Classifying sets of class $[1, q+1, 2q+1, q^2+q+1]_2$ in $PG(r, q)$, $r \geq 3$.

Dedicated to Prof. Osvaldo Ferri on the occasion of his 85th birthday

Remembering Prof. Giuseppe Tallini on the 25th anniversary of his death

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Abstract

In this paper we remove the solid incidence assumption in the characterization of J. Schillewaert (A characterization of quadrics by intersection numbers, Des. Codes Cryptogr., 47 (2008), 165-175) by proving that quadric plane incidence numbers implies quadric solid incidence numbers, except for the dual complete 11-cap of $PG(4, 3)$. Furthermore, new characterizations of the parabolic quadric $Q(4, q)$ and the ovoidal cone of $PG(4, q)$ are provided.

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1 Introduction and motivation

In $PG(r,q)$, the projective space of dimension r and order q, with $q = p^h$ a prime power, let K denote a k-set, i.e. a set of k points. A k-cap is a set of k points no three of which are collinear. A k-cap is complete if it is not contained in a $(k+1)$-cap. One can also consider (complete) k-caps in the dual projective spaces. Structures obtained in this way are called dual (complete) k-caps. Every k-cap of $PG(3,q)$, $q > 2$, satisfies $k \leq q^2+1$, and a (q^2+1)-cap of $PG(3,q)$ is called an ovoid. For each integer i such that $0 \leq i \leq \theta_d := \sum_{j=0}^{d} q^j$, let us denote by $t_i^d = t_i^d(K)$ the number of d-subspaces of $PG(r,q)$ meeting K in exactly i points. The numbers t_i^d are called the *characters* of K with respect to the d-subspaces, see [9]. Let m_1, m_2,..., m_s be s integers such that $0 \leq m_1 \leq m_2 \leq \cdots \leq m_s \leq \theta_d$. A set K is said to be of class $[m_1, m_2, \ldots, m_s]_d$ if $t_i^d \neq 0$ only if $i \in \{m_1, m_2, \ldots, m_s\}$. Moreover, K is said to be of type $(m_1, m_2, \ldots, m_s)_d$ if $t_i^d \neq 0$ if and only if $i \in \{m_1, m_2, \ldots, m_s\}$. The integers m_1,m_2,..., m_s are called *intersection numbers* with respect to the d-subspaces.

We call *lines*, *planes* and *solids* respectively the 1-dimensional, 2-dimensional and 3-dimensional subspaces. By an i-line we mean a 1-subspace meeting K in exactly i points; i-planes and i-solids are defined in a similar way.

It is surprising how little is known about point-sets of $PG(r,q)$ subjected to even simple intersection conditions with subspaces, see for instance [6] and [11]. A central investigation in finite geometry is to distinguish algebraic varieties by intersection numbers, see [10] and the recent article [1] of S. Barwick et al. In [8] J. Schillewaert gave the following classification

Result 1.1. *[Theorem 3.10 pag. 174] If a set of points K in $PG(r,q)$, $r \geq 4$, intersects planes and solids in the same number of points as quadrics, then K is one of the following:*

1. *the projective space $PG(r,q)$;*
2. *a hyperplane in $PG(r,q)$;*
3. *a quadric in $PG(r,q)$;*
4. *For q even*
   - *a cone with vertex an $(r-3)$-dimensional space and base an oval;*
   - *a cone with vertex an $(r-4)$-dimensional space and base an ovoid.*

In Section 2, we improve Result 1.1 by classifying sets which intersect planes as quadrics (here by quadrics we mean both singular or non singular ones). Indeed, we prove the following

Theorem 1.2. *Apart from one exceptional case, the dual complete 11-cap of* $PG(4,3)$*, if a set of points* K *in* $PG(r,q)$*,* $r \geq 4$*, intersects planes in the same number of points as quadrics, then* K *intersects solids in the same number of points as quadrics.*

So the solid incidence assumption in Result 1.1 can be removed and hence it could sound like this *"Apart from one exceptional case, the dual complete 11-cap of* $PG(4,3)$*, if a set of points* K *in* $PG(r,q)$*,* $r \geq 4$*, intersects planes in the same number of points as quadrics, then* K *is one of the following: ..."*.

In [2] Osvaldo Ferri and Giuseppe Tallini proved the following

Result 1.3. *In* $PG(r,q)$*,* $r \geq 4$*, a set* K *of points,* $|K| \geq q^3+q^2+q+1$*, of class* $[1,a,b]_2$*, where* $b \geq 2q+1$*, and of class* $[c,c+q,c+2q]_3$*, where* $c \leq q^2+1$*, such that* c*-solids and* $(c+q)$*-solids exist, is a parabolic quadric* $Q(4,q)$*.*

In [8] (Theorem 2.4 pag. 170) J. Schillewaert proved the following

Result 1.4. *In* $PG(4,q)$*, any set of points of class* $[1,q+1,2q+1]_2$ *and class* $[q^2+1,q^2+q+1,q^2+2q+1]_3$ *is a parabolic quadric* $Q(4,q)$*.*

We recall that in $PG(4,q)$, an *ovoidal cone* is a cone whose vertex is a point V and whose base is an ovoid in a solid not containing V, see [3] and [4] for backgrounds. In Section 3, we classify sets of class $[1,q+1,2q+1]_2$ in $PG(4,q)$ by proving the following

Theorem 1.5. *Let* K *be a set of class* $[1,q+1,2q+1]_2$ *in* $PG(4,q)$*. Then one of the following happens*

1. K *is a parabolic quadric* $Q(4,q)$*;*
2. K *is an ovoidal cone;*
3. K *is the dual complete 11-cap of* $PG(4,3)$*.*

Finally, as by-product of Theorem 1.5, we also obtain new, up to our best knowledge, characterizations for the parabolic quadric $Q(4,q)$ and the ovoidal cone of $PG(4,q)$.

2 Sets of class $[1, q+1, 2q+1, q^2+q+1]_2$ in $PG(3, q)$.

In this section we prove the following main theorem.

Theorem 2.1. *Let K be a set of class $[1, q+1, 2q+1, q^2+q+1]_2$ in $PG(r, q)$, $r \geq 3$. Then K is a set of class*
$[q+1, q^2+1, q^2+q+1, q^2+2q+1, 2q^2+q+1, q^3+q^2+q+1]_3$
or K is the dual complete 11-cap of $PG(4, 3)$.

Then Theorem 1.2 is an immediate consequence of Theorem 2.1.

We start by recalling three well-known equations and proving a Lemma which could be useful in general. If K is a k-set of $PG(r, q)$ of class $[m_1, m_2, ..., m_s]_d$, then (for instance see [10]) the following *characters equations* hold

$$\sum_{i=1}^{s} t_{m_i}^{d} = \prod_{u=0}^{d} \frac{\theta_{r-u}}{\theta_{d-u}} \tag{1}$$

$$\sum_{i=1}^{s} m_i t_{m_i}^{d} = k \prod_{u=0}^{d-1} \frac{\theta_{r-1-u}}{\theta_{d-1-u}} \tag{2}$$

$$\sum_{i=1}^{s} m_i(m_i-1) t_{m_i}^{d} = k(k-1) \prod_{u=0}^{d-2} \frac{\theta_{r-2-u}}{\theta_{d-2-u}} \tag{3}$$

Lemma 2.2. *In $PG(r, q)$ with $r \geq 2$, let K be a k-set of class $[m_1, m_2, \ldots, m_s]_d$ and of class $[n_1, n_2, \ldots, n_u]_{d+1}$ with $1 \leq d < d+1 \leq r$. If for any $m_i \in \{m_1, m_2, \ldots, m_s\}$ we have $m_i \equiv x$ mod q, then for any $n_j \in \{n_1, n_2, \ldots, n_u\}$ we have $n_j \equiv x$ mod q. Thus $k \equiv x$ mod q too, since K is of type $(k)_r$.*

Proof. If σ is a $(d+1)$-subspace of $PG(r, q)$, then there is $n_j \in \{n_1, n_2, \ldots, n_u\}$ such that $|K \cap \sigma| = n_j$. The n_j-set $H := K \cap \sigma$ is of class $[m_1, m_2, \ldots, m_s]_d$ in $PG(d+1, q)$. Now for any $m_i \in \{m_1, m_2, \ldots, m_s\}$ let us denote by $t_{m_i} = t_{m_i}^{d}(H)$ the number of d-subspaces of $PG(d+1, q)$ meeting H in exactly m_i points. By equations (1) and (2), where $r = d+1$, we obtain $\sum_{i=1}^{s} t_{m_i} = \theta_{d+1}$ and $\sum_{i=1}^{s} m_i t_{m_i} = n_j \theta_d$. By the hypothesis, for any $m_i \in \{m_1, m_2, \ldots, m_s\}$ there is an integer a_i such that $m_i = qa_i + x$. Putting $m_i = qa_i + x$ into the first equation we obtain $q \sum_{i=1}^{s} a_i t_{m_i} + x \sum_{i=1}^{s} t_{m_i} = n_j \theta_d$. In view of the second equation we have $q \sum_{i=1}^{s} a_i t_{m_i} = n_j \theta_d - x\theta_{d+1} = (n_j - x)\theta_d - xq^{d+1}$. So $n_j \equiv x$ *mod* q. □

In [5], the authors proved the following classification

Result 2.3. *Let K be a set of class $[1, q+1, 2q+1]_2$ in $PG(3, q)$. Then K is*

- *a line;*
- *an ovoid;*
- *a (q^2+q+1)-set of type $(1, q+1, 2q+1)_2$;*
- *a (q^2+2q+1)-set of type $(q+1, 2q+1)_2$;*
- *the complementary set of the symmetric difference of a hyperbolic quadric $Q(3, 3)^+$ and one of its tangent planes.*

Now we extend Result 2.3 by the following classification

Lemma 2.4. *Let K be a k-set of class $[1, q+1, 2q+1, q^2+q+1]_2$ in $PG(3, q)$. Then K is one of*

1. *a line;*
2. *an ovoid;*
3. *a (q^2+q+1)-set of type $(1, q+1, 2q+1)_2$;*
4. *a (q^2+2q+1)-set of type $(q+1, 2q+1)_2$;*
5. *a plane;*
6. *the pointset of two planes;*
7. *the whole space;*
8. *the complementary set of the symmetric difference of a hyperbolic quadric $Q(3, 3)^+$ and one of its tangent planes in $PG(3, 3)$.*

Proof. Let K be a k-set of class $[1, q+1, 2q+1, q^2+q+1]_2$ in $PG(3, q)$. If K contains all the points of a plane α, then any other plane meets K in at least $q+1$ points and, hence, there is no 1-plane. Viceversa, if there is a 1-plane β, then any other plane meets K in at most q^2+1 points and, hence, K contains no plane. So K is a set of class either $[1, q+1, 2q+1]_2$ or $[q+1, 2q+1, q^2+q+1]_2$. In the first case, by Result 2.3, we have cases (1), (2), (3), (4) and (8).

Now, let K be of class $[q+1, 2q+1, q^2+q+1]_2$. By Lemma 2.2, we have $k \equiv 1 \ (mod\ q)$. Putting $r = 3$, $d = 2$, $k = aq+1$, $s = 3$, $m_1 = q+1$, $m_2 = 2q+1$ and $m_3 = q^2+q+1$ into equations (1), (2) and (3) we get

$$qt_{q+1} = (a-\theta_2)[(q+1)(a-2q-1)-1]; \tag{4}$$

$$(q-1)t_{2q+1} = -(q+1)(a-\theta_1)(a-\theta_2) \tag{5}$$

$$q(q-1)t_{q^2+q+1} = (a-q-2)[(q+1)(a-2q+1)-2] \tag{6}$$

Being $t_{q+1} \geq 0$, $t_{2q+1} \geq 0$, $t_{q^2+q+1} \geq 0$, it is easy to prove that

$$a \in \{q+1, q+2, 2q+1, q^2+q+1\} \tag{7}$$

Furthermore, by equations (4), (5) and (6), we have that

- if $a = q+1$, then K is a (q^2+q+1)-set of type $(q+1, q^2+q+1)_2$; so K is a plane;
- if $a = q+2$, then K is a (q^2+2q+1)-set of type $(q+1, 2q+1)_2$;
- if $a = 2q+1$, then K is a $(2q^2+q+1)$-set of type $(q+1, 2q+1, q^2+q+1)_2$ with $t_{q^2+q+1} = 2$; so K is the pointset of two planes;
- if $a = q^2+q+1$, then K is a (q^3+q^2+q+1)-set of type $(q^2+q+1)_2$; so K is the whole space.

□

Lemma 2.5. *Let K be a k-set of class $[1, 4, 7, 13]_2$ in $PG(4, 3)$ such that there is at least one 19-solid R. Then K is the dual complete 11-cap.*

Proof. By Result 2.3, $R \cap K$ is the complementary set of the symmetric difference of a hyperbolic quadric $Q(3,3)^+$ and one of its tangent planes. Therefore, $R \cap K$ contains exactly one 1-plane and it meets any 4-plane in a conic, see [5]. By Lemma 2.4, K is of class $[4, 10, 13, 16, 19, 22, 40]_3$. It is easy to see that there is no 22-solid and no 40-solid, since there is a 19-solid. So K is of class $[4, 10, 13, 16, 19]_3$. By Result 2.3, $K \cap S$ is a line for any 4-solid S, $K \cap U$ is an ovoid for any 10-solid U, $K \cap T$ is of type $(4, 7)_2$ for any 16-solid T. Furthermore, if S is a 13-solid, then $S \cap K$ cannot be a plane. Hence, $S \cap K$ is of type $(1, 4, 7)_2$ and K is a set of class $[1, 4, 7]_2$.

Now let us suppose that there is a 4-solid S and a 16-solid T. Being S of type $(1,4)_2$ and T of type $(4,7)_2$, the plane $\alpha := S \cap T$ is a 4-plane. Moreover, the set $K \cap S$ is a line r. So α is a plane passing throug r. Into solid T, other three planes pass through r. Anyone of such three planes has at most three points outside r, then $K \cap T$ has at most 13 points, a contradiction. Thus, K is of class either $[4,10,13,19]_3$ or $[10,13,16,19]_3$.

Now, let us suppose that there is at least one 4-solid S. Let r be the line $K \cap S$ and α a plane of S passing through r. Only 4-solids and 13-solids can pass through α. One of these three solids has at most 9 points outside α. So $k \leq 31$. Now let β be a 7-plane of the solid R. Anyone of the other three solids passing through β has at least six points outside β. So $k \geq 37$, a contradiction. Thus, there is no 4-solid and, hence, K is of class $[10,13,16,19]_3$.

Now, let γ be a 1-plane of the 19-solid R. Anyone of the other three solids passing through β has at least nine points outside γ. So $k \geq 46$. Now, let us suppose that there is at least one 16-solid T and let α be a 7-plane of T. No 19-solid pass through α since α contains a line. Thus anyone of the other three solids passing through α has at most nine points outside α. So $k \leq 43$, a contradiction. Hence, there is no 16-solid and K is of class $[10,13,19]_3$.

Now, let us suppose that there is a 13-solid T and let γ a 7-plane of T. Only 13-solids and 19-solids can pass through γ. If another 13-plane pass through γ, then $k \leq 43$, a contradiction. So the other three planes passing through γ are 19-planes. Hence $k = 49$. By characters equations with respect to solids, we get $t_{10} = -11$, a contradiction. So there is no 13-solid and K is of class $[10,19]_3$.

Now, let α be a 7-plane of the 19-solid S. Only 19-solids can pass through α. Thus $k = 55$. By characters equations with respect to solids we have that there are 11 10-solids and 110 19-solids. So K is of type $(10,19)_3$.

By characters equations with respect to planes we have that there are 55 1-planes, 330 4-planes and 825 7-planes. So K is of type $(1,4,7)_2$.

Counting the points of K via the solids passing through a plane, we get

- any 1-plane is contained in two 10-solids and two 19-solids;
- any 4-plane is contained in one 10-solid and three 19-solids;
- any 7-plane is contained in four 19-solids.

Thus, since any 19-solid contains exactly one 1-plane, the 11-set of 10-solids is the complete 11-cap K^* of the dual space. □

Now, Theorem 2.1 is an immediate consequence of Lemmas 2.4 and 2.5.

Remark 2.6. *We recall (see [7] for instance) that the points of* K^* *and the corresponding solids form the unique Mathieu-Witt design* $4-(11,5,1)$ *with the sporadic Mathieu group* M_{11} *as point 4-transitive full group authomorphism. Moreover,* K^* *corresponds to the perfect ternary Golay* $[11,6,5]$*-code (the code corresponding to a cap is defined by its parity check matrix, whose columns are the points of the cap treated as 5-dimensional vectors).*

3 Sets of class $[1, q+1, 2q+1]_2$ in $PG(4, q)$.

In this section we will prove the following two main theorems.

Theorem 3.1. *Let* K *be a set of class* $[1, q+1, 2q+1]_2$ *in* $PG(4, q)$*. If any solid meets* K *in more than* $q+1$ *points, then* K *is a parabolic quadric* $Q(4, q)$ *or the dual complete 11-cap of PG(4,3).*

Theorem 3.2. *Let* K *be a set of class* $[1, q+1, 2q+1]_2$ *in* $PG(4, q)$*. If there is one solid meeting* K *in at most* $q+1$ *points, then* K *is an ovoidal cone.*

Then Theorem 1.5 is an immediate consequence of Theorems 3.1 and 3.2. Furthermore, we observe that, up to our knowledge, Theorem 3.1 (respectively Theorem 3.2) is a new characterization of $Q(4, q)$ (respectively of the ovoidal cone) in $PG(4, q)$.

Lemma 3.3. *Let* K *be a set of class* $[1, q+1, 2q+1]_2$ *in* $PG(4, q)$*. Then* K *is*

- *a set of class* $[q^2+1, q^2+q+1, q^2+2q+1]_3$ *or*
- *a set of class* $[q+1, q^2+1, q^2+q+1]_3$ *with* $t^3_{q+1} > 0$ *or*
- *the dual complete 11-cap of* $PG(4, 3)$*.*

Proof. If S is a solid of $PG(4, q)$, then $K \cap S$ is a set of class $[1, q+1, 2q+1]_2$ in $PG(3, q)$. By Result 2.3, K is of class $[q+1, q^2+1, q^2+q+1, q^2+2q+1]_3$ or, by Theorem 2.1, the dual complete 11-cap in $PG(4, 3)$.

If any solid meets K in more than $q+1$ points, i.e. there is no $(q+1)$-solid, then K is of class $[q^2+1, q^2+q+1, q^2+2q+1]_3$ or the dual complete 11-cap in $PG(4, 3)$.

Now let us suppose that there is at least one $(q+1)$-solid S, i.e. $t^3_{q+1} > 0$. By Result 2.3, the set $K \cap S$ is a line l. Each plane of S meets K either in 1 or $q+1$ points. We have to prove that there is no (q^2+2q+1)-solid. On the contrary let us suppose that there is at least one (q^2+2q+1)-solid T. By Result 2.3, each plane of T meets K either in $q+1$ or $2q+1$ points. So, if we denote by α the plane $S \cap T$, then α meets K in $q+1$ points and hence $\alpha \cap K = l$. In T each one of the other q planes passing through l contains at most other q points of K. So $K \cap T$ has at most q^2+q+1 points, a contradiction. Thus, K is of class $[q+1, q^2+1, q^2+q+1]_3$ with $t^3_{q+1} > 0$. □

Now, Theorem 3.1 is an immediate consequence of Lemma 3.3 and Result 1.4.

Proposition 3.4. *In $PG(4,q)$, let K be a k-set of class $[q+1, q^2+1, q^2+q+1]_3$. Let α be a plane meeting K in exactly h points. If we denote by x, respectively by y, the number of $(q+1)$-solids, respectively (q^2+1)-solids, passing through α, then $h = (2-x)q - y + q^2 + 2 - (k-1)/q$.*

Proof. Counting the points of K via the solids passing through α we have

$$k = h + x(q+1-h) + y(q^2+1-h) + (q+1-x-y)(q^2+q+1-h).$$

from which we get the statement. □

In Proposition 3.5 up to and including 3.8, K will ever be a set of class $[1, q+1, 2q+1]_2$ and of class $[q+1, q^2+1, q^2+q+1]_3$ with $t^3_{q+1} > 0$. By Result 2.3, $K \cap S$ is a line for any $(q+1)$-solid S. We denote by $\mathcal{L}$ the non empty set of the lines $K \cap S$ where S is a $(q+1)$-solid. Of course it is $|\mathcal{L}| = t^3_{q+1} > 0$.

Proposition 3.5. *In $PG(4,q)$ let K be a set of class $[1, q+1, 2q+1]_2$ and of class $[q+1, q^2+1, q^2+q+1]_3$ with $t^3_{q+1} > 0$. If S and T are two distinct $(q+1)$-solids and α is the h-plane $S \cap T$, then no other $(q+1)$-solid passes throuhg α. Furthermore, $h = 1$. So if l_1 and l_2 are two distinct lines of $\mathcal{L}$, then $|l_1 \cap l_2| = 1$.*

Proof. Let α be the h-plane $S \cap T$ and let us denote by $b \geq 0$ the number of the other $(q+1)$-solids passing through α. By Proposition 3.4 with $x = 2+b$ we get $h + bq = 2 - y - [k - (q^3+1)]/q$. Being $h \in \{1, q+1\}$, $y \geq 0$ and $k \geq q^3+1$ we necessarily have that $b = 0$ and $h = 1$. □

Proposition 3.6. *In $PG(4,q)$, if K is a set of class $[1, q+1, 2q+1]_2$ and of class $[q+1, q^2+1, q^2+q+1]_3$ with $t^3_{q+1} > 0$, then K has q^3+q+1 points and it is a cone.*

Proof. By Result 2.3, $K \cap S$ is a line for any $(q+1)$-solid S and $K \cap T$ is an ovoid for any (q^2+1)-solid T. Being $t^3_{q+1} > 0$, there are $(q+1)$-solids. Let S be a $(q+1)$-solid, l be the line $K \cap S$ and α be a plane of S passing through l. Of course, no (q^2+1)-solid passes through α. If no (q^2+q+1)-solid passes through α too, then K is the line l, a contradiction since in $PG(4,q)$ a line is a set of type $(0,1,q+1)_2$. So there is at least one (q^2+q+1)-solid U. By Result 2.3, the set $K \cap U$ is of type $(1, q+q, 2q+1)_2$. Thus there is a $(2q+1)$-plane β. Of course, both $(q+1)$-solids and (q^2+1)-solids do not pass through β. Hence, counting the points of K via the solids passing through β we have that $k = 2q+1+(q+1)[(q^2+q+1)-(2q+1)] = q^3+q+1$.

Substituting $r = 4$, $d = 3$, $k = q^3+q+1$, $s = 3$, $m_1 = q+1$, $m_2 = q^2+1$ and $m_3 = q^2+q+1$ into Equations (1), (2) and (3) we get $t^3_{q+1} = q^2+1 \geq 5$. So $\mathcal{L}$ contains at least five lines. Let l_1 and l_2 be two of them. By Proposition 3.5 they meet in a point. Thus there is exactly one plane α containing these two lines. By Proposition 3.4 with $k = q^3+q+1$, we have that $h = (2-x)q+1-y$ and hence α meets K in at most $2q+1$ points. Thus the plane α contains no other line of $\mathcal{L}$. By Proposition 3.5, the point-set C of $\mathcal{L}$ is a cone contained in K. Finally, K is the cone C since $|K| = q^3+q+1 = qt^3_{q+1}+1 = q|\mathcal{L}|+1$. □

Proposition 3.7. *In $PG(4,q)$ let K be a set of class $[1,q+1,2q+1]_2$ and of class $[q+1,q^2+1,q^2+q+1]_3$ with $t^3_{q+1} > 0$. If S is a (q^2+1)-solid, then $K \cap S$ is an ovoid of S. Furthermore, S does not contain the vertex of the cone and S meets each line of $\mathcal{L}$ in exactly one point.*

Proof. The set $K \cap S$ is a (q^2+1)-set of type $(1,q+1,2q+1)_2$ in $PG(3,q)$. So, by Result 2.3, it is an ovoid. If S contains the vertex V of the cone, then $K \cap S$ is the pointset of q lines passing through V, i.e. $K \cap S$ is a cone of S. So in S there is a h-plane α with $h \geq 2q+1$ points. Being $k = q^3+q+1$, by Proposition 3.4, we get $xq+y \leq 0$, a contradiction since $y \geq 1$. If S meets a line l of $\mathcal{L}$ in s points, then $s \leq 1$. Otherwise S contains l and hence the vertex V too, a contradiction. So $s = 1$ since $|\mathcal{L}| = q^2+1$. □

By Propositions 3.6 and 3.7 we immediately have the following

Proposition 3.8. *In $PG(4,q)$ if K is a set of class $[1,q+1,2q+1]_2$ and of class $[q+1,q^2+1,q^2+q+1]_3$ with $t^3_{q+1} > 0$, then K is an ovoidal cone.*

Theorem 3.2 is an immediate consequence of Lemma 3.3 and Proposition 3.8.

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