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Stability conditions for linear discrete-time switched systems in block companion form

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Abstract: Switched models whose dynamic matrices are in block companion form arise in theoretical and applicative problems such as representing switched ARX models in state-space form for control purposes. Inspired by some insightful results on the delay-independent stability of discrete-time systems with time-varying delays, in this work, the authors study the arbitrary switching stability for some classes of block companion discrete-time switched systems. They start from the special case in which the first block-row is made of permutations of non-negative matrices, deriving a simple necessary and sufficient stability condition under arbitrary switching. The condition is computationally less demanding than the sufficient-only existence of a linear common Lyapunov function. Then, both non-negativity and combinatorial assumptions are dropped, at the expense of introducing conservatism. Some implications on the computation of the joint spectral radius for the aforementioned families of matrices are illustrated.

1 Introduction

In the last decades an intensive research effort towards the stability analysis of dynamical systems has been carried out. While the stability of linear time-invariant systems is a classic and relatively simple problem, for which long standing results are available, tough technical challenges arise when time-varying systems or systems with time-delays are considered. In these cases, necessary and sufficient conditions are either not available or intractable from a computational viewpoint, and conservative results are often derived [1–3]. A remarkable simplification takes place considering positive systems, i.e. systems whose state trajectory stays non-negative as long as inputs and initial conditions are non-negative. This has proven influential in various classes of positive linear delay systems (see e.g. [4–7]) and switched systems (see [8, 9] and references therein).

In particular, the so-called comparison approach has provided useful results to export to not necessarily positive (briefly, *non positive* hereinafter) systems the aforementioned favourable properties, although at the expense of introducing some conservatism. In this work, we proceed a step forward in this direction: first of all, we discuss how existing results for positive delay systems naturally yield an interesting consequence on the stability of discrete-time switched systems in block companion form, for which the arbitrary switching stability becomes a trivial task. Then, we show how the positivity-based results for both delay and switched systems of the aforementioned classes can be exported to non positive systems of the same classes, yielding novel sufficient stability conditions.

Namely, we will focus on the stability of discrete-time switched systems described by

$$x(k+1) = \mathscr{A}_{s(k)}x(k) \tag{1}$$

where s is an arbitrary switching sequence taking values in $\{1, \dots, p\}$ such that at time k the matrix $\mathscr{A}_{s(k)} \in \mathbb{R}^{mn \times mn}$ is taken from a family $\mathbb{A} = \{\mathscr{A}_1, \dots, \mathscr{A}_p\}$ of block companion matrices, of the type

$$\mathscr{A}_j = \begin{bmatrix} A_{j,1} & A_{j,2} & \cdots & A_{j,m-1} & A_{j,m} \\ I_n & 0 & \cdots & 0 & 0 \\ 0 & I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & 0 \end{bmatrix}, \quad j = 1, \dots, p, \tag{2}$$

where $A_{j,i} \in \mathbb{R}^{n \times n}$ for all $i = 1, \dots, m$. Block companion switched systems arise in both theoretical and practical problems. For instance, they can serve as state-space representations of linear switched ARX models [10], whose estimation for control of large-scale systems is an active research area (see e.g. [11] and references therein). In particular, some of these estimation techniques have been successfully applied to experimental use-cases in energy efficient building automation [12], in structural health monitoring and control [13] and in control of Software Defined communication Networks [14]. While the learning approaches leveraged by the aforementioned works have exhibited impressive experimental results, they tend to produce unstable models even if the original system is stable. The impact of this limitation is negligible in a closed-loop regime, but preserving stability during an estimation procedure is a matter of consistency in terms of accuracy. The theoretical results of this work have direct implications on solving this drawback, as they can be used to constrain the learning procedure to only produce stable models from stable systems.

We will prove a number of stability results on block companion switching systems, starting from the case in which the first block row of $\mathscr{A}_j$ consists of non-negative permuting matrices, then relaxing these assumptions.

The starting point of our analysis comes from the fact that system (1) with block companion $\mathscr{A}_j$ can be shown equivalent, under some conditions, to the discrete-time delay system

$$z(k+1) = \sum_{i=1}^{r} M_i z(k - \delta_i(k)) \tag{3}$$

where $\delta_i(k)$ takes value in $\mathbb{N}_0$. This equivalence is certainly not new, as it has been instrumental in popular works such as [15], where the existence of multiple Lyapunov functions for switched systems has been shown equivalent to the existence of delay-dependent

Lyapunov–Krasovskii functionals for delay systems. Whilst this paper exploits the same link between delay and switched block companion systems, the approach is reversed and is largely based on delay-independent results for positive delay systems, as we aim to explain in detail.

Main contribution and paper organisation: This work goes beyond the preliminary results presented in [16], introducing novel developments on both delay systems and switched systems in the block companion form. In Section 2, we will discuss in some detail the stability properties of the delay system (3), starting from the positive framework, then relaxing this assumption to consider not necessarily positive systems. For the latter, a simple and easy-to-check sufficient condition for the delay-independent stability is presented in Theorem 7. Here, we extend the results of [16] showing that the condition is robust against unbounded time-varying delays. Section 3 will relate the stability of switched systems in the block companion form to the results illustrated in Section 2, presenting the main contributions of this work and comparing them to stability conditions of heterogeneous complexity. We start the section discussing the problem of arbitrary switching stability for general switching systems and the simplifications introduced by the positivity assumption. In Section 3.1, we illustrate the theoretical and applicative importance of block companion switched systems by a motivating example: state-space representation of switched autoregressive models. The remainder of Section 3 introduces the main results of the work, starting from the case of positive block companion systems with combinatorial assumptions on the matrices in their first block-row, for which we present necessary and sufficient conditions for arbitrary switching stability (Theorems 9 and 12). For simplicity and clarity, the results are introduced incrementally (i.e. the results of Section 3.2 are a special case of those of Section 3.3). Then, in Sections 3.4 and 3.5, we remove both the positivity (Theorem 14) and the combinatorial assumptions (Theorems 15 and 16), introducing sufficient conditions for the arbitrary switching stability of quite general classes of block companion switched systems. Moreover, we examine the relationship among the results of this work and previous contributions appeared in the literature. In particular, we discuss how the results of this work allow to decide whether the joint spectral radius (JSR) of the aforementioned families of matrices is less than one (meaning that a generally NP-hard problem becomes trivial in these cases). In the positive framework, we compared our results with the existence of linear co-positive common Lyapunov functions. The aforementioned comparison, the illustrative Section 3.1, the results of Section 3.3 and part of Section 3.5 are all novel contributions with respect to [16]. Section 4 provides four numerical examples in which several comparisons are carried out, showing how the novel results of the paper are both: (i) less conservative than linear common Lyapunov functions (for block companion non-negative matrices); (ii) considerably simpler, from a computational viewpoint, with respect to computing the JSR of either non-negative and arbitrary families of block companion matrices. Again, both the examples and their implications on the conservativeness of the theoretical results of the paper are novel with respect to our previous work [16]. Conclusions and ideas for future work close the paper.

Notation: $\mathbb{N}_0$ is the set of non-negative integers. $\mathbb{R}_+$ is the set of non-negative real numbers. $\mathbb{R}_+^n$ and $\mathbb{R}_{++}^n$, respectively, denote the non-negative and positive orthants of $\mathbb{R}^n$, i.e. $\mathbb{R}_+^n = \{v \in \mathbb{R}^n : v_i \geq 0, \forall i\}$, $\mathbb{R}_{++}^n = \{v \in \mathbb{R}^n : v_i > 0, \forall i\}$. $\mathbb{R}_+^{m \times n}$ is the cone of non-negative $m \times n$ matrices. I_n is the $n \times n$ identity matrix. Inequalities among vectors and matrices of the same dimensions have to be understood componentwise, i.e. $M \leq N$ if $m_{ij} \leq n_{ij}$ for all i,j. In this sense, a non-negative matrix $M \in \mathbb{R}_+^{m \times n}$ is also denoted by $M \geq 0$, where 0 is the matrix of appropriate dimensions whose entries are all zero. $\sigma(M)$ and $\rho(M)$, respectively, denote the spectrum and the spectral radius of a square matrix M, which is said to be *Schur-stable* if $\sigma(M) \subset \{z \in \mathbb{C} : |z| < 1\}$ or, equivalently, if $\rho(M) < 1$. For a set of matrices $\mathbb{M} = \{M_1, \ldots, M_p\}$, the JSR of $\mathbb{M}$ is defined and denoted as

$$\rho^*(\mathbb{M}) = \rho^*(M_1, \ldots, M_p) = \lim_{k \to \infty} \max_{B \in \mathbb{M}^k} \| B \|^{1/k}, \tag{4}$$

where $\mathbb{M}^k$ is the set of all products of length k (allowing for repetitions) of matrices in $\mathbb{M}$. Clearly, for a single matrix M, $\rho^*(M) = \rho(M)$. See [17] for further details.

2 Enabling results on delay systems

In this section, we are going to illustrate a number of results on discrete-time delay systems with time-varying delays which will be instrumental to establish various stability conditions on switched systems in the block companion form (Section 3).

We will start discussing how the stability analysis of delay systems is computationally hard for time-varying delays, with a noteworthy exception: positive delay systems, for which straightforward necessary and sufficient conditions are available. For positive systems, as proven in [6], the stability is independent of delays, i.e. if the system is stable for any values of the delays (e.g. zero-valued delays), it is stable for all their values (Theorem 2). Then, we will show how this strong result can be exported also to non positive systems, at the expense of losing necessity, via the internally positive representation (IPR) technique. The outcome of this procedure is Theorem 7, which gives a sufficient condition for the delay-independent stability without positivity constraints. The condition is simple and holds for possibly unbounded delays. Moreover, it is computationally efficient to check, as it only involves non-negative matrices (Remark 3).

Let us start considering a discrete-time delay system governed by the difference equation

$$\begin{aligned} z(k+1) &= \sum_{i=1}^{r} M_i z(k - \delta_i(k)), \quad k \geq 0, \\ z(k) &= \phi(k), \qquad\qquad\qquad k = -\delta, \ldots, -1, 0, \end{aligned} \tag{5}$$

where $\delta_i(k)$ is a time-varying delay with integer values in $[0, \delta]$, $z(k) \in \mathbb{R}^n$ is the state trajectory at time k and ϕ is the initial state function. Since the focus of our work is on stability, we do not model input and output functions to avoid unnecessary details.

It is instrumental to note that the trivial case of constant delays $\delta_i(k) = \delta_i \in \mathbb{N}_0$, with $0 \leq \delta_i \leq \delta$ can be rewritten as

$$z(k+1) = \sum_{j=0}^{\delta} \tilde{M}_j z(k-j), \quad k \geq 0, \tag{6}$$

where $\tilde{M}_j = \sum_{i \in I_j} M_i$, with $I_j = \{i : \delta_i = j\}$ accounting for possibly multiple delays with value j (note that I_j can be empty). It is clear that (6) admits an equivalent delay-free state-space representation

$$x(k+1) = \mathscr{A} x(k) \tag{7}$$

where $x(k) = \begin{bmatrix} z^T(k) & z^T(k-1) & \cdots & z^T(k-\delta) \end{bmatrix}^{\mathrm{T}}$ and

$$\mathscr{A} = \begin{bmatrix} \tilde{M}_0 & \tilde{M}_1 & \cdots & \tilde{M}_{\delta-1} & \tilde{M}_\delta \\ I_n & 0 & \cdots & 0 & 0 \\ 0 & I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & 0 \end{bmatrix} \tag{8}$$

which allows to conclude that the stability of a discrete-time delay system is a trivial question when constant delays are involved (i.e. $\rho(\mathscr{A}) < 1$ is a necessary and sufficient condition for the exponential stability). This explains why most of the literature has focused on the general case of time-varying delays (5), for which the stability analysis is far from being trivial. In this respect, the literature has mainly investigated Lyapunov approaches, yielding LMIs of growing complexity in order to reduce conservatism. Notable contributions can be found in [18–20]. For some general and updated results on the stability and stabilisability of various classes of delay systems the reader is referred to [21–23]. If time-varying delays are considered, it is still possible to represent a delay system as in (5) via an augmented delay-free representation. However, a switching dynamics will arise due to the time-varying delays, and the block companion $\mathscr{A}$ of (8) will be replaced by a switching block companion $\mathscr{A}_{s(k)}$ whose first block-row matrices will vary according to the values of the delays at time k. This idea will be the starting point of Section 3, and will lead to novel stability results for block companion switched systems.

2.1 Stability of positive delay systems

A truly remarkable simplification is achieved if system (5) is positive, i.e. if non-negative initial conditions (and non-negative input, if modelled) can only produce non-negative state at all time instants. This definition is trivially satisfied checking the system matrices, as proven in [6], from which Lemma 1 and Theorem 2 are taken.

Lemma 1: The delay system (5) is positive if and only if M_i is componentwise non-negative (i.e. $M_i \geq 0$) for all $i = 1, \dots, r$.

Then, if positivity is satisfied, checking the asymptotic stability of (5) is very simple, and a single check ensures that the system is delay-independent stable, i.e. is stable for all possible time-varying values of the delays $\delta_i(k)$, (see [6]).

Theorem 2: The delay system (5), with $M_i \geq 0$ for all i, is delay-independent asymptotically stable if and only if $\rho(\sum_{i=1}^{r} M_i) < 1$.

Remark 3: Whereas Theorem 2 requires to check that $\rho(\sum_{i=1}^{r} M_i) < 1$, there is no need to actually compute neither the eigenvalues of $\sum_{i=1}^{r} M_i$ nor its charachteristic polynomial in order to verify its Schur stability, since a number of simple and robust conditions are available for non-negative matrices. Indeed, all the following conditions are equivalent [5]:

- $$\rho\left(\sum_{i=1}^{r} M_i\right) < 1 \quad (9)$$

- all leading principal minors of $I_n - \sum_{i=1}^{r} M_i$ are positive, (10)

- $$I_n - \sum_{i=1}^{r} M_i \text{ is non-singular and } \left(I_n - \sum_{i=1}^{r} M_i\right)^{-1} \geq 0, \quad (11)$$

- $$\exists\, p \in \mathbb{R}_{++}^{n} \text{ s.t. } \left(\sum_{i=1}^{r} M_i\right) p < p, \text{ or } p^T \left(\sum_{i=1}^{r} M_i\right) < p^T. \quad (12)$$

Conditions (10)–(12) are computationally simpler than the more eloquent (9). This remark applies whenever condition (9) – or any similar condition requiring to test the Schur-stability of a non-negative matrix – will be used in the remainder of the work.

Remark 4: Note that the theorem states that the delay-independent stability can be verified checking the zero-valued delays stability (just substitute $\delta_i(k) \equiv 0$ in (5)), but this check is only the simplest way to say that the system is delay-independent stable if and only if it is stable for any set of constant values of the delays. Indeed, delay-dependent and delay-independent stability are equivalent for positive systems (the former collapses into the latter). We will use this equivalence later in the work. As a further remark, note that the stability is exponential for all possible constant delays, as shown in [24].

Now consider the special case of constant delays described by (6). Then Theorem 2 implies that if the system is positive ($\tilde{M}_j \geq 0$ for all j), its exponential stability is simply tested verifying that $\rho(\sum_{j=0}^{\delta} \tilde{M}_j) < 1$. Moreover, since the stability of (6) is equivalent to that of its augmented state-space representation (7) and (8), the following result is straightforward.

Lemma 5: Consider m non-negative $n \times n$ matrices M_i. Then, the following result holds:

$$\rho\left(\begin{bmatrix} M_1 & M_2 & \cdots & M_{m-1} & M_m \\ I_n & 0 & \cdots & 0 & 0 \\ 0 & I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & 0 \end{bmatrix}\right) < 1 \iff \rho\left(\sum_{i=1}^{m} M_i\right) < 1 \quad (13)$$

2.2 Stability of arbitrary delay systems via IPR

The results of the previous section are strong and efficient, but they only apply to positive systems. A possible way to overcome this limitation is resorting to the so-called ‘comparison principle’ (see e.g. [25]), whose general idea is to associate to a given non positive system an augmented positive representation whose state (and output, if modelled) trajectories always dominate those of the original system. The method has been systematically implemented for a number of different classes of delay systems by means of the internally positive representation (IPR) technique, see [26–31], which allowed to export to non positive systems stability results that only hold for positive ones, although at the expense of introducing some conservatism.

Avoiding unnecessary details which can be found in the referenced works, the IPR method can be readily applied to the class of systems described by (5) defining the following positive representations of a vector $v \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{n \times n}$:

$$\pi(v) = \begin{bmatrix} v^+ \\ v^- \end{bmatrix}, \quad \Pi(M) = \begin{bmatrix} M^+ & M^- \\ M^- & M^+ \end{bmatrix}, \quad (14)$$

where v^+ denotes the componentwise positive part of v, i.e. $v_i^+ = \max(0, v_i)$, and v^- denotes its componentwise negative part, i.e. $v_i^- = \max(0, -v_i)$. The same can be said for M^+ and M^-. Notice that $\pi(v)$ and $\Pi(M)$ are positive (non-negative, actually) representations of twice the dimensions of v and M. Moreover, $v = v^+ - v^-$ and $|v| = v^+ + v^-$ (and the same holds for M and $|M|$). Noteworthy, v can be obtained back from $\pi(v)$ simply defining the backwards operator $\Delta_n = [I_n \; -I_n]$, yielding $v = \Delta_n \pi(v)$, and $Mv = \Delta_n \Pi(M) \pi(v)$.

With these definitions in mind, it is easy to show that a simple IPR for system (5) is the following system:

$$\begin{aligned} \mathscr{Z}(k+1) &= \sum_{i=1}^{r} \Pi(M_i) \mathscr{Z}(k - \delta_i(k)), \quad && k \geq 0, \\ \mathscr{Z}(k) &= \pi(\phi(k)), && k \in [-\delta, 0], \\ z(k) &= \Delta_n \mathscr{Z}(k), && k \geq -\delta. \end{aligned} \quad (15)$$

The proof that (15) is a valid IPR for (5) consists of showing that the original system state trajectory $z(k)$, for a given initial condition ϕ can be obtained from the state trajectory $\mathscr{Z}(k)$ once started from $\pi(\phi)$, at every time step. We omit for brevity this straightforward proof, which is on the lines of that given for Theorem 6 in [27].

Now, the following result is the core point of this section in order to export to non positive systems the strong delay-independent stability criterion of Theorem 2.

Lemma 6: The asymptotic stability of the IPR (15) implies the asymptotic stability of the original system (5).

Proof: The result is a simple consequence of the fact that the state trajectory of the IPR always dominates the state trajectory of the original system, since

$$z(k) = \Delta_n \mathscr{Z}(k) \implies \| z(k) \| \leq \| \Delta_n \| \| \mathscr{Z}(k) \|. \quad (16)$$

Then, if the IPR is (asymptotically) stable, the original system is stable too.□

Clearly, the previous result only gives a sufficient stability condition, since it can happen that the IPR of a stable system is not stable. This is due to the fact that $\Pi(M)$ properly contains the spectrum of M, but it also contains the spectrum of $|M|$ (see [29] for details), i.e.

$$\begin{aligned}\sigma(\Pi(M)) &= \sigma(M^+ - M^-) \cup \sigma(M^+ + M^-) \\ &= \sigma(M) \cup \sigma(|M|),\end{aligned} \tag{17}$$

and since $\rho(M) \le \rho(|M|)$, it may happen that the spectrum of $\Pi(M)$ contains unstable eigenvalues even if M is stable.

Nevertheless, one can apply Theorem 2 to the IPR (15) and if the latter is proved stable, the original system (5) is stable as well. This leads to the following theorem, which concludes the section and is instrumental for the results in the next section.

Theorem 7: The delay system (5) is delay-independent asymptotically stable if $\rho(\sum_{i=1}^{r} |M_i|) < 1$.

Proof: The proof consists of noting that Theorem 2 gives a necessary and sufficient stability condition for the IPR (15) which requires to check that $\rho(\sum_{i=1}^{r} \Pi(M_i)) < 1$. However, since

$$\sigma\left(\sum_{i=1}^{r} \Pi(M_i)\right) = \sigma\left(\sum_{i=1}^{r} M_i\right) \cup \sigma\left(\sum_{i=1}^{r} |M_i|\right) \tag{18}$$

it would suffice to verify that both $\rho(\sum_{i=1}^{r} M_i) < 1$ and $\rho(\sum_{i=1}^{r} |M_i|) < 1$ in order to prove the stability of the IPR, which in turn implies the stability of the original system (5). Nevertheless, for matrices $P \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{n \times n}$, such that $|P| \le Q$, it holds that (see [32]):

$$\rho(P) \le \rho(|P|) \le \rho(Q). \tag{19}$$

Applying (19) with $P = \sum_{i=1}^{r} M_i$ and $Q = \sum_{i=1}^{r} |M_i|$ one has

$$\rho\left(\sum_{i=1}^{r} M_i\right) \le \rho\left(\left|\sum_{i=1}^{r} M_i\right|\right) \le \rho\left(\sum_{i=1}^{r} |M_i|\right), \tag{20}$$

and this means that $\rho(\sum_{i=1}^{r} |M_i|) < 1$ is a sufficient condition for the delay-independent asymptotic stability of (5), as stated in the theorem. □

Remark 8: As a further contribution, we point out that Theorem 7 still holds for the case of unbounded delays $\delta_i(k)$ satisfying, for some $T \in \mathbb{N}$:

$$0 \le \sup_{k > T} \frac{\delta_i(k)}{k} < 1, \quad i = 1, \ldots, r. \tag{21}$$

Indeed, this comes from the fact that Theorem 2 also holds for unbounded delays satisfying (21), as shown in [33].

3 Stability of block companion switched systems

Deciding the arbitrary switching stability for a general discrete-time switched system as in (1) is equivalent to verifying that the joint spectral radius of the family $\mathbb{A} = \{\mathscr{A}_1, \ldots, \mathscr{A}_p\}$ is less than one ([17]), a generally NP-hard problem, whose difficulty explains why a relevant effort has been devoted by the research community towards alternative approaches to stability, such as Lyapunov methods. Also, broader stability characterisations have widely been addressed in the literature (such as stability under dwell-time constraints, or stability under admissible switching sequences, see [3, 34, 35]), but in many practical cases, no such characterisation is available for the system under investigation, meaning that the only possible stability analysis is the arbitrary switching one. Nevertheless, for some special classes of switched systems huge simplifications are in order. The most relevant case is probably that of positive switched systems, whose dynamic matrices $\mathscr{A}_j$ can only admit non-negative coefficients, to guarantee that the state trajectory is forced to assume non-negative values for non-negative initial conditions. This simplification in the stability analysis comes from the fact that approximating the JSR of a family $\mathbb{A}$ of non-negative matrices is way simpler than addressing the general case. In this respect, [36] introduced some noteworthy inequalities. The simplest of them is

$$\frac{1}{p}\rho\left(\sum_{i=1}^{p} \mathscr{A}_i\right) \le \rho^*(\mathbb{A}) \le \rho\left(\sum_{i=1}^{p} \mathscr{A}_i\right) \tag{22}$$

while an approximating inequality is formulated resorting to Kronecker lifting

$$\frac{1}{p^{1/k}}\rho^{1/k}\left(\sum_{i=1}^{p} \mathscr{A}_i^{[k]}\right) \le \rho^*(\mathbb{A}) \le \rho^{1/k}\left(\sum_{i=1}^{p} \mathscr{A}_i^{[k]}\right) \tag{23}$$

where $[k]$ denotes the kth-order Kronecker power of a matrix, and the right inequality converges to the equality as $k \to \infty$.

It is clear that, while the aforementioned approximation holds for general non-negative matrices, its computational burden increases dramatically with the required accuracy. More details on both inequalities can be found in [36].

If sufficient-only stability conditions are looked for, positive switched systems allow for a remarkable simplification in Lyapunov methods. Indeed, it is known that the existence of a linear co-positive common Lyapunov function (LCLF) is a sufficient condition for the arbitrary switching stability and is equivalent to a number of other conditions of similar complexity (see [8]). The existence of a LCLF for the family $\mathbb{A} = \{\mathscr{A}_1, \ldots, \mathscr{A}_p\}$ is equivalent to solving the following linear problem:

$$\exists\, p \in \mathbb{R}_{++}^{mn} \text{ s.t. } p^T \mathscr{A}_j < p^T \quad j = 1, \ldots, p. \tag{24}$$

Unfortunately, such a condition is conservative with respect to the existence of a quadratic Lyapunov function (co-positive or not), as shown in [8].

In this section, we consider the class of discrete-time switched systems in block companion form presented in Section 1, and described by (1)–(2), to show that the stability analysis of such class is vastly simplified. We will proceed starting from special cases and then move towards general results. The analysis will begin from positive systems with combinatorial assumptions: to improve readability, and to gradually introduce the combinatorial notation, we will present the results incrementally. In fact, the results of Section 3.2 are special cases of those in Section 3.3. Then, in Sections 3.4 and 3.5, we will gradually remove the positivity and combinatorial assumptions, yielding sufficient stability conditions for quite general classes of block companion switched systems. A schematic overview is illustrated in Fig. 1. Before going into the technical details, we will present a motivating example to illustrate the relevance of the class of block companion switched systems in both theory and applications.

3.1 Motivating example: switched autoregressive models

System identification is the science of estimating dynamical models from data. Much of the early contributions in system identification focused on estimating linear time-invariant models, which to some extent were also able to provide good accuracy when confronted with mildly non-linear dynamics. More recently, the research community shifted to the problem of estimating more complex dynamics (i.e. time-varying and non-linear models) to cope with large-scale systems in environments such as cyber-physical systems. In this respect, switched models have offered an ideal choice due to their strong capability of arbitrarily approximating complex non-linear dynamics. Since the black-box

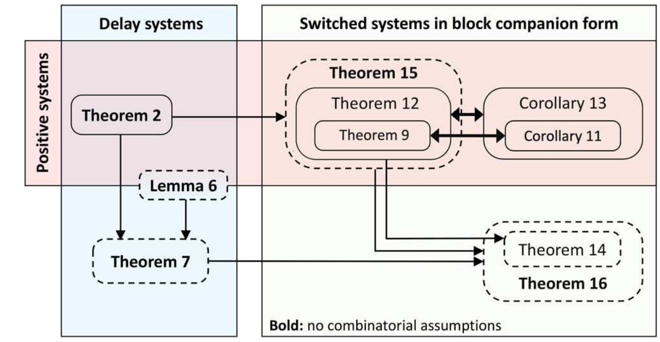

Fig. 1 *Stability conditions of Sections 2 and 3: schematic overview. Dashed edge: sufficient condition; solid edge: necessary and sufficient condition*

identification problem is typically formulated on input–output data (collected from an unknown system), a large amount of estimation methods focuses on input–output switched autoregressive models with exogenous inputs (switched ARX, or simply SARX). However, both for analysis and control purposes, it may be interesting to represent SARX models in state-space form. This leads to the problem of analysing the properties of block companion families of matrices, or even constraining a-priori their estimation procedure, as will be discussed below. Clearly, we will omit most technicalities and inessential details due to the illustrative nature of this section.

A switched ARX model is a regressive model in input–output form described by

$$y(k) = \Theta_{s(k)}\phi(k) \tag{25}$$

where $y \in \mathbb{R}^q$ and $u \in \mathbb{R}^p$ are measured, $\phi(k) = [y(k-1)^T \cdots y(k-\delta_y)^T u(k)^T \cdots u(k-\delta_u)^T]^\mathrm{T} \in \mathbb{R}^{q\delta_y + p(\delta_u+1)}$ is the vector of regressive terms, and $\Theta_{s(k)} \in \mathbb{R}^{q \times [q\delta_y + p(\delta_u+1)]}$ is the matrix of parameters to be estimated, which switches according to signal $s(k)$. Rewriting (25) as

$$y(k) = [\Theta^y_{s(k)}\Theta^u_{s(k)}]\begin{bmatrix} y(k-1) \\ \vdots \\ y(k-\delta_y) \\ u(k-1) \\ \vdots \\ u(k-\delta_u) \end{bmatrix} + \Theta^{u_0}_{s(k)}u(k) \tag{26}$$

and defining the state vector

$$x(k) = [y(k-1)^T \cdots y(k-\delta_y)^T u(k-1)^T \cdots u(k-\delta_u)^T]^\mathrm{T} \tag{27}$$

model (25) can be put in state-space form as follows:

$$\begin{aligned} x(k+1) &= \mathscr{A}_{s(k)}x(k) + \mathscr{B}_{s(k)}u(k) \\ y(k) &= \mathscr{C}_{s(k)}x(k) + \mathscr{D}_{s(k)}u(k) \end{aligned} \tag{28}$$

where

$$\mathscr{A}_{s(k)} = \begin{bmatrix} \Theta^y_{s(k)} & & \Theta^u_{s(k)} & \\ I_{q(\delta_y-1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{p(\delta_u-1)} & 0 \end{bmatrix}, \quad \mathscr{B}_{s(k)} = \begin{bmatrix} \Theta^{u_0}_{s(k)} \\ 0 \\ I_p \\ 0 \end{bmatrix} \tag{29}$$

$$\mathscr{C}_{s(k)} = \begin{bmatrix} \Theta^y_{s(k)} & \Theta^u_{s(k)} \end{bmatrix}, \quad \mathscr{D}_{s(k)} = \Theta^{u_0}_{s(k)}. \tag{30}$$

The identification problem is solved estimating the matrix $\Theta_{s(k)}$ from input–output measurements according to some technique. Some relevant problems in identification and realisation of SARX models are widely described in [10, 37]. For prediction and control purposes, the estimated SARX model can be represented in the state-space form, as shown in a number of recent works which leveraged machine learning methods for estimating switching models over a predictive horizon (see [11] and references therein). Nevertheless, assessing some important properties (such as stability and stabilisability) is challenging: the presence of switching dynamics requires sophisticated and computationally hard mathematical tools, as discussed above. In fact, most of SARX identification techniques tend to produce unstable models even when data are gathered from stable systems, a limitation that can be overcome in closed-loop regime, but is unpleasant in terms of faithfulness to the original system. The results of this section have direct application in approaching some of the problems described above, and most notably assessing the stability of system (28) and designing a strategy to constrain the estimation of $\Theta_{s(k)}$ to produce stable models. To see this, consider matrix $\mathscr{A}_j$ selected at time k by $s(k) = j$: it is an upper block-triangular matrix, whose stability is only determined by the top-left block $\mathscr{A}^y_j$:

$$\mathscr{A}^y_j = \begin{bmatrix} \Theta^y_j & \\ I_{q(\delta_y-1)} & 0 \end{bmatrix} = \begin{bmatrix} A_{j,1} & A_{j,2} & \cdots & A_{j,\delta_y-1} & A_{j,\delta_y} \\ I_q & 0 & \cdots & 0 & 0 \\ 0 & I_q & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_q & 0 \end{bmatrix} \tag{31}$$

since it is well-known that the JSR of block-triangular matrices is the maximum among the joint spectral radii of the diagonal blocks (see [17]). Hence, we face the problem of deciding the stability of a family of matrices in block companion form, possibly avoiding the computation of their JSR, which in case of large-scale systems with large autoregressive order may easily prove prohibitive. The results of the next sections will provide a novel approach to efficiently address this problem.

3.2 First block non-negative permutations

Consider the switching system in block companion form described by (1) and (2). We start addressing the special case in which the first block row of $\mathscr{A}_j$ consists, for each j, of a permutation (with no repetitions), of some matrices $M_1, \ldots, M_m$. As first discussed in Section 2, in this case, the first block-component of the state $x(k)$ can be written in a delay system form, and the arbitrary switching stability of the whole switched system in the block companion form is equivalent to the delay-independent stability of his first block-component. Before entering into the detail, we need to introduce an appropriate notation.

Defining $\mathscr{P}$ as the set of all possible permutations of $\{1, \ldots, m\}$, we denote the particular permutation (among the possible $m!$) selected at time k with $\mathscr{P}^{s(k)}$, and the single element with $\mathscr{P}^{s(k)}_i$, for $i = 1, \ldots, m$. Then, $\mathscr{A}_{s(k)}$ switches among the $m!$ matrices of the family $\mathbb{A} = \{\mathscr{A}_1, \ldots, \mathscr{A}_{m!}\}$ and the first block row of each $\mathscr{A}_j$ contains the matrices $A_{j,i} = M_{\mathscr{P}^j_i}$ for $i = 1, \ldots, m$.

Let us present a simple example in order to clarify the notation.

Example 1: Consider $m = 3$ matrices $M_1, M_2, M_3 \in \mathbb{R}^{n \times n}$. The set of the 3! possible permutations of $\{1, 2, 3\}$ is $\mathscr{P} = \{\mathscr{P}^1, \mathscr{P}^2, \mathscr{P}^3, \ldots, \mathscr{P}^6\} = \{\{1,2,3\}, \{1,3,2\}, \{2,1,3\}, \{2,3,1\}, \{3,1,2\}, \{3,2,1\}\}$. At each time k, one of the six permutations is possible. For example, consider at time $k = 2$, $s(2) = 3$, such that $\mathscr{P}^3 = \{2,1,3\}$ is selected. Then, $\mathscr{P}^3_1 = 2$, $\mathscr{P}^3_2 = 1$, and $\mathscr{P}^3_3 = 3$. In this case, the first block-row of the $\mathscr{A}_3$ matrix will be made of $A_{3,i} = M_{\mathscr{P}^3_i}$, yielding

$$\mathscr{A}_3 = \begin{bmatrix} M_2 & M_1 & M_3 \\ I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix}. \tag{32}$$

Bearing in mind the notation, the relationship among switched systems in the block companion form and delay systems with time-varying delays leads to the following result.

Theorem 9: Consider a switched system as in (1) and (2), where $\mathscr{A}_{s(k)}$ is arbitrarily taken from the family $\mathbb{A} = \{\mathscr{A}_1, \ldots, \mathscr{A}_{m!}\}$ of matrices whose first block-row is a permutation of m non-negative matrices $\mathbb{M} = \{M_1, \ldots, M_m\}$, i.e. $A_{j,i} = M_{\mathscr{P}_i^j}$. Then, the switched system is asymptotically stable for all possible switching sequences if and only if $\rho(\sum_{i=1}^m M_i) < 1$.

Proof: The result comes from the fact that, as detailed for the constant delay case (7) and (8), and remarked above, the first (block) state component of the switched system under examination, i.e. $x_1(k) \in \mathbb{R}^n$ is described at every time step k by the delay system

$$z(k+1) = \sum_{i=1}^{m} M_i z(k - \tilde{\mathscr{P}}_i^{s(k)}), \quad k > 0, \tag{33}$$

with $x_1(k) = z(k)$ and $\tilde{\mathscr{P}}_i^{s(k)} = \mathscr{P}_i^{s(k)} - 1$. Since M_i is non-negative for all i, system (33) is a positive delay system with time-varying delays $\tilde{\mathscr{P}}_i^{s(k)}$ with values in $\{0, 1, \ldots, m-1\}$. Then, by Theorem 2, we know that it is stable for all possible values of the delays if and only if $\rho(\sum_{i=1}^m M_i) < 1$, i.e. $z(k) \to 0$ as $k \to \infty$ for any initial condition ϕ, and since $x(k) = [z^T(k) \;\; z^T(k-1) \;\; \cdots \;\; z^T(k-m+1)]^{\mathrm{T}}$ one trivially has that $x(k) \to 0$ as $k \to \infty$ for any $x(0) \in \mathbb{R}^{mn}$. □

Remark 10: It is rather clear from the proof above that the case of permuting matrices is not the most general class of switching block companion systems which can be equivalently mapped into delay systems as in (5), since permutations (with no repetitions) do not cover the case of coinciding delays. We will show in Section 3.3 how Theorem 9 can be extended to cover a larger class of switching systems.

The simplification introduced by Theorem 9 on deciding the problem '$\rho^*(\mathbb{A}) < 1$?' even though restricted to the special class of block companion matrices with permuting entries, is noteworthy, as it directly implicates the following corollary.

Corollary 11: Consider a set of non-negative matrices $\mathbb{M} = \{M_1, \ldots, M_m\}$, and the family of block companion matrices $\mathbb{A} = \{\mathscr{A}_1, \ldots, \mathscr{A}_{m!}\}$ whose first block-row is a permutation of $\mathbb{M}$, i.e. $A_{j,i} = M_{\mathscr{P}_i^j}$. Then the following result holds:

$$\rho^*(\mathbb{A}) < 1 \iff \rho\left(\sum_{i=1}^{m} M_i\right) < 1\,. \tag{34}$$

The reader could wonder whether the existence of a LCLF, which is generally only sufficient for the stability of positive switched systems, is still only sufficient also for the special cases of non-negative block companion matrices presented in this and next sections, making it conservative with respect to the necessary and sufficient conditions of this work. The answer is affirmative, as it will be discussed in Examples 4.2 and 4.3. Moreover, searching for an LCLF for a positive switched system requires testing p inequalities as opposed to the single check required by Theorem 9 and Corollary 11 (see Remark 3). Both these points testify the remarkable efficacy and inexpensiveness of the novel results of this work with respect to the existent literature for the case of block companion switched systems.

3.3 First block broader non-negative combinations

We now proceed extending the previous results to a broader class of systems. In this case, the first block-row of $\mathscr{A}_j$ can consist of matrices obtained as permutations of combinations (sums), with no repetitions, of some matrices $M_1, \ldots, M_m$.

Let us define, for a given set $\{1, \ldots, m\}$, the set of all its partitions, denoted by P, with $|P|$ being its cardinality (the number of its subsets). Then, a particular partition P^j consists of at most m subsets, denoted by $P_1^j, \ldots, P_m^j$. Clearly, those subsets are exactly m only for the partition consisting of singletons, otherwise if P^j consists of $l < m$ subsets, we define $P_{l+1}^j, \ldots, P_m^j$ as empty sets.

Let us illustrate the notation with an example.

Example 2: The five possible partitions of $\{1, 2, 3\}$ are $P = \{\{\{1, 2, 3\}\}, \{\{1, 2\}, \{3\}\}, \{\{1\}, \{2, 3\}\}, \{\{1, 3\}, \{2\}\}, \{\{1\}, \{2\}, \{3\}\}\}$. Consider e.g. partition $P^4 = \{\{1, 3\}, \{2\}\}$, whose subsets are $P_1^4 = \{1, 3\}$, $P_2^4 = \{2\}$. It follows that P_3^4 is empty, since no further subset of P^4 exists for this particular partition. Similarly, if the partition $P^1 = \{\{1, 2, 3\}\}$ is considered, we have $P_1^1 = \{1, 2, 3\}$, and two empty sets P_2^1, P_3^1.

Now for every particular partition P^j of P defined as above, we denote with $\mathscr{P}^{Pj}$ the set of its permutations, excluding the mutual permutations among the $m - l$ empty subsets. We will call this restriction the 'essential permutations'. The whole set containing the essential permutations of every possible partition of $\{1, \ldots, m\}$ is then $\mathscr{P}^P$, and its cardinality (the number of its subsets) is denoted by $|\mathscr{P}|$.

Example 2 (continued): For the previously considered partition $P^4 = \{\{1, 3\}, \{2\}\}$ with $P_1^4 = \{\{1, 3\}\}$, $P_2^4 = \{2\}, P_3^4 = \varnothing$ we have six different permutations $\mathscr{P}^{P4} = \{\{P_1^4, P_2^4, P_3^4\}, \{P_1^4, P_3^4, P_2^4\}, \{P_2^4, P_1^4, P_3^4\}, \ldots\}$, with $\mathscr{P}_1^{P4} = \{P_1^4, P_2^4, P_3^4\}$ and so on. In this case, the cardinality of $\mathscr{P}^{P4}$ is $\left|\mathscr{P}^{P4}\right| = 6$. We will need another index to extract a particular subset of a partition, i.e. $\mathscr{P}_{2,i}^{P4}$, with, e.g. $\mathscr{P}_{2,3}^{P4} = P_2^4$. Whereas, for the case of multiple empty subsets, the permutations of the partition $P^1 = \{\{1, 2, 3\}\}$ (resulting in $P_1^1 = \{1, 2, 3\}, P_2^1 = P_3^1 = \varnothing$) are deprived of mutual empty sets permutations, i.e. the essential permutations are $\mathscr{P}^{P1} = \{\{P_1^1, P_2^1, P_3^1\}, \{P_2^1, P_1^1, P_3^1\}, \{P_2^1, P_3^1, P_1^1\}\}$ and $\mathscr{P}_{3,2}^{P1} = P_3^1$. Notice that $\left|\mathscr{P}^{P1}\right| = 3$.

Now let us see how this rather involved notation can be employed on block companion switching matrices.

Example 2 (continued): Consider three matrices M_1, M_2, M_3. The set of the five possible partitions of $\{1, 2, 3\}$ is P, as previously defined. We again consider the partition P^4 such that $P_1^4 = \{1, 3\}$, $P_2^4 = \{2\}$, $P_3^4 = \varnothing$. Then, six $\mathscr{A}_4^{(h)}$ matrices are possible, since the first block of the generic $\mathscr{A}_4$ can consist of all possible essential permutations of P_1^4, P_2^4, P_3^4, which we denoted by $\mathscr{P}^{P4}$, with $\left|\mathscr{P}^{P4}\right| = 6$. In detail, the first-row blocks of each $\mathscr{A}_4^{(h)}$ are $A_{4,i}^{(h)} = \sum_{q \in \mathscr{P}_{h,i}^{P4}} M_q, \quad i = 1, \ldots, m$. The possible matrices corresponding to $\mathscr{P}^{P4}$ are then

$$\mathscr{A}_4^{(1)} = \begin{bmatrix} M_1 + M_3 & M_2 & 0 \\ I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix}, \quad \mathscr{A}_4^{(2)} = \begin{bmatrix} M_1 + M_3 & 0 & M_2 \\ I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix},$$
$$\mathscr{A}_4^{(3)} = \begin{bmatrix} M_2 & M_1 + M_3 & 0 \\ I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix}, \quad \mathscr{A}_4^{(4)} = \begin{bmatrix} 0 & M_1 + M_3 & M_2 \\ I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix}, \tag{35}$$
$$\mathscr{A}_4^{(5)} = \begin{bmatrix} M_2 & 0 & M_1 + M_3 \\ I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix}, \quad \mathscr{A}_4^{(6)} = \begin{bmatrix} 0 & M_2 & M_1 + M_3 \\ I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix}$$

while the possible matrices corresponding to $\mathscr{P}^{P1} = \{\{P_1^1, P_2^1, P_3^1\}, \{P_2^1, P_1^1, P_3^1\}, \{P_2^1, P_3^1, P_1^1\}\}$, with $\left|\mathscr{P}^{P1}\right| = 3$, are

$$\mathscr{A}_1^{(1)} = \begin{bmatrix} M_1 + M_2 + M_3 & 0 & 0 \\ I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix},$$
$$\mathscr{A}_1^{(2)} = \begin{bmatrix} 0 & M_1 + M_2 + M_3 & 0 \\ I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix}, \quad (36)$$
$$\mathscr{A}_1^{(3)} = \begin{bmatrix} 0 & 0 & M_1 + M_2 + M_3 \\ I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix}.$$

At this point, bearing in mind the notation, the following relaxation of Theorem 9 is obtained. We will refer to the partitions of a set of matrices $\{M_1, \ldots, M_m\}$ meaning the partitions of the set of their indexes $\{1, \ldots, m\}$, with a little abuse of nomenclature. Moreover, we will denote with $\mathbb{A}$ the family of $|\mathscr{P}|$ block companion matrices $\{\mathscr{A}_1^{(1)}, \ldots, \mathscr{A}_1^{(|\mathscr{P}^{P1}|)}, \mathscr{A}_2^{(1)}, \ldots, \mathscr{A}_2^{(|\mathscr{P}^{P2}|)}, \ldots, \mathscr{A}_{|P|}^{(1)}, \ldots, \mathscr{A}_{|P|}^{(|\mathscr{P}^{P|P|}|)}\}$ built as above.

Theorem 12: Consider a switched system as in (1) and (2), where $\mathscr{A}_{s(k)}$ is arbitrarily taken from the family $\mathbb{A}$ of matrices whose first block-row is an essential permutation of the partitions of m non-negative matrices $\mathbb{M} = \{M_1, \ldots, M_m\}$, i.e. $A_{j,i}^{(h)} = \sum_{q \in \mathscr{P}_{h,i}^{Pj}} M_q$, $i = 1, \ldots, m$, as detailed above. Then, the system is asymptotically stable for all possible switching sequences if and only if $\rho(\sum_{i=1}^{m} M_i) < 1$.

Proof: The result is a consequence of Theorem 2, and the proof is the same of Theorem 9 when possibly coinciding delays are considered. For the sake of brevity, we omit further details. □

Corollary 13: Consider a set of non-negative matrices $\mathbb{M} = \{M_1, \ldots, M_m\}$, and the family of block companion matrices $\mathbb{A}$ whose first block-row is an essential permutation of the partitions of $\mathbb{M}$, i.e. $A_{j,i}^{(h)} = \sum_{q \in \mathscr{P}_{h,i}^{Pj}} M_q$, $i = 1, \ldots, m$, as detailed above. Then the following result holds:

$$\rho^*(\mathbb{A}) < 1 \iff \rho\left(\sum_{i=1}^{m} M_i\right) < 1 . \quad (37)$$

Clearly, Theorem 9 is a particular case of Theorem 12, whereas Corollary 11 is a particular case of Corollary 13.

3.4 Removing the positivity assumption

Bearing in mind Theorem 7, the results of Theorems 9 and 12, and their corresponding Corollaries 11 and 13, can easily be restated (introducing conservatism) for not necessarily positive systems (i.e. arbitrary set of matrices $\{M_1, \ldots, M_m\}$).

In this case, denoting $\mathbb{A}$ the family of block companion matrices satisfying the conditions of the aforementioned theorems and corollaries, we can easily state the following result.

Theorem 14: Consider a set of arbitrary matrices $\mathbb{M} = \{M_1, \ldots, M_m\}$ and the family $\mathbb{A}$ of block companion matrices built from $\mathbb{M}$ under the conditions of Theorem 12. Then, the following result holds:

$$\rho\left(\sum_{i=1}^{m} |M_i|\right) < 1 \implies \rho^*(\mathbb{A}) < 1. \quad (38)$$

3.5 More general results: removing positivity and combinatorial assumptions

It can be surely conceded that the previously illustrated results concern very special classes of switched systems. This restriction, however, should not be unexpected, since the arbitrary switching stability is known to be a very difficult problem in the general case. Nevertheless, in order to extend the previous results to a broader class of switched systems, we note that the comparison method readily gives us an idea to remove the special combinatorial structures of Theorems 9 and 12.

One just needs to note that, for non-negative matrices P and Q such that $P \le Q$ it follows that $\rho(P) \le \rho(Q)$ and clearly $x(k+1) = Px(k)$ is upper-bounded by $x(k+1) = Qx(k)$. For switched systems, we readily have that if $P_k \le Q_k$ for all k, the trajectory of $x(k+1) = Q_k x(k)$ always dominates that of $x(k+1) = P_k x(k)$. This trivially leads to the following Theorem.

Theorem 15: Consider a switched system $x(k+1) = \mathscr{A}_k x(k)$ with non-negative block companion $\mathscr{A}_k$ as in (2), consisting of possibly distinct blocks $A_{k,i}$ at every time instant. If there exist non-negative matrices $M_1, \ldots, M_m$ such that $\rho(\sum_{i=1}^{m} M_i) < 1$, and the associated family $\bar{\mathbb{A}}$ as in Theorem 12 (or Theorem 9) such that at every k there exists a $\bar{\mathscr{A}}_j$ in $\bar{\mathbb{A}}$ with $\mathscr{A}_k \le \bar{\mathscr{A}}_j$, then the system $x(k+1) = \mathscr{A}_k x(k)$ is asymptotically stable.

Let us present a simple example in order to illustrate the previous result.

Example 3: Consider a switched system $x(k+1) = \mathscr{A}_k x(k)$, where

$$\mathscr{A}_k = \begin{bmatrix} A_{k,1} & A_{k,2} \\ I_n & 0 \end{bmatrix} \ge 0, \quad \forall k . \quad (39)$$

If there exist non-negative M_1, M_2 such that, for all k, $A_{k,1} \le M_i$ and $A_{k,2} \le M_j$, $i \ne j$, then $\rho(M_1 + M_2) < 1$ ensures the asymptotic stability of the system under examination. Indeed, in this case, the family $\bar{\mathbb{A}}$ associated to M_1, M_2 is simply the family $\{\bar{\mathscr{A}}_1, \bar{\mathscr{A}}_2\}$ built as in Theorem 9, with

$$\bar{\mathscr{A}}_1 = \begin{bmatrix} M_1 & M_2 \\ I_n & 0 \end{bmatrix}, \quad \bar{\mathscr{A}}_2 = \begin{bmatrix} M_2 & M_1 \\ I_n & 0 \end{bmatrix}, \quad (40)$$

and $x(k+1) = \mathscr{A}_k x(k)$ is asymptotically stable because at each time k it holds $\mathscr{A}_k \le \bar{\mathscr{A}}_j$, for at least one $j = 1, 2$.

Again by the comparison method, the previous result can easily be extended to not necessarily positive systems (i.e. arbitrary block companion $\mathscr{A}_k$), just requiring that $|\mathscr{A}_k| \le \bar{\mathscr{A}}_j$ in Theorem 15. This comes from noting that the trajectories of system $x(k+1) = P_k x(k)$, with no sign restrictions on P_k, are dominated by those of $x(k+1) = Q_k x(k)$, with non-negative Q_k, provided that $|P_k| \le Q_k$ for all k. This is trivially due to the property $P_k \le |P_k| \le Q_k$ (which also holds on their spectral radii). The result is formalised as follows.

Theorem 16: Consider a switched system $x(k+1) = \mathscr{A}_k x(k)$ with arbitrary block companion $\mathscr{A}_k$ as in (2), consisting of possibly distinct blocks $A_{k,i}$ at every time instant. If there exist non-negative matrices $M_1, \ldots, M_m$ such that $\rho(\sum_{i=1}^{m} M_i) < 1$, and the associated family $\bar{\mathbb{A}}$ as in Theorem 12 (or Theorem 9) such that at every k there exists a $\bar{\mathscr{A}}_j$ in $\bar{\mathbb{A}}$ with $|\mathscr{A}_k| \le \bar{\mathscr{A}}_j$, then the system $x(k+1) = \mathscr{A}_k x(k)$ is asymptotically stable.

Remark 17: For companion-form non-negative matrices with scalar blocks, i.e. the first row of (2) is scalar with $A_{k,i} = a_{k,i} \in \mathbb{R}_+$, some interesting facts have been unveiled in [38], reducing the arbitrary switching stability criterion to $\sum_{i=1}^{m} a_{k,i} < 1$ for all $k = 1, \ldots, p$, and proving that the maximum growth rate of $x(k)$ is obtained staying on the $\mathscr{A}_{\bar{k}}$ system with largest spectral radius. The result does not require special structures (permutations etc.) on the $a_k = (a_{k,1}, \ldots, a_{k,m})$ sequences, but is derived resorting to successive rank-one corrections of a common matrix in a given uncertainty set. Clearly, the generalisation of this result to the class of block companion matrices considered in this work is not trivial, since multiple-rank corrections are involved.

A schematic overview of the results of Section 2 and 3 is illustrated in Fig. 1. For switched block companion systems, necessary and sufficient stability conditions are formulated resorting to positivity and combinatorial assumptions (Theorem 12 and its equivalent Corollary 13), exploiting the enabling delay-independent result of Theorem 2. Leveraging the comparison method, both positivity and combinatorial assumptions can be removed in various steps: Theorem 14 removes positivity, Theorem 15 removes combinatorics, Theorem 16 removes both assumptions, yielding the most general result of our work.

4 Numerical examples

4.1 Example 4.1

This first example illustrates Theorem 9 and Corollary 11, describing a simple positive switched system for which the results of this work allow to readily deduce its arbitrary switching stability. We also verify that the same conclusion cannot be obtained by the Joint Spectral Radius toolbox for MATLAB (see [39]) under standard settings.

Consider system $x(k+1) = \mathscr{A}_{s(k)}x(k)$, where $\mathscr{A}_{s(k)}$ switches in the family $\mathbb{A}$ of the $4! = 24$ matrices $\mathscr{A}_j$ whose first block-row is made of all the permutations of the following non-negative matrices:

$$M_1 = \begin{bmatrix} 0.08 & 0.10 & 0.12 \\ 0.03 & 0.02 & 0.11 \\ 0.07 & 0.02 & 0.10 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.12 & 0.22 & 0.11 \\ 0.03 & 0.07 & 0.01 \\ 0.16 & 0.20 & 0.17 \end{bmatrix},$$
$$M_3 = \begin{bmatrix} 0.05 & 0.10 & 0.13 \\ 0.10 & 0.10 & 0.03 \\ 0.02 & 0.06 & 0.05 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0.08 & 0.01 & 0.02 \\ 0.12 & 0.05 & 0.10 \\ 0.10 & 0.12 & 0.03 \end{bmatrix}, \tag{41}$$

i.e.

$$\mathscr{A}_j = \begin{bmatrix} A_{j,1} & A_{j,2} & A_{j,3} & A_{j,4} \\ I_3 & 0 & 0 & 0 \\ 0 & I_3 & 0 & 0 \\ 0 & 0 & I_3 & 0 \end{bmatrix} \quad j = 1, \dots, 4!, \tag{42}$$

with $A_{j,i} = M_{\mathscr{P}_i^j}$, as clarified in Example 1 (Section 3).

Then, by Theorem 9 we prove that the switching system is stable for all possible switching sequences, since $\rho(\sum_{i=1}^4 M_i) = 0.9959 < 1$, and Corollary 11 allows to conclude that $\rho^*(\mathbb{A}) < 1$.

It is noteworthy that those conclusion cannot be attained with the JSR toolbox, which after 12 min of computations on an Intel Core i7-9750H returns the following bounds: $0.9985 \le \rho^*(\mathbb{A}) \le 1.0277$.

4.2 Example 4.2

This example presents a very simple system which is proven stable by Theorem 9 and whose stability cannot be verified resorting to linear co-positive common Lyapunov functions.

Consider a system as in (1) and (2) switching in the family of two matrices $\mathbb{A} = \{\mathscr{A}_1, \mathscr{A}_2\}$, where

$$\mathscr{A}_1 = \begin{bmatrix} M_1 & M_2 \\ I_3 & 0 \end{bmatrix}, \quad \mathscr{A}_2 = \begin{bmatrix} M_2 & M_1 \\ I_3 & 0 \end{bmatrix}, \tag{43}$$

with

$$M_1 = \begin{bmatrix} 0.2 & 0.2 & 0.15 \\ 0.15 & 0.25 & 0.15 \\ 0.15 & 0.1 & 0.05 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.2 & 0.25 & 0.2 \\ 0.1 & 0.15 & 0.2 \\ 0.1 & 0.15 & 0.1 \end{bmatrix}. \tag{44}$$

It can easily be seen that $\rho(M_1 + M_2) = 0.9561$, allowing to conclude by Theorem 9 that the system is asymptotically stable for all switching sequences, as confirmed computing the joint spectral radius $\rho^*(\mathbb{A}) = 0.9707$.

Nevertheless, there exist no LCLF for $\mathscr{A}_1, \mathscr{A}_2$, i.e. no vector $p \in \mathbb{R}^6_{++}$ can be found such that $p^T\mathscr{A}_j < p^T$ for $j = 1, 2$, as can be easily verified by a linear solver (e.g. MATLAB's *linprog*).

4.3 Example 4.3

This third example illustrates Theorem 12, Corollary 13, and shows how they reduce conservatism with respect to LCLF for the class of positive systems addressed in Section 3.3. Consider a system arbitrarily switching in $\mathbb{A} = \{\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3, \mathscr{A}_4\}$, where the first block-row of each $\mathscr{A}_j$ is built on the essential permutations of the partitions of

$$M_1 = \begin{bmatrix} 0.17 & 0.25 & 0.31 \\ 0.17 & 0.21 & 0.22 \\ 0.04 & 0.12 & 0.22 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.08 & 0 & 0.49 \\ 0.17 & 0.06 & 0.22 \\ 0.21 & 0.16 & 0.02 \end{bmatrix} \tag{45}$$

i.e.,

$$\mathscr{A}_1 = \begin{bmatrix} M_1 & M_2 \\ I_3 & 0 \end{bmatrix}, \quad \mathscr{A}_2 = \begin{bmatrix} M_2 & M_1 \\ I_3 & 0 \end{bmatrix},$$
$$\mathscr{A}_3 = \begin{bmatrix} M_1 + M_2 & 0 \\ I_3 & 0 \end{bmatrix}, \quad \mathscr{A}_4 = \begin{bmatrix} 0 & M_1 + M_2 \\ I_3 & 0 \end{bmatrix}. \tag{46}$$

Again we can straightforwardly prove that the system is stable for all possible switching sequences, as $\rho(M_1 + M_2) = 0.9924$ (Theorem 12) implying $\rho^*(\mathbb{A}) < 1$ (Corollary 13). Actually, $\rho^*(\mathbb{A}) = 0.9962$, which coincides with the largest spectral value among the $\mathscr{A}_j$ matrices, that of $\mathscr{A}_4$, meaning that the worst convergence rate is attained if the system stays on the $\mathscr{A}_4$ subsystem. Also in this case it can be shown that no LCLF does exist for the family of non-negative matrices $\mathbb{A}$.

As a further illustration of the results of this work, notice that any sign variation in the first block row of the matrices of $\mathbb{A}$ still produces a system which is stable for any switching sequence. This is due to Theorem 14, and inequality $\rho(\sum_i M_i) < \rho(\sum_i |M_i|)$. Clearly, if the non-negativity of $\mathbb{A}$ is dropped, linear Lyapunov functions cannot be used, and the computation of the Joint Spectral Radius is more demanding.

4.4 Example 4.4

To conclude, we provide a simple example for Theorem 16. Consider the arbitrary (not positive) system $x(k+1) = \mathscr{A}_k x(k)$, where

$$\mathscr{A}_k = \begin{bmatrix} A_{k,1} & A_{k,2} \\ I_2 & 0 \end{bmatrix}, \tag{47}$$

with:

$$A_{k,1} = \begin{bmatrix} -0.1\sin\left(\frac{\pi}{2}k\right) & 0.3\cos\left(\frac{\pi}{2}k\right) \\ -0.3\cos\left(\frac{\pi}{2}k\right) & 0.1\sin\left(\frac{\pi}{2}k\right) \end{bmatrix},$$
$$A_{k,2} = \begin{bmatrix} 0.2\cos\left(\frac{\pi}{2}k\right) & -0.05 \\ -0.1 & -0.2\cos\left(\frac{\pi}{2}k\right) \end{bmatrix}. \tag{48}$$

Then, the non-negative matrices M_1, M_2:

$$M_1 = \begin{bmatrix} 0.2 & 0.05 \\ 0.1 & 0.2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0.3 \\ 0.3 & 0 \end{bmatrix}, \tag{49}$$

such that $\rho(M_1 + M_2) = 0.5742 < 1$, can be used to construct the 'dominating' family $\bar{\mathbb{A}} = \{\bar{\mathscr{A}}_1, \bar{\mathscr{A}}_2\}$, built with the two permutations of M_1 and M_2 on the first block-row

$$\bar{\mathscr{A}}_1 = \begin{bmatrix} M_1 & M_2 \\ I_2 & 0 \end{bmatrix}, \quad \bar{\mathscr{A}}_2 = \begin{bmatrix} M_2 & M_1 \\ I_2 & 0 \end{bmatrix}. \tag{50}$$

Since

$$\left.\begin{array}{l} |A_{k,1}| \le M_1 \\ |A_{k,2}| \le M_2 \end{array}\right\} \text{ i.e. } |\mathscr{A}_k| \le \bar{\mathscr{A}}_1, \quad \text{for } k = 1, 3, 5, \ldots \tag{51}$$

$$\left.\begin{array}{l} |A_{k,1}| \le M_2 \\ |A_{k,2}| \le M_1 \end{array}\right\} \text{ i.e. } |\mathscr{A}_k| \le \bar{\mathscr{A}}_2, \quad \text{for } k = 2, 4, 6, \ldots \tag{52}$$

we can conclude that the system is asymptotically stable, as confirmed computing the joint spectral radius of the family $\mathbb{A} = \{\mathscr{A}_k\}$: $\rho^*(\mathbb{A}) = 0.6276$.

5 Conclusion and future work

This work, taking inspiration from some recent results on discrete-time positive delay systems, has illustrated how the arbitrary switching stability can be easily and efficiently studied for some classes of switched systems in the block companion form. Starting from positive systems whose first block-row has special combinatorial structures, we removed both assumptions (even though at the expense of some conservatism). The results have been favourably compared with linear co-positive common Lyapunov functions, a widely adopted tool for the stability analysis of positive switched systems. The consequences on the problem of computing the joint spectral radius of matrices in the block companion form have been highlighted, describing how the NP-hard problem '$\rho^* < 1$?' becomes very simple in the aforementioned cases.

The results of this work can impact on both theory and applications, as the problem of deciding the stability of families of block companion matrices can arise, for example, in state-space representations of switched ARX models, a typical outcome of many identification methods.

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