A characterization of the set of internal points of a conic in PG(2,q), q odd

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Abstract

A point P not on a non-degenerate conic C in $PG(2,q)$, q odd, is called internal to C if no tangent line to C contains P, external otherwise. The set of internal points of C is a $\frac{q(q-1)}{2}$-set of type $(0, \frac{q-1}{2}, \frac{q+1}{2})$. In this paper, we classify all $\frac{q(q-1)}{2}$-sets of class $[0,m,n]$ having exactly two kinds of outer points.

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1 Introduction and motivation

In a Galois projective space $PG(r,q)$, of dimension $r \geq 2$ and order $q = p^h$ a prime power, let K be a k-set, i.e. a set of k points of $PG(r,q)$. Let d be

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an integer such that $1 \leq d \leq r-1$ and let us denote by $\theta_d := \sum_{i=0}^{d} q^i$ the number of points of a d-subspace of $PG(r,q)$. For each integer i such that $0 \leq i \leq min\{k, \theta_d\}$, let us denote by $t_i^d = t_i^d(K)$ the number of d-subspaces meeting the set K in exactly i points. The numbers t_i^d are called the *characters* of K with respect to the d-subspaces, see [7]. Now let $m_1, m_2, ..., m_s$ be s integers such that $0 \leq m_1 < m_2 < ... < m_s \leq min\{k, \theta_d\}$. A set K is said to be of *class* $[m_1, m_2, ..., m_s]_d$ if $t_i^d > 0$ only if $i \in \{m_1, m_2, ..., m_s\}$. Moreover, K is said to be of *type* $(m_1, m_2, ..., m_s)_d$ if $t_i^d > 0$ if and only if $i \in \{m_1, m_2, ..., m_s\}$. The integers $m_1, m_2, ..., m_s$ are called *intersection numbers* with respect to the d-subspaces. A large number of papers is devoted to the study of sets with given intersection numbers. For instance see [1], [3], [4], [5], [6], [8], [9], [10], [11], [14], [15], [16], [17], [18], and [19].

In a projective plane an algebraic curve can be thought as a set of points that has a certain behavior with respect to lines. In a finite projective plane π_q the curve contains a finite number of points and each line (i.e. a 1-subspace) meets the curve in a finite number of points too. By characterization of a curve we mean the classification of those sets which, for axiom, possess a certain number of properties of the given curve. The curve is intended to be characterized if one can prove that a set K satisfying the axiomatized properties is the curve except at most some exceptions for particular values of q (sporadic cases), see [13]. The characterization will be a good characterization when the required axioms are few and significant and few are also sporadic cases.

If K is a k-set of $PG(2,q)$ and a line r meets K in exactly i points, then we say that r is a *i-line* of K. A 0-line is also said an *external line*. For each integer i such that $0 \leq i \leq min\{q+1, k\}$, let us denote by $t_i = t_i^1(K)$ the characters of K with respect to the lines. Now, let us suppose that K is a set of type $(m_1, m_2, ..., m_s)$ with respect to the lines. If P is a point of $PG(2,q)$, then we denote by $v_{m_i} = v_{m_i}(P)$ the number of m_i-lines through P, for any $i \in \{1, 2, ..., s\}$, and we say that the point P is of *kind* $(v_{m_1}, v_{m_2}, ..., v_{m_s})$. A point P belonging to K is said an *inner point*, while a point P not belonging to K is called an *outer point*.

In $PG(2,q)$, q odd, the internal point set K_C of a non-degenerate conic C is a $\frac{q(q-1)}{2}$-set of type $(0, \frac{q-1}{2}, \frac{q+1}{2})$. We remark that there are at least $q+1$ external lines to K_C (any tangent line of C), and that the set K_C has exactly two kinds of outer points, the external points of C and the points on C. A natural question is to characterize, in $PG(2,q)$, q odd, the internal point-set of a non-degenerate conic among $\frac{q(q-1)}{2}$-sets of class $[0,m,n]$, having these two elementary properties. In $PG(2,q)$, another example of $\frac{q(q-1)}{2}$-set of class $[0,m,n]$ having at least two external lines and exactly two kinds of outer points, is the point-set of $\frac{q-1}{2}$ concurrent lines, except the point of intersection. Indeed, it is a $\frac{q(q-1)}{2}$-set of type $(0,\frac{q-1}{2},q)$.

In [2], (Theorem 6.1, page 1920) F. De Clerck, N. De Feyter proved the following

Result 1.1. *If K is a set of type $(0,\frac{q-1}{2},\frac{q+1}{2})$ in $PG(2,q)$, q odd, then K is the set of internal points of a non-degenerate conic.*

In this paper, we classify all $\frac{q(q-1)}{2}$-sets of $PG(2,q)$, q odd of class $[0,m,n]$ having exactly two kinds of outer points. In particular, we prove the following

Theorem 1.2. *Let K be a $\frac{q(q-1)}{2}$-set of class $[0,m,n]$ in $PG(2,q)$, q odd. Then*

- *K is of type $(0,m,n)$ with $m \le \frac{q-1}{2} < \frac{q+1}{2} \le n \le q$;*
- *K has at least two kinds of outer points;*
- *$t_0 \le q+1$; furthermore*
  1. *$t_0 = 1$ if and only if $q = p^{2s}$ is a square, $m = \frac{p^s(p^s-1)}{2}$, $n = m + p^s$; moreover K has exactly two kinds of outer points;*
  2. *if $1 < t_0 < q+1$ and K has exactly two kinds of outer points, then K is the pointset of $\frac{q-1}{2}$ concurrent lines, except the point of intersection;*
  3. *$t_0 = q+1$ if and only if K is the set of internal points of a non-degenerate conic; so K has exactly two kinds of outer points.*

We point out that Theorem 1.2 will be an immediate consequence of Theorem 2.3, Corollary 2.4, Theorem 3.1, and Theorem 4.2.

2 Some preliminary results

Let us start with a property of subsets of $PG(2,q)$ with $1 \le t_0 \le q+1$ external lines.

Proposition 2.1. *Let K be a k-set of points in $PG(2,q)$. If $1 \le t_0 \le q+1$, then there is at least one point $A \notin K$ such that $v_0(A) = 1$. Furthermore,*

1. *K has only one external line r_0 (i.e. $t_0 = 1$) if and only if $v_0(B) = 0$ for any point $B \notin K \cup r_0$; let us note that in such a case it is $v_0(Q) \in \{0,1\}$ for any point $Q \notin K$; so K has exactly two kinds of outer points;*
2. *if K has more than one external line (i.e. $t_0 \ge 2$), then*
   - *there is at least one point $C \notin K$ such that $v_0(C) \ge 2$; so K has at least two kinds of outer points;*
   - *for any point $Q \notin K$ it is $v_0(Q) \le q+1-\frac{k}{q}$.*

Proof. Being $t_0 \ge 1$ the set K has at least one external line r_0. Since there are other $t_0 - 1 \le q$ external lines different from r_0 and there are $q+1$ points on r_0, there is at least one point $A \in r_0$ (hence $A \notin K$) such that no other external line passes through A, so $v_0(A) = 1$.

If $t_0 = 1$, then $v_0(Q) = 1$ for any point $Q \in r_0$ and $v_0(B) = 0$ for any point $B \notin K \cup r_0$.

If $t_0 \ge 2$, then there is at least one external line s_0 different from r_0. If $\{C\} = r_0 \cap s_0$, then $C \notin K$ and $v_0(C) \ge 2$.

Now, let us suppose that $Q \notin K$ is a point such that $v_0(Q) = b \ge 2$. If we consider the b external lines passing through Q we have that there are at least $bq+1$ points not in K. So $q^2+q+1-k \ge bq+1$ from which we get $v_0(Q) = b \le q+1-\frac{k}{q}$. □

Now let K be a k-set of class $[0,m,n]$ in $PG(2,q)$. Hence $0 < m < n \le q+1$. By counting in double way the number of lines, the number of pairs (P,l) where $P \in K$ and l is a line through P, and the number of pairs $((P,Q),l)$ where P and Q are two distinct points of K and l is a line through P and Q, we get the following equations on the integers $t_i = t_i^1(K)$

$$t_0 + t_m + t_n = q^2 + q + 1 \tag{1}$$

$$mt_m + nt_n = k(q+1) \quad (2)$$

$$n(m-1)t_m + n(n-1)t_n = k(k-1) \quad (3)$$

Proposition 2.2. *If K is a k-set of type $(0, m, n)$ in $PG(2, q)$, then $n \leq q$.*

Proof. If $n = q + 1$, then K contains a line r, since $t_n = t_{q+1} > 0$. Since any line meets r, K has no external line, i.e. $t_0 = 0$, a contradiction. □

Theorem 2.3. *Let K be a k-set of class $[0, m, n]$ in $PG(2, q)$, with q odd. If $k = q(q-1)/2$, then K is of type $(0, m, n)$ with $m \leq \frac{q-1}{2} < \frac{q+1}{2} \leq n \leq q$. Furthermore, $t_0 \leq q+1$ and equality holds if and only if $m = \frac{q-1}{2}$ and $n = \frac{q+1}{2}$.*

Proof. If $k = q(q-1)/2$, then by (1), (2) and (3) we obtain

$$4mn(q^2 + q + 1 - t_0) = q(q+1)(q-1)(2m + 2n - q) \quad (4)$$

$$4m(n-m)t_m = q(q-1)(q+1)(2n-q) \quad (5)$$

$$4n(n-m)t_n = q(q-1)(q+1)(q-2m) \quad (6)$$

Being $0 < m < n$, $t_m \geq 0$ and $t_n \geq 0$, by (5) and (6) we get $m \leq q/2 \leq n$. So $m \leq \frac{q-1}{2} < \frac{q+1}{2} \leq n$ since q is odd. Again by (5) and (6) we get $t_m > 0$ and $t_n > 0$. Let us note that $2mn = (2m)n \leq (q-1)(q+1) = q^2 - 1$.

If $t_0 = 0$, then by (4) we get

$$2mn(q^2 + q + 1) = q(q+1)(\frac{q-1}{2})(2m + 2n - q) \quad (7)$$

Being $(q^2 + q + 1)$ and $q(q+1)$ coprime, $q(q+1)$ divides $2mn$. So $q(q+1) \leq 2mn \leq q^2 - 1$, a contradiction. Thus $t_0 > 0$ too and K is of type $(0, m, n)$. Finally, by Proposition 2.2 we have that $n \leq q$.

Now let us note that (4) can be rewritten in the following way
$4mn(q+1-t_0) = q[(q-1-2m)(2n-q-1)q + (q-1-2m)(q+1) + (2n-q-1)(q-1)]$
Being $q - 1 - 2m \geq 0$ and $2n - q - 1 \geq 0$, we have that $q + 1 - t_0 \geq 0$ and equality holds if and only if $2m = q - 1$ and $2n = q + 1$. □

By Result 1.1 and Theorem 2.3 we immediately have the following

Corollary 2.4. *Let K be a $\frac{q(q-1)}{2}$-set of class $[0, m, n]$ in $PG(2, q)$, with q odd. Then $t_0 = q+1$ if and only if K is the set of internal points of a non-degenerate conic.*

Corollary 2.5. *Let K be a k-set of class $[0, m, n]$ in $PG(2, q)$, with q odd and $k = q(q-1)/2$. Then $n - m = 1$ if and only if $t_0 = q + 1$.*

Proof. By Theorem 2.3 we have that $m = -x+(q-1)/2$ and $n = y+(q+1)/2$ with $x \geq 0$ and $y \geq 0$. So $1 = n - m = 1 + x + y$ if and only if $x + y = 0$ or, equivalently, $x = y = 0$. □

Now, let P be a point of K and Q a point not in K. Counting the points of K by the lines through P and through Q we have that

$$(m-1)v_m(P) + (n-1)v_n(P) + 1 = mv_m(Q) + nv_n(Q) = k \tag{8}$$

Being

$$v_m(P) + v_n(P) = v_0(Q) + v_m(Q) + v_n(Q) = q + 1 \tag{9}$$

we obtain

$$(n-m)v_n(P) = k + q - m(q+1) = k - m - q(m-1) \tag{10}$$

and

$$(n-m)v_n(Q) = k - m(q+1) + mv_0(Q) \tag{11}$$

Finally, by (10) and (11) we immediately get

$$(n-m)(v_n(P) - v_n(Q)) = q - mv_0(Q) \tag{12}$$

By Proposition 2.1 and Formula (12) we immediately have the following

Proposition 2.6. *Let K be a k-set of class $[0, m, n]$ in $PG(2, q)$, with q odd and $k = q(q-1)/2$. Then $n - m$ divides $q - m$. So $n - m$ divides $q - n$ too. Furthermore, if there is at least one point $Q \notin K$ such that $v_0(Q) = 0$, then $n - m$ divides q. So $n - m$ divides both m and n too.*

3 Sets with exactly one external line

Theorem 3.1. *Let K be a $\frac{q(q-1)}{2}$-set of class $[0, m, n]$ in $PG(2, q)$, q odd. Then $t_0 = 1$ if and only if $q = p^{2s}$ is a square, $m = p^s(p^s-1)/2$ and $n = p^s(p^s+1)/2$. Furthermore, K has exactly two kinds of outer points.*

Proof. If $m = p^s(p^s - 1)/2$ and $n = p^s(p^s + 1)/2$, then by (4) we get $t_0 = 1$. Now let us suppose that $t_0 = 1$. Equation (4) becomes

$$2mn + m + n = q[m + n - (q - 1)/2] \tag{13}$$

If $t_0 = 1$, then by Proposition 2.6 we have that $n - m$ divides $q = p^h$. So there is an integer s such that $n - m = p^s$. If $s = 0$, then by Corollary 2.5 we get $t_0 = q + 1$, a contradiction. If $s = h$, then $q \geq n = q + m \geq q + 1$, a contradiction. So $0 < s < h$. There is a positive integer x and a non-negative integer t such that p does not divide x and $m = xp^t$. Being $t_0 = 1$, then by Propositions 2.1 and 2.6 we have that $n - m$ divides m. So $t \geq s > 0$. Furthermore, $t < h$ since $m < q = p^h$. Substituting $m = xp^t$ and $n = xp^t + p^s$ in (13) we obtain

$$2x(xp^t + p^s + 1)p^{t-s} + 1 = p^{h-s}[2xp^t + p^s - (p^h - 1)/2] \tag{14}$$

Being $s < h$, if $t > s$, then p divides 1, a contradiction. So $t = s$ and equation (14) becomes

$$4p^s x^2 - 4(p^h - p^s - 1)x + (p^h - 1)(p^{h-s} - 2) = 0 \tag{15}$$

If we consider the previous equation as an equation of second degree in x we can calculate its discriminant $\Delta = 16(p^{2s} - p^h + 1)$. Let us note that $\Delta \neq 0$, otherwise p divides 1. So $\Delta > 0$ if and only if $h \leq 2s$. Now, let us suppose that $h < 2s$. If Δ is a square then there is a positive integer c such that $c^2 = (p^{2s} - p^h + 1)$. So $(c-1)(c+1) = c^2 - 1 = q(p^{2s-h} - 1) \equiv 0 \ (mod\ q)$. So $c-1 \equiv 0 \ (mod\ q)$ aut $c+1 \equiv 0 \ (mod\ q)$ since q is odd and $GCD(c-1, c+1) \leq 2$. Hence there is a positive integer d such that $c = dq \pm 1$. Now we have that $q(p^{2s-h} - 1) = c^2 - 1 = dq(dq \pm 2)$. Thus $p^{2s-h} - 1 = d(dq \pm 2)$. Being $s < h$ we have that $s+1 \leq h$ and so $2s-h+1 < 2s-h+2 \leq h$. So $p^{2s-h+1} < p^h = q$. Let us note that $d(dq \pm 2) = p^{2s-h} - 1 < p(p^{2s-h} - 1) = p^{2s-h+1} - p < p^{2s-h+1} - 2 < q-2 \leq dq-2 \leq d(dq-2) < d(dq+2)$, a contradiction. So $h = 2s$ and equation (15) becomes

$$4p^s x^2 - 4(p^{2s} - p^s - 1)x + (p^{2s} - 1)(p^s - 2) = 0 \tag{16}$$

which is equivalent to

$$[2x - (p^s - 1)][2p^s x - (p^s + 1)(p^s - 2)] = 0 \tag{17}$$

whose solutions are $x = \frac{p^s-1}{2}$ and $x = \frac{p^s-1}{2} - \frac{1}{p^s}$. Being $s > 0$ and p odd, only the first solution is an integer. So $m = p^s(p^s-1)/2$ and $n = p^s(p^s+1)/2$.

Finally, by Proposition 2.1 K has exactly two kinds of outer points. □

Remark 3.2. *Let $q = p^{2h}$, p odd, and put $s = p^2$. A general construction of $\frac{q(q-1)}{2}$-sets of type $(0, \frac{s(s-1)}{2}, \frac{s(s+1)}{2})$ is an open problem. There are examples for such sets in $PG(2,9)$, see [11], and in $PG(2,81)$, see [12].*

4 On sets with at least two external lines

Lemma 4.1. *Let $q = p^h$ be an odd prime power and b an integer such that $2 \leq b \leq \frac{q+3}{2}$. Then $\Delta = [(b-1)(q+1)]^2 - q(q+2) \geq 1$ is a square if and only if $b = 2$ or $b = \frac{q+3}{2}$.*

Proof. Put $\delta = (b-1)(q+1)$. Then $(\delta - q)^2 \leq \Delta \leq (\delta-1)^2$ if and only if $2 \leq b \leq \frac{q+3}{2}$. Indeed

- $(\delta-q)^2 \leq \Delta = \delta^2 - q(q+2) \Leftrightarrow q^2 - 2q\delta \leq -q^2 - 2q \Leftrightarrow 2q(q+1) \leq 2q\delta$ $\Leftrightarrow 1 \leq b-1 \Leftrightarrow 2 \leq b$.

- $\delta^2 - q(q+2) = \Delta \leq (\delta-1)^2 \Leftrightarrow -q^2 - 2q \leq -2\delta + 1 \Leftrightarrow 2\delta \leq (q+1)^2 \Leftrightarrow$ $2(b-1) \leq q+1 \Leftrightarrow b \leq (q+3)/2$.

So Δ is a square if and only if there is an integer x such that $\Delta = (\delta - x)^2$ with $1 \leq x \leq q$. From $\delta^2 - q(q+2) = \Delta = (\delta-x)^2$ we get

$$2x(b-1)(q+1) = x^2 + q(q+2) \tag{18}$$

By (18) we have

- if $b = 2$, then $(x-q)[x-(q+2)] = 0$ and so $x = q$, since $x \leq q$; viceversa if $x = q$, then $b = 2$; furthermore, in such a case $\Delta = 1$ is a square;

- if $b = (q+3)/2$, then $(x-1)[x-q(q+2)] = 0$ and so $x = 1$, since $x \leq q$; viceversa if $x = 1$, then $b = (q+3)/2$; furthermore, in such a case $\Delta = [(q^2+2q-1)/2]^2$ is a square.

Now we prove that if $3 \leq b \leq (q+1)/2$, then Δ is not a square or, equivalently, we prove that there is no integer x such that $2 \leq x \leq q-1$ and Equation

(18) holds. Assume by contradiction that there is an integer x such that $2 \leq x \leq q-1$ and Equation (18) holds.

Equation (18) can be rewritten in the two following equivalent ways

$$x(b-1) = \frac{(q+1)}{2} + \frac{(x-1)(x+1)}{2(q+1)} \tag{19}$$

$$\{2[(b-1)x-1]-q\}q = x[x-2(b-1)] \tag{20}$$

Being $2 \leq x \leq q-1$, by (19) we have $\frac{q+1}{2} + \epsilon \leq x(b-1) \leq q-1+\epsilon$ with $\epsilon = \frac{3}{2(q+1)}$ and, hence, $\frac{q+3}{2} \leq x(b-1) \leq q-1$ since $x(b-1)$ is an integer. So $2[x(b-1)-1]-q > 0$. Now by (20) we have that $x-2(b-1) > 0$, since $q > 0$ and $x > 0$. Again by (20) we have that $q = p^h$ divides $x[x-2(b-1)]$. Finally, p divides both x and $x-2(b-1)$, since $0 < x < q$ and $0 < x-2(b-1) < x < q$. Hence, p divides $b-1$ too, since p is odd. So there are four positive integers a, c, r and s such that p does not divide neither a nor c, $0 < r < h$, $0 < s < h$, $x = ap^r$ and $b-1 = cp^s$. Being $(b-1)x < q = p^h$ we get $r+s < h$. Substituting $x = ap^r$ and $b-1 = cp^s$ in (18) we obtain

$$2ac(q+1)p^{r+s} = a^2p^{2r} + (q+2)p^h \tag{21}$$

If $2r \geq h$, then $r+s \geq h$, a contradiction. So $2r < h$. If $r > s$, then p divides $2ac(q+1)$, a contradiction. So $r \leq s$. If $r < s$, then p divides a^2, a contradiction. So $s = r$. Being $(a-2c)p^r = x-2(b-1) > 0$ we get $a-2c > 0$. Now, if we put $t = h-2r > 0$ and we substitute $x = ap^r$ and $b-1 = cp^r$ in (20), then we obtain

$$[2(acp^{2r}-1)-p^h]p^t = a(a-2c) \tag{22}$$

So p^t divides $a-2c > 0$, since p does not divide a. Thus $p^t \leq a-2c < a$ and, hence, $cp^t < ca$. Being $acp^{2r} = x(b-1) < q = p^h$, we have that $ac < p^{h-2r} = p^t$. Finally, we get $cp^t < ca < p^t$ and, thus, $c < 1$, a contradiction. □

Theorem 4.2. *Let K be a $\frac{q(q-1)}{2}$-set of class $[0,m,n]$ in $PG(2,q)$, q odd. If $1 < t_0 < q$ and K has exactly two kinds of outer points, then K is the pointset of $\frac{q-1}{2}$ concurrent lines, except the point of intersection.*

Proof. If $1 < t_0 < q$ and K has exactly two kinds of outer points, then by Proposition 2.1 $v_0(Q) \in \{1,b\}$ with $2 \leq b \leq (q+3)/2$ for any point $Q \notin K$. Now, let us denote by z_i the number of points $Q \notin K$ such that $v_0(Q) = i$

with $i \in \{1, b\}$. By counting in double way the number of outer points, the number of pairs (Q, l), where $Q \notin K$ and l is an external line through Q, and the number of pairs $((Q, Q'), l)$ where Q and Q' are two distinct points not in K and l is an external line through Q and Q', we get the following equations

$$z_1 + z_b = q^2 + q + 1 - q(q-1)/2 = 1 + q(q+3)/2 \tag{23}$$

$$z_1 + bz_b = (q+1)t_0 \tag{24}$$

$$b(b-1)z_b = (t_0 - 1)t_0 \tag{25}$$

From the previous equations we obtain

$$t_0^2 - [b(q+1)+1]t_0 + b(q+1)(q+2)/2 = 0 \tag{26}$$

whose discriminant is $\Delta = (b-1)^2(q+1)^2 - q(q+2)$. By Lemma 4.1 Δ is a square if and only if $b = 2$ or $b = \frac{q+3}{2}$. If $b = 2$, then Equation (26) becomes $(q+1-t_0)(q+2-t_0) = 0$; so $t_0 > q$, a contradiction. If $b = (q+3)/2$, then Equation (26) becomes $(q+3-2t_0)(q^2+3q+2-2t_0) = 0$; so $t_0 = (q+3)/2 = b$ since $t_0 < q$. By (25) it is $z_b = 1$, i.e. there is exactly one point $Q \notin K$ such that all the $t_0 = b$ external lines pass through Q; so the $1 + q(q+3)/2$ outer points of K are exactly the $1+q(q+3)/2$ ones on those $(q+3)/2$ external lines; finally K is the point-set of the other $\frac{q-1}{2}$ lines passing through Q, except the point Q. □

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