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Identification of Forward and Feedback Transfer Functions in Closed-Loop Systems with Feedback Delay

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Abstract: The subject of this paper is the identification of closed-loop continuous-time systems, with delayed feedback action, from sampled input-output measurements. In particular, a method for the identification of both forward and feedback subsystems is presented that requires only the knowledge of their orders and of the time-delay introduced in the feedback loop. The identification procedure is divided in two parts. The first step captures the behavior of the whole closed-loop system, estimating its transfer function. In the second step two different approaches are presented to separate the contributions of the forward and feedback subsystems in the loop. One of these approaches exploits a system theoretical method to compute the approximate greatest common divisor between polynomials. Numerical results validate the effectiveness of the proposed technique.

Keywords: Closed loop identification, Continuous time system estimation, Delay systems.

1. INTRODUCTION

The identification of closed-loop systems from input-output data, due to its considerable industrial applications, has been extensively studied in the control systems literature since the seventies. Besides classical methods, usually described as direct, indirect and joint input-output approaches (see Söderström and Stoica (1988), Van den Hof (1998), Ljung (1999), Forssell and Ljung (1999)), in recent years the application of subspace identification methods has shed new light on the problem, resulting in a renewed interest in the field (see for example Van der Klauw et al. (1991), Verhaegen (1993), Ljung and McKelvey (1996), and refer to Katayama (2006), van der Veen et al. (2013) for comprehensive overviews). As pointed out in van der Veen et al. (2013), Garnier and Wang (2008), much of the literature examines discrete-time models, while many real applications concern continuous-time models with sampled data. The presence of known or unknown time-delays has been taken into account by several authors: a nice overview can be found in O'Dwyer (2000).

In this work we will focus on a particular closed-loop identification problem which is relevant in some applications, such as telecontrol systems, where the time-delay acts in the feedback loop. We propose a simple and effective indirect method to estimate the forward and feedback subsystems in a closed-loop continuous-time linear system using sampled input-output measurements, assuming only the knowledge of the subsystems orders and the time-delay in the feedback action. Although the proposed approach deals with a deterministic framework, numerical examples show a certain robustness to measurement noise. The advantage of the proposed technique relies in the explicit modeling of the feedback delay, which allows to effectively separate the forward and feedback subsystems, and also in the fact that no other assumption is needed on the operating conditions of the system.

The paper is organized as follows. Section 2 gives the problem formulation. The proposed approach is described in Section 3, where two distinct methods are introduced to separate the forward and feedback subsystems. A numerical example is then presented in Section 4. Conclusions and some ideas for future works are given in Section 5.

2. PROBLEM FORMULATION

Consider the following closed-loop linear continuous time-delay system:

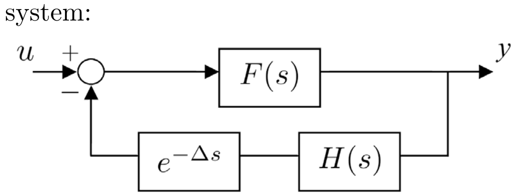

Fig. 1. Structure of the system under investigation.

where $u, y \in \mathbb{R}$ are measured input and output signals, $\Delta \in \mathbb{R}$ is a known time-delay and $F(s), H(s)$ are the unknown forward and feedback transfer functions, respectively, that we assume strictly proper:

$$F(s) = \frac{b^F_{n-1}s^{n-1} + b^F_{n-2}s^{n-2} + \cdots + b^F_1 s + b^F_0}{s^n + a^F_{n-1}s^{n-1} + a^F_{n-2}s^{n-2} + \cdots + a^F_1 s + a^F_0},$$

$$H(s) = \frac{b^H_{m-1}s^{m-1} + b^H_{m-2}s^{m-2} + \cdots + b^H_1 s + b^H_0}{s^m + a^H_{m-1}s^{m-1} + a^H_{m-2}s^{m-2} + \cdots + a^H_1 s + a^H_0}.$$

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We consider the problem of estimating $F(s)$ and $H(s)$ using sampled input-output data $\{u(k\delta), y(k\delta)\}$, $k = 0, 1, \ldots, N$, with sampling interval $\delta \in \mathbb{R}$.

The orders of the forward and feedback transfer functions, i.e. the degrees n and m, are assumed to be known. Moreover, rewriting the transfer functions as

$$F(s) = \frac{N_F(s)}{D_F(s)}, \qquad H(s) = \frac{N_H(s)}{D_H(s)}$$

we also assume that all the polynomials $N_F(s)$, $D_F(s)$, $N_H(s)$, $D_H(s)$ are mutually coprime.

The closed–loop transfer function is given by:

$$\begin{aligned} W(s) &= \frac{F(s)}{1 + F(s)H(s)e^{-\Delta s}} \\ &= \frac{N_F(s)D_H(s)}{D_F(s)D_H(s) + N_F(s)N_H(s)e^{-\Delta s}}. \end{aligned}$$

Defining the polynomials

$$\begin{aligned} \alpha(s) &= N_F(s)D_H(s) \\ \beta(s) &= D_F(s)D_H(s) \\ \gamma(s) &= N_F(s)N_H(s) \end{aligned} \tag{1}$$

$W(s)$ can be written as

$$W(s) = \frac{\alpha(s)}{\beta(s) + \gamma(s)e^{-\Delta s}}$$

where:

$$\begin{aligned} \alpha(s) &= \alpha_{n+m-1}s^{n+m-1} + \cdots + \alpha_1 s + \alpha_0 \\ \beta(s) &= s^{n+m} + \beta_{n+m-1}s^{n+m-1} + \cdots + \beta_1 s + \beta_0 \\ \gamma(s) &= \gamma_{n+m-2}s^{n+m-2} + \cdots + \gamma_1 s + \gamma_0. \end{aligned} \tag{2}$$

The input-output relation is given by $Y(s) = W(s)U(s)$, which is equivalent to:

$$Y(s)\left(\beta(s) + \gamma(s)e^{-\Delta s}\right) = \alpha(s)U(s).$$

We want to obtain an input-output relation in the time domain that can be used together with the sampled measurements for estimating the coefficients of the polynomials $\alpha(s)$, $\beta(s)$ and $\gamma(s)$. To this purpose, in order to avoid the use of time derivatives of the input and output signals, we multiply both members by $s^{-(n+m)}$, obtaining

$$\begin{aligned} Y(s)&\Big(1+\frac{\beta_{n+m-1}}{s} + \cdots + \frac{\beta_0}{s^{n+m}} + \big(\frac{\gamma_{n+m-2}}{s^2} + \cdots + \frac{\gamma_0}{s^{n+m}}\big)e^{-\Delta s}\Big) \\ &= \Big(\frac{\alpha_{n+m-1}}{s} + \cdots + \frac{\alpha_0}{s^{n+m}}\Big)U(s) \end{aligned}$$

which applying the inverse Laplace transform to both members, is equal to:

$$\begin{aligned} y(t) &+ \beta_{n+m-1}\int_0^t y(t_1)dt_1 + \cdots \\ &+ \beta_0 \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n+m-1}} y(t_{n+m})dt_{n+m}\cdots dt_2 dt_1 \\ &+ \gamma_{n+m-2}\int_0^{t-\Delta}\int_0^{t_1} y(t_2)dt_2 dt_1 + \cdots \\ &+ \gamma_0 \int_0^{t-\Delta}\int_0^{t_1} \cdots \int_0^{t_{n+m-1}} y(t_{n+m})dt_{n+m}\cdots dt_2 dt_1 \\ &= \alpha_{n+m-1}\int_0^t u(t_1)dt_1 + \cdots \\ &+ \alpha_0 \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n+m-1}} u(t_{n+m})dt_{n+m}\cdots dt_2 dt_1. \end{aligned} \tag{3}$$

Defining the repeated integrals

$$\begin{aligned} v_1(t) &= \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n+m-1}} y(t_{n+m})dt_{n+m}\cdots dt_2 dt_1 \\ w_1(t) &= \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n+m-1}} u(t_{n+m})dt_{n+m}\cdots dt_2 dt_1 \end{aligned}$$

and their derivatives

$$\begin{aligned} v_2 &= \dot{v}_1 & w_2 &= \dot{w}_1 \\ v_3 &= \dot{v}_2 & w_3 &= \dot{w}_2 \\ &\vdots & &\vdots \\ v_{n+m-1} &= \dot{v}_{n+m-2} & w_{n+m-1} &= \dot{w}_{n+m-2} \\ v_{n+m} &= \dot{v}_{n+m-1} & w_{n+m} &= \dot{w}_{n+m-1} \end{aligned}$$

from (3) we get the following relation in the time-domain

$$\begin{aligned} y(t) = &- \beta_{n+m-1}v_{n+m}(t) + \cdots - \beta_0 v_1(t) \\ &- \gamma_{n+m-2}v_{n+m-1}(t-\Delta) + \cdots - \gamma_0 v_1(t-\Delta) \\ &+ \alpha_{n+m-1}w_{n+m}(t) + \cdots + \alpha_0 w_1(t). \end{aligned} \tag{4}$$

Writing equation (4) at the sampling times $k\delta$ we get

$$\begin{aligned} y(k\delta) = &-\beta_{n+m-1}v_{n+m}(k\delta) + \cdots - \beta_0 v_1(k\delta) \\ &- \gamma_{n+m-2}v_{n+m-1}(k\delta-\Delta) + \cdots - \gamma_0 v_1(k\delta-\Delta) \\ &+ \alpha_{n+m-1}w_{n+m}(k\delta) + \cdots + \alpha_0 w_1(k\delta), \end{aligned} \tag{5}$$

that is a set of equations where the left-hand side terms are the sequence of measured outputs, and are known, and all terms $w_i(k\delta)$, $v_i(k\delta)$ and $v_i(k\delta - \Delta)$, $i = 1, \ldots, n+m$, in the right-hand side can be computed exploiting the sequence of input-output measurements $\{u(k\delta), y(k\delta)\}$. The computation of these variables can be performed by means of a couple of Brunovsky canonical systems:

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \vdots \\ \dot{v}_{n+m-1} \\ \dot{v}_{n+m} \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n+m-1} \\ v_{n+m} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} y(t)$$

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \vdots \\ \dot{w}_{n+m-1} \\ \dot{w}_{n+m} \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{n+m-1} \\ w_{n+m} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

which can be denoted, in compact form, respectively as:

$$\dot{V}(t) = A_b V(t) + B_b y(t), \tag{6}$$

$$\dot{W}(t) = A_b W(t) + B_b u(t). \tag{7}$$

From the available sequence of sampled measurements $\{u(k\delta), y(k\delta)\}$ the time-evolutions $V(t)$ and $W(t)$ can be computed by replacing $y(t)$ and $u(t)$ in (6)–(7) with continuous piecewise linear functions $\tilde{y}(t)$ and $\tilde{u}(t)$:

$$\begin{aligned} \tilde{y}(t) &= y(k\delta) + \bar{y}_k \frac{t - k\delta}{\delta}, \\ \tilde{u}(t) &= u(k\delta) + \bar{u}_k \frac{t - k\delta}{\delta}, \end{aligned} \qquad t \in [k\delta, (k+1)\delta)$$

where

$$\bar{y}_k = y((k+1)\delta) - y(k\delta), \quad \bar{u}_k = u((k+1)\delta) - u(k\delta).$$

Of course, the accuracy of the approximations $\tilde{y}(t) \approx y(t)$ and $\tilde{u}(t) \approx u(t)$ depends on the frequency content of the signals and on the sampling time δ. Exact integration of the systems (6)–(7) with inputs $\tilde{y}(t)$ and $\tilde{u}(t)$ yields at discrete times $k\delta$:

$$V\big((k+1)\delta\big) = A_\delta V(k\delta) + B_\delta y(k\delta) + \overline{B}_\delta \bar{y}_k \quad (8)$$
$$W\big((k+1)\delta\big) = A_\delta W(k\delta) + B_\delta u(k\delta) + \overline{B}_\delta \bar{u}_k \quad (9)$$

where $A_\delta = e^{A_b\delta}$ and

$$B_\delta = \int_0^\delta e^{A_b\theta} B_b \, d\theta, \quad \overline{B}_\delta = \int_0^\delta e^{A_b\theta}\left(1 - \frac{\theta}{\delta}\right) B_b \, d\theta.$$

The entries of $V(k\delta)$ and $W(k\delta)$ are the components $v_i(k\delta)$ and $w_i(k\delta)$ required in (5). The computation of the terms at times $k\delta - \Delta$ can be similarly obtained using a recursion of the type (8), replacing A_δ, B_δ and $\overline{B}_\delta$ with suitable matrices that depend on $\Delta \bmod \delta$. However, in order to have a simpler notation, in the following we assume that the feedback-delay Δ is a multiple of the sampling time δ, i.e. $\Delta = p\delta$ for a suitable integer p. Thus, (5) becomes:

$$\begin{aligned} y(k\delta) = &-\beta_{n+m-1} v_{n+m}(k\delta) + \cdots - \beta_0 v_1(k\delta) \\ &- \gamma_{n+m-2} v_{n+m-1}\big((k-p)\delta\big) + \cdots - \gamma_0 v_1\big((k-p)\delta\big) \\ &+ \alpha_{n+m-1} w_{n+m}(k\delta) + \cdots + \alpha_0 w_1(k\delta) \end{aligned} \quad (10)$$

which is the starting point of the proposed technique.

Remark 1. We point out that the integral-based technique described in this section to avoid time derivatives may give biased estimates when the initial conditions are not null, and can also result in numerical ill-conditioning. More refined techniques can be chosen to avoid these drawbacks (based, for example, on the Linear Integral Filter approach, see Sagara and Zhao (1989)). However, being this section only preliminary for the two-steps method proposed in the following, we have chosen to present this preparatory material in its simplest formulation.

3. PROPOSED APPROACH

We conceived a two-steps procedure for the identification of the forward and feedback dynamics of the closed-loop system under investigation. The first step exploits the equations (10) to obtain an estimate of the parameters of the closed-loop transfer function $W(s)$, i.e. the coefficients of the polynomials $\alpha(s), \beta(s), \gamma(s)$. The second step aims at obtaining the parameters of $F(s)$ and $H(s)$ from the estimated closed-loop parameters. For the second step two alternatives are presented and discussed.

3.1 Estimating the parameters of $W(s)$

Let's define a compact notation for (10). First of all, we collect the unknown coefficients of the polynomials $\alpha(s)$, $\beta(s)$ and $\gamma(s)$ in row vectors:

$$\begin{aligned} \boldsymbol{\beta} &= [\, \beta_{n+m-1} \ \beta_{n+m-2} \ \cdots \ \beta_1 \ \beta_0 \,] \\ \boldsymbol{\gamma} &= [\, \gamma_{n+m-2} \ \gamma_{n+m-3} \ \cdots \ \gamma_1 \ \gamma_0 \,] \\ \boldsymbol{\alpha} &= [\, \alpha_{n+m-1} \ \alpha_{n+m-2} \ \cdots \ \alpha_1 \ \alpha_0 \,] \end{aligned} \quad (11)$$

and we define, for some integer $q \leq n + m$:

$$\boldsymbol{v}_{q:1}(\cdot) = [\, v_q(\cdot) \ \ v_{q-1}(\cdot) \ \cdots \ v_2(\cdot) \ v_1(\cdot) \,]$$
$$\boldsymbol{w}_{q:1}(\cdot) = [\, w_q(\cdot) \ w_{q-1}(\cdot) \ \cdots \ w_2(\cdot) \ w_1(\cdot) \,].$$

From now on we will omit the δ in the time dependencies to lighten the notation, writing k for $k\delta$ and so on.

With these simplifications (10) is rewritten, for $k = p, \ldots, N$, as:

$$\begin{aligned} y(k) = &[-\boldsymbol{v}_{n+m:1}(k) \,|\, -\boldsymbol{v}_{n+m-1:1}(k-p) \,|\, \boldsymbol{w}_{n+m:1}(k)] \\ &\cdot [\boldsymbol{\beta} \mid \boldsymbol{\gamma} \mid \boldsymbol{\alpha}]^T. \end{aligned} \quad (12)$$

Defining the vector of unknown parameters

$$\theta = [\boldsymbol{\beta} \mid \boldsymbol{\gamma} \mid \boldsymbol{\alpha}]^T \in \mathbb{R}^{3(n+m)-1},$$

and the sequence of row vectors

$$C(k) = [-\boldsymbol{v}_{n+m:1}(k) \,|\, -\boldsymbol{v}_{n+m-1:1}(k-p) \,|\, \boldsymbol{w}_{n+m:1}(k)],$$

for $k = p, \ldots, N$, equation (12) can be written as

$$y(k) = C(k)\theta.$$

Note that $C(k)$ is made of the entries in reverse order of the vectors $V(k)$, $V(k-p)$ and $W(k)$ obtained by the recursions (8)–(9).

An offline estimate of θ after $N > 3(n+m) + p - 2$ measurements can be obtained solving the system:

$$Y_{p:N} = \begin{bmatrix} y(p) \\ y(p+1) \\ \vdots \\ y(N-1) \\ y(N) \end{bmatrix} = \begin{bmatrix} C(p) \\ C(p+1) \\ \vdots \\ C(N-1) \\ C(N) \end{bmatrix} \theta = C_{p:N}\theta,$$

with the least-squares criterion, thus obtaining the least-squares estimate of θ:

$$\hat{\theta} = \left(C_{p:N}^T C_{p:N}\right)^{-1} C_{p:N}^T Y_{p:N}.$$

An online estimate of θ can be achieved using recursive least-squares (see Ljung (1999)):

$$\begin{aligned} L(k) &= \frac{P(k-1)\, C(k)^T}{C(k)\, P(k-1)\, C(k)^T + \lambda} \\ P(k) &= \frac{1}{\lambda}\left[P(k-1) - \frac{P(k-1)C(k)^T C(k) P(k-1)}{C(k)P(k-1)C(k)^T + \lambda}\right] \\ \hat{\theta}(k) &= \hat{\theta}(k-1) + L(k)\left[y(k) - C(k)\hat{\theta}(k-1)\right]. \end{aligned} \quad (13)$$

The parameter λ is a forgetting factor, usually chosen in $[0.98, 1]$. The recursion must be initialized with an initial value of $\hat{\theta}(0)$ and a suitable choice of $P(0)$ (symmetric, positive definite, and sufficiently large), and provides the sequence of recursive least-squares estimates of the parameters vector θ at time instants $k = p, \ldots, N$, i.e.

$$\hat{\theta}(k) = \left[\hat{\boldsymbol{\beta}}(k) \ \hat{\boldsymbol{\gamma}}(k) \ \hat{\boldsymbol{\alpha}}(k)\right]^T.$$

3.2 Extracting the parameters of $F(s)$ and $H(s)$

In the previous section we have seen how to obtain an online least-squares estimate of $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, parameters of the closed-loop transfer function $W(s)$, denoted $\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}}$.

From the definitions (1) we know that

$$\frac{\alpha(s)}{\beta(s)} = \frac{N_F(s) D_H(s)}{D_F(s) D_H(s)} = F(s), \quad (14)$$
$$\frac{\gamma(s)}{\alpha(s)} = \frac{N_F(s) N_H(s)}{N_F(s) D_H(s)} = H(s). \quad (15)$$

Thus, using the identified coefficients we can define the estimates of the forward and backward transfer functions:

$$\widetilde{F}(s) = \frac{N_{\widetilde{F}}(s)}{D_{\widetilde{F}}(s)} = \frac{\hat{\alpha}(s)}{\hat{\beta}(s)} \quad (16)$$
$$\widetilde{H}(s) = \frac{N_{\widetilde{H}}(s)}{D_{\widetilde{H}}(s)} = \frac{\hat{\gamma}(s)}{\hat{\alpha}(s)} \quad (17)$$

whose orders are $n + m$ and $n + m - 1$ respectively, and therefore are not minimal, because we know that the orders of $F(s)$ and $H(s)$ are n and m, respectively. However, although $\alpha(s)$ and $\beta(s)$ have a common divisor, which is $D_H(s)$ (it is in fact the GCD, greatest common divisor, because by assumption $N_F(s)$ and $D_F(s)$ are coprime) the estimated polynomials $\hat{\alpha}(s)$ and $\hat{\beta}(s)$, in general, do not have a common divisor, as a result of various approximations and numerical errors in the estimation process. The same happens to $\gamma(s)$ and $\alpha(s)$, whose greatest common divisor is $N_F(s)$, while $\hat{\gamma}(s)$ and $\hat{\alpha}(s)$ may not admit a common divisor. Thus, we can formulate the problem of finding $\widehat{F}(s)$ of order n that approximates $\widetilde{F}(s)$, and $\widehat{H}(s)$ of order m that approximates $\widetilde{H}(s)$. This goal is achieved by finding *approximate* common divisors of given orders between the identified polynomials. Computing approximate-GCD of polynomials is an active research topic in Numerical Algebra (see for example Eckstein and Zítko (2015) and Stoica and Söderström (1997) for a system theoretical discussion). We will now introduce two possible ways to obtain an estimate of the coefficients of minimal realizations of $F(s)$ and $H(s)$ from the identified coefficients of $W(s)$, i.e. $\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}}$.

Least-squares approach A straightforward solution can be obtained imposing the identified coefficients of $\hat{\alpha}(s)$, $\hat{\beta}(s)$, $\hat{\gamma}(s)$ to be the discrete convolutions of the unknown $\{N_F(s), D_H(s)\}$, $\{D_F(s), D_H(s)\}$, $\{N_F(s), N_H(s)\}$. To this end, we formulate a least-squares optimization problem using the Kronecker product (see Van Loan (2000)). First of all, let's define the vector containing all the powers of s up to a fixed degree $q \in \mathbb{N}$ as

$$\boldsymbol{\Sigma_q} = \begin{bmatrix} s^q & s^{q-1} & \cdots & s^1 & 1 \end{bmatrix}^T.$$

Then for $N_F(s), N_H(s), D_F(s), D_H(s)$ one has:

$$\begin{aligned} N_F(s) &= \begin{bmatrix} 0 & b^F_{n-1} & b^F_{n-2} & \cdots & b^F_1 & b^F_0 \end{bmatrix} \boldsymbol{\Sigma_n} = \begin{bmatrix} 0 & \boldsymbol{b^F} \end{bmatrix} \boldsymbol{\Sigma_n} \\ N_H(s) &= \begin{bmatrix} 0 & b^H_{m-1} & b^H_{m-2} & \cdots & b^H_1 & b^H_0 \end{bmatrix} \boldsymbol{\Sigma_m} = \begin{bmatrix} 0 & \boldsymbol{b^H} \end{bmatrix} \boldsymbol{\Sigma_m} \\ D_F(s) &= \begin{bmatrix} 1 & a^F_{n-1} & a^F_{n-2} & \cdots & a^F_1 & a^F_0 \end{bmatrix} \boldsymbol{\Sigma_n} = \begin{bmatrix} 1 & \boldsymbol{a^F} \end{bmatrix} \boldsymbol{\Sigma_n} \\ D_H(s) &= \begin{bmatrix} 1 & a^H_{m-1} & a^H_{m-2} & \cdots & a^H_1 & a^H_0 \end{bmatrix} \boldsymbol{\Sigma_m} = \begin{bmatrix} 1 & \boldsymbol{a^H} \end{bmatrix} \boldsymbol{\Sigma_m} \end{aligned}$$

and from (11):

$$\begin{aligned} \alpha(s) &= [0\ \boldsymbol{\alpha}]\, \boldsymbol{\Sigma_{n+m}} & \hat{\alpha}(s) &= [0\ \hat{\boldsymbol{\alpha}}]\, \boldsymbol{\Sigma_{n+m}} \\ \beta(s) &= [1\ \boldsymbol{\beta}]\, \boldsymbol{\Sigma_{n+m}} & \hat{\beta}(s) &= [1\ \hat{\boldsymbol{\beta}}]\, \boldsymbol{\Sigma_{n+m}} \\ \gamma(s) &= [0\ 0\ \boldsymbol{\gamma}]\, \boldsymbol{\Sigma_{n+m}} & \hat{\gamma}(s) &= [0\ 0\ \hat{\boldsymbol{\gamma}}]\, \boldsymbol{\Sigma_{n+m}}. \end{aligned} \tag{18}$$

With this notation, we can rewrite the equations (1) as:

$$\begin{aligned} [0\ \boldsymbol{\alpha}]\, \boldsymbol{\Sigma_{n+m}} &= ([0\ \boldsymbol{b^F}]\, \boldsymbol{\Sigma_n}) \cdot ([1\ \boldsymbol{a^H}]\, \boldsymbol{\Sigma_m}) \\ [1\ \boldsymbol{\beta}]\, \boldsymbol{\Sigma_{n+m}} &= ([1\ \boldsymbol{a^F}]\, \boldsymbol{\Sigma_n}) \cdot ([1\ \boldsymbol{a^H}]\, \boldsymbol{\Sigma_m}) \\ [0\ 0\ \boldsymbol{\gamma}]\, \boldsymbol{\Sigma_{n+m}} &= ([0\ \boldsymbol{b^F}]\, \boldsymbol{\Sigma_n}) \cdot ([0\ \boldsymbol{b^H}]\, \boldsymbol{\Sigma_m}). \end{aligned} \tag{19}$$

The products between scalar terms in the right hand side of each one of equations (19) can be replaced by the Kronecker product, and using the well-known property

$$(A \cdot B) \otimes (C \cdot D) = (A \otimes C) \cdot (B \otimes D)$$

we get:

$$\begin{aligned} [0\ \boldsymbol{\alpha}]\, \boldsymbol{\Sigma_{n+m}} &= ([0\ \boldsymbol{b^F}] \otimes [1\ \boldsymbol{a^H}]) \cdot (\boldsymbol{\Sigma_n} \otimes \boldsymbol{\Sigma_m}) \\ [1\ \boldsymbol{\beta}]\, \boldsymbol{\Sigma_{n+m}} &= ([1\ \boldsymbol{a^F}] \otimes [1\ \boldsymbol{a^H}]) \cdot (\boldsymbol{\Sigma_n} \otimes \boldsymbol{\Sigma_m}) \\ [0\ 0\ \boldsymbol{\gamma}]\, \boldsymbol{\Sigma_{n+m}} &= ([0\ \boldsymbol{b^F}] \otimes [0\ \boldsymbol{b^H}]) \cdot (\boldsymbol{\Sigma_n} \otimes \boldsymbol{\Sigma_m}). \end{aligned} \tag{20}$$

It is easy to verify that, given $n, m \in \mathbb{N}$, there exists a reduction matrix $\mathcal{T}_{red} \in \mathbb{R}^{(n+1)(m+1) \times (n+m+1)}$ such that

$$\boldsymbol{\Sigma_n} \otimes \boldsymbol{\Sigma_m} = \mathcal{T}_{red} \cdot \boldsymbol{\Sigma_{n+m}}$$

with

$$\mathcal{T}_{red} = \begin{bmatrix} \mathcal{T}_1 \\ \vdots \\ \mathcal{T}_{n+1} \end{bmatrix}, \quad \text{where: } \mathcal{T}_i = [0_{m+1 \times i-1}\ \ I_{m+1}\ \ 0_{m+1 \times n+1-i}] \text{ for } i = 1, \ldots, n+1.$$

Applying the reduction (20) become:

$$\begin{aligned} [0\ \boldsymbol{\alpha}]\, \boldsymbol{\Sigma_{n+m}} &= ([0\ \boldsymbol{b^F}] \otimes [1\ \boldsymbol{a^H}])\mathcal{T}_{red}\, \boldsymbol{\Sigma_{n+m}} \\ [1\ \boldsymbol{\beta}]\, \boldsymbol{\Sigma_{n+m}} &= ([1\ \boldsymbol{a^F}] \otimes [1\ \boldsymbol{a^H}])\mathcal{T}_{red}\, \boldsymbol{\Sigma_{n+m}} \\ [0\ 0\ \boldsymbol{\gamma}]\, \boldsymbol{\Sigma_{n+m}} &= ([0\ \boldsymbol{b^F}] \otimes [0\ \boldsymbol{b^H}])\mathcal{T}_{red}\, \boldsymbol{\Sigma_{n+m}}. \end{aligned}$$

The above identities between polynomials reduce to identities between coefficients:

$$\begin{aligned} [0\ \boldsymbol{\alpha}] &= ([0\ \boldsymbol{b^F}] \otimes [1\ \boldsymbol{a^H}])\mathcal{T}_{red} \\ [1\ \boldsymbol{\beta}] &= ([1\ \boldsymbol{a^F}] \otimes [1\ \boldsymbol{a^H}])\mathcal{T}_{red} \\ [0\ 0\ \boldsymbol{\gamma}] &= ([0\ \boldsymbol{b^F}] \otimes [0\ \boldsymbol{b^H}])\mathcal{T}_{red}. \end{aligned} \tag{21}$$

Taking into account the identities (21), the estimates $\hat{\boldsymbol{\alpha}}$, $\hat{\boldsymbol{\beta}}$, $\hat{\boldsymbol{\gamma}}$ of the closed-loop parameters obtained in the previous step can be used to get least-squares estimates of $\boldsymbol{a^F}, \boldsymbol{a^H}, \boldsymbol{b^F}, \boldsymbol{b^H}$ (the desired parameters of $\widehat{F}(s), \widehat{H}(s)$) by defining the cost function:

$$\begin{aligned} J(\boldsymbol{a^F}, \boldsymbol{a^H}, \boldsymbol{b^F}, \boldsymbol{b^H}) = {} & \| [0\ \hat{\boldsymbol{\alpha}}] - ([0\ \boldsymbol{b^F}] \otimes [1\ \boldsymbol{a^H}])\mathcal{T}_{red} \|^2 \\ & + \| [1\ \hat{\boldsymbol{\beta}}] - ([1\ \boldsymbol{a^F}] \otimes [1\ \boldsymbol{a^H}])\mathcal{T}_{red} \|^2 \\ & + \| [0\ 0\ \hat{\boldsymbol{\gamma}}] - ([0\ \boldsymbol{b^F}] \otimes [0\ \boldsymbol{b^H}])\mathcal{T}_{red} \|^2 \end{aligned}$$

and solving the minimization problem

$$(\hat{\boldsymbol{a}}^{\boldsymbol{F}}, \hat{\boldsymbol{a}}^{\boldsymbol{H}}, \hat{\boldsymbol{b}}^{\boldsymbol{F}}, \hat{\boldsymbol{b}}^{\boldsymbol{H}}) = \arg \min_{\boldsymbol{a^F}, \boldsymbol{a^H}, \boldsymbol{b^F}, \boldsymbol{b^H}} J.$$

Approximate GCD via Kalman Decomposition and SVD As previously discussed, if the polynomials $\alpha(s)$, $\beta(s)$, $\gamma(s)$ were exactly identified, they would contain common factors that could be simplified in the ratios (14)–(15), giving back the true $F(s)$ and $H(s)$. When considering the ratios of the identified polynomials $\hat{\alpha}(s)$, $\hat{\beta}(s)$, $\hat{\gamma}(s)$ no common divisor is expected, so that no simplification can be applied and the order of $\widetilde{F}(s)$ in (16) remains $m+n$, while the one of $\widetilde{H}(s)$ in (17) remains $m+n-1$. A straightforward idea is to find approximate common divisors to perform simplifications and get estimates $\widehat{F}(s)$ and $\widehat{H}(s)$ of desired orders. We propose here a system theoretical solution to neglect the approximate-GCD factors in $\widetilde{F}(s)$ and $\widetilde{H}(s)$ combining Kalman Decomposition and Singular Value Decomposition, leading to a simple model order reduction technique. We proceed for $F(s)$. First of all, we construct the controllability canonical form associated to the estimated $\widetilde{F}$:

$$A_{\widetilde{F}} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\hat{\beta}_0 & -\hat{\beta}_1 & \cdots & -\hat{\beta}_{n+m-1} \end{bmatrix} \quad B_{\widetilde{F}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C_{\widetilde{F}} = [\ \hat{\alpha}_0 \quad \hat{\alpha}_1 \quad \cdots \quad \hat{\alpha}_{n+m-1}\ ]. \tag{22}$$

We then compute the observability matrix

$$Q_{\widetilde{F}} = \left[C^T_{\widetilde{F}}\ \ (C_{\widetilde{F}} A_{\widetilde{F}})^T\ \cdots\ (C_{\widetilde{F}} A^{n+m-1}_{\widetilde{F}})^T \right]^T.$$

Ideally, the rank of $Q_{\widetilde{F}}$ would be exactly equal to the order of $F(s)$, i.e. n, but since we are dealing with estimated coefficients $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ the rank is expected to be different from n (most likely $Q_{\widetilde{F}}$ is full-rank). We can manage to isolate an approximate minimal form of system (22) by applying a convenient Kalman observability decomposition to $(A_{\widetilde{F}}, B_{\widetilde{F}}, C_{\widetilde{F}})$. More precisely:

- The transfer function of the *approximately* observable part of the system $(A_{\widetilde{F}}, B_{\widetilde{F}}, C_{\widetilde{F}})$ will be the desired order n estimate of $F(s)$.
- The characteristic polynomial of the *approximately* unobservable subsystem will be the approximate common divisor of $\hat{\alpha}(s)$ and $\hat{\beta}(s)$ to be eliminated in the ratio.

From the Kalman decomposition (see Kalman (1963)) it is well known that the system matrices can be written as:

$$A_K = TA_{\widetilde{F}}T^{-1} = \begin{bmatrix} A_u & A_{uo} \\ 0 & A_o \end{bmatrix} \quad B_K = TB_{\widetilde{F}} = \begin{bmatrix} B_u \\ B_o \end{bmatrix}$$
$$C_K = C_{\widetilde{F}}T^{-1} = [0 \ C_o] \tag{23}$$

where $T^{-1} = [T_u \ T_o]$ is a suitable change of basis matrix in which the columns of T_u form a basis for the unobservable subspace $ker(Q_{\widetilde{F}})$ and the columns of T_o form a basis for the observable subspace obtained as a complement of $ker(Q_{\widetilde{F}})$. Moreover, the transfer function of the observable subsystem is the minimal form of the transfer function of the whole system:

$$F(s) = F_o(s) = C_o(sI - A_o)^{-1}B_o.$$

Now, attempting the computation of the exact Kalman decomposition of the system (22) we would likely find that the whole state space is observable, being $Q_{\widetilde{F}}$ full rank as a consequence of numerical errors and approximations introduced by the identification procedure. In this case, no order reduction of $\widetilde{F}(s)$ would be possible. On the other hand, we can enforce a model reduction to order n by defining an approximately unobservable subsystems of (22) of order m (the degree of the common factor between the true polynomials $\alpha(s)$ and $\beta(s)$). To achieve this goal we need to define a subspace of order m of *approximately unobservable* states. This is done by suitably partitioning the matrices of the SVD of $Q_{\widetilde{F}}$ as follows:

$$Q_{\widetilde{F}} = USV^T = [U_1 \ U_2] \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \tag{24}$$

where U and V are the unitary matrices of left and right singular vectors such that $U_1, V_1 \in \mathbb{R}^{n+m \times n}, U_2, V_2 \in \mathbb{R}^{n+m \times m}$ and $S = diag\{\sigma_1, \ldots, \sigma_{n+m}\}$ is the matrix of singular values of $Q_{\widetilde{F}}$, with $S_1 = diag\{\sigma_1, \ldots, \sigma_n\} \in \mathbb{R}^{n \times n}$, $S_2 = diag\{\sigma_{n+1}, \ldots, \sigma_{n+m}\} \in \mathbb{R}^{m \times m}$ and $\sigma_1 \geq \cdots \geq \sigma_{n+m} \geq 0$. The singular values in S_2 would be zero if $\hat{\alpha}(s) = \alpha(s)$ and $\hat{\beta}(s) = \beta(s)$, because in this case they would have a common divisor of degree m (the polynomial $D_H(s)$). However, since $\hat{\alpha}(s)$ and $\hat{\beta}(s)$ are the outcome of an identification process, most likely they have no common divisor, and the singular values in S_2 are nonzero. However, if the estimates $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$ are not far from the true values $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, the singular values in S_2 are expected to be rather small, if compared to those in S_1. Stated in other terms, if the identification procedure were ideal, m would be exactly the rank loss of $Q_{\widetilde{F}}$. In this case, as it is well known (see Kalman (1996)), the m vectors of V_2, i.e. $v_{n+1}, \ldots, v_{n+m}$, corresponding to the zero-valued singular values in S_2, would form a basis of $ker(Q_{\widetilde{F}})$, and the n vectors of V_1, i.e. $v_1, \ldots, v_n$, corresponding to the non-zero singular values in S_1, would form a basis for $Im(Q_{\widetilde{F}}^T) = ker(Q_{\widetilde{F}})^{\perp}$.

In our method we proceed as if all the m lowest singular values of $Q_{\widetilde{F}}$ were actually zero, and enforce an approximate observability decomposition choosing $T^{-1} = [T_u \ T_o] = [V_2 \ V_1]$ (recall that U, V are unitary, so $V_1^T V_1 = I_n, V_2^T V_2 = I_m$). Taking into account the form of the exact Kalman decomposition (23), we can write the approximately observable and unobservable subsystems as follows:

$$\widehat{A}_o = V_1^T A_{\widetilde{F}} V_1 \quad \widehat{B}_o = V_1^T B_{\widetilde{F}} \quad \widehat{C}_o = C_{\widetilde{F}} V_1$$
$$\widehat{A}_u = V_2^T A_{\widetilde{F}} V_2 \quad \widehat{B}_u = V_2^T B_{\widetilde{F}} \quad \widehat{C}_u = 0.$$

Concluding, the minimal estimate of $F(s)$ is given by

$$\widehat{F}(s) = \widehat{C}_o(sI_n - \widehat{A}_o)^{-1}\widehat{B}_o$$

and the canceled approximated common divisor of degree m between $\hat{\alpha}(s)$ and $\hat{\beta}(s)$ (see (14) and (16)), is the characteristic polynomial of $\widehat{A}_u$:

$$\widetilde{D}_H(s) = \det(sI_m - \widehat{A}_u).$$

The same method can be applied to $\widetilde{H}(s)$ to estimate and cancel the approximate-GCD between $\hat{\gamma}(s)$ and $\hat{\alpha}(s)$, i.e. $\widetilde{N}_F(s)$, obtaining the desired minimal order estimate $\widehat{H}(s)$.

4. EXAMPLE

The proposed identification approach has been tested by means of computer simulation. In this section we report simulation results for a closed–loop system whose forward and feedback dynamics are described by the following third-order transfer functions:

$$F(s) = \frac{2s^2 + s + 2}{s^3 + 3s^2 + 2s + 3} \quad H(s) = \frac{s^2 + 5s + 1}{s^3 + 3s^2 + 2s + 2}$$

with $n = m = 3$, feedback time-delay $\Delta = 0.5s$ and sampling interval $\delta = 0.05s$, for a total simulation time of $T = 5s$. An identification input function with a sufficiently rich harmonic content has been generated using a Fourier basis on $L^2(0, T)$:

$$u(t) = 1 + \sum_{k=1}^{2(n+m)} \left[\sqrt{2} \sin\left(\frac{2\pi}{T}kt\right) + \sqrt{2} \cos\left(\frac{2\pi}{T}kt\right) \right].$$

Recursive least-squares (13) have been initialized with:

$$\hat{\theta}(0) = 0_{[3(m+n)-1] \times 1}, \quad P(0) = I_{3(m+n)-1}, \quad \lambda = 0.99.$$

We compared the identification results obtained with both approaches proposed in Section III-B. We will refer to the minimization method results as *min(J)-estimate* and to the approximate-GCD as *GCD-estimate*. We obtained:

$$\widehat{F}_{min(J)}(s) = \frac{2.025s^2 - 1.125s + 0.269}{s^3 + 2.051s^2 - 1.570s + 0.241}$$
$$\widehat{H}_{min(J)}(s) = \frac{1.523s^2 + 1.486s - 0.355}{s^3 + 0.491s^2 + 0.915s + 0.003}$$
$$\widehat{F}_{GCD}(s) = \frac{2.001s^2 + 1.510s + 2.039}{s^3 + 3.332s^2 + 2.503s + 2.325}$$
$$\widehat{H}_{GCD}(s) = \frac{1.531s^2 + 2.032s + 0.254}{s^3 + 0.901s^2 + 1.169s + 0.290}.$$

While the identified coefficients can significantly differ from the true ones, it must be noted that the identification of the input-output behavior of both transfer functions is very accurate. To verify this, we compared the responses of the true $F(s), H(s)$ and the estimated ones to a white noise validating input with zero mean and unit variance, for $T = 5s$, as shown in Figures 2–3. Moreover, we computed the mean squared error (MSE), comparing the

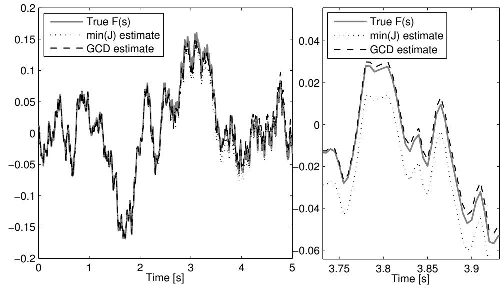

Fig. 2. Response to white noise input for $F(s)$ (with detail).

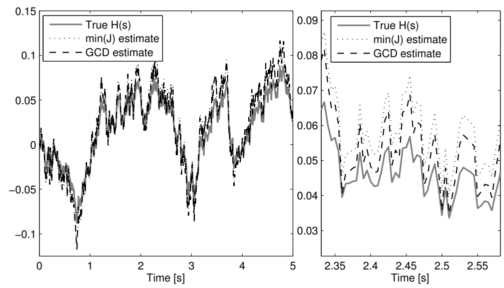

Fig. 3. Response to white noise input for $H(s)$ (with detail).

true (y_F, y_H) and estimated $(\hat{y}_F, \hat{y}_H)$ output, for both subsystems, using:

$$MSE = \frac{1}{N}\sum_{k=1}^{N}(\hat{y}(kt_c) - y(kt_c))^2$$

with $N = T/t_c$, $t_c = 0.005$, obtaining:

$$MSE^{F(s)}_{min(J)} = 1.783 \cdot 10^{-4} \quad MSE^{H(s)}_{min(J)} = 7.877 \cdot 10^{-4}$$
$$MSE^{F(s)}_{GCD} = 1.626 \cdot 10^{-4} \quad MSE^{H(s)}_{GCD} = 3.337 \cdot 10^{-4}.$$

It is clear that, while both the identification procedures give good results, the approximate-GCD estimate has the lowest MSE, and is confirmed in all simulations made. Although our approach is deterministic, to obtain a more realistic simulation we added measurement noise on the closed–loop system output, and repeated the identification procedure for a 15% noise-to-signal ratio, with

$$NSR = \frac{mean(abs(noise))}{mean(abs(signal))}.$$

The following estimates have been attained:

$$\widehat{F}_{min(J)}(s) = \frac{2.077s^2 + 2.572s - 1.218}{s^3 + 3.936s^2 + 2.062s + 0.580}$$
$$\widehat{H}_{min(J)}(s) = \frac{1.109s^2 + 0.094s - 0.279}{s^3 + 0.050s^2 + 0.632s - 0.417}$$
$$\widehat{F}_{GCD}(s) = \frac{1.981s^2 + 3.087s + 0.359}{s^3 + 3.900s^2 + 2.920s + 2.472}$$
$$\widehat{H}_{GCD}(s) = \frac{1.097s^2 + 2.698s + 0.745}{s^3 + 2.051s^2 + 1.597s + 1.323}.$$

We then repeated the validation using a white noise input signal with zero mean and unit variance (Figures 4–5) obtaining the following MSE in a 5 seconds test:

$$MSE^{F(s)}_{min(J)} = 1.967 \cdot 10^{-2} \quad MSE^{H(s)}_{min(J)} = 5.656 \cdot 10^{-3}$$
$$MSE^{F(s)}_{GCD} = 1.211 \cdot 10^{-3} \quad MSE^{H(s)}_{GCD} = 1.286 \cdot 10^{-3}.$$

The results prove the effectiveness of the proposed approach even in a noisy framework, and also confirm the overall major accuracy of the approximate-GCD method.

5. CONCLUSIONS AND FUTURE WORKS

In this work, we have proposed a simple but effective approach to estimate the forward and feedback dynamics of a linear continuous-time system with delayed feedback

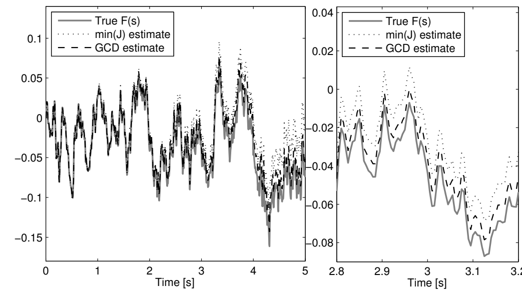

Fig. 4. Response to white noise input for $F(s)$, noisy case.

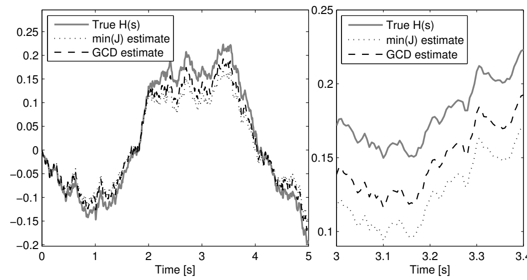

Fig. 5. Response to white noise input for $H(s)$, noisy case.

using sampled closed-loop data. The first step of the technique concerns the estimation of the whole closed-loop behavior, while the second step focuses on the separation of the forward and feedback subsystems. We illustrated two different ways to solve the second step: a least-squares optimization procedure and a system theoretical method to cancel the greatest common divisor between polynomials, the latter showing better performances and noise robustness in the validating numerical simulations. Future works will be devoted to the explicit modeling of the noisy case and to the possibility of estimating the feedback delay and the orders of the subsystems.

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