the free boundary is expressed by a second order nonlinear differential equation. The boundary conditions of this equation are a priori unknown [20]. After proposing

[1] The authors dedicate this paper to the memory of Prof. Agostino Farroni.

a procedure to define these boundary conditions, this brief note provides the degree of the approximation and the range of validity of Boussinesq's method. Within the framework of the free overfall under critical flow conditions [6,7], the computational results are validated by comparison with test data available in literature [1,7]. The validation is obtained in terms of the free boundary location and bed pressure.

2. The free boundary problem

Consider the steady 2D inviscid potential fluid flow in the vicinity of the abrupt end of a very wide rectangular channel (Fig. 1). The bed of channel is horizontal, non-erodible, impermeable and hydraulically smooth. The fluid flow is referred to the Cartesian coordinates system (x,z), with the horizontal x-axis located on the channel bed and the vertical z-axis oriented upwards.
Let $\boldsymbol{v} = (v_x,v_z)$ be the velocity vector field, g the acceleration of gravity, ρ the constant fluid density, h the vertical flow depth, and p the (relative) pressure.
Given the constant unit discharge q, the fluid flow problem consists in finding the location of the free boundary $h(x)$, the distribution of the velocity components $v_x(x,z)$ and $v_z(x,z)$, and the pressure distribution $p(x,z)$ in the unbounded flow domain $(-\infty,0]\times[0,h(x)]$.

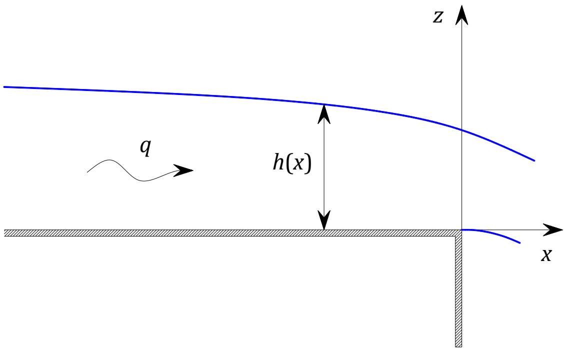

Fig. 1. Definition sketch of the flow configuration and coordinate system.

The fluid flow problem is ruled by the following momentum and continuity equations:

$$\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v}+g\nabla z+\nabla p=0 \tag{1}$$
$$\nabla\cdot\boldsymbol{v}=0 \tag{2}$$

Eqs. (1) and (2) are completed by the kinematic conditions at the bed $v_z(x,0) = 0$ and at the free boundary $v_z(x,h) = v_x(x,h)\mathrm{d}h/\mathrm{d}x$, by the dynamic condition at the free boundary $p(x,h) = 0$, and by the auxiliary boundary condition at the end of channel $p_b(0) = 0$, where $p_b(x) = p(x,0)$ is the bed pressure.
By using the continuity equation and the irrotationality condition:

$$\nabla\times\boldsymbol{v}=0 \tag{3}$$
Eq. (1) becomes:

$$\nabla H = 0 \tag{4}$$

where *H* is the total energy head expressed as:

$$H = z + \frac{p}{\rho g} + \frac{v_x^2 + v_z^2}{2g} \tag{5}$$

From Eq. (4), it follows that the first integral of the momentum equation (4) is *H* = *constant*.

3. The Boussinesq-type approximation

By adopting the Boussinesq-type model for the solitary wave [12-19], the components of the velocity vector field ***v*** can be expressed in approximate form as [21-23] (see Appendix 1):

$$v_x = \frac{q}{h}\left[1 + \frac{1}{3}\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2 - \frac{h}{6}\frac{\mathrm{d}^2 h}{\mathrm{d}x^2}\right] - \frac{1}{2}\frac{q}{h^2}\left[\frac{2}{h}\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2 - \frac{\mathrm{d}^2 h}{\mathrm{d}x^2}\right]z^2 \tag{6}$$

$$v_z = \frac{q}{h^2} z \frac{\mathrm{d}h}{\mathrm{d}x} \tag{7}$$

With Eqs. (6) and (7), Eq. (3) and the kinematic condition at the bed are satisfied, while Eq. (2) and the kinematic condition at the free boundary are satisfied if and only if the following terms are nil [21,22]:

$$\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^3 = 0,\ \frac{\mathrm{d}^2 h}{\mathrm{d}x^2}\frac{\mathrm{d}h}{\mathrm{d}x} = 0,\ \frac{\mathrm{d}^3 h}{\mathrm{d}x^3} = 0 \tag{8}$$

Eq. (8) defines the closure hypotheses associated to Eqs. (6) and (7).
In the approximation degree related to Eqs. (6), (7), and (8), the pressure head distribution $p/(\rho g)$ and the total energy head *H* are given as [16,21,22] (see Appendix 1):

$$\frac{p}{\rho g} = h - z + \frac{q^2}{2gh^2}\left(1 - \frac{z^2}{h^2}\right)\left[h\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} - \left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2\right] \tag{9}$$

$$H = h + \frac{1}{2g}\frac{q^2}{h^2}\left[1 - \frac{1}{3}\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2 + \frac{2}{3}h\frac{\mathrm{d}^2 h}{\mathrm{d}x^2}\right] \tag{10}$$

Accordingly, by using the Boussinesq-type approximation, the free boundary problem is reduced to find the free boundary $h(x)$.

4. Model validation and results analysis

Within the framework of the free overfall under critical flow conditions [6,7], the constant value of the total energy head is given as $H = 3h_c/2$, where h_c is the critical depth defined as $h_c = (q^2/g)^{1/3}$.

By setting $H = 3h_c/2$ in Eq. (10), the free boundary is expressed by the following second order nonlinear differential equation [6]:

$$\frac{d^2 h}{d x^2} = \frac{3gh}{q^2}\left(\frac{3}{2}h_c - h\right) - \frac{3}{2h} + \frac{1}{2h}\left(\frac{d h}{d x}\right)^2 \tag{11}$$

For the numerical integration of Eq. (11), the theoretical boundary conditions $h = h_c$, $dh/dx = 0$ imposed at $x \to -\infty$ [7] must be substituted with the relations:

$$h(x = \underline{x}) < h_c - \varepsilon_1 \tag{12}$$

$$\frac{d h}{d x}(x = \underline{x}) < -\varepsilon_2 \tag{13}$$

where the flow section $\underline{x}$ is a priori unknown, and $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are two fixed tolerances. The flow section $\underline{x}$ can be determined iteratively by the following procedure. By setting $\underline{x} = \underline{x}^{(1)}$, with $\underline{x}^{(1)}$ the first attempt value, Eq. (11) can be integrated with the conditions (12) and (13); once this free boundary $h_1(x)$ is available, the bed pressure profile is obtained from Eq. (9) as:

$$p_b(x) = \rho g h - \rho \frac{q^2}{2gh^2}\left[h\frac{d^2 h}{d x^2} - \left(\frac{d h}{d x}\right)^2\right] \tag{14}$$

The free boundary $h_1(x)$ is the solution of the posed problem if and only if the auxiliary boundary condition is verified $p_b(0) = 0$. In this computational context, the auxiliary boundary condition must be expressed as:

$$|p_b(0)| < \varepsilon_3 \tag{15}$$

where $\varepsilon_3 > 0$ is a fixed tolerance. If the condition (16) is violated, the procedure is repeated by setting $\underline{x} = \underline{x}^{(2)}$, with $\underline{x}^{(2)}$ the second attempt value. This procedure can be automated by using the dichotomous method.

Fig. 2 shows the comparison between the computational (Eq. (11)) and experimental [1,7] free boundary locations. Eq. (11) is numerically solved by using the fourth-order Runge-Kutta method. The same figure shows that the computational bed pressure (Eq. (14)) tends to zero at the end of the channel.

Fig. 3 shows the comparison between the computational (Eq. 9) and numerical [5] bed pressure profiles. The numerical test data are obtained by Montes (1992) [5] by using the potential flow theory.

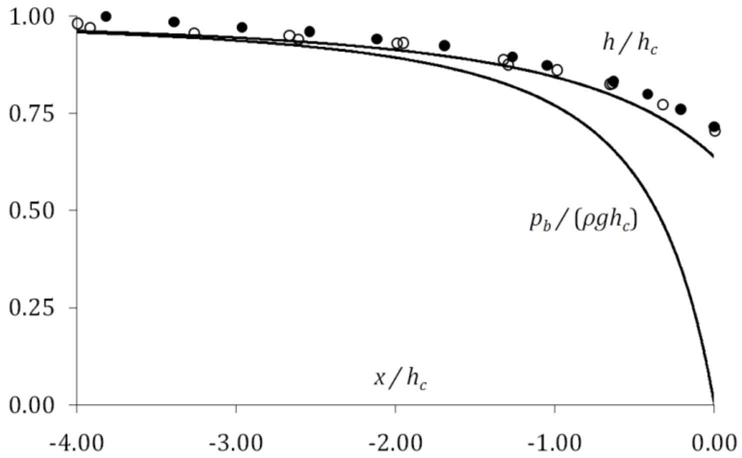

Fig. 2. Free boundary: – computational results (Eq. (11)), ● experimental data [1], ○ experimental data [7]; bed pressure: – computational results (Eq. (14)).

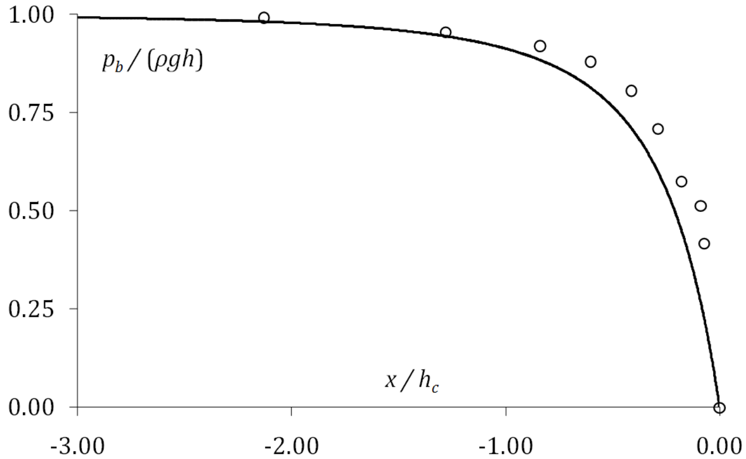

Fig. 3. Bed pressure profile: – computational results (Eq. (14)), ○ numerical data [5].

As shown in Figs. 2 and 3, in the approximation degree related to Eqs. (6), (7), (8), (9), (10), and (11), Boussinesq's method gives good results in terms of the free boundary location and bed pressure profile. However, in the same approximation, the theoretical pressure distribution at the end of the channel is physically unrealistic: by setting $x = 0$ in Eq. (9), it follows that $p(0,z) \leq 0$. This result can be attributed to the higher order effects of curvilinear flow: close to the end of the channel, the free boundary present the surface slopes and curvature that cannot be properly represented by adopting the approximation related to Eqs. (6), (7), (8), (9), (10), and (11). From the conceptual point of view, this weakness can be overcome by employing the higher order equations for the velocity vector field $\boldsymbol{v}$, the pressure distribution p, and the total energy head H. As shown in the Appendix 1, these higher order equations can be deduced by using a method based on the Picard iteration [24-26].

Within the framework of the Boussinesq-type approximation, Castro-Orgaz and Hager (2010) [11] have used Eqs. (11) and (14) to study the free overfall under critical flow conditions.

These authors have numerically integrated Eq. (11) imposing the boundary condition $h = h_c$ at the section $x = -3h_c$; the boundary free surface slope was determined iteratively until the downstream boundary condition at the brink section $h = 0.71h_c$ was reached. By using this procedure, the computational free boundary is in excellent agreement with the experimental data, but the bed pressure (computed by using Eq. (14)) is poorly simulated. According to Castro-Orgaz and Hager, this last result indicates that the Boussinesq-type approximation expressed by Eq. (11) and (14) is not suitable to represent the curvilinear flow. For this purpose, the authors provide a novel Boussinesq-type approximation. Di Nucci and Russo Spena (2011) [22] have shown that this novel Boussinesq-type approximation is physically and mathematically unfounded; on the other hand, the procedure proposed in this note highlights that the computational results obtained by Castro-Orgaz and Hager (2010) [11] are a consequence of an inexpert application of the Boussinesq-type approximation.

5. Conclusions

This brief note proposes a one-dimensional Boussinesq-type model for the free overfall under critical flow conditions. The free boundary is expressed by a second order nonlinear differential equation with unknown boundary conditions. After proposing a procedure to define these boundary conditions, this brief note provides the degree of the approximation and the range of validity of the proposed solution. In terms of the free boundary location and bed pressure profile, the computational results are in good agreement with the test data available in literature. Finally, this note shows that the erroneous use of the Boussinesq-type approximation gives misleading results.

Appendix 1

The components of the potential velocity vector field $\boldsymbol{v}$ can be determined by using Boussinesq's procedure [12-19] or Matthew's procedure [27].

To study the solitary wave, Boussinesq (1871) [12] proposed a method based on the power series expansion of the harmonic stream function related to a potential velocity vector field.

To obtain higher-order equations for steady 2D potential fluid flow, Matthew (1991) [27] provides a method based on Picard iteration [24-26].

Di Nucci and Russo Spena (2011) [22] have shown that Boussinesq's procedure is equivalent to Matthew's procedure. In addition, both of these procedures are equivalent to the following method based on the following novel Picard iteration.

Setting in first approximation:

$$v_x = \frac{q}{h} \tag{A1}$$

$$v_z = 0 \tag{A2}$$

Eq. (3) is verified, while Eq. (2) is verified if and only if the following first-order term is nil:

$$\frac{\mathrm{d}h}{\mathrm{d}x}=0 \tag{A3}$$

Eq. (A3) defines the closure hypotheses related to Eqs. (A1) and (A2).
The second approximation can be deduced by putting:

$$v_x=\frac{q}{h} \tag{A4}$$

$$v_z=f_2(x,z) \tag{A5}$$

where $f_2(x,z)$ is a function a priori unknown. From Eq. (2), it follows that:

$$\frac{\partial f_2}{\partial z}=\frac{q}{h^2}\frac{\mathrm{d}h}{\mathrm{d}x} \tag{A6}$$

from which:

$$f_2=\frac{q}{h^2}\frac{\mathrm{d}h}{\mathrm{d}x}z+g_2(x) \tag{A7}$$

The kinematic condition at the bed ($v_z(x,0)=0$) corresponds to $g_2(x)=0$. In conclusion, the second approximation is given as [28,29]:

$$v_x=\frac{q}{h} \tag{A8}$$

$$v_z=\frac{q}{h^2}\frac{\mathrm{d}h}{\mathrm{d}x}z \tag{A9}$$

With Eqs. (A8) and (A9), Eq. (3) is verified if and only if the following second-order terms are nil:

$$\frac{\mathrm{d}^2h}{\mathrm{d}x^2}=0,\ \left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2=0 \tag{A10}$$

Eq. (A10) defines the closure hypotheses related to Eqs. (A8) and (A9).
The third approximation can be deduced by putting:

$$v_x=\frac{q}{h}+f_3(x,z) \tag{A11}$$

$$v_z=\frac{q}{h^2}\frac{\mathrm{d}h}{\mathrm{d}x}z \tag{A12}$$

From Eq. (3), it follows that:

$$\frac{\partial f_3}{\partial z} = \frac{q}{h^2}\left[\frac{d^2 h}{d x^2} - \frac{2}{h}\left(\frac{d h}{d x}\right)^2\right] z \tag{A13}$$

from which:

$$f_3 = \frac{q}{h^2}\left[\frac{d^2 h}{d x^2} - \frac{2}{h}\left(\frac{d h}{d x}\right)^2\right]\frac{z^2}{2} + g_3(x) \tag{A14}$$

From the relation:

$$q = \int_0^{h(x)} v_x \, d z \tag{A15}$$

it follows that:

$$g_3(x) = \frac{1}{3}\frac{q}{h}\left(\frac{d h}{d x}\right)^2 - \frac{1}{6} q \frac{d^2 h}{d x^2} \tag{A16}$$

In conclusion, the third approximation is given as [21,22]:

$$v_x = \frac{q}{h}\left[1 + \frac{1}{3}\left(\frac{d h}{d x}\right)^2 - \frac{h}{6}\frac{d^2 h}{d x^2}\right] - \frac{1}{2}\frac{q}{h^2}\left[\frac{2}{h}\left(\frac{d h}{d x}\right)^2 - \frac{d^2 h}{d x^2}\right] z^2 \tag{A17}$$

$$v_z = \frac{q}{h^2} z \frac{d h}{d x} \tag{A18}$$

With Eqs. (A17) and (A18), Eq. (2) is verified if and only if the following third-order terms are nil:

$$\left(\frac{d h}{d x}\right)^3 = 0, \; \frac{d^2 h}{d x^2}\frac{d h}{d x} = 0, \; \frac{d^3 h}{d x^3} = 0 \tag{A19}$$

Eq. (A19) defines the closure hypotheses related to Eqs. (A17) and (A18).
This procedure can be extended to any required degree of accuracy. For example, the fourth approximation is given as [16]:

$$v_x = \frac{q}{h}\left\{1 + \left[\frac{h}{2}\frac{d^2 h}{d x^2} - \left(\frac{d h}{d x}\right)^2\right]\left(\frac{z^2}{h^2} - \frac{1}{3}\right)\right\} \tag{A20}$$

$$v_z = \frac{q}{h}\left\{\left[\frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}x}+\frac{1}{3h}\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^3-\frac{2}{3}\frac{\mathrm{d}h}{\mathrm{d}x}\frac{\mathrm{d}^2h}{\mathrm{d}x^2}+\frac{h}{6}\frac{\mathrm{d}^3h}{\mathrm{d}x^3}\right]z+\left[\frac{1}{h^2}\frac{\mathrm{d}h}{\mathrm{d}x}\frac{\mathrm{d}^2h}{\mathrm{d}x^2}-\frac{1}{h^3}\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^3-\frac{1}{6h}\frac{\mathrm{d}^3h}{\mathrm{d}x^3}\right]z^3\right\} \qquad \text{(A21)}$$

while the closure hypotheses related to Eqs. (20) and (21) are expressed as:

$$\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^4=0,\ \frac{\mathrm{d}^2h}{\mathrm{d}x^2}\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2=0,\ \frac{\mathrm{d}^4h}{\mathrm{d}x^4}=0,\ \frac{\mathrm{d}^3h}{\mathrm{d}x^3}\frac{\mathrm{d}h}{\mathrm{d}x}=0,\ \left(\frac{\mathrm{d}^2h}{\mathrm{d}x^2}\right)^2=0 \qquad \text{(A22)}$$

Once the velocity vector field $\boldsymbol{v}$ and the closure hypotheses are available, the pressure head distribution is obtained from the vertical momentum equation [16]:

$$\frac{1}{\rho g}\frac{\partial p}{\partial z}=-1-\frac{1}{g}v_x\frac{\partial v_z}{\partial x}-\frac{1}{g}v_z\frac{\partial v_z}{\partial z} \qquad \text{(A23)}$$

while the constant total energy head can be evaluated at the free boundary $z = h$ where the (relative) pressure vanishes $p(x,h) = 0$ [16].

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