# Revenue Maximization Envy-free Pricing for Homogeneous Resources

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## Abstract

Pricing-based mechanisms have been widely studied and developed for resource allocation in multiagent systems. One of the main goals in such studies is to avoid envy between the agents, i.e., guarantee fair allocation. However, even the simplest combinatorial cases of this problem is not well understood. Here, we try to fill these gaps and design polynomial revenue maximizing pricing mechanisms to allocate homogeneous resources among buyers in envy-free manner. In particular, we consider *envy-free* outcomes in which all buyers' utilities are maximized. We also consider *pair envy-free* outcomes in which all buyers prefer their allocations to the allocations obtained by other agents. For both notions of envy-freeness, we consider item and bundle pricing schemes. Our results clearly demonstrate the limitations and advantages in terms of revenue between these two different notions of envy-freeness.

## 1 Introduction

Pricing-based mechanisms become a key approach to allocate resources in multi-agent systems. For example, in sponsored search ad-slots are sold in online auctions [Nisan *et al.*, 2009]. Main objectives in designing such mechanisms could include social welfare maximization, revenue maximization, strategy-proofness and/or fairness. In this paper, we consider the problem of assigning $m$ homogeneous items or resources among $n$ buyers without budgets, where agents valuations depend only on the number of items they get. This multi-unit settings allows us to highlight and concentrate on the simplest case where allocating different number of items to different agents plays a crucial role, i.e., it is arguably the most basic variant of combinatorial auctions. Somewhat astonishingly, this very simple settings generates a rich set of problems to study. Still, this limited setting seems to be of practical importance as these items could be homogeneous server grids or supercomputers in computer networks, power supply in manufacture systems, cargo space in transportation industry and etc. Our goal is to design polynomial pricing mechanisms that maximize the seller's revenue and assign items to buyers in a fair manner. One of the common notions used to model fair division is envy-freeness, where each agent believes that he is treated the best. Envy-freeness dates back to twenty century [Foley, 1967; Varian, 1974] and is still an intense research topic in mathematics, computer science and economics. Different notions of envy-freeness have been proposed and studied in the literature. For example, envy-freeness was defined to denote the outcomes where all buyers receive the allocations they prefer the most at given prices [Bilò *et al.*, 2014; Briest, 2008; Chen and Deng, 2010; Guruswami *et al.*, 2005; Hartline and Koltun, 2005]. As the prices for unsold items are not necessarily zero, this is a relatively weaker notion than Walrasian equilibrium (WE) [Walras, 1954]. On the other hand, following suggestions from classical papers [Foley, 1967; Varian, 1974], envy-freeness has also been defined in outcomes where buyers prefer their allocations to the allocations received by other agents [Colini-Baldeschi *et al.*, 2014; Feldman *et al.*, 2012; Fiat and Wingarten, 2009]. Throughout this paper, we will call the latter notion *pair envy-freeness*. It is easy to see that WE $\subset$ envy-freeness $\subset$ pair envy-freeness.

With respect to pricing mechanisms, Guruswami *et al.* [2005] showed that the Vickrey-Clarke-Groves (VCG) mechanism could return envy-free outcomes in polynomial time when buyers can buy only single items, i.e., unit-demand valuations. However, VCG not only fails to deliver envy-free outcomes in our setting, but also could result in poor revenues. Hence, we need to propose different mechanisms for the case studied here. First of all, we observe that for both our notions of envy-freeness we can restrict our attention to *bundle pricing schemes*, where the seller sets up a price for each bundle size and the payment of buyer $i$ is the price of the bundle assigned. If a price for a given bundle size is infinite it can be seen as not selling items at this quantity at all, e.g., in many cases we are offered to buy a single item, or a fixed multi-pack of items at a discounted price. In other words, envy-freeness implies that we have to avoid price discrimination [Carlton and Perloff, 1990], i.e., sell the same bundles to different agents at different prices. In some cases we might want to be even less discriminatory and apply *item pricing* [Feldman *et al.*, 2012; Colini-Baldeschi *et al.*, 2014], where the seller sets up an uniform price $p \in R$ for each item and the payment of buyer $i$ is proportional to the number of items he receives.

Bundle pricing is more general and powerful and so al-

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<tr><th></th><th></th><th></th><th>single-minded valuations</th><th>non-decreasing valuations</th><th>general valuations</th></tr>
<tr><td rowspan="4">item pricing</td><td rowspan="2">Envy-free</td><td>lower bound</td><td>NP-hard</td><td rowspan="2" colspan="2">polytime</td></tr>
<tr><td>upper bound</td><td>FPTAS</td></tr>
<tr><td rowspan="2">Pair Envy-free</td><td>lower bound</td><td>NP-hard</td><td></td><td></td></tr>
<tr><td>upper bound</td><td colspan="3">$O(\log n)$</td></tr>
<tr><td rowspan="2">bundle pricing</td><td rowspan="2">(Pair) Envy-free</td><td>lower bound</td><td>NP-hard</td><td></td><td>$\Omega(\log^{\epsilon} n)$</td></tr>
<tr><td>upper bound</td><td>FPTAS</td><td>$O(\log n)$</td><td>$O(\log n \log m)$</td></tr>
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Table 1: The table summarizes our results in different cases.

lows to extract more revenue than item pricing. However, as we will present, envy-free or pair envy-free bundle pricing is more difficult to compute. Since revenue maximizing envy-free or pair envy-free pricing is trivial for unit-demand valuations in multi-unit settings, we consider the following common types of valuations: *single-minded valuations with free disposal* – each buyer wants a particular number of items and when he receives more items his valuation remains the same; *non-decreasing valuations* – the valuation depends on the number of items, but cannot decrease with increasing number of items; *general valuations* – there is no restriction on how the valuation depends on the number of items.

**Our results** We provide upper and lower bounds for item and bundle pricing for the two notions of envy-freeness. The results are summarized in Table 1. Although, the problems studied here are rather basic, some of them are quite challenging. Our paper should be seen as opening this research direction and naturally it leaves a few open problems for further research as exemplified by empty cells in Table 1.

**Related Work** Revenue maximizing envy-free pricing has drawn a lot of attention recently. The differences between our study and previous work are summarized as follows. First, most of the existing work studies the case of heterogeneous items (aka, multi-good settings) [Briest, 2008; Chen and Deng, 2010; Cheung and Swamy, 2008; Guruswami *et al.*, 2005; Hartline and Koltun, 2005], or the case that buyers have budgets [Colini-Baldeschi *et al.*, 2014; Feldman *et al.*, 2012]. There are also some studies of unlimited supply, e.g., [Balcan and Blum, 2007; Chalermsook *et al.*, 2012; Fiat and Wingarten, 2009; Briest and Krysta, 2006]. This is the first paper to consider the problem in multi-unit setting without budgets. Second, compared to existing works that usually have assumptions on the valuations of buyers such as single-minded or non-decreasing valuations, we consider general valuations as well. Third, our study is related to combinatorial walrasian equilibrium, a generalization of Walrasian equilibrium, as given in [Feldman *et al.*, 2013] where authors consider revenue maximization problems in which buyers have non-decreasing valuations and the seller could package the items into indivisible bundles, but in contrast to the classical notion of Walrasian equilibrium they allow unsold bundles to have nonzero prices and do not require market clearance. This notion is very similar to the envy-free bundle pricing in our paper. In comparison to [Feldman *et al.*, 2013] we give a simpler $O(\log n)$ approximation mechanism that is especially tailored to our case. The $O(\log n)$ approximation mechanism could also apply to their settings. In addition, recently Dobzinski *et al.* [2014] studied welfare and revenue guarantees on competitive bundling equilibrium that is a similar notion of combinatorial walrasian equilibrium but requires market clearance.

## 2 Preliminaries

A multi-unit pricing problem $\mathcal{A} = \langle n, m, \mathbf{V} \rangle$ consists of $n$ buyers, $m$ homogeneous items and a valuation profile $\mathbf{V} = \langle \mathbf{v_1}, \ldots, \mathbf{v_n} \rangle$, where $\mathbf{v_i} = \langle v_i(0), v_i(1), \ldots, v_i(m) \rangle$ and $v_i(j) \in \mathbb{R}^+$ is the value buyer $i$ has for $j$ items. We assume that for any $i \in \{1, \ldots, n\}$, $v_i(0) = 0$. We assume that $v_i(j)$ is also the maximum payment buyer $i$ is willing to pay for $j$ items.

In this paper, we consider *single-minded valuations with free disposal, non-decreasing valuations* and *general valuations*. We say buyers have single-minded valuations with free disposal if each buyer $i$ has a fixed value $w_i$ for receiving at least $k_i$ items, and has value 0 for receiving less than $k_i$ items. Formally, the valuations of any buyer $i$ for different numbers of items are represented as $v_i(k') = 0$ for all $0 \le k' < k_i$ and $v_i(k') = w_i$ for all $k_i \le k' \le m$. We say buyers have non-decreasing valuations if for any $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m-1\}$, it holds that $v_i(j) \le v_i(j+1)$. Finally, we say buyers have general valuations if there is no assumption on the valuations of buyers.

We consider both *item pricing* and *bundle pricing* schemes. In item pricing scheme, a fixed price $p \in R$ for each additional item is to be determined and the payment of buyer $i$ is proportional to the number of items he receives, that is, $p_i = x_i p$. W.l.o.g., we write outcomes in item pricing scheme as $\langle \mathbf{X}, p \rangle$, where buyer $i$ gets $x_i \in \mathbf{X}$ items and the price per item is $p$. An item pricing outcome $\langle \mathbf{X}, p \rangle$ is *feasible* if and only if all the following conditions hold:

1. supply constraint: it holds $x_i \in \{1, \ldots, m\}$ for every $i \in \{1, \ldots, n\}$ and $\sum_{i=1}^{n} x_i \le m$;
2. individual rationality: for every $i \in \{1, \ldots, n\}$, it holds $v_i(x_i) \ge p \cdot x_i$;
3. non-negative payments: $p \ge 0$.

Given an item pricing outcome $\langle \mathbf{X}, p \rangle$, the utility $u_i(x_i, p)$ of buyer $i$ is given by $v_i(x_i) - x_i \cdot p$. The revenue $r(\mathbf{X}, p)$ of the outcome is the total payment of buyers, i.e., $r(\mathbf{X}, p) = \sum_{i=1}^{n} x_i p$. Given an item pricing outcome, we say buyer $i$ *envies the allocation* if the utility of buyer $i$ is not maximized, that is, $x_i \notin \operatorname{argmax}_y v_i(y) - p \cdot y$. An item pricing outcome $\langle \mathbf{X}, p \rangle$ is *envy-free* if no buyer envies the allocation. Given an item pricing outcome, we also say buyer $i$ *envies buyer* $i'$ if buyer $i$ prefers buyer $i'$'s assignment, that is, $u_i(x_i, p) < u_i(x_{i'}, p)$. An item pricing outcome $\langle \mathbf{X}, p \rangle$ is *pair envy-free* if no buyer envies other buyers.

In bundle pricing scheme, prices for different sizes of bundles are to be determined and the payment of buyer $i$ is equal to the price of the bundle he gets. W.l.o.g., we write outcomes in bundle pricing scheme as $\langle \mathbf{X}, \mathbf{P} \rangle$, where buyer $i$ gets $x_i \in \mathbf{X}$ items and the price for the bundle of $j$ items is

$p_j \in \mathbf{P}$. From now on, we will use $p(j)$ instead of $p_j$. W.l.o.g, we assume $p(0) = 0$. A bundle pricing outcome $\langle \mathbf{X}, \mathbf{P} \rangle$ is *feasible* if and only if the following conditions hold:

1. supply constraint: it holds $x_i \in \{1, \dots, m\}$ for every $i \in \{1, \dots, n\}$ and $\sum_{i=1}^n x_i \le m$;
2. individual rationality: for every $i \in \{1, \dots, n\}$, it holds $v_i(x_i) \ge p(x_i)$;
3. non-negative payment: $\forall j \in \{0, 1, \dots, m\}, p(j) \ge 0$.

Given a bundle pricing outcome $\langle \mathbf{X}, \mathbf{P} \rangle$, the utility $u_i(x_i, p(x_i))$ of buyer $i$ is given by $v_i(x_i) - p(x_i)$. The revenue $r(\mathbf{X}, \mathbf{P})$ for the outcome is the total payment of buyers, i.e., $r(\mathbf{X}, \mathbf{P}) = \sum_{i=1}^n p(x_i)$. Similar to item pricing scheme, given a bundle pricing outcome $\langle \mathbf{X}, \mathbf{P} \rangle$, we say buyer $i$ *envies the allocation* if the utility of buyer $i$ is not maximized, that is, $x_i \notin \text{argmax}_y \, v_i(y) - p(y)$. A bundle pricing outcome $\langle \mathbf{X}, \mathbf{P} \rangle$ is *envy-free* if no buyer envies the allocation. Given a bundle pricing outcome, we also say buyer $i$ *envies buyer* $i'$ if buyer $i$ prefers buyer $i'$'s assignment, that is, $u_i(x_i, p(x_i)) < u_i(x_{i'}, p(x_{i'}))$. A bundle pricing outcome $\langle \mathbf{X}, \mathbf{P} \rangle$ is *pair envy-free* if no buyer envies other buyers.

In this paper we study the following problems of revenue maximizing item pricing and bundle pricing outcomes in multi-unit settings under these two notions of envy-freeness.

**Definition 1.** *The envy-free item pricing multi-unit (EFIP-MUL) problem: Given $\langle n, m, \mathbf{V} \rangle$, compute a feasible and envy-free item pricing outcome $\langle \mathbf{X}, p \rangle$ that maximizes the revenue $r(\mathbf{X}, p)$.*

**Definition 2.** *The pair envy-free item pricing multi-unit (PEFIP-MUL) problem: Given $\langle n, m, \mathbf{V} \rangle$, compute a feasible and pair envy-free item pricing outcome $\langle \mathbf{X}, p \rangle$ that maximizes the revenue $r(\mathbf{X}, p)$.*

**Definition 3.** *The envy-free bundle pricing multi-unit (EFBP-MUL) problem: Given $\langle n, m, \mathbf{V} \rangle$, compute a feasible and envy-free bundle pricing outcome $\langle \mathbf{X}, \mathbf{P} \rangle$ that maximizes the revenue $r(\mathbf{X}, \mathbf{P})$.*

**Definition 4.** *The pair envy-free bundle pricing multi-unit (PEFBP-MUL) problem: Given $\langle n, m, \mathbf{V} \rangle$, compute a feasible and pair envy-free bundle pricing outcome $\langle \mathbf{X}, \mathbf{P} \rangle$ that maximizes the revenue $r(\mathbf{X}, \mathbf{P})$.*

We call the outcomes that maximize the revenue the optimal outcomes. Optimal outcomes are denoted by $\mathsf{OPT} = \langle \mathbf{X}^{\mathsf{OPT}}, p^{\mathsf{OPT}} \rangle$ in item pricing schemes and $\mathsf{OPT} = \langle \mathbf{X}^{\mathsf{OPT}}, \mathbf{P}^{\mathsf{OPT}} \rangle$ in bundle pricing schemes, respectively.

## 2.1 A Key Lemma

In this section, we present a key lemma that will be heavily used when we study the revenue for pair envy-freeness. Let us first define a weaker notion of feasibility.

**Definition 5.** *An outcome is nearly-feasible if it satisfies only individual rationality and non-negative payment (i.e., it does not satisfy supply constraint).*

The following key lemma transforms a nearly-feasible and pair envy-free outcome selling at item pricing $y$ to a feasible and pair envy-free outcome selling at least $\frac{m}{2}$ items at item pricing $p \ge y$.

**Lemma 1.** *Consider the PEFIP-MUL problem, given a nearly-feasible and pair envy-free outcome $\langle \mathbf{O}, y \rangle$, there exists a feasible and pair envy-free outcome $\langle \mathbf{X}, p \rangle$ such that $2 \cdot r(\mathbf{X}, p) \ge m \cdot y$.*

*Proof.* Let $n_j$ be the number of buyers who receive $j$ items in $\mathbf{O}$. We give a constructive proof where two cases are considered as follows.

First, suppose there exists a $j \in \{1, \dots, m\}$ such that $j \cdot n_j \ge \frac{m}{2}$. In this case, the first step is to construct an outcome where only bundles of size $j$ are sold at price $y$ per item. Hence, by the property of pair envy-freeness, if any buyer receives $j$ items, all buyers with positive $u_i(j, y)$ would also demand $j$ items. In addition, buyers with $u_i(j, y) = 0$ are indifferent between receiving $j$ items and nothing, and buyers with negative $u_i(j, y)$ demand nothing. Now let $m'$ be the number of items demanded by buyers with positive $u_i(j, y)$. If $m' \le m$, we sell $j$ items to all buyers with positive $u_i(j, y)$ and as many buyers with $u_i(j, y) = 0$ as possible under the supply constraint. In this way, we get a feasible outcome that extracts a revenue of at least $\frac{m}{2}y$. If $m' > m$, we increase the price from $y$ to $p$ which is the minimum price that we have enough items to satisfy the demands of buyers with positive $u_i(j, p)$. Similarly at price $p$, we get a feasible outcome that extracts a revenue of at least $\frac{m}{2}p$ by selling $j$ items to all buyers with positive $u_i(j, p)$ and as many buyers with $u_i(j, p) = 0$ as possible.

Otherwise, we repeat the following process until there are enough items to satisfy the demand of all buyers or the first case is reached. Arbitrarily pick a bundle size $j$ that is sold, remove $j$ from the bundles sold in the outcome and then recompute the best bundles for all buyers given the fact that bundles of size $j$ are not available. Since we are not in the first case, it implies that at most half of the revenue is lost when we remove bundle $j$. Considering the first time that the total demand for the best bundles of all buyers is smaller than $m/2$. In such case in the previous round there is a bundle such that the total demand for it is more than $m/2$ items. We reach a contradiction since the process would end before. It concludes that by this process we will obtain a feasible outcome that extracts a revenue of at least $\frac{m}{2}y$. □

# 3 Envy-free Item Pricing

In this section, we consider the EFIP-MUL problem. The envy-freeness requires that, given an uniform price per item, each buyer gets the number of items that maximizes his utility. The main result is to solve the EFIP-MUL problem optimally via a dynamic programming for general valuations. The technique also applies to non-decreasing valuations since the input size is the same. Additionally, we also consider the EFIP-MUL problem when buyers are single-minded.

## 3.1 General Valuations

In this section we assume that the input size of the EFIP-MUL problem is $\Theta(nm)$. Note that it is also essential to know all $v_i(j)$ in order to check if an outcome is envy-free or not. First, we introduce some notations. Given a buyer $i$ and a price $p$, let $D_i(p)$ be the set of bundles that maximize the utility of buyer $i$, i.e., $D_i(p) = \text{argmax}_{j' \in \{0, 1, \dots, m\}} u_i(j', p)$. We call $D_i(p)$ the *demand set* of buyer $i$ given $p$. Let $s_i$ be the size of the demand set of $D_i(p)$. It is clear that there are at most $O(m)$ elements in $D_i(p)$, i.e., $s_i \leq m+1$. We denote $D_i(p) = \{D_i^1(p), \ldots, D_i^{s_i}(p)\}$. Let $\mathbb{P}_i$ include the set of prices such that $D_i(p) \neq D_i(p+\epsilon)$ where $\epsilon$ is a small quantity. An easy way to get $\mathbb{P}_i$ is to take any two valuations $v_i(j)$ and $v_i(j')$, where $j \neq j'$, compute the price $p$ such that $v_i(j) - p \cdot j = v_i(j') - p \cdot j'$ and include $p$ in $\mathbb{P}_i$ if $p \geq 0$. For example, when $j = 0$ we have all the prices $p$ such that there exists a bundle $j$ such that $v_i(j) - p \cdot j = 0$. Hence, the size of $\mathbb{P}_i$ is polynomial in $m$. The following lemma gives us the set of possible optimal prices:

**Lemma 2.** *Consider the EFIP-MUL problem, given an optimal outcome, the price $p^{\mathsf{OPT}}$ is in $\bigcup_{i\in\{1,\ldots,n\}} \mathbb{P}_i$.*

For each $p \in \bigcup_{i\in\{1,\ldots,n\}} \mathbb{P}_i$, the maximum revenue given price $p$ is computed through a dynamic programming. Let $z(i,d)$ be the maximum revenue that at most $d$ copies of the items are sold to the first $i$ buyers. Initially $z(0,d) = 0$ for all $d = 1, \ldots, m$. To compute $z(i,d)$ for $i = 1, \ldots, n, d = 1, \ldots, m$, we use the recursion as follows: [1]

$$z(i,d) = \max_{k\in[s_i]} (z(i-1, d - D_i^k(p)) + D_i^k(p) \cdot p)[d - D_i^k(p)]_+$$

where $[s_i] = \{1, \ldots, s_i\}$. The maximum revenue at price $p$ is given as $z(n,m)$. The memory required in the dynamic programing is $O(nm)$, and the number of operations is $O(nm^2)$. By Lemma 2, we can compute the maximum revenue for every candidate price and then output the optimal revenue.

**Theorem 1.** *The envy-free item pricing multi-unit (EFIP-MUL) problem is solvable in polynomial time.*

### 3.2 Single-minded Valuations with Free Disposal

Let us now discuss the EFIP-MUL problem when buyers have single-minded valuations with free disposal. It is necessary to have a different algorithm for this case due to the fact that the input size is much smaller than for general valuations. As single-minded buyers could be described by two parameters, $w_i$ and $k_i$, the input of this problem has size $\Theta(n \log m)$ only. Hence, we are interested in revenue maximizing item pricing schemes with the running time polynomial in $n$ and $\log m$, but not in $n$ and $m$ as it was the case in the previous section.

One can show that the EFIP-MUL problem cannot be solved efficiently unless $P = NP$ by the reduction from the Subset Sum problem. For the upper bound, we argue that there exists an FPTAS for the EFIP-MUL problem. The idea of the FPTAS is similar to the optimal envy-free item pricing for general valuations. First, there still exists a polynomial number $O(n)$ of possible optimal prices. We need the following arguments to prove this. Given any price $p > 0$, the demand set of buyer $i$ contains either 0 items, or $k_i$ items or both. By the similar arguments as Lemma 2, the set of possible optimal price is $\bigcup_{i=1}^{n} \frac{w_i}{k_i}$. Second, given any possible price, if there are not enough items to satisfy the buyers whose demand sets do not contain 0, then the price is not feasible. Otherwise, besides satisfying the demand of such buyers, one could simply use the FPTAS for knapsack problem to assign the remaining items. This process would give $(1-\epsilon)$-approximation to the optimal revenue at the given price. Together with the fact the number of choices for the optimal price is polynomial in $n$, this gives an FPTAS for the EFIP-MUL problem in the case of single-minded buyers with free disposal.

**Theorem 2.** *When buyers have single-minded valuations with free disposal, envy-free item pricing multi-unit (EFIP-MUL) problem is NP-hard but there exists an FPTAS.*

[1]For a number $a \in R$, $[a]_+$ returns 1 if $a \geq 0$, and 0 otherwise.

**Algorithm 1:** A logarithmic approximation algorithm for general valuations in PEFIP-MUL.

```
for p ∈ ℙ do
    For any j ∈ {1, ..., m}, let n_j be the number of
    buyers such that the bundle of j items is the largest
    one they have non-negative utilities for at price p;
    Sell bundles with sizes at least z where
    z = argmax_j j Σ_{k=j}^{m} n_k;
    Let n̄_j be the number of buyers who demand j items
    at price p if only bundles of at least z items are sold;
    if Σ_{j≥z}^{m} j · n̄_j ≤ m then
        Assign j ≥ z items to the buyers who demand j
        items and charge each of them p · j;
    else
        Apply Lemma 1 to obtain a feasible outcome;
Output the outcome with the maximum revenue.
```

## 4 Pair Envy-free Item Pricing

In this section, we start our study of pair envy-freeness in the item pricing scheme. An immediate thought is that the optimal revenue in pair envy-free outcomes could be higher than the optimal revenue in envy-free outcomes. One could actually show that the gap could unbounded.[2]

### 4.1 General Valuations

Our main result in PEFIP-MUL problem is a $O(\log n)$-approximation algorithm. Let $\mathbb{P}$ contain all prices $p$ such that there exists a buyer $i$, two different sizes of bundles $j$ and $j' \neq j$ such that $v_i(j) - p \cdot j = v_i(j') - p \cdot j'$. Similar to Lemma 2 we have the following.

**Lemma 3.** *Consider the PEFIP-MUL problem, given an optimal outcome, the price $p^{\mathsf{OPT}}$ is in $\mathbb{P}$.*

Algorithm 1 approximates the revenue for every possible price and outputs the best one. For each price, Algorithm 1 figures out the best way to sell all bundles larger than a fixed size. When total demand is greater than $m$, it uses the technique from Lemma 1.

**Theorem 3.** *Algorithm 1 always returns a feasible and pair envy-free item pricing outcome that $O(\log n)$-approximates the optimal revenue in the PEFIP-MUL problem.*

[2]Consider two buyers with $v_1(1) = 1+\epsilon$ and $v_2(1) = 1+\epsilon$, $v_i(j) = j$ for any $i \in \{1,2\}$ and $j \in \{2, \ldots, m\}$.

*Proof.* Given a price $p$, the maximum revenue of any (unconstrained) allocation is at most $p\sum_{j=1}^{m} n_j \cdot j$. Algorithm 1 picks bundles with at least $z$ items to sell, where $z$ maximizes $z\sum_{j=z}^{m} n_j$. In this case, buyers, whose largest bundles with positive utilities at price $p$ are at least $z$, will demand at least $z$ items. Since $\sum_{j=z}^{m} j\bar{n}_j \geq z\sum_{j=z}^{m} n_j$, $\sum_{j=z}^{m} j\bar{n}_j$ $\log n$-approximates $\sum_{j=1}^{m} n_j \cdot j$. Hence, if there are enough items to satisfy the demand of all buyers (lines 5-6 in Algorithm 1), it gives a $\log n$-approximation to the maximum revenue at price $p$. Otherwise, Algorithm 1 performs the operations in Lemma 1 and we lose at most another half of the revenue. In this case, Algorithm 1 gives a $2\log n$-approximation of the maximum revenue at price $p$. Finally, by Lemma 3, the best revenue from all possible prices in Algorithm 1 gives a $O(\log n)$-approximation to the optimal revenue. □

### 4.2 Other Valuations

It is trivial to see that Algorithm 1 also works when buyers have non-decreasing valuations. On the other hand, similarly as for item pricing when buyers have single-minded valuations it can be shown that the problem is NP-hard. Since there are $n$ candidates for $z$, with a little tweak, it can also give an $O(\log n)$ approximation in polynomial time in $n$ and $\log m$ when buyers have single-minded valuations.

## 5 Envy-free or Pair Envy-free Bundle Pricing

In Section 4, we have seen that pair envy-freeness could extract more revenue in the item pricing scheme. However, it is not true for the bundle pricing scheme. Indeed the two different notions of envy-freeness produce exactly the same optimal revenues. In fact there exist a transformation between optimal allocation in the two notions of envy-freeness in the bundle pricing scheme. On one hand, in an envy-free outcome the utilities of buyers are maximized, which directly implies that the outcome is also pair envy-free. On the other hand, in a pair envy-free outcome it is possible that some sizes of bundles are not assigned to any buyer, and the envy-free outcome producing the same revenue could be achieved by the following. To transform the solution we just need to price these unassigned bundles extremely high, so that no one buys them. Under this transformation, buyers purchase the same bundles of items and the payments remain unchanged. This observation is summarized by the following theorem.

**Theorem 4.** *The envy-free bundle pricing multi-unit (EFBP-MUL) problem is equivalent to the pair envy-free bundle pricing multi-unit (PEFBP-MUL) problem.*

By this result, in bundle pricing schemes, we could only consider envy-freeness or pair envy-freeness for different valuations. Before we proceed, we would like to start by showing some gaps between item pricing and bundle pricing. First, it is possible to show that a (pair) envy-free bundle pricing is able to extract nearly $m$ times the revenue obtained in EFIP-MUL. Second, we also observe that a (pair) envy-free bundle pricing could extract at least $\Omega(\log n)$ times of the optimal revenue of PEFIP-MUL, where $n \approx \sqrt{m}$. Let us consider the instance with $n$ buyers and $m = \sum_{j=1}^{n} j$ items where the valuation of buyer $i$, for any $i = 1, \ldots, n$, is $v_i(i) = \frac{1}{i}$ and $v_i(j) = 0$ for any $j \neq i$. In the PEFBP-MUL the optimal revenue is $H_n$ ($H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is the harmonic number and it is well known to be about as large as the natural logarithm of $n$) that we can get by selling the bundle of size $j$ at buyer $j$ at price $\frac{1}{j}$, for any $j = 1, \ldots, n$. Let us now prove that the optimal revenue for the PEFIP-MUL is a constant and this finishes the proof. From Lemma 3 we get that the optimal item price belongs to the set $\mathbb{P} = \{1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{i^2}, \ldots, \frac{1}{n^2}\}$. W.l.o.g., let us consider the price $p = \frac{1}{i^2}$ for some $i = 1, \ldots, n$. It is easy to see that at price $p$ only buyers $j : j \leq i$ have interest on buying (i.e., have non negative utility) a bundle of size $j$. Therefore the total revenue would be $\sum_{k=1}^{i} \frac{1}{i^2} \cdot k = \frac{1}{i^2} \cdot (1+2+\ldots, i) = \frac{O(i^2)}{i^2} = O(1)$.

We first provide a hardness result for general valuations in the EFBP-MUL and PEFBP-MUL problems, followed by a $O(\log n \log m)$ approximation algorithm. Then, we present a FPTAS when buyers have single-minded valuations, and a $O(\log n)$ algorithm when buyers have increasing valuations.

### 5.1 Hardness

**Theorem 5.** *The EFBP-MUL or PEFBP-MUL problem cannot be approximated within $\mathcal{O}(\log^{\epsilon} n)$ unless UDP-MIN can be approximated within $\mathcal{O}(\log^{\epsilon} |\mathcal{C}|)$, for some $\epsilon > 0$.*

The technique is to show a reduction from uniform-budget unit-demand min-buying pricing problem (UDP-MIN) to EFBP-MUL or PEFBP-MUL problem. Briest [Briest, 2008] showed that UDP-MIN is R3SAT$^*(poly(n))$-hard to be approximated within $\mathcal{O}(\log^{\epsilon} |\mathcal{C}|)$ for some $\epsilon > 0$.

### 5.2 General Valuations

In this section, we consider EFBP-MUL and PEFBP-MUL problems for general valuations. The main result is the proof that the optimal revenue in PEFBP-MUL is at most $O(\log m)$ times of the optimal revenue in PEFIP-MUL. Given the results in PEFIP-MUL, we could obtain an $O(\log n \log m)$ approximation algorithm for EFBP-MUL and PEFBP-MUL.

**Lemma 4.** *Given a feasible and pair envy-free bundle pricing outcome, there exists a feasible and pair envy-free item pricing outcome in which the revenue is at least $O(\log m)$-fraction of the revenue of the feasible and pair envy-free bundle pricing outcome.*

*Proof.* We give a constructive proof as follows. Suppose the feasible and pair envy-free bundle pricing outcome sells the bundles with sizes $S = \{s_1, \ldots, s_l\}$. For each $s_j \in S$, let $p_j$ be the price of the bundle and $\bar{n}_j$ be the number of buyers who are assigned to the bundles of size $s_j$. Hence, there are $s_1 \cdot \bar{n}_1$ items sold at price $p_1/s_1$, $s_2 \cdot \bar{n}_2$ items sold at price $p_2/s_2$, and so on. W.l.o.g., we assume that $p_1/s_1 \geq \ldots \geq p_l/s_l$.

Let $z$ be the index that maximizes $p_j/s_j \sum_{k \leq j} \bar{n}_k s_k$. Now consider the outcome that uses a price $p_z/s_z$ and sells the bundles $S' = \{s_1, \ldots, s_z\}$. There are two cases. The first case is that given price $p'$ and bundles $S'$, there are enough items to satisfy all the demand of buyers. Then we can bound the optimal revenue as follows.

$$r^{OPT} = \sum_{j\in[l]} \frac{p_j}{s_j} s_j \bar{n}_j \le \log(\sum_{j\in[l]} s_j \bar{n}_j) \frac{p_z}{s_z} \sum_{k\le z} \bar{n}_k s_k \le \log m \frac{p_z}{s_z} \sum_{k\le z} \bar{n}_k k_z \le r^{ALG}$$

The last inequality holds because all buyers who purchase bundles in $S' = \{s_1, \dots, s_z\}$ in the feasible and pair envy-free bundle pricing outcome would demand at least as much when the unit price is $p_z/s_z$.

The other case is that given price $p'$ and bundles $S'$, there are not enough items to satisfy all the demand of buyers. But notice that if we assign all buyers their demands it is a nearly-feasible outcome. By Lemma 1, we can obtain a feasible and pair envy-free item pricing with no more than half of the revenue. Therefore, there exists a feasible and pair envy-free item pricing outcome in which the revenue is at least $O(\log m)$-fraction of the revenue in any feasible and pair envy-free bundle pricing outcome. □

The analysis above is almost tight since as we mentioned the gap between EFBP-MUL (or PEFBP-MUL) and PEFIP-MUL is at least $\Omega(\log n)$.

From Lemma 4 and Theorem 3 we immediately get the following theorem.

**Theorem 6.** *There exists an algorithm that always returns a feasible and (pair) envy-free bundle pricing outcome that $O(\log n \log m)$-approximates the optimal envy-free revenue.*

## 5.3 Single-minded Valuations with Free Disposal

Next, we consider the EFBP-MUL problem when buyers have single-minded valuations with free disposal. As in Section 3.2, single-minded valuations could be described by two parameters $w_i$ and $k_i$. We are interested in revenue maximizing bundle pricing algorithms whose running time are polynomial in $n$ and $\log m$. We first state the NP-hardness of the problem. Then we show a pseudo-polynomial time algorithm and argue how to transfer it to an FPTAS.

**Theorem 7.** *The EFBP-MUL or PEFBP-MUL problem when buyers have single-minded valuations with free disposal is NP-hard but there exists an FPTAS.*

**A Pseudo-polynomial Algorithm** We assume that buyers are sorted by their $k_i$, that is, $k_1 \le \dots \le k_n$. Given buyer $i$, we denote by $L_i$ the set of buyers before $i$ (including $i$) having greater valuations, i.e., $L_i = \{j | j < i \text{ and } w_j > w_i\} \cup \{i\}$. The pesudo-polynomial algorithm largely depends on the following condition for envy-freeness. Consider any two buyers with $k_1 \le k_2$ and $w_1 > w_2$, in any feasible envy-free allocation, if the demand of buyer 2 is satisfied, then the demand of buyer 1 must be also satisfied. In other words, if the seller sells at least $k_i$ items to buyer $i$ at price $p$, it must also sell at least $k_j$ items to every buyer $j$ in $L_i$ at price at most $p$. Otherwise, the envy-freeness is violated. Hence, the pesudo-polynomial algorithm considers one buyer at each round. Given a target revenue, it computes the minimum number of items required in order to achieve an envy-free allocation. If there are not enough items an infinite number is returned.

Denote by $S(i,t)$ the minimum number of items sold to the first $i$ buyers such that the revenue is exactly $t$. Initially we set $S(0,t) = 0$ for $t = 0$, and $S(0,t) = +\infty$ for all $t = 1, \dots, \max_i w_i \cdot n$. To compute $S(i,t)$ for $i = 1, \dots, n$ and $t = 1, \dots, \max_i w_i \cdot n$, we use the recursion:

$$S(i,t) = \min \begin{cases} S(i-1,t); \\ \min_{j<i} \Big( \sum_{j' \in L_{i\setminus j}} k_{j'} + S\big(j, t - w_i|L_{i\setminus j}| + \\ \qquad \max(w_j - w_i, 0)|L_{i\cap j}|\big)\Big); \end{cases}$$

where $L_{i\setminus j} = L_i \setminus L_j$ and $L_{i\cap j} = L_i \cap L_j$.

The first formula in the recursion describes the case when the buyer $i$ receives nothing. The second formula is more involved. First, when the seller decides to satisfy the demand of a buyer $i$, by the envy-freeness, it must also satisfy the demands of all buyers in $L_i$. Let $j$ be the last buyer before buyer $i$ such that the seller decides to sell. To ensure envy-freeness, the seller must have at least $\sum_{j'\in L_{i\setminus j}} k_{j'}$ items to fulfill the demand of these extra buyers. Now let us turn to the revenue. Each buyers in $L_{i\setminus j}$ gives the seller a revenue of $w_i$. In addition, for buyers in $L_{i\cap j}$, it is possible that the seller gets less revenue from these buyers when $w_i$ is smaller than $w_j$. Therefore, in order to extract target revenue $t$, the sell must receive a revenue of at least $t - w_i|L_{i\setminus j}| + \max(w_j - w_i, 0)|L_{i\cap j}|$ from the first $j$ buyers. The optimal revenue $r$ is given as the maximum $t$ such that $S(n,t) \le m$.

**FPTAS** The algorithm given above is polynomial in $W$. We show that applying the standard technique allows us to compute a nearly optimal revenue in time that is polynomial in $nand1/\epsilon$, where $\epsilon$ is the error bound. Let $K = \epsilon W/n$. Set $w'_i = \lfloor w_i/K \rfloor$, and call the obtained problem with $w'_i$ the *scaled* problem. Now using the dynamic programming from the previous section, we can compute the optimal revenue for the scaled problem. The running time of the dynamic programming is $O(n^3 \cdot \lfloor W/K \rfloor) = O(n^4/\epsilon)$. Let $\mathbf{X}$ and $\mathbf{P}$ be the optimal allocation and payment in the original problem, and $\mathbf{X}'$ and $\mathbf{P}'$ the optimal allocation and payment in the scaled problem. For each buyer, because the rounding down, we can charge buyer $i$ price $K \cdot p'(x'_i)$ in the original problem. Since buyers are single-minded with free disposal, the envy-freeness in the scaled problem implies the envy-freeness in the original problem. The revenue is bounded by:

$$r(\mathbf{X}', \mathbf{P}') \ge r(\mathbf{X}, \mathbf{P}) - nK = r(\mathbf{X}, \mathbf{P}) - \epsilon W \ge (1-\epsilon) r(\mathbf{X}, \mathbf{P}).$$

## 5.4 Non-decreasing Valuations

Finally, we consider a variant of the EFBP-MUL problem in which buyers have non-decreasing valuations. In this section, we present a simple and intuitive algorithm that extracts at least $\Omega(\frac{1}{\log n})$ fraction of the optimal revenue. Essentially, Algorithm 2 only sells the bundles of a particular size by setting high prices for other bundles. Denote $\bar{n}(i,j)$ be the number of buyers who have value at least $v_i(j)$ for $j$ items. Given $i$ and $j$, let $r = v_i(j) \cdot \bar{n}(i,j)$. Under the supply constraint, Algorithm 2 finds the optimal $\hat{i}$ and $\hat{j}$ that maximize $r$ and sells $\hat{j}$

**Algorithm 2:** A $O(\log n)$ approximation algorithm for non-decreasing valuations.

1 Compute $\hat{i}$ and $\hat{j}$ such that $r$ is maximized and $\hat{j} \cdot \bar{n}(\hat{i}, \hat{j}) \leq m$;
2 For any $j \in \{1, \ldots, m\}$, set $p(j) = +\infty$ except $p(\hat{j}) = v_{\hat{i}}(\hat{j})$;
3 For any $i \in \{1, \ldots, n\}$, if $v_i(j) \geq v_{\hat{i}}(\hat{j})$, set $x_i = \hat{j}$. Otherwise, set $x_i = 0$.

items to buyers having at least $v_i(\hat{j})$ for $\hat{j}$ items at price $v_{\hat{i}}(\hat{j})$.

**Theorem 8.** *Algorithm 2 always returns a feasible and (pair) envy-free bundle pricing outcome that $O(\log n)$-approximates the optimal envy-free revenue.*

## 6 Conclusion and Future Work

To the best of our knowledge this is the first paper that investigates revenue maximizing envy-free pricing without budgets in multi-unit settings. We were able to demonstrate the revenue difference on two different notions of envy-freeness, and shed some light on the capabilities and limitations of both notions of envy-freeness for general, non-decreasing and single-minded valuations. Our paper should be seen as opening this research direction and it leaves a few open problems for further research, i.e., to close the lower and upper bounds of revenue maximizing envy-free (resp. pair envy-free) item or bundle pricing in different settings.

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