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ATTRACTING SETS OF IMPULSE-PERTURBED HEAT EQUATION IN THE SPACE OF CONTINUOUS FUNCTIONS

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Abstract. The paper deals with the impulsive infinite-dimensional problem generated by solutions of the thermal conductivity equation under the condition of impulse "pumping" of heat. Moments of impulses are not fixed and are determined by the amount of total heat in the system. It is proved that such a problem generates impulsive dynamical system in the space of continuous functions and its ω-limits sets are investigated.

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1. INTRODUCTION

Many papers [1, 3, 10–13, 16] are devoted to the qualitative behavior of impulsive dynamical systems, in particular to the study of ω-boundary sets. The number of papers establish the solvability and properties of solutions for various types systems such that Navier-Stokes systems [9], linear differential equation systems with a deviating argument [14], systems of functional differential equations in both linear and nonlinear cases [15, 17], nonlinear systems of viscoelasticity with a memory term [6].

In infinite-dimensional phase spaces, such studies are recently carried out in papers [2, 4, 5, 8] and concerned the study of uniform attraction sets for evolutionary impulse-perturbed systems in Hilbert spaces. In particular, using L^2-theory, in [4, 5] it is investigated qualitative behaviour of some parabolic impulse-perturbed system, where impulsive set was given as seminorm in phase space. As authors know, only in paper [16] the limit modes of impulse-perturbed heat equation in the space of continuous functions are studied.

In this paper we treat the impulsive infinite-dimensional problem generated by solutions of the heat equation under the condition of impulsive "pumping" of heat. Moments of impulse are not fixed and are determined by the amount of total heat in

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the system. We prove that such a problem generates impulsive dynamical system in the space of continuous functions and its ω-limit sets are investigated.

2. Problem Setting

Let a function $u = u(t,x)$, $t > 0$, $x \in (0,\pi)$ be defined from the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + h(x), \\ u|_{x=0} = u|_{x=\pi} = 0, \\ u|_{t=0} = u_0(x), \end{cases} \tag{2.1}$$

where $h \in C([0,\pi])$ is given, $u_0 \in X = \{u \in C([0,\pi]) | u(0) = u(\pi) = 0\}$ is a phase space of (2.1) with the norm $\|u\| = \max\limits_{x\in[0,\pi]} |u(x)|$.

Let's consider a functional $\psi : X \to \mathbb{R}$ in form

$$\psi(u) = \int_0^\pi u(x)dx,$$

that determines the total amount of heat in the system.

We formulate the following impulse problem [13]: if at time t the functional ψ reaches a fixed value $\psi_0 > 0$ on the non-negative solution of problem (2.1), then at this moment there is an instantaneous "pumping" of heat by α, and system (2.1) is in a new position

$$u(t,x) + \alpha(x), \tag{2.2}$$

where $\alpha(x) \geq 0$ is given, and $\int_0^\pi \alpha(x)dx = \alpha_0 > 0$.

We show that problem (2.1), (2.2) generates an impulsive dynamical system [13], i.e. its dynamics is given by a continuous semigroup $V : \mathbb{R}_+ \times X \to X$, the trajectories of which, when a fixed set $M \subset X$ (impulse set) is reached, are transferred by mapping $I : M \to X$ (impulse mapping) to the new position Iu. In this case, for the correct setting of such a system requires that the following conditions are met

$$\begin{cases} V : \mathbb{R}_+ \times X \to X \ \text{ is continouos semigroup}, \\ M \subset X \text{ is closed}, \ M \bigcap IM = \varnothing, \\ \forall u_0 \in M \ \ \exists \tau = \tau(u_0) > 0 \ \ \forall t \in (0,\tau) \ \ V(t,u_0) \notin M. \end{cases} \tag{2.3}$$

Introduce the following notations:

$$u^+ := Iu \text{ for } u \in M,$$

$$M^+(u_0) = \left(\bigcup_{t>0} V(t,u_0) \right) \cap M \text{ for } u_0 \in X.$$

If the property (2.3) holds and $M^+(u_0) \neq \varnothing$, then

$$\exists s = s(u_0) > 0 \ \ \forall t \in (0,s) \ \ V(t,u_0) \notin M, \ \ V(s,u_0) \in M. \tag{2.4}$$

Then for every $u_0 \in X$ the impulse trajectory $\{\widetilde{V}(t,u_0), t \geq 0\}$ is constructed in such way [5]:

- if $M^+(u_0) = \varnothing$, then $\widetilde{V}(t,u_0) = V(t,u_0),\ t \geq 0$,
- if $M^+(u_0) \neq \varnothing$ then for $s_0 := s(u_0),\ u_1 = V(s_0,u_0)$ and

$$\widetilde{V}(t,u_0) = \begin{cases} V(t,u_0),\ t \in [0,s_0), \\ u_1^+,\ t = s_0, \end{cases}$$

- if $M^+(u_1^+) = \varnothing$, then $\widetilde{V}(t,u_0) = V(t-s_0,u_1^+),\ t \geq s_0$,
- if $M^+(u_1^+) \neq \varnothing$, then for $s_1 := s(u_1^+),\ u_2 = V(s_1,u_1^+)$ and

$$\widetilde{V}(t,u_0) = \begin{cases} V(t-s_0,u_1^+),\ t \in [s_0,s_0+s_1), \\ u_2^+,\ t = s_0+s_1, \end{cases}$$

and etc. As a result we obtain finite or infinite number of impulse points $\{u_{n+1}^+ = IV(s_n,u_n^+), u_0^+ = u_0\}$ and corresponding time moments

$$T_{n+1} := \sum_{k=1}^{n} s_k,\ T_0 := 0,$$

and $\widetilde{V}$ is defined by formula

$$\widetilde{V}(t,u_0) = \begin{cases} V(t-T_n,u_n^+),\ t \in [T_n,T_{n+1}), \\ u_{n+1}^+,\ t = T_{n+1}. \end{cases} \tag{2.5}$$

There is another condition

$$\forall u_0 \in X \ \text{ trajectory } \ t \mapsto \widetilde{V}(t,u_0) \ \text{ is defined on } \ [0,+\infty), \tag{2.6}$$

i.e. either the number of impulses is not more than finite or

$$T_n \to \infty,\ n \to \infty.$$

When the conditions (2.3), (2.6) are fulfilled, $\widetilde{V} : \mathbb{R}_+ \times X \to X$ is a semigroup with the right-continuous trajectories [7], which is called impulsive dynamical system. In this paper we prove that problem (2.1), (2.2) generates impulsive dynamical system (2.5) and study its ω-limit sets

$$\omega(u_0) = \bigcap_{T>0} \overline{\bigcup_{t>T} \widetilde{V}(t,u_0)}.$$

3. Results

The solutions of problem (2.1) generate the continuous semigroup $V:\mathbb{R}_+\times X\to X$ by formula

$$V(t,u_0)=\sum_{k=1}^{\infty}\{u_k^0e^{-k^2t}+\frac{h_k}{k^2}(1-e^{-k^2t})\}\sin kx, \tag{3.1}$$

where $u_k^0=\frac{2}{\pi}\int_0^{\pi}u_0(x)\sin kx dx$.

Impulsive set has a form

$$M=\{u\in X|u(x)\geq 0,\ \ \Psi(u)=\Psi_0\}. \tag{3.2}$$

Impulsive mapping $I\colon M\to X$ has a form

$$(Iu)(x)=u(x)+\alpha(x). \tag{3.3}$$

Theorem 1. *Let*

$$h(x)=\sum_{k=1}^{\infty}h_k\sin kx,\ \ h_{2k-1}\leq 0,\ \ \sum_{k=1}^{\infty}|h_k|<H, \tag{3.4}$$
$$\alpha(x)=O(x),\ x\to 0,\ \ \alpha(x)=O(\pi-x),\ x\to\pi.$$

Then problem (3.1)-(3.3) generates impulsive dynamical system, i.e. the conditions (2.3), (2.6) hold.

Proof. Since the semigroup $V:\mathbb{R}_+\times X\to X$ is continuous, the set $M\subset X$ is closed, for $u\in M$

$$\int_0^{\pi}(Iu)(x)dx=\Psi_0+\alpha_0>\Psi_0,$$

then $M\bigcap IM=\varnothing$.

For further thinking, we will show that for $u_0(x)\geq 0$ (but $u_0(x)\not\equiv 0$) the function $t\mapsto\Psi(V(t,u_0))$ decreases at interval $[0,+\infty)$.

Let us denote $u(t,x)=V(t,u_0)$. Then

$$u(t,x)=V(t,u_0)=v(t,x)+\omega(t,x),$$

where

$$v(t,x)=\sum_{k=1}^{\infty}u_k^0e^{-k^2t}\sin kx \text{ is the solution of (2.1) with } h\equiv 0,$$

$$\omega(t,x)=\sum_{k=1}^{\infty}\frac{h_k}{k^3}(1-e^{-k^2t})\sin kx \text{ is the solution of (2.1) with } u_0\equiv 0.$$

Since

$$\Psi(\omega(t))=\sum_{k=1}^{\infty}\frac{h_k}{k^2}(1-e^{-k^2t})(1-(-1)^k)=2\sum_{k=1}^{\infty}\frac{h_{2k-1}}{(2k-1)^3}(1-e^{-(2k-1)^2t}),$$

then for $t_2 > t_1$

$$\Psi(\omega(t_2)) - \Psi(\omega(t_1)) = 2\sum_{k=1}^{\infty} \frac{h_{2k-1}}{(2k-1)^3}\left(e^{-(2k-1)^2 t_1} - e^{-(2k-1)^2 t_2}\right) \le 0.$$

So, it's sufficient to prove that $\forall t_2 > t_1 \;\; \Psi(v(t_2)) < \Psi(v(t_1))$.

If this is not the case, then for some $t_2 > t_1$

$$\int_0^{\pi} v(t_2,x)dx = \int_0^{\pi} v(t_1,x)dx.$$

Then, by integrating the equation $\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$, we obtain

$$v(t_2,x) - v(t_1,x) = \int_{t_1}^{t_2} \frac{\partial^2 v}{\partial x^2},$$

$$\int_0^{\pi} (v(t_2,x) - v(t_1,x))dx = \int_{t_1}^{t_2} (v_x(t,\pi) - v_x(t,0))dt. \qquad (3.5)$$

Since the maximum principle the inequality $v(t,x) \ge 0$ holds $\forall t \ge 0$, then

$$v(t,0) = v(t,\pi) = 0 \;\Rightarrow\; v_x(t,\pi) \le 0, \;\; v_x(t,0) \ge 0.$$

Therefore the left part of equality (3.5) equal to zero:

$$0 = \int_0^{\pi} (v(t_2,x) - v(t_1,x))dx,$$

and for the difference from the right side of (3.5), the inequality holds

$$v_x(t,\pi) - v_x(t,0) \le 0.$$

Hence

$$\forall t \in [t_1,t_2] \qquad v_x(t,\pi) = v_x(t,0) \qquad \Rightarrow$$
$$v_x(t,\pi) = 0, \qquad v_x(t,0) = 0 \qquad \forall t \in [t_1,t_2] \qquad \Rightarrow\; u_0(x) \equiv 0.$$

Thus, the function $t \mapsto \Psi(V(t,u_0))$ strictly decreases.

Then, if $u_0 \in M$, then $\forall t > 0$

$$\Psi(V(t,u_0)) < \Psi(u_0) = \Psi_0 \;\Rightarrow\; V(t,u_0) \notin M.$$

Thus, the conditions (2.3) hold. Let us prove the condition (2.6).

If $M^+(u_0) = \varnothing$ (in particular, if $u_0 \in M$), then trajectory is not subject to impulse perturbations, and hence it exists for $\forall t \ge 0$. The same conclusion about global existence can be made for trajectories with a finite number of perturbations. Therefore, we will assume that the trajectory has an infinite number of perturbations.

Suppose that $M^+(u_0) \ne \varnothing$. Then $\exists s_0 > 0$: $u(s_0) \in M$. Therefore for $u_1^+ = IV(s_0,u_0)$ we obtain

$$u_1^+(x) = u(s_0) + \alpha(x), \;\; \Psi(u_1^+) = \Psi_0 + \alpha_0 > \Psi_0.$$

Let $s_1 = s(u_1^+)$. Due to (3.4) we get

$$\exists A > 0: \quad \alpha(x) \le A \sin x, \quad x \in [0,\pi].$$

Further, $\forall t \ge 0 \;\; \omega(t,\cdot) \in \mathbb{C}^1([0,\pi])$, moreover

$$\forall t \ge 0 \;\; \forall x \in [0,\pi] \;\; |\omega(t,x)| \le \sum_{k=1}^{\infty} \frac{|h_k|}{k}(1-e^{-k^2 t}) \le H.$$

So, $\forall s \ge 0$

$$|\omega(s,x)| \le \overline{H} \cdot \sin x, \quad \text{where} \quad \overline{H} \ge H. \tag{3.6}$$

Finally, since $v(s_0,\cdot) \in \mathbb{C}^1([0,\pi])$, then there exists $C_0 = C_0(u_0) > 0$ that

$$v(s_0,x) \le C_0 \cdot \sin x. \tag{3.7}$$

therefore

$$u_1^+(x) \le (C_0 + \overline{H} + A) \cdot \sin x. \tag{3.8}$$

From the maximum principle for solution of problem

$$\begin{cases} \frac{\partial v}{\partial t} & = \frac{\partial^2 v}{\partial x^2}, \\ v|_{x=0} & = v|_{x=\pi} = 0, \\ v|_{t=0} & = u_1^+(x) \end{cases}$$

we obtain

$$v(t,x) \le (C_0 + \overline{H} + A)e^{-t} \cdot \sin x.$$

Then

$$\max\{v_x(t,0) - v_x(t,\pi)\} \le (C_0 + \overline{H} + A)e^{-t}.$$

So, for $t > 0$ and for $u(t,x) = v(t,x) + \omega(t,x)$

$$\begin{aligned} \frac{d}{dt}\Psi(u(t)) &= v_x(t,\pi) - v_x(t,0) + 2\sum_{k=1}^{\infty} \frac{h_{2k-1}}{2k-1} e^{-(2k-1)^2 t} \\ &\ge -2(C_0 + \overline{H} + A)e^{-t} - 2\overline{H}. \end{aligned} \tag{3.9}$$

If $u(s_1) \in M$, then from inequality (3.9) we deduce

$$\Psi_0 \ge \Psi_0 + \alpha_0 - 2(C_0 + \overline{H} + A)(1 - e^{-s_1}) - 2\overline{H} \cdot s_1$$

$$(C_0 + \overline{H} + A)(1 - e^{-s_1}) + \overline{H} \cdot s_1 \ge \frac{\alpha_0}{2}. \tag{3.10}$$

From (3.10) we deduce that

$$s_1 \ge \tau_1,$$

where τ_1 is the solution of equation

$$(C_0 + \overline{H} + A)(1 - e^{-\tau}) + \overline{H} \cdot \tau = \frac{\alpha_0}{2}. \tag{3.11}$$

Considering the estimate $1 - e^{-x} \le x, \; x \ge 0$, we get from the formula (3.11) that

$$s_1 \ge \frac{\alpha_0}{2(C_0 + \overline{H} + A)}.$$

Then

$$u_2^+ = IV(s_1, u_1^+) = v(s_1) + \omega(s_1) + \alpha(x)$$
$$u_2^+(x) \le (C_0 + \overline{H} + A)e^{-s_1} \cdot \sin x + \overline{H}\sin x + A\sin x$$
$$\le (C_0 + 2(\overline{H} + A))\sin x.$$

Then, repeating the previous considerations, we obtain

$$s_2 \ge \frac{\alpha_0}{2(C_0 + \overline{H} + 2(\overline{H} + A))}.$$

In the nth step we get an estimate

$$s_n \ge \frac{\alpha_0}{2(C_0 + \overline{H} + n(\overline{H} + A))}, \tag{3.12}$$

which means that $\sum_{n=0}^{\infty} s_n = \infty$. Thus, the condition 2.6 holds. □

The following theorem is the main result of this article.

Theorem 2. *Let the conditions of Theorem 1 hold and moreover, for the function*

$$\alpha(x) = \sum_{k=1}^{\infty} \alpha_k \sin kx$$

we get

$$\forall n \ge 2 \ \ \alpha_{2n-1} \le 0, \ \ \sum_{n=2}^{\infty} \frac{1}{2n-1} |\alpha_{2n-1}| < \alpha_1.$$

Then for $\forall u_0 \in X$ ω*-limit set*

$$\omega(u_0) = \bigcap_{T>0} \overline{\bigcup_{t \ge T} \widetilde{V}(t, u_0)}$$

is nonempty, compact subset of X *and*

$$dist(\widetilde{V}(t, u_0), \omega(u_0)) \to 0, \ \ t \to \infty. \tag{3.13}$$

Proof. We will prove that

$$\forall t_m \nearrow \infty \ \ \text{a sequence} \ \ \xi_m = \widetilde{V}(t_m, u_0) \ \ \text{is precompact } X. \tag{3.14}$$

From (3.14) it evidently follows that $\omega(u_0) \ne \varnothing$, it is compact and satisfies the condition (3.13). To do this, we first show that for the sequence $\{s_n\}_{n \ge 0}$, except the estimate (3.12), the following estimate is valid:

$$\exists \bar{s} = \bar{s}(u_0) > 0 : \ \inf_{k \ge 1} s_k \ge \bar{s}. \tag{3.15}$$

Based on the formula (3.1), for $T_{k+1} = \sum_{i=0}^{k} s_i$ we have equality

$$\Psi_0 = 2\sum_{j=1}^{\infty} \frac{1}{2j-1} \left\{ \alpha_{2j-1} \cdot \sum_{i=0}^{k} e^{-(2j-1)^2(T_{k+1} - T_i)} + u_{2j-1}^0 \cdot e^{-(2j-1)^2 T_{k+1}} \right\}$$

$$+2\sum_{j=1}^{\infty}\frac{h_{2j-1}}{(2j-1)^3}(1-e^{-(2j-1)^2T_{k+1}}). \tag{3.16}$$

According [13], we will rewrite the formula (3.16) in the form

$$\frac{\Psi_0}{2}=\sum_{j=1}^{\infty}\frac{1}{2j-1}\left\{\alpha_{2j-1}\cdot\sum_{i=0}^{k}e^{-(2j-1)^2\sum_{r=i}^{k}s_r}+u^0_{2j-1}\cdot e^{-(2j-1)^2\sum_{r=i}^{k}s_r}\right\}$$
$$+\sum_{j=1}^{\infty}\frac{h_{2j-1}}{(2j-1)^3}(1-e^{-(2j-1)^2\sum_{r=i}^{k}s_r}).$$

With this value $k\geq 2$, we multiply this equality by a multiplier e^{s_k} and subtract equality (3.16) with k reduced by 1. Then we get

$$\frac{\Psi_0}{2}(e^{s_k}-1)=\alpha_1+\sum_{j=2}^{\infty}\frac{1}{2j-1}\alpha_{2j-1}\Big\{e^{-4j(j-1)s_k}-(1-e^{-4j(j-1)s_k})\times$$
$$\times\sum_{i=1}^{k-1}e^{-(2j-1)^2\sum_{r=i}^{k-1}s_r}\Big\}-\sum_{j=1}^{\infty}\frac{1}{2j-1}u^0_{2j-1}\cdot e^{-(2j-1)^2T_k}(1-e^{-4j(j-1)s_k}) \tag{3.17}$$
$$+\sum_{j=1}^{\infty}\frac{h_{2j-1}}{(2j-1)^3}\cdot\Big((1-e^{-(2j-1)^2T_{k+1}})\cdot e^{s_k}-(1-e^{-(2j-1)^2T_k})\Big).$$

Suppose that up to some subsequence there is convergence $s_k\to 0+$, $k\to\infty$. Then the left side of the equation (3.17) tends to $0+$. According that under Theorem 1 $T_k\to\infty$, we deduce that the term with the value u^0_{2j-1} tends to zero. For the same reasons, the last term with the value h^0_{2j-1} also tends to zero. Let's analyze the second term. Here

$$\beta_j^k:=e^{-4j(j-1)s_k}-(1-e^{-4j(j-1)s_k})\cdot\sum_{i=1}^{k-1}e^{-(2j-1)^2\sum_{r=i}^{k-1}s_r}$$
$$\leq e^{-4j(j-1)s_k}-(1-e^{-4j(j-1)s_k})\cdot e^{-(2j-1)^2\cdot s_{k-1}}.$$

Hence, $\varliminf_{k\to\infty}\beta_j^k\leq 1$. If $\varliminf_{k\to\infty}\beta_j^k\in[0,1]$, then

$$\varliminf_{k\to\infty}\Big(\alpha_1+\sum_{j=2}^{\infty}\frac{1}{2j-1}\cdot\alpha_{2j-1}\cdot\beta_j^k\Big)\geq\alpha_1-\sum_{j=2}^{\infty}\frac{1}{2j-1}\cdot|\alpha_{2j-1}|>0$$

and we have a contradiction.

If $\varliminf_{k\to\infty}\beta_j^k\leq 0$, then

$$\varliminf_{k\to\infty}\Big(\alpha_1+\sum_{j=2}^{\infty}\frac{1}{2j-1}\cdot\alpha_{2j-1}\cdot\beta_j^k\Big)\geq\alpha_1>0$$

and also we have a contradiction. Thus, estimate (3.15) holds.

Now we show that $\forall u_0 \in X$ the set $\bigcup\limits_{t\geq 0} \widetilde{V}(t,u_0)$ is bounded. From the estimate

$$\forall t \geq s \geq 0 \ \ \|V(t,u_0)\|_\infty \leq M_1\|V(s,u_0)\|e^{-(t-s)} + M_2 \tag{3.18}$$

we obtain the required statement for non-impulsive trajectories, or for trajectories with a finite number of impulsive perturbations. Hence, let the trajectory with an infinite number of perturbations start from the point u_0. Repeating the arguments (3.6) - (3.8) from the proof of Theorem 1, as well as using the estimate (3.15), we obtain

$$\begin{aligned}
u_1^+(x) &\leq (C_0+\overline{H}+A)\cdot\sin x \ \Rightarrow \ \|u_1^+\| \leq C_0+\overline{H}+A,\\
u(s_1,x) &\leq \big((C_0+\overline{H}+A)e^{-\bar{s}}+\overline{H}\big)\sin x,\\
u_2^+(x) &\leq \big((C_0+\overline{H}+A)e^{-\bar{s}}+H+A\big)\sin x \ \Rightarrow \ \|u_2^+\| \leq C_0e^{-\bar{s}}+(H+A)(1+e^{-\bar{s}}),\\
u(s_2,x) &\leq \Big(\big((C_0+\overline{H}+A)e^{-\bar{s}}+H+A\big)e^{-\bar{s}}+H\Big)\sin x,\\
u_3^+(x) &\leq \Big(\big((C_0+\overline{H}+A)e^{-\bar{s}}+H+A\big)e^{-\bar{s}}+H+A\Big)\sin x \ \Rightarrow\\
&\|u_3^+\| \leq C_0e^{-2\bar{s}}+(H+A)(1+e^{-\bar{s}}+e^{-2\bar{s}}).
\end{aligned}$$

Continuing this process, at the nth step we obtain that

$$\begin{aligned}
\|u_3^+\| &\leq C_0e^{-n\bar{s}}+(H+A)(1+e^{-\bar{s}}+e^{-2\bar{s}}+\ldots+e^{-n\bar{s}})\\
&\leq C_0e^{-n\bar{s}}+(H+A)(1-e^{-\bar{s}})^{-1}.
\end{aligned} \tag{3.19}$$

From estimates (3.18) and (3.19) we derive

$$\exists M = M(u_0) > 0 \ \ \sup_{t\geq 0}\|\widetilde{V}(t,u_0)\| \leq M. \tag{3.20}$$

Now we prove (3.14). It is known [7] that the following estimate holds:

$$\|V(t,u_0)\|_{\mathbb{C}^1} \leq \frac{K_1}{t^\delta}\|u_0\| + \frac{K_2t^{1-\delta}}{1-\delta}\|h\|, \tag{3.21}$$

where constants $\delta \in (0,1)$, $K_1>0$, $K_2>0$ don't depend on t, u_0. Then from (3.15), (3.20), (3.21) we obtain that

$$\exists K = K(u_0) \ \ \sup_{n\geq 1}\|u_0(s_n)\|_{\mathbb{C}^1} \leq K. \tag{3.22}$$

This means that sequence $\{u_n^+ = u(s_n)+\alpha\}_{n=1}^\infty$ is precompact in X.

Then for $t_m \nearrow +\infty$ $\exists n = n(m) \geq 1$, $n(m)\to\infty$, $m\to\infty$ such that $t_m \in [T_{n(m)}, T_{n(m)+1})$.

Therefore, according to (2.5), $\xi_m = \widetilde{V}(t_m,u_0) = V(t_m - T_{n(m)}, u_{n(m)}^+)$.

If $t_m - T_{n(m)} \to \infty$, $m\to\infty$, then for sufficient large m

$$\xi_m = V(1, V(t_m - T_{n(m)} - 1, u_{n(m)}^+)) = V(1, \widetilde{V}(t_m - T_{n(m)} - 1, u_{n(m)}^+)),$$

thus,

$$\|\xi_m\|_{\mathbb{C}^1} \le K \cdot M + \frac{K_2}{1-\delta}\|h\|,$$

that means the precompactness of $\{\xi_m\}$.

Otherwise up to some subsequence $t_m - T_{n(m)} \to \tau \ge 0$ and then precompactness of $\{\xi_m\}$ follows from precompactness of sequence $\{u^+_{n(m)}\}$ and continuity of semigroup V. Theorem is proven. □

Remark 1. In the case of condition $M^+(u_0) \ne \varnothing$ the structure of the function $\omega(u_0)$ can be complicated. In particular, [13] states that under conditions of Theorems 1 and 2 there is a discontinuous impulsive cycle in the system (2.1), (2.2).

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