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Full Length Article

Contractivity of stochastic θ-methods under non-global Lipschitz conditions

Helena Biščević [a], Raffaele D'Ambrosio [b,*], Stefano Di Giovacchino [b]

[a] Gran Sasso Science Institute, L'Aquila, Italy

[b] Department of Information Engineering and Computer Science and Mathematics, University of L'Aquila, L'Aquila, Italy

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ABSTRACT

The paper is devoted to address the numerical preservation of the exponential mean-square contractive character of the dynamics of stochastic differential equations (SDEs), whose drift and diffusion coefficients are subject to non-global Lipschitz assumptions. The conservative attitude of stochastic θ-methods is analyzed both for Itô and Stratonovich SDEs. The case of systems with linear drift is also analyzed in terms of spectral properties of the coefficient matrix of the drift. Numerical evidence on selected test problems confirms the effectiveness of the approach.

1. Introduction

We focus our interest on numerical issues concerning the computation of approximate solutions to the following stochastic differential problem

$$\begin{cases} \mathrm{d}X(t) = f(X(t))\mathrm{d}t + g(X(t))\mathrm{d}W(t), & t \in [0,T], \\ X(0) = X_0, \end{cases} \tag{1}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ is the *drift* of the problem, while $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is the *diffusion* of the stochastic differential equation (SDE) and X_0 is assumed in $L^{2p}(\Omega)$, for any $p \geq 1$, being Ω the probability space. The stochastic process $W(t)$ governing the random part of the dynamics described by the right-hand side of (1) is a m-dimensional Wiener process. It is well-known that the differential formulation (1) has to be intended as a short-hand notation for the integral equation

$$X(t) = X_0 + \int_0^t f(X(s))\mathrm{d}s + \int_0^t g(X(s))\mathrm{d}W(s), \quad t \in [0,T], \tag{2}$$

where the integral of the drift is the classical Riemann integral along paths, while the integral in $\mathrm{d}W(s)$ is here meant in Itô setting, defined as follows:

* Corresponding author.
E-mail addresses: helena.biscevic@gssi.it (H. Biščević), raffaele.dambrosio@univaq.it (R. D'Ambrosio), stefano.digiovacchino@univaq.it (S. Di Giovacchino).

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$$\int_0^t g(X(s))\mathrm{d}W(s) = \lim_{n\to\infty} \sum_{j=0}^{n-1} g(X(\tau_j))\Big(W(\tau_{j+1}) - W(\tau_j)\Big),$$

where the above limit has to be meant in the mean-square sense, on the set of equidistant grid points

$$\mathcal{I}_h = \left\{ \tau_\ell = \ell h, \quad \ell = 0, 1, \ldots, n, \quad h = \frac{T}{n} \right\}.$$

Above Wiener increments $W(\tau_{j+1}) - W(\tau_j)$ are independent and identically distributed as $\sqrt{h} \cdot \mathcal{N}(0,1)$, being $\mathcal{N}(0,1)$ a standard normal random variable.

A large number of contributions regarding the qualitative analysis of SDEs, including existence, uniqueness, stability issues and so on, has been provided in the scientific literature (see, for instance, [2,10,17,18,22,31,34,38,39] and references therein). Correspondingly, a theory of numerical methods for SDEs has been well established in the last decades (see, for instance [10,17,22,31,37,38,40] and references therein), but the interest in analyzing nonlinear stability issues of stochastic numerical methods is a more recent research topic that still deserves a specific attention [3,10–13,16,23,25,33].

Specifically, we address our attention to the general family of θ-Maruyama methods [4,5,10,12,13,21,22,41,42] for (1) that, with reference to the discretized domain

$$\mathcal{I}_{\Delta t} = \{t_n = n\Delta t, \quad n = 0, 1, \ldots, N, \quad \Delta t = T/N\},$$

assume the following form:

$$X_{n+1} = X_n + (1-\theta)\Delta t f(X_n) + \theta \Delta t f(X_{n+1}) + g(X_n)\Delta W_n, \tag{3}$$

where $\theta \in [0,1]$, X_n is the approximate value for $X(t_n)$ and the Wiener increment $\Delta W_n = W(t_{n+1}) - W(t_n)$ is distributed as a Gaussian random variable with zero mean and variance Δt.

The analysis of nonlinear stability properties specifically characterizing θ-methods has been object of [10,12,13,16,25] and the results have relied on the global Lipschitz assumption for the diffusion coefficient of (1). At the best of our knowledge, contributions on the analysis of nonlinear stability for stochastic numerical methods have not made use of weaker assumptions as it happened, for instance, in the convergence analysis of such methods (see, for instance, [24,29,30,32,35,43]). It is worth highlighting that SDEs with non-global Lipschitz coefficients are very relevant also under a modeling point of view (see, for instance, [29] and references therein): this evidence additionally motivates the importance of putting proper efforts also in the direction of their numerical treatment. Inspired by aforementioned contributions on convergence analysis under non-global Lipschitz conditions (in particular, [43] and references therein), we aim to provide a generalization of the results contained in [12,25], relaxing the assumptions needed to provide an exponential mean-square contractive behavior, i.e.,

$$\mathbb{E}\,\|X(t) - Y(t)\|^2 \le \mathbb{E}\,\|X_0 - Y_0\|^2\, e^{\alpha t}, \tag{4}$$

with $\alpha < 0$, both along exact and numerical solutions computed by (3). The two solutions $X(t)$ and $Y(t)$ of (1), are computed from two different initial states, respectively given by X_0 and Y_0.

The paper is organized as follows: Section 2 contains a brief review of existing results regarding nonlinear stability analysis of stochastic numerical methods and, in particular, of θ-Maruyama methods; in Section 3, we provide the analysis of exponential mean-square contractivity along the exact dynamics, under weak hypothesis on the coefficients of (1); Section 4 analyzes this behavior for the numerical trajectories in terms of proper stepsize restrictions; Section 5 reports selected numerical results confirming the theoretical expectations; an extension of the results also to Stratonovich problem is discussed in Section 6; concluding remarks and future perspectives are highlighted in Section 7.

2. Brief review of existing results

As aforestated, existing results regarding exponential stability properties for SDEs (1), leading to (4), rely on the hypothesis that f and g are globally Lipschitz. According to [23,25], an estimate for the parameter α in (4) is given by $\alpha = 2\mu + L$, where μ is the one-sided Lipschitz constant of the drift, i.e.,

$$\langle x - y, f(x) - f(y)\rangle \le \mu \,\|x - y\|^2 \tag{5}$$

and L is the global Lipschitz constant of the diffusion, i.e.,

$$\|g(x) - g(y)\|^2 \le L\,\|x - y\|^2, \tag{6}$$

for C^1 continuous drift and diffusion coefficients. We observe that $\|\cdot\|$ stands for both the Euclidean norm in $\mathbb{R}^n$ and the Frobenius one in $\mathbb{R}^{n\times m}$.

Inequality (4) provides an *exponential mean-square stability inequality* for the dynamics described by Equation (1). If $\alpha < 0$, Equation (4) suggests an exponential decay of the mean-square gap between two distinct solutions of (1). Inspired by an analogous property occurring in the deterministic case (see, for instance, [8,20]) the following definition is given [12].

Definition 1. A nonlinear SDE (1) whose solutions satisfy the exponential stability inequality (4) with $\alpha < 0$ is said to be exponentially mean-square dissipative and generates an exponentially mean-square contractive dynamics.

As anticipated, for purely deterministic problems (i.e., Equation (1) with null diffusion), above definition gives a generalization of the deterministic case where the condition $\mu < 0$ guarantees the contractive behavior of the solutions to the corresponding deterministic problem, with respect to a given norm. The discretization of deterministic differential equations with one-sided Lipschitz vector field with negative one-sided Lipschitz constant gave rise to the notion of G-stability of numerical methods and related numerical issues, introduced by Dahlquist in [8].

A natural attention is then devoted to the ability of numerical methods to inherit exponential mean-square contractivity along the generated numerical dynamics approximating that of an exponentially mean-square dissipative problem, according to Definition 1. As proved in [10,12], exponential mean-square contractivity is numerically maintained if certain stepsize restrictions are respected. However, these restrictions have been studied so far only in the globally Lipschitz case, so a generalization under weaker assumptions is object of next section.

3. Exponential mean-square stability under weaker assumptions

We now aim to prove an exponential mean-square stability inequality for the continuous problem (1), assuming weaker conditions than those formerly used by [3,10–13,16,23,25,33], mostly relying on global Lipschitz continuity for the diffusion of the SDE. In our analysis, we consider the following two assumptions, inspired by [43].

Assumption 3.1. For the vector fields f and g, we consider the following conditions.

(a) There exists a constant $c < 0$, such that

$$\langle x-y\,, f(x)-f(y)\rangle + \frac{1}{2}\,\|g(x)-g(y)\|^2 \le c\|x-y\|^2, \quad \forall x,y\in\mathbb{R}^n.$$

(b) There exists a sufficiently large $p_0 > 1$ such that

$$\langle x-y\,, f(x)-f(y)\rangle + \frac{2p_0-1}{2}\,\|g(x)-g(y)\|^2 \le \bar{c}\|x-y\|^2, \quad \forall x,y\in\mathbb{R}^n,$$

for some $\bar{c} < 0$.

For null diffusion coefficients, both inequalities reduce to the hypothesis of one-sided Lipschitz continuity of the drift. As a consequence, we cannot expect an exponential mean-square dissipative character without a one-sided Lipschitz continuity with negative constant for the vector field of the underlying deterministic problem (i.e., Equation (1) with $g \equiv 0$).

Then, the following result holds true, whose proof proceeds similarly as in [25] for globally Lipschitz diffusion coefficients.

Theorem 1. *Any given SDE (1) fulfilling item (a) in Assumption 3.1 satisfies the following exponential mean-square stability inequality*

$$\mathbb{E}\|X(t)-Y(t)\|^2 \le \mathbb{E}\|X_0-Y_0\|^2 e^{2ct}, \tag{7}$$

being $X_0 \neq Y_0$ initial values assumed in $L^2(\Omega)$.

Proof. A direct application of Itô Lemma to $Z(t) := \|X(t)-Y(t)\|^2$ yields

$$\begin{aligned}\frac{1}{2}\mathrm{d}Z(t) &= \frac{1}{2}\mathrm{d}\left(\langle X(t)-Y(t), X(t)-Y(t)\,\rangle\right)\\ &= \left(\langle X(t)-Y(t), f(X(t))-f(Y(t))\,\rangle + \frac{1}{2}\,\|g(X(t))-g(Y(t))\|^2\right)\mathrm{d}t + M(t),\end{aligned}$$

where $M(t)$ is a martingale satisfying $\mathbb{E}\,[M(0)] = 0$. Then, Assumption 3.1 yields

$$\mathrm{d}Z(t) \le 2cZ(t)\mathrm{d}t + M(t). \tag{8}$$

The inequality in (7) then comes by taking the side-by-side expectation in (8) and exploiting Gronwall's inequality. □

Remark 1. We observe that the result in Theorem 1 clearly holds true also when item (a) in Assumption 3.1 is replaced by the stronger version (b) in 3.1, since $\frac{1}{2} \le \frac{2p_0-1}{2}$, when $p_0 > 1$.

The following result provides the numerical counterpart of 1, obtained along the numerical dynamics computed by θ-Maruyama methods (3).

Theorem 2. *Under the assumptions given in Theorem 1, the numerical dynamics generated by θ-Maruyama method (3) applied to (1), satisfies the inequality*

$$\mathbb{E}\left\|X_n - Y_n\right\|^2 \leq \mathbb{E}\left\|X_0 - Y_0\right\|^2 e^{\nu(\theta,\Delta t)t_n}, \tag{9}$$

where

$$\nu(\theta,\Delta t) = \frac{1}{\Delta t}\ln\beta(\theta,\Delta t) \tag{10}$$

and

$$\beta(\theta,\Delta t) = \frac{1+(1-\theta)^2\Delta t^2 M - 2|c|\Delta t + \theta\Delta t(1+M)}{1+2\theta\Delta t|\mu|}, \tag{11}$$

being

$$M = \max_{n=1,2,\dots,N} \frac{\mathbb{E}\left\|f(X_n)-f(Y_n)\right\|^2}{\mathbb{E}\left\|X_n - Y_n\right\|^2}, \quad X_n \neq Y_n. \tag{12}$$

Proof. θ-Maruyama numerical solutions for Equation (1), with initial values $X_0 \neq Y_0$, can be recast as

$$\Delta^{n+1} = \Delta^n + (1-\theta)\Delta t\Delta_f^n + \theta\Delta t\Delta_f^{n+1} + \Delta_g^n\Delta W_n,$$

being

$$\Delta^{n+1} = X_{n+1} - Y_{n+1}, \quad \Delta_f^n = f(X_n) - f(Y_n), \quad \Delta_g^n = g(X_n) - g(Y_n).$$

Using Lemma 3.4 in [24], we obtain

$$(1-2\theta\mu\Delta t)\left\|\Delta^{n+1}\right\|^2 \leq \left\|\Delta^n + (1-\theta)\Delta t\Delta_f^n + \Delta_g^n\Delta W_n\right\|^2,$$

further leading to

$$\begin{aligned}(1-2\theta\mu\Delta t)\left\|\Delta^{n+1}\right\|^2 \leq{}& \|\Delta^n\|^2 + (1-\theta)^2\Delta t^2\left\|\Delta_f^n\right\|^2 + \left\|\Delta_g^n\Delta W_n\right\|^2 + 2(1-\theta)\Delta t\langle\Delta^n,\Delta_f^n\rangle + 2\langle\Delta^n,\Delta_g^n\Delta W_n\rangle \\ &+ 2(1-\theta)\Delta t\langle\Delta_f^n,\Delta_g^n\Delta W_n\rangle.\end{aligned} \tag{13}$$

Passing to the expectation in (13), a Taylor series argument and Cauchy-Schwarz inequality then yield

$$\begin{aligned}(1-2\theta\mu\Delta t)\mathbb{E}\left[\left\|\Delta^{n+1}\right\|^2\right] \leq{}& \left(1+(1-\theta)^2\Delta t^2 M\right)\mathbb{E}\left[\|\Delta^n\|^2\right] + \Delta t\mathbb{E}\left[\left\|\Delta_g^n\right\|^2\right] \\ &+ 2\Delta t\mathbb{E}\left[\left\langle\Delta^n,\Delta_f^n\right\rangle\right] + 2\theta\Delta t\mathbb{E}\left[\left|\left\langle\Delta^n,\Delta_f^n\right\rangle\right|\right].\end{aligned} \tag{14}$$

By Assumption 3.1, we have

$$\begin{aligned}\Delta t\mathbb{E}\left[\left\|\Delta_g^n\right\|^2\right] + 2\Delta t\mathbb{E}\left[\left\langle\Delta^n,\Delta_f^n\right\rangle\right] &= 2\Delta t\mathbb{E}\left[\frac{1}{2}\left\|\Delta_g^n\right\|^2 + \left\langle\Delta^n,\Delta_f^n\right\rangle\right] \\ &\leq 2c\Delta t\mathbb{E}\left[\|\Delta^n\|^2\right].\end{aligned} \tag{15}$$

By Cauchy-Schwarz inequality and Young's inequality we also get

$$\begin{aligned}2\theta\Delta t\mathbb{E}\left[\left|\left\langle\Delta^n,\Delta_f^n\right\rangle\right|\right] &\leq 2\theta\Delta t\mathbb{E}\left[\|\Delta^n\|\left\|\Delta_f^n\right\|\right] \\ &\leq \theta\Delta t\mathbb{E}\left[\|\Delta^n\|^2 + \left\|\Delta_f^n\right\|^2\right] \\ &\leq \theta\Delta t(1+M)\mathbb{E}\left[\|\Delta^n\|^2\right].\end{aligned} \tag{16}$$

Plugging (15)-(16) into (14) we finally obtain

$$\mathbb{E}\left[\left\|\Delta^{n+1}\right\|^2\right] \leq \beta(\theta,\Delta t)\mathbb{E}\left[\|\Delta^n\|^2\right],$$

where

$$\beta(\theta,\Delta t) = \frac{1+(1-\theta)^2\Delta t^2 M - 2|c|\Delta t + \theta\Delta t(1+M)}{1+2\theta\Delta t|\mu|}. \quad \square \tag{17}$$

It is worth noting that the definition of M in (12) can be provided independently on the numerical simulation if the drift f is upper bounded via a polynomial growth condition [31,34]. This can be achieved by coupling, indeed, such assumption together with the well-posedness of the SDE (1) and by exploiting the mean-value theorem.

The following corollary of Theorem 2 provides an estimate of the deviation between the exact and numerical rates of the exponential terms in (7) and (9).

Corollary 1. *When the assumptions of Theorem 2 are fulfilled,*

$$|2c - \nu(\theta, \Delta t)| = \mathcal{O}(\Delta t), \tag{18}$$

for a given $\theta \in [0,1]$*, being* $2c$ *and* $\nu(\theta, \Delta t)$ *the exact and numerical rates of the exponential terms in (7) and (9), respectively.*

Proof. The result immediately follows by Taylor series arguments for $\nu(\theta, \Delta t)$ around Δt. □

If the vector fields f and g satisfy condition (b) in Assumption 3.1, we then have the following result.

Theorem 3. *Let us assume condition (b) in Assumption 3.1 and that*

$$\theta \in \left[0, 1 - \frac{1}{2p_0 - 1}\right].$$

Then, the numerical dynamics generated by θ*-Maruyama method (3) applied to (1), satisfies the inequality*

$$\mathbb{E}\left\|X_n - Y_n\right\|^2 \leq \mathbb{E}\left\|X_0 - Y_0\right\|^2 e^{\bar{\nu}(\theta, \Delta t) t_n}, \tag{19}$$

where

$$\bar{\nu}(\theta, \Delta t) = \frac{1}{\Delta t} \ln \bar{\beta}(\theta, \Delta t) \tag{20}$$

and

$$\bar{\beta}(\theta, \Delta t) = \frac{1 + (1-\theta)^2 \Delta t^2 M + 2\Delta t(1-\theta)\bar{c}}{1 + 2\theta \Delta t |\mu|}, \tag{21}$$

Proof. The proof relies on taking the side-by-side expectation of (13) and using the condition (b) of Assumption 3.1. □

Remark 2. Note that the condition (b) in Assumption 3.1 allows us to couple the third and fourth terms in (13), obtaining a more polite estimate. In absence of this condition and, hence, under condition (a) in Assumption 3.1, we have to bound the aforementioned terms separately.
Moreover, it is worth observing that both conditions (a) and (b) in Assumption 3.1 lead to the same qualitative behavior in terms of exponential mean-square contractivity, as clarified by the analogous structure of inequalities (9) and (19).

4. Exponential mean-square contractivity of θ-Maruyama methods

According to Theorem 2, for θ-Maruyama methods (3), exponential mean-square stability inequality (9) provides exponential mean-square contractivity if

$$\nu(\theta, \Delta t) < 0$$

or, equivalently, if the constant $\beta(\theta, \Delta t)$ defined in (11) satisfies

$$0 < \beta(\theta, \Delta t) < 1.$$

Hence, we give the following corollary of Theorem 2.

Corollary 2. *Under the assumptions of Theorem 2, for the stochastic* θ*-Maruyama methods (3), applied to a mean-square dissipative SDE, the following results hold true.*

(i) *If the constants* $M, c,$ *and* μ *satisfy* $(1 + M) < 1(|\mu| + |c|)$*, then the numerical solution provided by the stochastic implicit Euler method, i.e., method (3) with* $\theta = 1$*, is mean-square contractive for all* $\Delta t > 0$*.*
(ii) *For* $\theta \in [0, 1)$*, if the following relation is satisfied*

$$2(|c| + \theta|\mu|) > \theta(1 + M),$$

then the corresponding numerical solution generates mean-square contractive dynamics if

$$\Delta t < \frac{2(|c| + \theta|\mu|) - \theta(1+M)}{(1-\theta)^2 M}. \tag{22}$$

Proof. The proof directly descends from the definition of $\beta(\theta, \Delta t)$ in (17). □

We also give the following corollary. We omit its proof since is similar to the one of the previous corollary.

Corollary 3. *Let us assume the setting of Theorem 3. Then, the method produces mean-square contractive numerical solutions if*

$$\Delta t \le \frac{2(\theta|\mu| + (1-\theta)|\bar{c}|)}{(1-\theta)^2 M}. \tag{23}$$

As discussed in [12], the quantities in (22) are fully computable provided that a suitable estimation of the parameters involved is given. This can be made possible via suitable algorithms introduced in the context of global optimization [45]. Based on a strategy also developed in [16], we propose here an alternative methodology whose computation relies directly on the numerical solution to the SDE under investigation.

To perform this estimation step, given an exponentially mean-square dissipative SDE (1), let us apply a stochastic θ-method with $\theta = \theta^*$, provided that $0 \le \theta^* \le 1$, as usual. Let us monitor the numerical dynamics occurring along a chosen grid of uniform stepsize $\widetilde{h} := T/N$, sufficiently small. Then, we estimate the constant c as follows:

$$\tilde{c} := \max_{n=0,1,\dots,N} \frac{\mathbb{E}\left[\langle X_n - Y_n, f(X_n) - f(Y_n)\rangle\right] + \frac{1}{2}\mathbb{E}\left[\|g(X_n) - g(Y_n)\|^2\right]}{\mathbb{E}\left[\|X_n - Y_n\|^2\right]}, \tag{24}$$

where the expectations can be approximated over P independent paths, with P sufficiently large. In other terms, for any $n = 0, 1, 2, \dots, N$, we have

$$\mathbb{E}\left[\langle X_n - Y_n, f(X_n) - f(Y_n)\rangle\right] + \frac{1}{2}\mathbb{E}\left[\|g(X_n) - g(Y_n)\|^2\right] \le \tilde{c}\, \mathbb{E}\left[\|X_n - Y_n\|^2\right],$$

that is, Assumption 3.1 averaged along the numerical solutions provided by the stochastic θ^*-methods, with stepsize $\widetilde{h}$. As a benchmark for the sharpness of the estimate, this step can eventually be repeated with a stepsize $\bar{h} := R\widetilde{h}$, where R is a suitable positive integer, keeping the data over the same aforecomputed P Brownian paths.

5. Numerical experiments

The section is devoted to present a selection of numerical experiments to confirm the effectiveness of the presented results and the robustness of θ-Maruyama methods in approaching exponentially mean-square dissipative SDEs.

5.1. Ginzburg-Landau equation

We first consider the following Ginzburg-Landau equation [19]

$$\mathrm{d}X(t) = \left(\sigma X(t) + \tau X(t)^3\right)\mathrm{d}t + \rho X(t)\mathrm{d}W(t), \tag{25}$$

where $\sigma, \tau, \rho \in \mathbb{R}$ and we choose $\sigma = -4$, $\tau = -1$, $\rho = 1$. In this case, the value of the parameter α in (4) is negative, since $\mu = -4$ and $L = 1$. According to the results obtained in [12,25] in the globally Lipschitz case, the corresponding problem is exponentially mean square contractive, according to the inequality

$$\mathbb{E}\,\|X(t) - Y(t)\|^2 \le \mathbb{E}\left\|X_0 - Y_0\right\|^2 \mathrm{e}^{-7t}, \quad t \in [0, T].$$

This exponential rate is also coherent with (9) and, as a consequence, the stepsize of θ-Maruyama methods has to be chosen according to Equation (22), in order to maintain exponential mean-square contractivity along the numerical dynamics. Choosing $X_0 = 0.01$ and $Y_0 = 0.02$, Figs. 1 and 2 are coherent with the results in [12] and show the ability of θ-Maruyama methods to catch the exponentially mean-square contractive behavior. Both plots, depicted in semilogarithmic scale, show the coherence between numerical exact exponential decays with rate $2c$. In particular, as regards the stochastic trapezoidal method, values of the stepsize chosen according to Equation (22) provide a decay that looks coherent with the exact one and the estimate for the requested value of the stepsize also appears to be pretty sharp; the value of M gained from the experiments and useful to compute the restriction given in Equation (22) is approximately equal to 16. For the unconditionally contractive implicit Euler-Maruyama method, the decay is faster than e^{-t} also for large values of the stepsize and, clearly, according to Theorem 1, the exact decay rate is well recovered as long as Δt is decreased.

5.2. Modified Ginzburg-Landau equation

The following alternative version of the above problem is considered, namely

$$\mathrm{d}X(t) = \left(\sigma X(t) + \tau X(t)^3\right)\mathrm{d}t + \rho X(t)^2\mathrm{d}W(t), \tag{26}$$

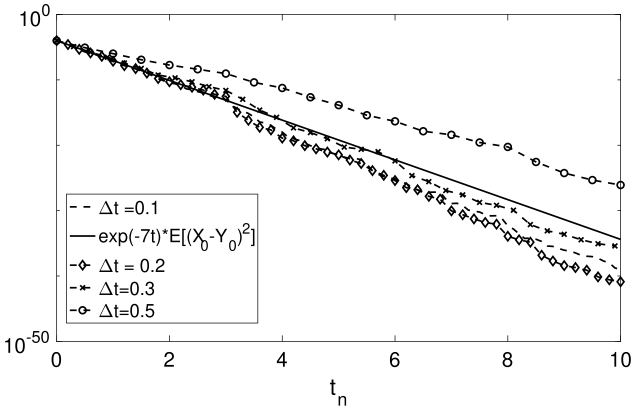

Fig. 1. Exponential mean-square stability pattern in semilogarithmic scale for Ginzburg-Landau equation (25), computed via stochastic trapezoidal method for chosen Δt. The values of the parameters are chosen according to Section 5.1.

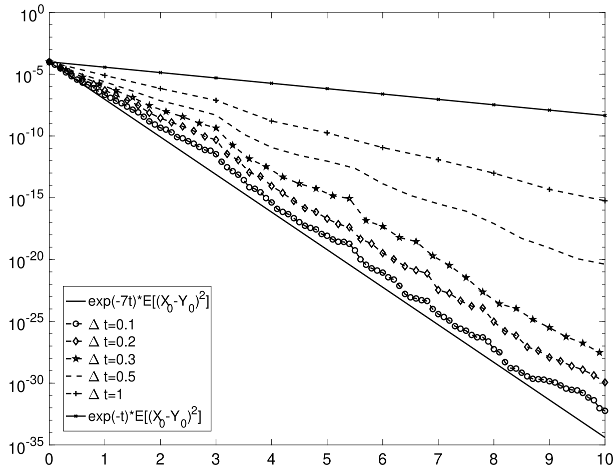

Fig. 2. Exponential mean-square stability pattern in semilogarithmic scale for Ginzburg-Landau equation (25), computed via the implicit Euler-Maruyama method for chosen Δt. The values of the parameters are chosen according to Section 5.1.

with $\sigma = -4$, $\tau = -1$, $\rho = 1$, whose diffusion, turned into a quadratic function, is not globally Lipschitz, so this case is not covered by the existing theory of numerical conservation of contractive behaviors, as for (25). Since the drift function is kept as before, one-sided Lipschitz condition is still satisfied.

With $\sigma = -4$, $\tau = -1$, $\rho = 1$, the drift and diffusion of (26) satisfy Assumption 3.1 with $c = -4$. One indeed has that

$$\langle X - Y, f(X) - f(Y)\rangle + \frac{1}{2}\|g(X) - g(Y)\|^2 = -4\|X - Y\|^2 - (X - Y)(X^3 - Y^3) + \frac{1}{2}(X^2 - Y^2)^2. \tag{27}$$

Since

$$\begin{aligned}(X - Y)(X^3 - Y^3) &= (X - Y)^2(X^2 + Y^2 + XY)\\ &= \frac{1}{2}(X - Y)^2(X + Y)^2 + \frac{1}{2}(X - Y)^2(X^2 + Y^2)\\ &\geq \frac{1}{2}(X^2 - Y^2)^2, \qquad \forall X, Y \in \mathbb{R},\end{aligned}$$

we then get that the term in (27) is bounded by $-4\|X - Y\|^2$, i.e.,

$$\langle X - Y, f(X) - f(Y)\rangle + \frac{1}{2}\|g(X) - g(Y)\|^2 \leq -4\|X - Y\|^2, \qquad \forall X, Y \in \mathbb{R}.$$

Choosing $X_0 = 0.01$ and $X_0 = 0.02$, Figs. 3 and 4 provide a numerical evidence of the exponentially mean-square dissipative character of the discretized problem, provided by θ-Maruyama methods, also in the non-globally Lipschitz case. As before, the stochastic trapezoidal method behaves coherently with the exact dynamics of the problem, whenever the stepsize satisfies the restriction in Equation (22). As in the previous case, the estimated values for the stepsize are sharp. Implicit Euler-Maruyama method also performs in a similar way, as described in the previous section.

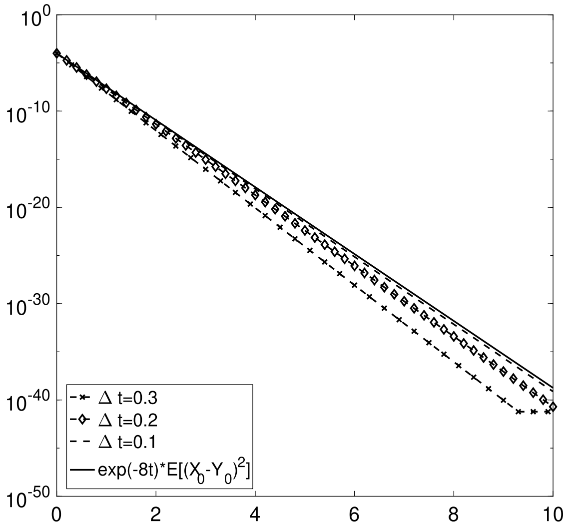

Fig. 3. Exponential mean-square stability pattern in semilogarithmic scale for the modified Ginzburg-Landau equation (26), computed via the stochastic trapezoidal method for chosen Δt. The values of the parameters are chosen according to Section 5.2.

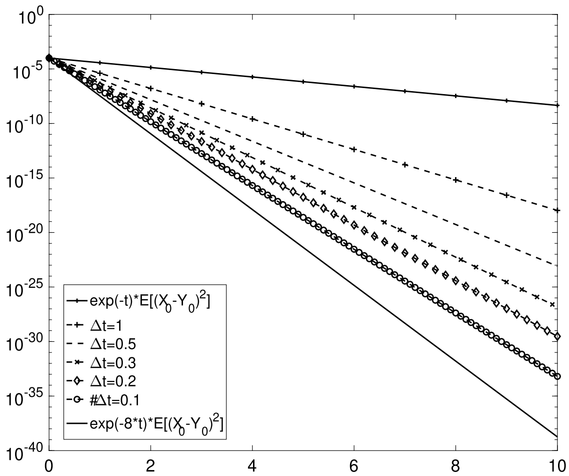

Fig. 4. Exponential mean-square stability pattern in semilogarithmic scale for the modified Ginzburg-Landau equation (26), computed via the implicit Euler-Maruyama method for selected values of the stepsize. The values of the parameters are chosen according to Section 5.2.

5.3. A case study: systems of SDEs with linear drift

We finally focus our attention to systems of SDEs of the form

$$\mathrm{d}X = -AX\mathrm{d}t + G(X)\mathrm{d}W(t), \quad X(0) = X_0 \in \mathbb{R}^d, \tag{28}$$

whose drift is linear and depending on the matrix $A \in \mathbb{R}^d$, that is assumed to be symmetric positive definite, whose diffusion is given by $G : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ and in dependence on the m-dimensional Wiener process $W(t)$. We denote by $\lambda_{\min}$ and $\lambda_{\max}$ the minimum and the maximum eigenvalues of A, respectively. In addition, let us assume that the condition (b) in Assumption 3.1 is satisfied. Then, we have

$$\begin{aligned}\langle X-Y, f(X)-f(Y)\rangle &= -\langle X-Y, A(X-Y)\rangle \\ &= -\left\langle \sum_{i=1}^{d} \rho_i v_i, \sum_{i=1}^{d} A\rho_i v_i \right\rangle \\ &= -\sum_{i,j=1}^{d} \rho_i\rho_j\lambda_j \left\langle v_i, v_j\right\rangle \\ &= -\sum_{i=1}^{d} \rho_i^2\lambda_i,\end{aligned} \tag{29}$$

where λ_i are eigenvalues of A for an orthonormal basis of eigenvectors v_i of A, for $i = 1, \dots . d$. Hence, since λ_i are all positive, plugging (29) into condition (b) of Assumption 3.1 yields

$$\begin{aligned}\|G(X) - G(Y)\|^2 &\le \frac{2}{2p_0 - 1}\left(\bar{c}\,\|X - Y\|^2 + \sum_{i=1}^{d} \rho_i^2 \lambda_i\right)\\ &\le \frac{2}{2p_0 - 1}\left(\bar{c}\,\|X - Y\|^2 + \lambda_{\max} \sum_{i=1}^{d} \rho_i^2\right)\\ &\le \frac{2}{2p_0 - 1}\left(\bar{c} + \lambda_{\max}\right)\|X - Y\|^2,\end{aligned}$$

where $X - Y = \sum_{i=1}^{d} \rho_i v_i$, being $\lambda_{\max}$ the maximum among all the eigenvalues of A. Hence, G has to satisfy

$$\|G(X(t)) - G(Y(t))\|^2 \le \frac{2}{2p_0 - 1}\left(\bar{c} + \lambda_{\max}\right)\|X - Y\|^2.$$

Let us suppose to apply θ-Maruyama method (3) to problem SDE (28) with two distinct initial values $X_0 \neq Y_0$, leading to

$$X_{n+1} - Y_{n+1} = X_n - Y_n - \Delta t(1-\theta)A(X_n - Y_n) - \theta\Delta t A(X_{n+1} - Y_{n+1}) + (G(X_n) - G(Y_n))\Delta W_n,$$

i.e.,

$$X_{n+1} - Y_{n+1} = (I + \theta\Delta t A)^{-1}\left[(I - \Delta t(1-\theta)A)(X_n - Y_n) + (G(X_n) - G(Y_n))\Delta W_n\right].$$

Hence, we get

$$\begin{aligned}\|X_{n+1} - Y_{n+1}\|^2 &= \left\|(I + \theta\Delta t A)^{-1}(I - \Delta t(1-\theta)A)(X_n - Y_n)\right\|^2 + \left\|(I + \theta\Delta t A)^{-1} G(X_n) - G(Y_n))\Delta W_n\right\|^2 + M_g\\ &\le \left\|(I + \theta\Delta t A)^{-1}(I - \Delta t(1-\theta)A)\right\|^2\|X_n - Y_n\|^2 + \left\|(I + \theta\Delta t A)^{-1}(G(X_n) - G(Y_n))\right\|^2\|\Delta W_n\|^2\\ &\le \left\|(I + \theta\Delta t A)^{-1}(I - \Delta t(1-\theta)A)\right\|^2\|X_n - Y_n\|^2 + \frac{2}{2p_0 - 1}\left(\bar{c} + \lambda_{\max}\right)\left\|(I + \theta\Delta t A)^{-1}\right\|^2\|X_n - Y_n\|^2\|\Delta W_n\|^2.\end{aligned}$$

Consequently, we get

$$e_{n+1} \le K(\Delta t, \theta)e_n, \qquad e_n := \mathbb{E}\|X_n - Y_n\|^2,$$

with

$$K(h, \theta) := \left\|(I + \theta\Delta t A)^{-1}(I - \Delta t(1-\theta)A)\right\|^2 + \frac{2}{2p_0 - 1}\left(\bar{c} + \lambda_{\max}\right)\Delta t\left\|(I + \theta\Delta t A)^{-1}\right\|^2.$$

Then, the sequence $\{e_n\}_{n\in\mathbb{N}}$ is monotonically decreasing if

$$K(\Delta t, \theta) < 1. \tag{30}$$

Remark 3. If the function G in (28) is globally Lipschitz with Lipschitz constant L, then Assumption 3.1 is satisfied for $\bar{c} = \frac{2p_0-1}{2}L - \lambda_{\min}$, i.e., problem (28) is exponentially mean-square dissipative if

$$\lambda_{\min} > \frac{(2p_0 - 1)L}{2}.$$

Moreover, it is easy to show that the constant $M = \lambda_{\max}^2$. Hence, in agreement with (23), the numerical dynamics computed by θ-Maruyama method, for $\theta \in [0, 1)$, maintains exponential mean-square contractivity if

$$\Delta t \le \frac{2\left(\theta\lambda_{\min} + (1-\theta)\left|\frac{(2p_0 - 1)L}{2} - \lambda_{\min}\right|\right)}{(1-\theta)^2\lambda_{\max}^2}. \tag{31}$$

As a test case, let us consider as function G the diagonal matrix

$$G(X(t)) = \begin{bmatrix} X_1(t) & & & \\ & X_2(t) & & \\ & & \ddots & \\ & & & X_d(t)\end{bmatrix}$$

and construct (at random) a symmetric positive definite matrix A of dimension $d = 4$, with $\lambda_{\min} = 10.6$ and $\lambda_{\max} = 11$. Condition (30) reveals that the numerical mean-square dissipation is achieved provided that $\Delta t < 0.432$. The numerical result has been displayed in

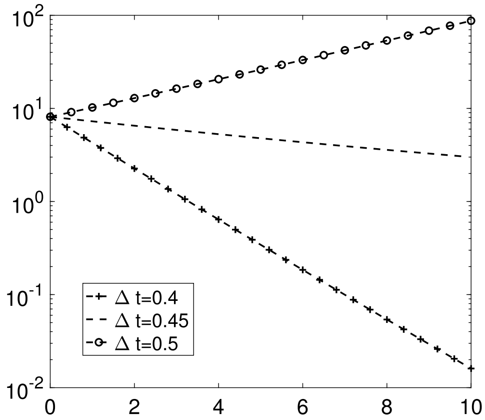

Fig. 5. Mean-square deviations of two numerical solutions to (28), computed with $\theta = 0.3$, for selected values of the stepsize Δt.

Fig. 5, where we recognize the sharpness of the aforementioned bound on Δt, ensuring mean-square dissipativity along the numerical dynamics given by the application of the θ-Maruyama method, with $\theta = 0.3$.

5.4. Systems of SDEs with nonlinear coefficients

In the previous subsection, we considered in more detail the case of linear systems which occurs as an extension to the central theory presented in this paper. The question naturally arises whether this generalizes to the case of nonlinear systems of SDEs. In the following, we will explain the problem and why it is meaningful to our exposition. It certainly regards the sharpness of the bound presented in Theorem 2 in the presence of multiple noisy terms.

Let us indeed consider the following nonlinear system of SDEs

$$\mathrm{d}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = -\left(A \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} X_1^3 \\ X_2^3 \end{bmatrix} \right)\mathrm{d}t + \begin{bmatrix} 0 & 0 \\ 0 & X_1^2 \end{bmatrix} \mathrm{d}W, \tag{32}$$

where $A \in \mathbb{R}^{2\times 2}$ is a symmetric positive definite matrix and $W = [W_1 \ W_2]^T$, with W_1 and W_2 i.i.d. standard Wiener processes. It is worth pointing out the similarity of the above-mentioned equation to the Ginzburg-Landau equation and its modified version presented in the previous sections. Similar calculations as in Section 5.2 reveal that the assumptions of Theorem 2 are satisfied for $c = -\lambda_{\min}$, where $\lambda_{\min}$ is the minimum of eigenvalues of A. In Fig. 6, the behavior of the implicit Euler-Maruyama method has been reported for

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 0.02 \\ 0.02 \end{bmatrix}.$$

The figure shows the exponential decay of the mean square deviation visible along the numerical dynamics, for selected values of the stepsize. When the stepsize diminishes, the numerical decay rate approaches a value (around -5), that is smaller than the actual value (equal to -4). In other terms, the conservative character of stochastic θ-methods is visible from a qualitative point of view, but in this setting it is not quantitatively sharp. As a consequence, the sharpness of the bound in Theorem 2 or the construction of novel numerical schemes that are conservative in a more strict quantitative sense remains an open issue of this research.

6. A remark on Stratonovich case

As aforestated, above results are related to SDEs in Itô sense. Then, a natural question regards extending above arguments to Stratonovich SDEs. Since Itô lemma does not apply to Stratonovich calculus, we check if Theorem 1 may naturally extend to Stratonovich SDEs. Let us then consider a Stratonovich SDE in the following form

$$\mathrm{d}X(t) = f(X(t))\mathrm{d}t + \sum_{i=1}^{m} g_i(X(t)) \circ \mathrm{d}W_i(t), \tag{33}$$

where $g_i : \mathbb{R}^n \to \mathbb{R}^n$, for any $i = 1, \dots, m$, and W_i are i.i.d. standard Brownian motions. Using the same notation as in Theorem 1, let us differentiate the function $Z(t)$ along $X(t)$ and $Y(t)$ solutions to (33). We get

$$\begin{aligned} \mathrm{d}Z(t) &= 2\langle X(t) - Y(t), f(X(t)) - f(Y(t))\rangle\,\mathrm{d}t \\ &\quad + 2\sum_{i=1}^{m} \langle X(t) - Y(t), g_i(X(t)) - g_i(Y(t))\rangle \circ \mathrm{d}W_i(t). \end{aligned} \tag{34}$$

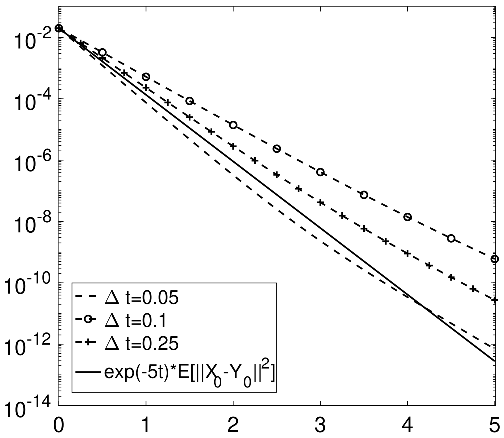

Fig. 6. Exponential mean-square stability pattern in semilogarithmic scale for the implicit Euler-Maruyama method, applied to system (32), for selected values of the stepsize.

The right-hand side of this equality cannot be upper bounded via one-sided Lipschitz continuity of the drift, because above Stratonovich integral can eventually be negative. A safe way to proceed is given by approximating Wiener process as suggested, for instance, in [46]. For $t \in [t_n, t_{n+1}]$, using the Wong-Zakai approximation of the Wiener processes $W_i(t)$, $i = 1, \ldots, m$, and integrating both sides of (34), we get

$$Z(t) = Z(0) + 2\int_0^t \langle X(s) - Y(s), f(X(s)) - f(Y(s))\rangle \, \mathrm{d}s$$
$$+ \frac{2}{h_W} \sum_{i=1}^m \Delta W_i^n \int_0^t \langle X(s) - Y(s), g_i(X(s)) - g_i(Y(s))\rangle \, \mathrm{d}s,$$

where ΔW_i^n is distributed as $\sqrt{h_W}\mathcal{N}(0,1)$, being h_W the stepsize chosen for the discretization of the Wiener processes. The first integral can be easily upper bounded by imposing the one-sided Lipschitz continuity of the drift, leading to

$$Z(t) \le Z(0) + 2\mu \int_0^t Z(s)\,\mathrm{d}s + \frac{2\Delta W^n}{h_W} \sum_{i=1}^m \int_0^t \left|\langle X(s) - Y(s), g_i(X(s)) - g_i(Y(s))\rangle\right| \mathrm{d}s,$$

being $\Delta W^n := \max_{i=1,\ldots,m} \left|\Delta W_i^n\right|$. As a consequence, one-sided Lipschitz continuity of the diffusion coefficient and suitable truncation of the Wiener increments ΔW_i^n [46] would be enough to set a similar upper bound for the second summand too, and an additional dependence on the chosen Wiener approximation is also included making the inequality accurate enough for small values of the stepsize chosen for Wiener discretization. To get rid of it, an alternative way to deal with Stratonovich SDEs could also be set via the usual transformation formula carrying from Stratonovich to Itô SDEs [17,31]. In this direction, an exponential mean-square inequality of type (7) appears if the assumption of Theorem 1 is satisfied by $\tilde{f}(X) \in \mathbb{R}^n$ and $G(X) \in \mathbb{R}^{n\times m}$, where

$$\tilde{f}(X) = f(X) + \frac{1}{2}\sum_{j=1}^n \sum_{i=1}^m \frac{\partial g^i(X)}{\partial X_j} g^{j,i}(X), \quad G(X) = [g^1(X) \ \cdots \ g^m(X)].$$

An evidence of the behavior of θ-methods applied to a transformed Stratonovich problem is here reported by considering the following Stratonovich SDE

$$\mathrm{d}X = -\left(4X + \rho X^3 + X^5\right)\mathrm{d}t + \gamma X^2 \circ \mathrm{d}W, \tag{35}$$

where $\rho > 0$ and $\gamma \in \mathbb{R}$. We note that similar nonlinearity has been taken, e.g., in [43]. Similar calculations as in Section 5.2 show that if $|\gamma| \le \sqrt{\frac{\rho}{2}}$, then the exact dynamics of (35) exhibits (7) with $c = -4$. It also directs to verify that with this choice, the assumptions of Theorem 2 are satisfied by $\tilde{f}$ and g. In Fig. 7, we display the behavior of the stochastic trapezoidal method applied to Stratonovich SDE (35), with $\rho = 1$ and $\gamma = 1/\sqrt{2}$. Also, we use $X_0 = 0.01$ and $Y_0 = 0.02$.

7. Conclusions and open issues

The focus of the paper has been oriented to the numerical preservation of the dissipative character observed in (7) under hypothesis of non-globally Lipschitz continuity for the coefficients of the SDE (1). The conservative character of θ-Maruyama methods has been

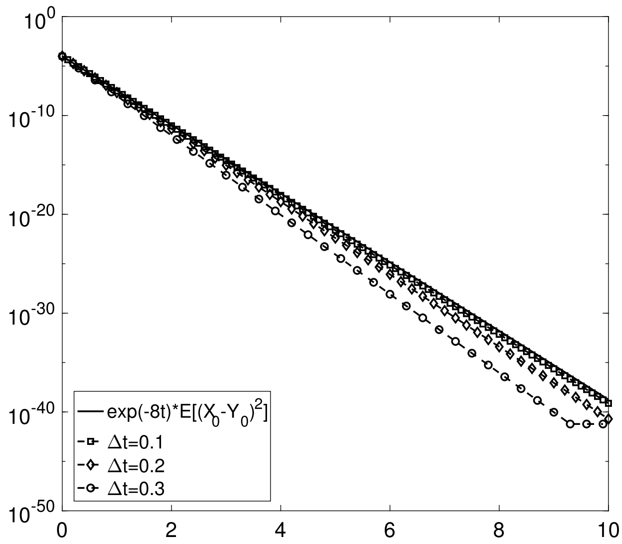

Fig. 7. Exponential mean-square stability pattern in semilogarithmic scale for the Stratonovich SDE (35), computed via the stochastic trapezoidal method for selected values of the stepsize. Here, $\rho = 1$ and $\gamma = 1/\sqrt{2}$.

analyzed and tested, assessing the robustness of this class of numerical methods. The investigation revealed that such a preservation of the exponential mean-square character along the discretized dynamics translates into suitable bounds on the choice of the stepsize used for the numerical solutions.

The paper opens up scenarios for further developments of this research, first of all moving in the direction of stochastic partial differential equations, dissipativity and further time-conservation issues that might be investigated along the dynamics of stochastic θ-methods applied to the spatially discretized problem. This research falls in the larger scenario of stochastic geometric numerical integration; a non-exhaustive list of contributions in this direction is given, for instance, by [1,6,7,9,10,14,15,26–28,36,44]. Indeed, the structure-preserving numerical integration of complex systems described by nonlinear stochastic ordinary and partial differential equations still deserves a significant future attention.

In Subsection 5.4, we pointed out an open problem regarding nonlinear systems of SDEs. We explained and provided numerical experiments showing that the bound from Theorem 2, though formally and qualitatively correct, may suffer from sharpness issues, giving rise to an open problem worth investigating further.

Author contribution

The authors contributed equally to this work.

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Data availability

No data was used for the research described in the article.

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